Chapter 8

Area in Neutral, Euclidean and Hyperbolic Geometry

8.1 Introduction

Up to this time we have not yet defined area. It is a measurement, like distance and angle measure, so it is a function that assigns a real number to a geometric object. We want to see what common properties area functions should have and see how much of that we can study in a neutral setting. We know that we must have some real differences between area in the Euclidean and hyperbolic settings — as we know that rectangles do not exist in the hyperbolic plane and thus *square units* will not be possible.

Euclid uses area from very early in his development of geometry. He states that two triangles are equal when he means that they have the same area. His development of area is then not as explicit as we might want for an axiomatic approach. What would an axiomatic approach to area offer us that we have not already seen?

i) An axiomatic approach shows us that in many ways area, distance, and angle measure are alike.

ii) An axiomatic approach will allow us to give a more unified treatment of area that will apply to both Euclidean and hyperbolic geometries.

iii) By using an axiomatic method we can avoid some of the technical difficulties that arise when we show that the area function is well defined.

iv) An axiomatic approach is consistent with the approach taken in most high school textbooks.

Now, in Hilbert's axiom system there was no Ruler Postulate or Protractor Postulate, but there is a careful development of the necessary results that show that there is a one-to-one correspondence between the lines in geometry and the real line and between the interval $[0, 180)$ and the measurements of angles.

In this treatment of geometry we will deal only with polygonal regions. For other regions we would need to consider a limiting process as in calculus or, as Eudoxus and Archimedes used, a method of exhaustion.

8.2   The Neutral Area Postulate

We need to define the region of which we want to find the area.

Definition 8.1 *Let $\triangle ABC$ be a triangle. The* interior of $\triangle ABC$, *denoted* Int($\triangle ABC$), *is the intersection of the interiors of the three interior angles $\angle ABC$, $\angle BCA$, and $\angle CAB$.*

Now, this gives us the points inside the triangle, but not the points on the triangle itself.

Definition 8.2 *Let $\triangle ABC$ be a triangle. The* associated triangular region *is the subset T of the plane consisting of all points that lie on the triangle or in its interior*

$$T = \triangle ABC \bigcup \text{Int}(\triangle ABC) = \blacktriangle ABC.$$

Note that the triangle and the associated triangular region are not the same thing. A triangular region is the set of points that includes not only the points on the triangle itself but also the points that lie inside the triangle. Intuitively a triangular region is a two-dimensional region with positive area, while the triangle is a one-dimensional object that has zero area - but does have length.

A *polygonal region* is a plane figure which can be expressed as the union of a finite number of triangular regions, in such a way that if two of the triangular regions intersect, their intersection is an edge or a vertex of each of them.

Note that every triangular region is a polygonal region, but not *vice versa.*

Let R be a polygonal region. A *triangulation* of R is a finite collection,

$$K = \{T_1, T_2, \ldots, T_n\}$$

of triangular regions T_i, such that

1. the T_i's intersect only at edges and vertices, and

2. their union is R.

Note that one polygonal region can have may different triangulations.

Definition 8.3 *Two polygonal regions R_1 and R_2 are* nonoverlapping *if $R_1 \bigcap R_2$ consists only of subsets of edges of each. Specifically, if T_1 is one of the triangular regions in R_1 and T_2 is one of the triangular regions in R_2, then $\text{Int}(T_1) \bigcap T_2 = \emptyset$ and $T_1 \bigcap \text{Int}(T_2) = \emptyset$.*

Axiom 7 (The Neutral Area Postulate) *Associated with each polygonal region R there is a positive number $\alpha(R)$, called the area of R, such that the following conditions are satisfied*

    i) (Congruence) If two triangles are congruent, then there associated triangular regions have equal areas.

    ii) (Additivity) If $R - R_1 \bigcup R_2$ is the union of two nonoverlapping polygonal regions, then $\alpha(R) = \alpha(R_1) + \alpha(R_2)$.

Theorem 8.1 *If $\triangle ABC$ is a triangle and E is a point on the interior of AC, then $\blacktriangle ABC = \blacktriangle ABE \cup \blacktriangle EBC$. Furthermore, $\blacktriangle ABE$ and $\blacktriangle EBC$ are nonoverlapping regions.*

8.3 Area in Euclidean Geometry

What choices do we have for area in Euclidean geometry? We have our usual choice for the area of a rectangle, but is it uniquely determined? For this section we will assume the Euclidean Parallel Postulate.

For a rectangle $\square ABCD$ we know that $\square ABCD$ is convex and so the diagonals AC and BD intersect in a point E. This then will allow us to associate a particular polygonal region for each rectangle.

Definition 8.4 *Let $\square ABCD$ be a rectangle and let E be the point of intersection of the diagonals. The* rectangular region *associated with $\square ABCD$ is the polygonal region R is*

$$R = \blacktriangle ABE \cup \blacktriangle BCE \cup \blacktriangle CDE \cup \blacktriangle ADE.$$

The length *of R is AB and the* width *is BC.*

Axiom 8 (Euclidean Area Postulate) *If R is a rectangular region, then*

$$\alpha(R) = length(R) \times width(R).$$

Now, it is easy to determine the area of a triangle.

Theorem 8.2 *If T is the triangular region corresponding to the right triangle $\triangle ABC$ with right angle at C, then $\alpha(T) = \frac{1}{2}(AC \times BC)$.*

Definition 8.5 *Let T be a triangular region corresponding to $\triangle ABC$. The* base *of T is AB. Drop a perpendicular from C to geolineAB and call the foot of that perpendicular D. The* height *of T is the length of CD.*

Theorem 8.3 *The area of a triangular region is one-half the length of the base times the height; that is,*

$$\alpha(T) = \frac{1}{2}base(T) \times height(T).$$

Theorem 8.4 *If two triangles are similar, then the ratio of their areas is the square of the ratio of the lengths of any two corresponding sides; i.e., if $\triangle ABC \sim \triangle DEF$ and $DE = k \cdot AB$, then $\alpha(\triangle DEF) = k^2 \cdot \alpha(\triangle ABC)$.*

8.4 Finite Decomposition

Let R_1 and R_2 be polygonal regions. Suppose that they have triangulations

$$\begin{aligned} K_1 &= \{T_1, T_2, \ldots, T_n\}, \\ K_2 &= \{T_1', T_2', \ldots, T_n'\}, \end{aligned}$$

such that for each i we have $T_i \cong T_i'$. Then we say that R_1 and R_2 are *equivalent by finite decomposition*, and we write $R_1 \equiv R_2$.

The process of decomposing a region into triangles and reassembling them to form a different region is usually called *dissection*. The main problem is:

Dissection Problem: *Given two regions R and $R\prime$ with $\alpha(R) = \alpha(R\prime)$, find triangulations T and $T\prime$ which show that $R \equiv R\prime$.*

Theorem 8.5 (Fundamental Theorem of Decomposition Theory) *If R and $R\prime$ are two polygonal regions such that $\alpha(R) = \alpha(R\prime)$, then $R \equiv R\prime$.*

This seems to be obvious, but it is not as simple as it seems at first. Max Dehn proved in 1902 that it is not possible to cut a tetrahedron into a finite number of polyhedral pieces and reassemble them to form a cube of equal volume. Thus, the analogous statement for three-dimensions is not true.

Theorem 8.6 *Equivalence by finite decomposition is an equivalence relation:*

i) *(Reflexive) $R \equiv R$ for every polygonal region R.*

ii) *(Symmetric) If $R_1 \equiv R_2$, then $R_2 \equiv R_1$.*

iii) *(Transitive) If $R_1 \equiv R_2$ and $R_2 \equiv R_3$, then $R_1 \equiv R_3$.*

PROOF: Reflexivity and symmetry are clear from the definition.

Suppose that R_1, R_2 and R_3 are polygonal regions such that $R_1 \equiv R_2$ and $R_2 \equiv R_3$. We need to show that $R_1 \equiv R_3$. Now the problem is that there may be different decompositions that give the equivalence of R_2 to R_1 and R_2 to R_3. Say that the triangles in the decomposition for the first equivalence are $T_1, \ldots, T_n$ and those for the second equivalence are $T_1\prime, \ldots, T_m\prime$. In order to get one common subdivision for R_2 into regions of the form $T_1 \bigcap T_j\prime$ where T_i and $T_j\prime$ overlap. Each is either a triangle or a quadrilateral region. If it is the latter, we would decompose it into triangles.
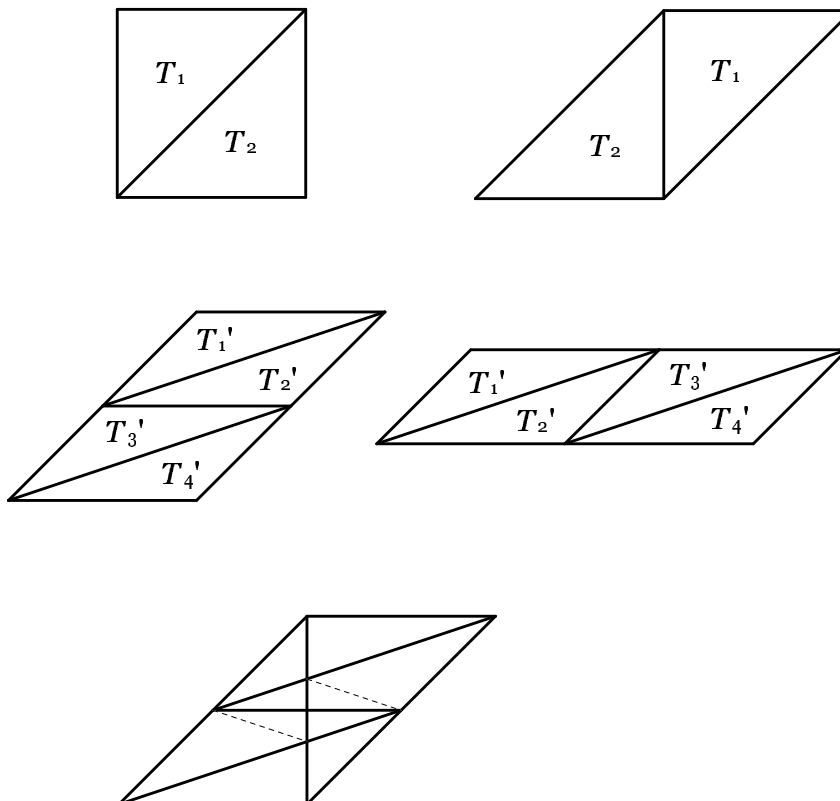
The result is a subdivision of R_2 into small triangles such that each small triangle $T_k\prime\prime$ is contained in both a T_i and a $T_j\prime$. Since $T_k\prime\prime \subset T_i$ there is a corresponding congruent

triangle in R_1. Since $T_k'' \subset T_j'$ there is a corresponding congruent triangle in R_2. In this way the triangles T_k'' induce triangulations of R_1 and R_3 that show $R_1 \equiv R_3$. ∎

Given $\triangle ABC$, with AB considered as the base. Let M and N be the midpoints of AC and BC, respectively. Let D, E, and F be the feet of the perpendiculars from B, A, and C, respectively, to $\ell = \overleftrightarrow{DE}$. As you will prove in the homework, $\square ABDE$ is a Saccheri quadrilateral. It is known as the quadrilateral *associated* with $\triangle ABC$. It depends on the choice of the base, but it should be clear which base we mean.
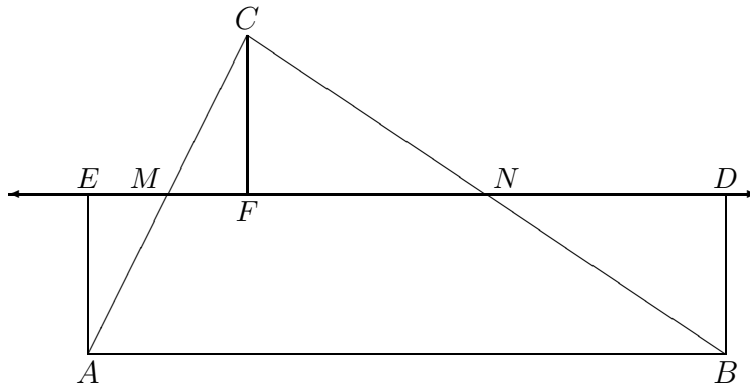

Figure 8.1: The Saccheri quadrilateral *associated* with $\triangle ABC$

Theorem 8.7 *Every triangular region is equivalent by finite decomposition to its associated Saccheri quadrilateral region.*

PROOF: We will look at the special case given in Figure 8.1. There are other cases that should be considered, but we will not undertake it here. Our special case is the situation where F lies between M and N. The more general case uses an intermediate parallelogram, but this case will give us a good feeling for why this should be true.

Let R be the quadrilateral region corresponding to $\square ABNM$, let $T_1 = \blacktriangle CMF$, $T_2 = \blacktriangle CNF$, $T_1' = \blacktriangle AME$, and $T_2' = \blacktriangle BND$. Then

$$\triangle ABC = R \bigcup T_1 \bigcup T_2 \text{ and } \square ABDE = R \bigcup T_1' \bigcup T_2'.$$

Since $T_1 \cong T_1'$, and $T_2 \cong T_2'$, it follows that $\triangle ABC \equiv \square ABDE$. ∎

Theorem 8.8 *Let $\triangle ABC$ be a triangle and let $\square ABDE$ be its associated Saccheri quadrilateral. If H is a point so that AH crosses $\overleftrightarrow{DE}$ at the midpoint of AH, then $\square ABDE$ is also the Saccheri quadrilateral associated with $\triangle ABH$.*

8.5 Finite Decomposition in Euclidean Geometry

We will prove the Fundamental Theorem of Decomposition Theory (Theorem 8.5) in Euclidean geometry. We will do this in such a way that we might see a method to construct a similar proof in hyperbolic geometry. Again, we will assume the Euclidean Parallel Postulate for this section.

First, let's restate the theorem.

Theorem 8.5: [**Fundamental Theorem of Decomposition Theory**] *If R and $R\prime$ are two polygonal regions such that $\alpha(R) = \alpha(R\prime)$, then $R \equiv R\prime$.*

We will approach the proof of this in three steps.

Step 1. Show that if $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\alpha(\triangle ABC) = \alpha(\triangle DEF)$ and $AB \cong DE$, then $\triangle ABC \equiv \triangle DEF$.

Step 2. Use Step 1 to show that if $\triangle ABC$ and $\triangle DEF$ are any two triangles such that $\alpha(\triangle ABC) = \alpha(\triangle DEF)$, then $\triangle ABC \equiv \triangle DEF$.

Step 3. Use Step 2 and an inductive argument to show that if $\alpha(R_1) = \alpha(R_2)$, then $R_1 \equiv R_2$.

First we will restate Theorem 8.7 in Euclidean geometry, in which a Saccheri quadrilateral is a rectangle.

Lemma 8.1 *If $\triangle ABC$ is a triangle, then there are points $A\prime$ and $B\prime$ such that $\square ABB\prime A\prime$ is a rectangle and $\triangle ABC \equiv \square ABB\prime A\prime$.*

We should then refer to $\square ABB\prime A\prime$ as the *associated rectangle*.

Lemma 8.2 *IF $\square ABCD$ and $\square EFGH$ are two rectangles such that $\alpha(\square ABCD) = \alpha(\square EFGH)$ and $AB = EF$, then $\square ABCD \cong \square EFGH$.*

PROOF: We know that the area of the rectangle is the length times the height. Since the triangles have the same area and the same length, they must have heights of equal length, so the rectangles are congruent. ∎

Now, we can prove Step 1.

Theorem 8.9 *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\alpha(\triangle ABC) = \alpha(\triangle DEF)$ and $AB = DE$, then $\triangle ABC \equiv \triangle DEF$.*

PROOF: Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\alpha(\triangle ABC) = \alpha(\triangle DEF)$ and $AB = DE$. Suppose that $\square ABB\prime A\prime$ and $\square DEE\prime D\prime$ are the associated rectangles to the two triangles. Since these two rectangles have the same area and congruent bases, by the above lemma, they are congruent. Thus, $\triangle ABC \equiv \square ABB\prime A\prime \cong \square DEE\prime D\prime \equiv \triangle DEF$, and the theorem follows by the transitivity of equivalence. ∎

Now, for Step 2 we have the following theorem.

Theorem 8.10 *If $\triangle ABC$ and $\triangle DEF$ are any two triangles such that $\alpha(\triangle ABC) = \alpha(\triangle DEF)$, then $\triangle ABC \equiv \triangle DEF$.*

PROOF: Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\alpha(\triangle ABC) = \alpha(\triangle DEF)$. We may assume that if corresponding sides are congruent, then the triangles are congruent and we are done. Thus, we may assume that $DF \geq AC$. Let M bet the midpoint of AC and let N be the midpoint of BC.

Now $AM \leq \frac{1}{2}DF$ and there are points $P \in \overrightarrow{MN}$ so that $MP > \frac{1}{2}DF$. Now, there is a point $G \in \overleftrightarrow{MN}$ such that $AG = \frac{1}{2}DF$. Choose a point $H \in \overline{AG}$ so that $GH \cong AG$. By Theorem 8.8 $\triangle ABC$ and $\triangle ABH$ share the same associated rectangle, so $\triangle ABC \equiv \triangle ABH$. Furthermore, $AH = DF$ so that $\triangle ABH \equiv \triangle DEF$. Therefore, $\triangle ABC \equiv \triangle DEF$. ∎
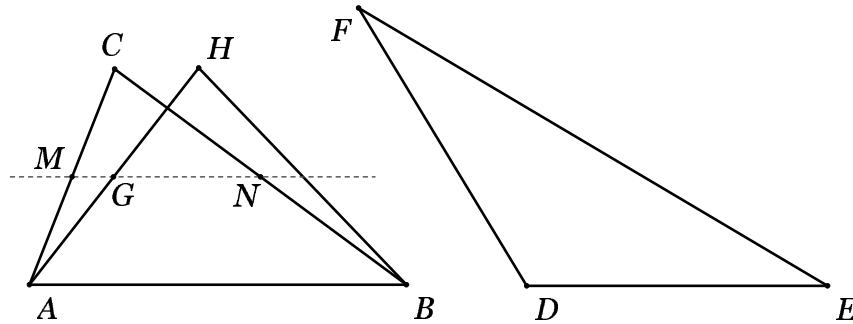
To complete this proof in Step 3 we need one more fact about the areas of Euclidean triangles.

Lemma 8.3 *If $\triangle ABC$ is a triangle and $a \in \mathbb{R}$ so that $0 < a < \alpha(\triangle ABC)$, then there exists a point $D \in AB$ so that $\alpha(\triangle ADC) = a$.*

PROOF: Use the Ruler Postulate to choose a point $D \in AB$ such that

$$AD = \frac{a}{\alpha(\triangle ADC)} AB.$$

Since $\triangle ABC$ and $\triangle ADC$ share the same height, if follows that $\alpha(\triangle ADC) = a$. ∎

Finally, we can finish the Fundamental Theorem of Decomposition Theory.

Theorem 8.11 *If R and $R\prime$ are two polygonal regions such that $\alpha(R) = \alpha(R\prime)$, then $R \equiv R\prime$.*

PROOF: Let R and $R\prime$ be two polygonal regions such that $\alpha(R) = \alpha(R\prime)$. Choose triangulations $T_1, \ldots, T_n$ and $T_1\prime, \ldots, T_m\prime$ of R and $R\prime$, respectively.

Consider two triangles T_1 and $T_1\prime$. We may assume that T_1 has the smaller area. If it should happen that $\alpha(T_1) = \alpha(T_1\prime)$, then delete T_1 from R and $T_1\prime$ from $R\prime$. If $\alpha(T_1) \neq \alpha(T_1\prime)$, then $\alpha(T_1) < \alpha(T_1\prime)$. By the above lemma, $T_1\prime$ can be subdivided into two triangles T_1'' and T_1''' such that $\alpha(T_1'') = \alpha(T_1)$. Delete T_1 from R and delete T_1'' from R'.

The result of this operation is a new pair of polygonal regions R_1 and R_1'. Now the deleted triangles have the same area, we know that $\alpha(R_1) = \alpha(R_1')$. If we could prove that $R_1 \equiv R_1'$, then it would follow that $R \equiv R'$ because $T_1 \equiv T_1''$.

Now the new regions R_1 and R_1' have smaller areas than the originals and the total number of triangles in the two triangulations has been reduced. The number is reduced by 2 in the first case and by 1 in the second case. We apply the previous process a finite number of times and we reduce to regions that are triangular. Then apply Theorem 8.10 to those triangular regions. Then you can the triangles back in and apply Theorem 8.10 each time to give the equivalence between the original regions. ∎

8.6   Area in Hyperbolic Geometry

We want to follow the above outline to study the concept of area in hyperbolic geometry. We need to look for an analogous result to Theorem 8.9. We don't have that type of theorem, but what we do have is that from Theorem 7.5 if two Saccheri quadrilaterals have

the same defect and same summit, then they are congruent. This will lead down a path to understanding area in the hyperbolic plane. It says that somehow area and defect will be related.

Let $\delta T = defect(\triangle(T))$ for any triangular region T. What will be prove is the following theorem due to Janos Bolyai.

Theorem 8.12 (Bolyai's Theorem for the Hyperbolic Plane) *If T_1 and T_2 are triangular regions, and $\delta T_1 = \delta T_2$, then $T_1 \equiv T_2$.*

NOTE: Defect applies to triangles and quadrilaterals while area applies to triangular regions and polygonal regions. We will not make such a strict delineation of the terms and will talk about the defect of a triangular region, knowing that it means the defect of the associated triangle.

Lemma 8.4 *If K_1 and K_2 are triangulations of the same polygonal region R, then $\delta K_1 = \delta K_2$.*

Lemma 8.5 *Every triangular region has the same defect as its associated Saccheri quadrilateral region.*

Lemma 8.6 *If $\triangle ABC$ and $\triangle DEF$ have the same defect and a pair of congruent sides, then the two triangular regions are equivalent by finite decomposition.*

PROOF: Assume that $\delta(\triangle ABC) = \delta(\triangle DEF)$ and that $AB \cong DE$. Let $\square ABB'A'$ and $DEE'D'$ be the associated Saccheri quadrilaterals corresponding to $\triangle ABC$ and $\triangle DEF$, respectively. By the above lemma $\delta(\square ABB'A') = \delta(\square DEE'D')$. Since AB is the summit of $ABB'A'$ and DE is the summit of $DEE'D'$ and $AB \cong DE$, by Theorem 7.5 we have that $\square ABB'A' \cong \square DEE'D'$. By the transitivity of equivalence and our previous result, we are done. ∎

Proof of Bolyai's Theorem: This is not that much different from the proof of Theorem 8.10.

Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\delta(\triangle ABC) = \delta(\triangle DEF)$. We may assume that if corresponding sides are congruent, then the triangles are congruent and we are done. Thus, we may assume that $DF \geq AC$. Let M bet the midpoint of AC and let N be the midpoint of BC.

Choose a point $G \in \overleftrightarrow{MN}$ such that $AG = \frac{1}{2}DF$. Choose a point $H \in \overrightarrow{AG}$ so that $A*G*H$ and $GH \cong AG$. By Theorem 8.8 $\triangle ABC$ and $\triangle ABH$ share the same associated Saccheri quadrilateral, so $\triangle ABC \equiv \triangle ABH$. Furthermore, $AH = DF$ so that $\triangle ABH \equiv \triangle DEF$. Therefore, $\triangle ABC \equiv \triangle DEF$.

This leads us to the following result in $\mathbb{H}^2$. It says that in hyperbolic geometry defect and area are essentially the same. Why should we have expected this? Well, we should not have expected it at all. Their definitions do not lead us to believe that this might be the case.

Theorem 8.13 (Area and Defect Theorem) *There exists a constant k such that*

$$\alpha(\triangle ABC) = k \cdot \delta(\triangle ABC)$$

for every triangle $\triangle ABC$ in the hyperbolic plane.

To prove this we need a few preliminary results. First though, a couple of corollaries:

Corollary 1 *There is an upper bound on the areas of triangles.*

Corollary 2 *Let $\triangle ABC$ and $\triangle DEF$ be two triangles. If $\alpha(\triangle ABC) = \alpha(\triangle DEF)$, then $\triangle ABC \equiv \triangle DEF$.*

Now, for the preliminaries for the proof of Theorem **??**.

Lemma 8.7 *For any two triangles $\triangle ABC$ and $\triangle DEF$,*

$$\frac{\alpha(\triangle ABC)}{\delta(\triangle ABC)} = \frac{\alpha(\triangle DEF)}{\delta(\triangle DEF)}.$$

PROOF: First, consider the case when $\delta(\triangle ABC) = \delta(\triangle DEF)$. Then, by Bolyai's Theorem, $\triangle ABC \equiv \triangle DEF$, so $\alpha(\triangle ABC) = \alpha(\triangle DEF)$, and the lemma holds.

Now, consider the case in which $\delta(\triangle ABC)/\delta(\triangle DEF)$ is a rational number, so that

$$\frac{\delta(\triangle ABC)}{\delta(\triangle DEF)} = \frac{p}{q},$$

where both p and q are positive integers and $p < q$. Since defect is a continuous function, there are points $P_0, P_1, \ldots, P_q$ on DE so that $D = P_0$, $P_{i-1} * P_i * P_{i+1}$ for all i, $P_q = E$ and

$$\delta(\triangle FP_iP_{i+1}) = \frac{1}{q}\delta(\triangle DEF)$$

for each i.

Note that by the Additivity of the Defect $\delta(\triangle ABC) = \delta(\triangle FP_0P_p)$. By our work at the beginning of the proof, all of the small triangles $\triangle FP_iP_{i+1}$ have the same area. Therefore, by the Additivity of the Defect and the Additivity of the Area, we are done.

To show this for any real number, we rely on the properties of the real numbers stated in the following two theorems, and we are done. ∎

Theorem 8.14 *If a and b are real numbers such that $a < b$, then there exists a rational number x such that $a < x < b$ and there exists an irrational number y so that $a < y < b$.*

Theorem 8.15 *IF x and y are real numbers such that*

  i) every rational number that is less than x is less than y, and

  ii) every rational number that is less than y is also less than x,

then $x - y$.

Now, we are in a position to prove Theorem **??**.

PROOF: Fix one triangle $\triangle DEF$ and let

$$k = \frac{\alpha(\triangle DEF)}{\delta(\triangle DEF)}.$$

By the above lemma,

$$\frac{\alpha(\triangle ABC)}{\delta(\triangle ABC)} = k$$

for every triangle $\triangle ABC$ and so $\alpha(\triangle ABC) = k\delta\triangle ABC)$ for every $\triangle ABC$. ∎

Note that we have not yet defined an area function for the hyperbolic plane. The easiest way to do so would be to take $k = 1$ thus making $\alpha(\triangle ABC) = \delta(\triangle ABC)$. This is not always what we want to do though. We will see indications of why as we move through the course.

We can now complete the proof of the Fundamental Theorem of Decomposition Theory in the hyperbolic plane. The proof is done much as it was in the Euclidean case, so adding a proof here will not add anything new.

Instead, let us look at one more bit of information about the area. Why did we not define the area for the hyperbolic plane much as we did for the Euclidean plane? We cannot define the area in terms of rectangles, but most assuredly we could have defined the area of a triangle using the same formula as we did in the Euclidean plane. Isn't true that

$$\text{Area} = \frac{1}{2}\text{base} \times \text{height?}$$

Actually, no! We need to see why.

8.7 The Hyperbolic Area Function

Theorem 8.16 *If $\delta\triangle ABC > \delta\triangle DEF$, then there is a point P between A and C such that $\delta\triangle ABP = \delta\triangle DEF$.*

Now, how do we define the area of a polygonal region in the hyperbolic plane? Should we define the area in the same way that we do in the Euclidean plane? If so, what are the minimum requirements for an area function of the Euclidean plane? Minimally it should satisfy the following: Let $\mathscr{R}$ be the set of all polygonal regions in H^2. An area function should be a function

$$\alpha : \mathscr{R} \to R$$

such that

1. $\alpha R > 0$ for every R;

2. if R_1 and R_2 intersect only in edges and vertices, then

$$\alpha(R_1 \cup R_2) = \alpha R_1 + \alpha R_2;$$

3. if T_1 and T_2 are triangular regions with the same base and altitude, then $\alpha T_1 = \alpha T_2$.

If there is such a function α, then by (2) and (3) it will satisfy

4. if $R_1 \equiv R_2$, then $\alpha R_1 = \alpha R_2$, because congruent triangles have the same bases and altitudes.

In Euclidean geometry we can show that this area function is unique and it must satisfy the formula $\alpha T = \frac{1}{2}bh$ for each triangular region, T, where b is the length of the base and h is the length of the altitude.

Theorem 8.17 *There is **no** such function $\alpha : \mathscr{R} \to R$ satisfying (1), (2), (3), and, hence, (4).*
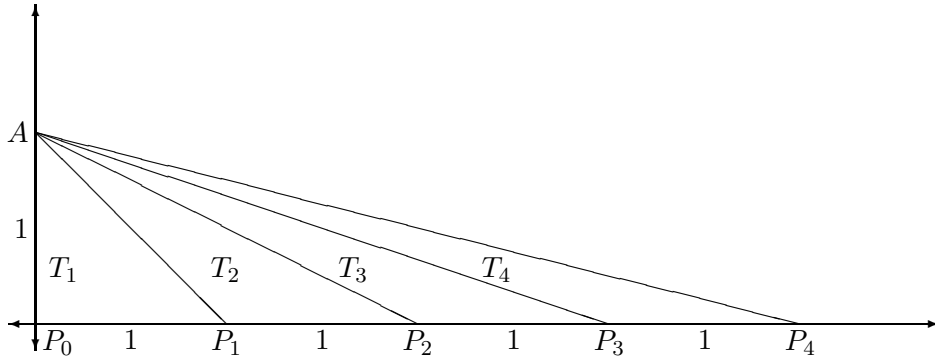
PROOF: Consider the right angle $\angle AP_0P_1$, with $AP_0 = P_0P_1 = 1$. For each n, let P_n be the point of $\overrightarrow{P_0P_1}$ such that $P_0P_n = n$. This gives a sequence of triangles

$$\triangle AP_0P_1, \ \triangle AP_1P_2, \ \ldots,$$

and a corresponding sequence of triangular regions

$$T_1, \ T_2, \ \ldots.$$

By condition (3) all the regions T_i have the same "area" $\alpha T_i = A$.

Now consider the defects of these triangles and let $d_i = \delta T_i$. For each n

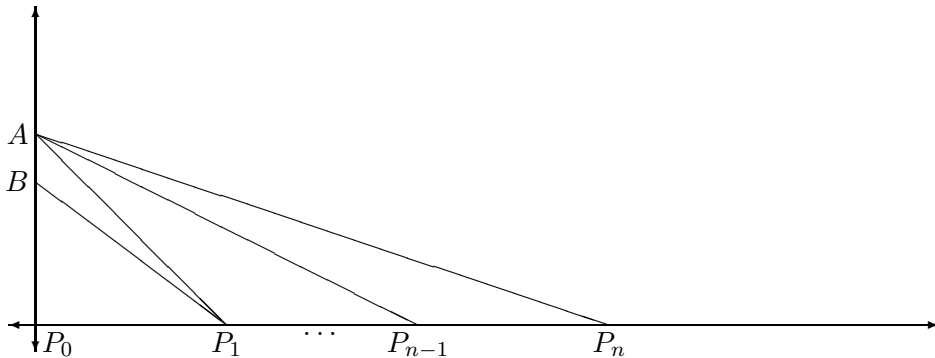$$d_1 + d_2 + \cdots + d_n = \delta \triangle AP_0P_n < 180.$$

Since the partial sums $d_1 + d_2 + \cdots + d_n$ are bounded, we have that the infinite series,

$$\sum_{n=0}^{\infty} d_n$$

is convergent. Therefore,

$$\lim_{n \to \infty} d_n = 0.$$

Hence $d_n < d_1$ for some n.

By Theorem 7 there is a point, B, between A and P_0 such that

$$\delta \triangle BP_0P_1 = \delta \triangle AP_{n-1}P_n = d_n.$$

Therefore by Bolyai's Theorem the regions T and T_n determined by these triangles are equivalent by finite decomposition. By condition (4) this means that $\alpha T = \alpha T_n$. But

$$\alpha T_n = \alpha T_1 = A.$$

Therefore,
$$\alpha T = \alpha T_1.$$

Because $\alpha \triangle ABP_1 > 0$ and
$$\alpha T_1 = \alpha T + \alpha \triangle ABP_1.$$

this must be impossible.   ∎

Note that we have proven the following result.

Theorem 8.18 *There exist triangles in $\mathbb{H}^2$ that have the same base and height but different areas.*

8.8 The Uniqueness of Hyperbolic Area Theory

Any reasonable area function α should have the following properties:

1. $\alpha R > 0$ for every R;

2. if R_1 and R_2 intersect only in edges and vertices, then

$$\alpha(R_1 \cup R_2) = \alpha R_1 + \alpha R_2;$$

3. if $R_1 \equiv R_2$, then $\alpha R_1 = \alpha R_2$.

We normally replace the Area-Defect Theorem result with the following constant:

Theorem 8.19 *There is a positive constant $k > 0$ such that*

$$\alpha(\triangle ABC) = \frac{\pi}{180} k^2 \delta(\triangle ABC)$$

for every triangle.

This changes our earlier result to

Corollary 3 *In $\mathbb{H}^2$ the area of any triangle is at most πk^2.*

There is no finite triangle whose area equals the maximal value πk^2, although you can approach this area as closely as you wish (and achieve it with a trebly asymptotic triangle). J. Bolyai proved that you can construct a *circle* of area πk^2 and a regular 4-sided polygon with a 45° angle that also has this area.