CHAPTER 9

THREE GEOMETRIES

9.1 INTRODUCTION

This chapter is purely informal; it does not form a part of the deductive sequence of the rest of the book. In fact, in this chapter we shall not prove anything at all; but everything that we discuss will be taken up more fully later. It may be of some help, however, to sketch in advance the kinds of geometry to which our theorems are going to apply.

For the sake of simplicity, we shall limit ourselves to geometry in a plane. The ideas that we shall discuss can be generalized to three dimensions, but only at the cost of considerable labor.

Two lines are called *parallel* if they lie in the same plane but do not intersect. In a *Euclidean* plane, the familiar parallel postulate holds.

The Euclidean Parallel Postulate. Given a line L and a point P not on L, there is one and only one line L' which contains P and is parallel to L.

This says that parallels always exist and are always unique.

For quite a while—for a couple of millennia, in fact—this proposition was regarded as a law of nature. In the nineteenth century, however, it was discovered by Lobachevski, Bolyai, and Gauss that you could get a perfectly consistent mathematical theory by starting with a postulate which states that parallels always exist, but denies that they are unique.

The Lobachevskian Parallel Postulate. Given a line L and a point P not on L, there are at least two lines L', L'' which contain P and are parallel to L.

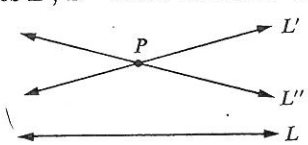


FIGURE 9.1

The picture looks implausible, because we are accustomed to thinking of the plane of the paper as Euclidean. But it is a fact, as we shall see, that a mathematical theory can be based on Lobachevski's postulate. And such a theory actually has applications in physics.

There is yet a third alternative. We can deny not the uniqueness of parallels but their existence.

The Riemannian Parallel Postulate. No two lines in the same plane are ever parallel.

These postulates give us three kinds of "plane geometry," the Euclidean, the Lobachevskian, and the Riemannian. In each of the three theories, of course, many other postulates are needed; we have merely been singling out their crucial difference. In this book, we shall be concerned mainly with the first of these geometries, incidentally with the second, and hardly at all with the third. In the following sections, we give concrete examples, or *models*, of these kinds of geometry, and indicate the most striking differences between them. In going through the rest of this book, you should have one of these models in mind most of the time; and at some points you should have in mind two of them.

9.2 THE POINCARÉ MODEL FOR LOBACHEVSKIAN GEOMETRY

In this section we shall assume that there is a mathematical system satisfying the postulates of Euclidean plane geometry, and we shall use Euclidean geometry to describe a mathematical system in which the Euclidean parallel postulate fails, but in which the other postulates of Euclidean geometry hold.

Consider a fixed circle C in a Euclidean plane. We assume, merely for the sake of convenience, that the radius of C is 1. Let E be the interior of C.

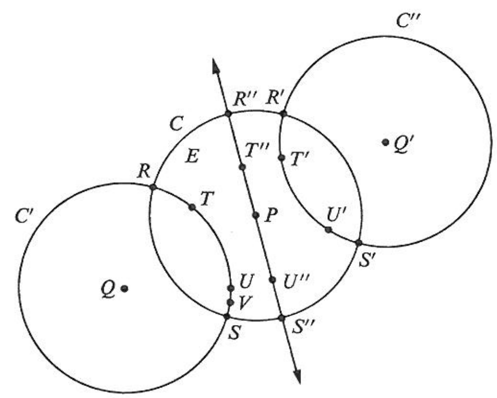


FIGURE 9.2

By an *L-circle* (L for Lobachevski) we mean a circle C' which is orthogonal to C. When we say that two circles are orthogonal, we mean that their tangents at each intersection point are perpendicular. If this happens at one intersection point R, then it happens at the other intersection point S. But we shall not stop to prove this, or, for that matter, to prove anything else; this chapter is purely descriptive and proofs will come later.

The *points* of our L-plane will be the points of the interior E of C. By an *L-line* we mean (1) the intersection of E and an L-circle, or (2) the intersection of E and a diameter of C.

It is a fact that

I-1. Every two points of E lie on exactly one L-line.

We are going to define a kind of "plane geometry," in which the "plane" is the set E and the lines are the L-lines. In our new geometry we already know what is meant by *point* and *line*. We need next to define distance and angular measure.

For each pair of points X, Y, either on C or in the interior of C, let XY be the usual Euclidean distance. Notice that if R, S, T and U, are as in the figure, then R and S are *not* points of our L-plane, but they *are* points of the Euclidean plane that we started with. Therefore, all of the distances TS, TR, US, UR are defined, and (I) tells us that R and S are determined when T and U are named. There is one and only one L-line through T and U, and this L-line cuts the circle C in the points R and S. We shall use these four distances TS, TR, US, UR to define a new distance $d(T, U)$ in our "plane" E, by the following formula:

$$d(T, U) = \left|\log_e \frac{TR/TS}{UR/US}\right|.$$

Evidently we have the following postulate.

D-0. d is a function

$$d: E \times E \to \mathbf{R}.$$

Let us now look at the ruler postulate D-4. On any L-line L, take a point U and regard this point as fixed. For every point T of L, let

$$f(T) = \log_e \frac{TR/TS}{UR/US}.$$

That is, $f(T)$ is what we get by omitting the absolute value signs in the formula for $d(T, U)$. We now have a function,

$$f: L \to \mathbf{R}.$$

We shall show that f is a coordinate system for L.

If V is any other point of L, then

$$f(V) = \log_e \frac{VR/VS}{UR/US}.$$

Let $x = f(T)$ and $y = f(V)$. Then

$$|x - y| = \left|\log_e \frac{TR/TS}{UR/US} - \log_e \frac{VR/VS}{UR/US}\right| = \left|\log_e \frac{TR/TS}{UR/US}\right|,$$

because the difference of the logarithms is the logarithm of the quotient. Therefore

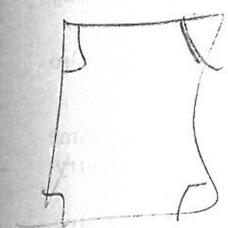
$$|x - y| = d(T, V),$$

which means that our new distance function satisfies the ruler postulate.

Since D-4 holds, the other distance postulates automatically hold. (See Problem 1, Section 3.3.)

We define betweenness, segments, rays, and so on, exactly as in Chapter 3. All of the theorems of Chapter 3 hold in our new geometry, because the new geometry satisfies the postulates on which the proofs of the theorems were based. The same is true of Chapter 4; it is rather easy to convince yourself that the plane-separation postulate holds in E.

To discuss congruence of angles, we need to define an angular-measure function. Given an "L-angle" in our new geometry, we form an angle in the old geometry by using the two tangent rays:



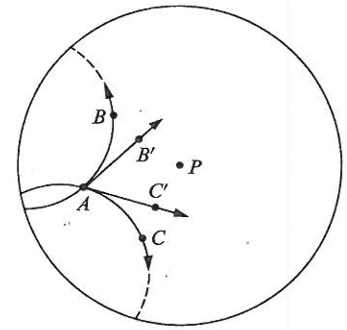


FIGURE 9.3

We then define the measure $m\angle BAC$ of $\angle BAC$ to be the measure (in the old sense) of the Euclidean angle $\angle B'AC'$.

It is a fact that the resulting structure

$$[E, L, d, m]$$

satisfies all the postulates of Chapters 2 through 6, including the SAS postulate. The proof of this takes time, however, and it requires the use of more Euclidean geometry that we know so far. Granted that the postulates hold, it follows that the theorems also hold. Therefore, the whole theory of congruence, and of geometric inequalities, applies to the Poincaré model of Lobachevskian geometry.

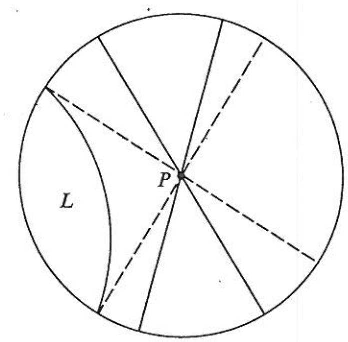


FIGURE 9.4

On the other hand, the Euclidean parallel postulate obviously does not hold for the Poincaré model. Consider, for example, an L-line L which does not pass through the center P of C (Fig. 9.4). Through P there are infinitely many L-lines which are parallel to L.

Lobachevskian geometry (also called *hyperbolic geometry*) is the kind represented by the Poincaré model. In such a geometry, when the familiar parallel postulate fails, it pulls down a great many familiar theorems with it. A few samples of theorems in hyperbolic geometry which are quite different from the analogous theorems of Euclidean geometry follow.

(1) No quadrilateral is a rectangle. In fact, if a quadrilateral has three right angles, the fourth angle is always acute.

(2) For any triangle, the sum of the measures of the angles is always *strictly less* than 180.

(3) No two triangles are ever similar, except in the case where they are also congruent.

The third of these theorems means that two figures cannot have exactly the same shape, unless they also have exactly the same size. Thus, in hyperbolic geometry, *exact scale models are impossible.*

In fact, each of the above three theorems characterizes hyperbolic geometry. If the angle-sum inequality,

$$m\angle A + m\angle B + m\angle C < 180,$$

holds, even for *one* triangle, then the geometry is hyperbolic; and if the angle-sum equality holds, even for *one* triangle, then the geometry is Euclidean; similarly for (1) and (3).

This has a rather curious consequence in connection with our knowledge of physical space. If physical space is hyperbolic, which it may be, it is theoretically possible for the fact to be demonstrated by measurement. For example, suppose that you measure the angles of a triangle, with an error less than $0.0001''$ for each angle. Suppose that the sum of the measures turns out to be $179°59'59.999''$. The difference between this and $180°$ is $0.001''$. This discrepancy could not be due to errors in measurement, because the greatest possible cumulative error is only $0.0003''$. Our experiment would therefore prove that the space that we live in is hyperbolic. (Granted, of course, that it satisfies the other postulates.)

On the other hand, no measurement however exact can prove that space is Euclidean. The point is that every physical measurement involves *some* possible error. Therefore we can never show by measurement that an equation,

$$r + s + t = 180,$$

holds exactly; and this is what we would have to do to prove that the space we live in is Euclidean.

Thus there are two possibilities: (1) The Euclidean parallel postulate does not hold in physical space, or (2) The truth about physical space will forever be unknown.

9.3 THE SPHERICAL MODEL FOR RIEMANNIAN GEOMETRY

Let V be the surface of a sphere in space. We may as well assume that the radius of V is $= 1$. A *great circle* is a circle which is the intersection of V with a plane through its center. If T and U are any points of V, then the shortest path on the surface joining T to U is an arc of a great circle.

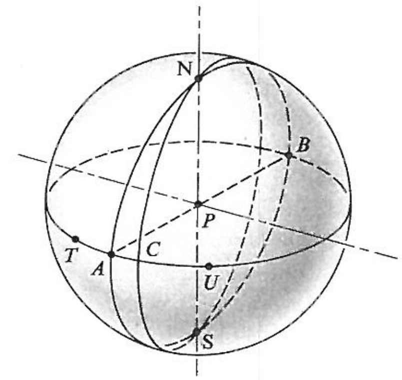


FIGURE 9.5

We might start to define a kind of "plane geometry" on V by taking the great circles as our lines. In this scheme we would take the length of the shortest path between each pair of points as the distance between the two points. The resulting system has some of the properties that we expect in plane geometry. For example, every "line" separates our "plane" into two "half planes," each of which is convex. But the Euclidean parallel postulate fails badly; i.e., every two lines intersect. Our "geometry" has many other peculiar properties.

(1) Two points do not necessarily determine a "line." For example, the north and south poles N and S lie on infinitely many great circles.

The same is true for the end points of any diameter of the sphere V. Such points are called *antipodal.* (More precisely, two points A and B of V are *antipodal* if the segment $\overline{AB}$ passes through the center of V.)

(2) While our "lines" never come to an end at any point, they are nevertheless finite in extent. In fact, if the radius of V is $= 1$, then the maximum possible distance between any two points is π. Thus the ruler postulate cannot possibly hold.

(3) Betweenness, in the form in which we are accustomed to it, collapses completely. In fact, given three points of a line it is not necessarily true that one of them is between the other two. We may have $AB = BC = AC$.

(4) The perpendicular to a line, from an external point, always exists, but is not necessarily unique. For example, any line joining the North Pole to a point of the equator is perpendicular to the equator.

(5) Some triangles have two right angles. (In the Fig. 9.5 at the start of this section, $\triangle ANC$ has right angles at both A and C.)

(6) The exterior angle theorem fails. (See the same example.)

The only one of these peculiarities that we can avoid is the first. We do this by altering the model in the following way. If two points A, B are antipodal, then we shall regard them as being the same. To be more precise, a *point* of our new geometry will be a pair of antipodal points of the sphere V. If A is a point of the sphere V, then $\overline{A}$ denotes the pair $\{A, A'\}$, where A' is the other end of the diameter that contains A. The *points* of our Riemannian plane E will be the pairs $\overline{A}$.

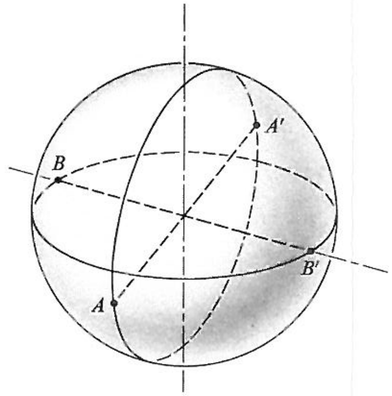


FIGURE 9.6

If L is a great circle on V, then $\overline{L}$ is the set of all points $\overline{A}$ for which A is on L. The sets $\overline{L}$ will be the *lines* in E.

The *distance* $d(\overline{A}, \overline{B})$ between two points $\overline{A}$ and $\overline{B}$ is the length of the shortest arc from A (or A') to B (or B'). Notice that this may easily be less than the length of the shortest arc from A to B.

In our new geometry, two points $\overline{A}$, $\overline{B}$ always determine a unique "line." The reason is that if A and B were antipodal on the sphere, $\overline{A}$ and $\overline{B}$ would be the same.

The Euclidean parallel postulate still fails, of course; two of our new lines always intersect in exactly one point. Lines are still of finite extent; the maximum possible distance between two points is now $\pi/2$. Betweenness still does not work. Perpendiculars still are not unique; we still have triangles with two right angles, and the exterior angle theorem still fails.

In fact, in arranging for two points to determine a line, we have introduced a new peculiarity: no line separates our Riemannian plane. In fact, if $\overline{L}$ is a line, and $\overline{A}$ and $\overline{B}$ are any two points not on $\overline{L}$, then there is always a segment which goes from $\overline{A}$ to $\overline{B}$ without intersecting $\overline{L}$.

In this book, we shall be concerned mainly with Euclidean geometry, but we shall devote considerable attention to hyperbolic geometry, mainly because it throws light on Euclidean geometry. The point is that these two kinds of geometry have so much in common that at the points where they do differ the differences are instructive. On the other hand, the differences between Riemannian and Euclidean geometry are so fundamental that it really forms a technical specialty, which is remote from our main purpose. We shall not be concerned with it hereafter in this book.

9.4 SOME QUESTIONS FOR LATER INVESTIGATION

In this chapter, we have raised more questions than we have answered.

(1) We have said that the Poincaré model for hyperbolic geometry satisfies all of the postulates of Euclidean geometry, with the sole exception of the Euclidean parallel postulate. This needs to be proved, and we surely haven't proved it with our conversational discussion in Section 9.2.

To check these postulates is a rather lengthy job. The reader is warned that this sort of verification is discussed rather casually in much of the literature. If the models for hyperbolic geometry had in common with Euclidean geometry merely the trivial properties that are discussed in semipopular books, they would not have the significance which is commonly and rightly attributed to them.

(2) When the postulates are checked, for the Poincaré model, we will know that hyperbolic geometry is just as good, logically, as Euclidean geometry. We constructed the model on the basis of Euclidean geometry. Therefore, *if* there is a mathematical system satisfying the Euclidean postulates, it follows that there is a system satisfying the Lobachevskian postulates.

(3) There remains the *if* in (2). Is there a system satisfying the Euclidean postulates? To prove this, we need to set up a model. We shall see that this can be done, assuming that the real number system is given.