MATH 148 Fall 2006 Lecture Notes, Friday, 6 October 2006

Limits

What does "$\lim_{x \to a} f(x) = L$" mean?

What does "The number L is the limit of the function $f(x)$ as x goes to a" mean?

Answer "For any prescribed, given (positive) distance, if x is near enough to a, then the distance between $f(x)$ and the number L is less than the given distance."

So, if the number L is 15 and the given distance is $1/10$, then the distance between $f(x)$ and the number 15 is supposed to be less than 0.1, if x is near enough to a. This means that $f(x)$ is between 14.9 and 15.1, namely $14.9 < f(x) < 15.1$. Written differently this says that $|f(x) - 15| < 0.1$.

Note that "near enough" means that x is only near a, not actually equal to a, that is, $x \neq a$.

Now the distance between $f(x)$ and the number L may be written as $|f(x) - L|$. So, if the prescribed, given distance is some number d, then we have that $|f(x) - L| < d$ or $L - d < f(x) < L + d$. That means that $f(x)$ is between the two numbers $L - d$ and $L + d$, that $f(x)$ lies in the open interval $(L - d, L + d)$.

So now what does "x is near enough to a" mean?

This just means that the number x is in some (possibly small?) interval around the number a. For example, it might be the case that x is near enough to a when the distance between x and a is less than 3, say. So it might be the case that if x is in the open interval $(a - 3, a + 3)$, then the distance between $f(x)$ and the number L is less than the given distance. Note this means that for *any* x between $a - 3$ and $a + 3$ that is different from a the distance between $f(x)$ and the number L would be less than the given distance. More generally, if ℓ is the length of the interval around a for which x is near enough to a, then we have $0 < |x - a| < \ell$. This says that x is near enough to a if either $a - \ell < x < a$ or if $a < x < a + \ell$. For any such x, if L is the limit of $f(x)$ as x goes to a, then the distance between $f(x)$ and the number L will be less than the prescribed, given distance.

For one-sided limits, such as limits from the left, we have a slightly different understanding of what "near enough" means. What the one-sided limit

$$\lim_{x \to a-} f(x) = L$$

is telling us is that the number L is the limit of the values of the function $f(x)$ when x is near a, *but less than* a. Here the minus ($-$) sign is telling us that x is to the left of (more negative than) the number a.

So suppose that all numbers that are within $\frac{1}{20}$-th of a are near enough, *as long as they are less than* a. That would mean that x would be near enough if $a - 0.05 < x < a$, that is, if x were in the open interval $(a - 0.05, a)$. This set is just "half" of that for the "normal" (two-sided) limit notion, namely the left half.

Taking limits from the right is handled similarly. The notation is $\lim_{x \to a+} f(x) = L$, with the plus ($+$) sign telling us that we are only interested when x is to the right of a, namely only when $a < x$.

What about infinite limits?

That is, suppose that there is no (finite) real number L that is the limit of $f(x)$ as x goes to a. What does

$$\lim_{x\to a} f(x) = \infty$$

mean ?

Answer For any given (prescribed) number M whatsoever, if x is near enough to a then $f(x)$ is greater than M. So, for example, if the given number is 11 or 1000 or whatever, then $f(x)$ will be greater than 11 or 1000 or whatever the number M is, as long as x is near enough to a.

Example $\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$

Remarks

(1) The notion of a one-sided limit is handled here in the same way as before, namely the notion of x being near enough is treated in the same way. For example, the one-sided limit from the right is indicated with a plus sign, such as $\lim_{x\to 1+} \frac{1}{(x-1)} = \infty$, telling us that only values of x greater than 1 are being considered.

(2) The notion of minus infinity as the value of a limit, namely, $\lim_{x\to a} f(x) = -\infty$, just means that $f(x)$ is less than any given number, as long as x is near enough to a. For example, $\lim_{x\to 1} \frac{-1}{(x-1)^2} = -\infty$.

Some Limit Facts

(1) If $f(x) \le g(x)$ for x near a, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

(2) (*Squeeze Theorem*) If $f(x) \le h(x) \le g(x)$ for x near a and if $L = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then $\lim_{x\to a} h(x) = L$.

(3) (*Direct Substitution for Polynomials*) If $P(x)$ is a polynomial, then $\lim_{x\to a} P(x) = P(a)$.

(4) (*Direct Substitution for Rational Functions*) If $f(x) = P(x)/Q(x)$, P and Q are polynomials, and a is in the domain of f, that is, if $Q(a) \neq 0$, then $\lim_{x\to a} f(x) = f(a) = P(a)/Q(a)$.

Two 0/0 Examples

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(1) $\frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x+2$ implies $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} x+2 = 4$.

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(2) $\frac{\sqrt{x+5}-3}{x-4} = \frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} = \frac{(x+5)-9}{(x-4)(\sqrt{x+5}+3)} = \frac{1}{\sqrt{x+5}+3}$ so $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{4+5}+3} = \frac{1}{6}$.

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