Theorem 3. Given two lines and a transversal. If a pair of alternate interior angles are congruent, then the lines are parallel.

The proof is exactly like that of Theorem 1.

In the figure below, $\angle 1$ and $\angle 1'$ are corresponding angles, $\angle 2$ and $\angle 2'$ are corresponding angles, and so on.

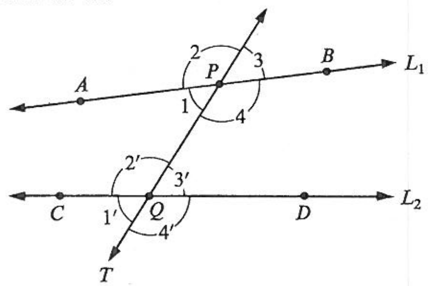

FIGURE 10.5

DEFINITION. If $\angle x$ and $\angle y$ are alternate interior angles, and $\angle y$ and $\angle z$ are vertical angles, then $\angle x$ and $\angle z$ are corresponding angles.

Theorem 4. Given two lines and a transversal. If a pair of corresponding angles are congruent, then a pair of alternate interior angles are congruent.

Theorem 5. Given two lines and a transversal. If a pair of corresponding angles are congruent, then the lines are parallel.

10.2 THE POLYGONAL INEQUALITY

The triangular inequality states that for any triangle ABC we have

$$AB + BC > AC.$$

If A, B, and C are not required to be noncollinear or even different, we get a weaker result.

Theorem 1. For any points A, B, C,

$$AB + BC \geqq AC.$$

Proof. If A, B, and C are noncollinear, this follows from the triangular inequality. If A, B, and C are collinear, we take a coordinate system on the line that contains them and let their coordinates be x, y, and z. Let

$$a = x - y, \qquad b = y - z.$$

By Theorem 12, Section 1.4, we know that

$$|a| + |b| \geqq |a + b|,$$

therefore

$$|x - y| + |y - z| \geqq |x - z|.$$

Hence

$$AB + BC \geqq AC,$$

which was to be proved. From this we get the following theorem.

Theorem 2. *The Polygonal Inequality.* If $A_1, A_2, \ldots, A_n$ are any points ($n > 1$), then

$$A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n \geqq A_1A_n.$$

The proof is by induction.

We shall need this result in the following section. For the first time, we are also about to use the Archimedean postulate for the real number system, given in Section 1.8. This says that if $e > 0$ and $M > 0$, then $ne > M$ for some positive integer n.

10.3 SACCHERI QUADRILATERALS

We recall, from Section 4.4, the definition of a quadrilateral. Given four points A, B, C, and D, such that they all lie in the same plane, but no three of them are collinear. If the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ intersect only at their end points, then their union is called a *quadrilateral*, and is denoted by $\square ABCD$. The segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are the *sides* of $\square ABCD$, and the segments $\overline{AC}$, $\overline{BD}$ are the *diagonals*. The *angles of* $\square ABCD$ are $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$; they are often denoted briefly as $\angle B$, $\angle C$, $\angle D$, $\angle A$. If all four of the angles are right angles, then the quadrilateral is a *rectangle*.

On the basis of the postulates that we have so far, without the use of the parallel postulate, it is impossible to prove that any rectangles exist. If we try, in a plausible fashion, to construct a rectangle, we get what is called a *Saccheri quadrilateral*.

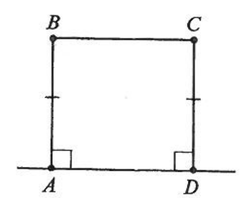

FIGURE 10.6

The definition is suggested by the markings on the figure above. To be precise, $\square ABCD$ is a *Saccheri quadrilateral* if $\angle A$ and $\angle D$ are right angles and $AB = CD$. The segment $\overline{AD}$ is called the *lower base;* and $\overline{BC}$ is called the *upper base*. The *lower base* angles are $\angle A$ and $\angle D$; and $\angle B$ and $\angle C$ are the *upper base* angles.

Theorem 1. The diagonals of a Saccheri quadrilateral are always congruent.

Proof. By SAS, we have $\triangle BAD \cong \triangle CDA$. Therefore $\overline{BD} \cong \overline{AC}$.

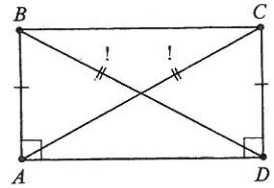

FIGURE 10.7

Roughly speaking, the following theorem states that a Saccheri quadrilateral is completely described, geometrically, by the distances AD and AB.

Theorem 2. Let $\square ABCD$ and $\square A'B'C'D'$ be Saccheri quadrilaterals, with lower bases $\overline{AD}$ and $\overline{A'D'}$. If $\overline{A'D'} \cong \overline{AD}$ and $\overline{A'B'} \cong \overline{AB}$, then $\overline{BC} \cong \overline{B'C'}$, $\angle B' \cong \angle B$ and $\angle C' \cong \angle C$.

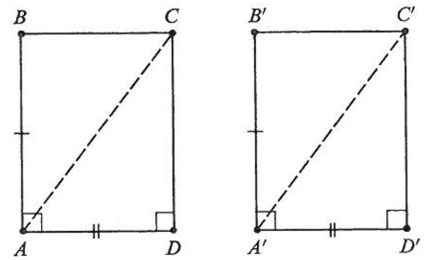

FIGURE 10.8

Proof. The main steps in the proof are as follows.

(1) $\triangle ACD \cong \triangle A'C'D'$ (by SAS).
(2) $\angle A \cong \angle A'$ (all right angles are congruent).
(3) $\angle BAC \cong \angle B'A'C'$.
(4) $\overline{AC} \cong \overline{A'C'}$.
(5) $\triangle ABC \cong \triangle A'B'C'$.
(6) $\angle B \cong \angle B'$.
(7) $\overline{BC} \cong \overline{B'C'}$.
(8) $\angle C \cong \angle C'$.

Applying this theorem to the Saccheri quadrilaterals $\square ABCD$, $\square DCBA$, we get $\angle B \cong \angle C$. Thus we have the following theorem.

Theorem 3. In any Saccheri quadrilateral, the upper base angles are congruent.

Theorem 4. In any Saccheri quadrilateral, the upper base is congruent to or longer than the lower base.

RESTATEMENT. Given a Saccheri quadrilateral $\square A_1B_1B_2A_2$, with lower base $\overline{A_1A_2}$. Then $B_1B_2 \geqq A_1A_2$.

Proof. Let us set up a sequence of n Saccheri quadrilaterals, end to end, starting with the given one, like this:

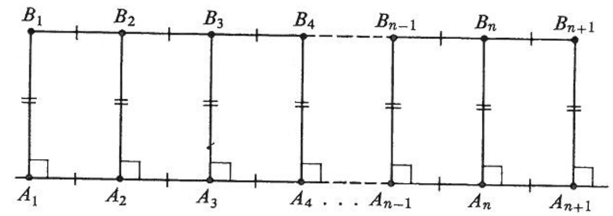

FIGURE 10.9

That is, $A_3, A_4, \ldots, A_{n+1}$ are points of the line $\overleftrightarrow{A_1A_2}$, appearing in the stated order on $\overleftrightarrow{A_1A_2}$; the angles $\angle B_2A_2A_3$, $\angle B_3A_3A_4, \ldots$ and so on are right angles;

$$A_1A_2 = A_2A_3 = A_3A_4 = \cdots = A_{n-1}A_n = A_nA_{n+1},$$

and

$$A_2B_2 = A_3B_3 = \cdots = A_nB_n = A_{n+1}B_{n+1}.$$

By Theorem 2, we have

$$B_1B_2 = B_2B_3 = \cdots = B_{n-1}B_n = B_nB_{n+1}.$$

We don't happen to know anything about the question of collinearity for the points $B_1, B_2, \ldots, B_{n+1}$. But we know by the polygonal inequality that

$$B_1B_{n+1} \leqq B_1B_2 + B_2B_3 + \cdots + B_{n-1}B_n + B_nB_{n+1}.$$

Since all of the distances on the right are $= B_1B_2$, we have

$$B_1B_{n+1} \leqq n \cdot B_1B_2.$$

By the same principle, we get

$$A_1A_{n+1} \leqq A_1B_1 + B_1B_{n+1} + B_{n+1}A_{n+1} \leqq A_1B_1 + nB_1B_2 + A_1B_1.$$

Since $A_1A_{n+1} = nA_1A_2$, we have

$$nA_1A_2 \leqq nB_1B_2 + 2A_1B_1,$$

and this conclusion holds *for every n*.

Now suppose that our theorem is false. Then $A_1A_2 > B_1B_2$, so that $A_1A_2 - B_1B_2$ is a positive number. Obviously, $2A_1B_1$ is a positive number. Let

$$e = A_1A_2 - B_1B_2, \quad \text{and} \quad M = 2A_1B_1.$$

Then $e > 0$ and $M > 0$, but $ne \leqq M$ for every positive integer n. This contradicts the Archimedean postulate, and so completes the proof.