MATH 148 Fall 2006 Lecture Notes, Friday, 17 November 2006

Cohen, 8.4, Trigonometric Equations

Example: Find all real numbers x, $0 \le x < 2\pi$, for which $2\sin^2 13\,x = 1$.

The left hand side (LHS) is $2(\sin 13\,x)^2$. Before we can find values for x we need to know what values $\sin 13\,x$ can have. To do this, set $z = \sin 13\,x$. Then the equation to be solved becomes

$$2z^2 = 1\,.$$

We can rewrite this equation as $2z^2 - 1 = 0$ and solve by factoring using the AC-method or by using the quadratic formula. Using the quadratic formula for $az^2 + bz + c = 0$ gives us two solutions, namely

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0 \pm \sqrt{0^2 - 4(2)(-1)}}{2(2)} = \frac{\pm\sqrt{4(2)}}{4} = \frac{\pm 2\sqrt{2}}{4} = \pm\frac{\sqrt{2}}{2}\,.$$

Now we have *two* equations to solve:

$$\sin 13\,x = +\frac{\sqrt{2}}{2}\,,$$
$$\sin 13\,x = -\frac{\sqrt{2}}{2}\,.$$

So the question now is, what to do with the $13\,x$?

Before worrying about x, what about considering a simpler, related pair of equations? Set $y = 13\,x$ and then consider the pair of equations

$$\sin y = +\frac{\sqrt{2}}{2}\,,$$
$$\sin y = -\frac{\sqrt{2}}{2}\,.$$

But these two equations we already know how to solve. There are two numbers in the interval $0 \le y < 2\pi$ that satisfy the first equation, namely $y = \pi/4$ and $y = 3\pi/4$ coming from the $45 - 45 - 90$ degree right triangle, one of our friendly triangles. (Recall that the other friendly triangle is the $30 - 60 - 90$ degree right triangle.) For the second equation there are two more numbers that satisfy it, namely $y = 5\pi/4$ and $y = 7\pi/4$, corresponding to angles $180 + 45$ degrees and $270 + 45$ degrees.

Since the sine function is periodic with period 2π, what we really have is *four* families of solutions to these two equations in y, namely, where k is any integer,

$$y = \frac{\pi}{4} + 2k\pi\,,$$
$$y = \frac{3\pi}{4} + 2k\pi\,,$$
$$y = \frac{5\pi}{4} + 2k\pi\,,$$
$$y = \frac{7\pi}{4} + 2k\pi\,.$$

But $y = 13\,x$. Which of these values for y correspond to x values in the interval $0 \le x < 2\pi$? First, *all* the x values satisfying the pair of equations, hence satisfying the original equation $2\sin^2 13\,x = 1$, are given by

$$\begin{aligned} 13\,x &= \frac{\pi}{4} + 2k\pi\ , \\ 13\,x &= \frac{3\pi}{4} + 2k\pi\ , \\ 13\,x &= \frac{5\pi}{4} + 2k\pi\ , \\ 13\,x &= \frac{7\pi}{4} + 2k\pi\ , \end{aligned}$$

where k is any integer. Thus *all* solutions to $2\sin^2 13\,x = 1$ are given by

$$\begin{aligned} x &= \frac{1}{13}\left(\frac{\pi}{4} + 2k\pi\right)\ , \\ x &= \frac{1}{13}\left(\frac{3\pi}{4} + 2k\pi\right)\ , \\ x &= \frac{1}{13}\left(\frac{5\pi}{4} + 2k\pi\right)\ , \\ x &= \frac{1}{13}\left(\frac{7\pi}{4} + 2k\pi\right)\ , \end{aligned}$$

where k is any integer.

To see which of these values for x lie within the interval $[0, 2\pi)$, look at the first family

$$x = \frac{1}{13}\left(\frac{\pi}{4} + 2k\pi\right) = \frac{\pi}{13(4)} + \frac{2k\pi}{13} = \frac{\pi}{52} + \frac{2k\pi}{13}\,.$$

The period of $A\sin B\,x$ is $2\pi/B$, so the period of $\sin 13\,x$ is $2\pi/13$. That means there are thirteen full periods of $f(x) = \sin 13\,x$ in the interval $0 \le x < 2\pi$. This means that the first solution family gives us thirteen numbers in the interval $[0, 2\pi)$ that satisfy the original equation, namely

$$x = \frac{\pi}{52}\,,\ \frac{\pi}{52} + \frac{2(1)\pi}{13}\,,\ \frac{\pi}{52} + \frac{2(2)\pi}{13}\,,\ \cdots\,,\ \frac{\pi}{52} + \frac{2(12)\pi}{13}\,.$$

Proceding with this line of reasoning with each of the remaining three families of solutions gives $3 \times 13 = 39$ more numbers in the interval $[0, 2\pi)$ that satisfy the original equation, for a grand total of $4 \times 13 = 52$ values for x that satisfy $2\sin^2 13\,x = 1$.

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