NUMERICAL SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATION

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ABSTRACT. In this article, I attempt to provide a systematic framework for an understanding of the numerical solution of linear (or nonlinear) stochastic differential equations. After that, I will try to use parallel computer to get some numerical solutions of the some classical models and compare different arithmetic with those equations too.

1. INTRODUCTION

Firstly, in order to give the reader a general idea what is stochastic equation and why it is useful, let us mention some situations where such equations appear and can be used [?]:

If we allow for some randomness in some of the coefficients of a differential equation we often obtain a more realistic mathematical model of the situation.

Problem 1: Consider the simple population growth model

$$\frac{dN}{dt} = a(t)N(t),\ N(0) = N_0\ (constant) \tag{1}$$

where N(t) is the size of the population at time t, and a(t) is the relative rate of growth at time t. It might happen that a(t) is not completely known, but subject to some random environmental effects, so that we have

$$a(t) = r(t) + "noise",$$

where we do not know the exact behaviour of the noise term, only its probability distribution. The function r(t) is assumed to be non-random. How do we solve (1) in this case?

Problem 2: Suppose a person has an asset or resource (e.g. a house, stocks, oil...) that she is planning to sell. The price X_t at time t of her asset on the open market varies according to a stochastic differential equation of the same type as in Problem 1:

$$\frac{dX_t}{dt} = rX_t + \alpha X_t \cdot "noise" \tag{2}$$

where r,α are known constants, The discount rate is a known constant p. At what time should she decide to sell?
We assume that she knows the behaviour of X_s up to the present time t, but because of the noise in the system she can not course never be sure at time of the sale if her choice of time will turn out to be the best. So what we are searching for is a stopping strategy that gives the best result in

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the long run, i.e. maximizes the expected profit when the inflation is taken into account.

More generally, the equation we obtain by allowing randomness in the coefficients of a differential equation is called a *stochastic differential equation.* This will be made more precise later. It is clear that any solution of a stochastic differential equation must involve some randomness, i.e. we can only hope to be able to say something about the probability distributions of the solutions.

2. Construction of the *Itô* Integral

We now turn to the question of finding a resonable mathematical interpretation of the "noise" term in the equation of Problem 1 in the Introduction:

$$\frac{dN}{dt} = (r(t) + \text{"}noise\text{"})N(t) \tag{3}$$

or more generally in equations of the form

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{"}noise\text{"}, \tag{4}$$

where b and σ are some given functions. Let us first concentrate on the case when the noise is 1-dimensional. It is reasonable to look for some stochastic process W_t to represent the noise term, so that

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot W_t, \tag{5}$$

Based on many situations, for example in engineering, one is let to assume that W_t hav, at least approximately, these properties:

(i) $t_1 \neq t_2 \Rightarrow \ W_{t_1}\ and\ W_{t_2}$ are independent.
(ii) W_t is stationary, i.e. the (joint) distribution of $W_{t_1+t}, \ldots, W_{t_k+t}$ does not depend on t.
(iii) E[W_t]=0 for all t.

However, it turns out there does not exist any "reasonable" stochastic process satisfying (i) and (ii): Such a W_t can not have continuous paths. Nevertheless it is possible to represent W_t as a generalized stochastic process called the *white noise process.* In my project, according the above three properties, I can use computer to simulate white noise and after that simulate the numerical solution of the Stochastic Differential Equation.

References

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