Chapter 3

NON-EUCLIDEAN GEOMETRIES

In the previous chapter we began by adding Euclid's Fifth Postulate to his five common notions and first four postulates. This produced the familiar geometry of the 'Euclidean' plane in which there exists precisely one line through a given point parallel to a given line not containing that point. In particular, the sum of the interior angles of any triangle was always 180° no matter the size or shape of the triangle. In this chapter we shall study various geometries in which parallel lines need not exist, or where there might be more than one line through a given point parallel to a given line not containing that point. For such geometries the sum of the interior angles of a triangle is then always greater than 180° or always less than 180°. This in turn is reflected in the area of a triangle which turns out to be proportional to the difference between 180° and the sum of the interior angles.

First we need to specify what we mean by a geometry. This is the idea of an *Abstract Geometry* introduced in Section 3.1 along with several very important examples based on the notion of *projective geometries*, which first arose in Renaissance art in attempts to represent three-dimensional scenes on a two-dimensional canvas. Both Euclidean and hyperbolic geometry can be realized in this way, as later sections will show.

3.1 ABSTRACT AND LINE GEOMETRIES. One of the weaknesses of Euclid's development of plane geometry was his 'definition' of points and lines. He defined a point as "... that which has no part" and a line as "... breadthless length". These really don't make much sense, yet for over 2,000 years everything he built on these definitions has been regarded as one of the great achievements in mathematical and intellectual history! Because Euclid's definitions are not very satisfactory in this regard, more modern developments of geometry regard points and lines as undefined terms. A *model* of a modern geometry then consists of specifications of points and lines.

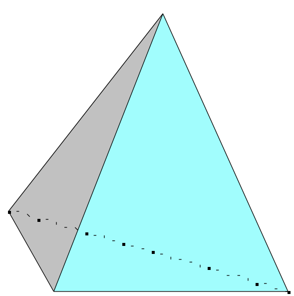
3.1.1 Definition. An *Abstract Geometry* G consists of a pair $\{\mathcal{P}, \mathcal{L}\}$ where $\mathcal{P}$ is a set and $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$. The elements of $\mathcal{P}$ are called *Points* and the elements of $\mathcal{L}$ are called *Lines*. We will assume that certain statements regarding these points and lines are true at the outset. Statements like these which are assumed true for a geometry are called *Axioms* of the geometry. Two Axioms we require are that each pair of points P, Q in $\mathcal{P}$ belongs to at least one line l in $\mathcal{L}$, and that each line l in $\mathcal{L}$ contains at least two elements of $\mathcal{P}$.

We can impose further geometric structure by adding other axioms to this definition as the following example of a *finite* geometry - finite because it contains only finitely many points - illustrates. (Here we have added a third axiom and slightly modified the two mentioned above.)

3.1.2 Definition. A 4-POINT geometry is an abstract geometry $\mathcal{G} = \{\mathcal{P}, \mathcal{L}\}$ in which the following axioms are assumed true:

- **Axiom 1:** $\mathcal{P}$ contains exactly four points;
- **Axiom 2:** each pair of distinct points in $\mathcal{P}$ belongs to exactly one line;
- **Axiom 3:** each line in $\mathcal{L}$ contains exactly two distinct points.

The definition doesn't indicate what objects points and lines are in a 4-Point geometry, it simply imposes restrictions on them. Only by considering a model of a 4-Point geometry can we get an explicit description. Look at a tetrahedron.



It has 4 vertices and 6 edges. Each pair of vertices lies on exactly one edge, and each edge contains exactly 2 vertices. Thus we get the following result.

3.1.3 Example. A tetrahedron contains a model of a 4-Point geometry in which

$\mathcal{P}$ = {vertices of the tetrahedron} and $\mathcal{L}$ = {edges of the tetrahedron}.

This example is consistent with our usual thinking of what a point in a geometry should be and what a line should be. But points and lines in a 4-Point geometry can be anything so long as they satisfy all the axioms. Exercise 3.3.2 provides a very different model of a 4-Point geometry in which the points are opposite faces of an octahedron and the lines are the vertices of the octahedron!

Why do we bother with models? Well, they give us something concrete to look at or think about when we try to prove theorems about a geometry.

3.1.4 Theorem. In a 4-Point geometry there are exactly 6 lines.

To prove this theorem synthetically all we can do is use the axioms and argue logically from those. A model helps us determine what the steps in the proof should be. Consider the tetrahedron model of a 4-Point geometry. It has 6 edges, and the edges are the lines in the geometry, so the theorem is correct for this model. But there might be a different model of a 4-Point geometry in which there are more than 6 lines, or fewer than 6 lines. We have to show that there will be exactly 6 lines whatever the model might be. Let's use the tetrahedron model again to see how to prove this.

- Label the vertices *A*, *B*, *C*, and *D*. These are the 4 points in the geometry.
- Concentrate first on *A*. There are 3 edges passing through *A*, one containing *B*, one containing *C*, and one containing *D*; these are obviously distinct edges. This exhibits 3 distinct lines containing *A*.
- Now concentrate on vertex *B*. Again there are 3 distinct edges passing through *B*, but we have already counted the one passing also through *A*. So there are only 2 new lines containing *B*.
- Now concentrate on vertex *C*. Only the edge passing through *C* and *D* has not been counted already, so there is only one new line containing *C*.
- Finally concentrate on *D*. Every edge through *D* has been counted already, so there are no new lines containing *D*.

Since we have looked at all 4 points, there are a total of 6 lines in all. This proof applies to any 4-Point geometry if we label the four points *A*, *B*, *C*, and *D*, whatever those points are. Axiom 2 says there must be one line containing *A* and *B*, one containing *A* and *C* and one containing *A* and *D*. But the Axiom 3 says that the line containing *A* and *B* must be distinct from the line containing *A* and *C*, as well as the line containing *A* and *D*. Thus there will always be 3 distinct lines containing *A*. By the same argument, there will be 3 distinct lines containing *B*, but one of these will contain *A*, so there are only 2 new lines containing *B*. Similarly, there will be 1 new line containing *C* and no new lines containing *D*. Hence in any 4-Point geometry there will be exactly 6 lines.

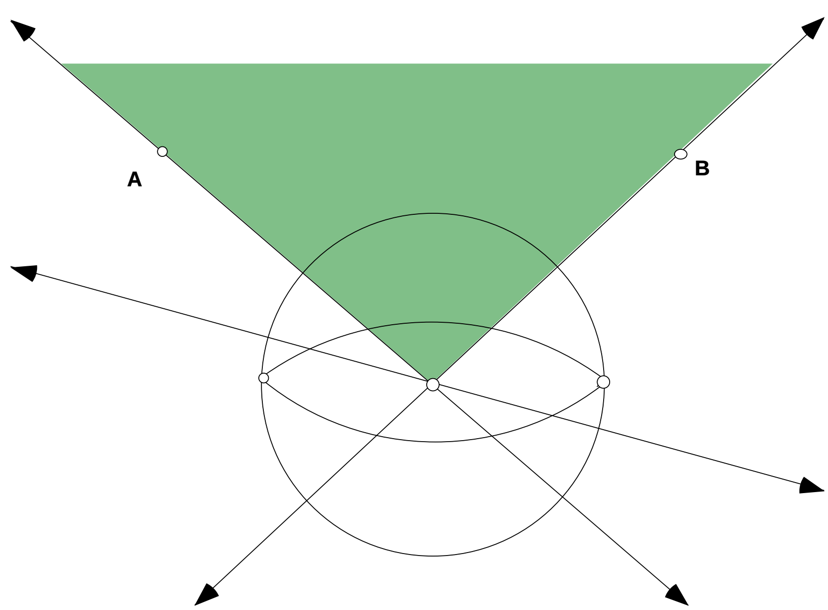
This is usually how we prove theorems in Axiomatic Geometry: look at a model, check that the theorem is true for the model, then use the axioms and theorems that follow from these axioms to give a logically reasoned proof. For Euclidean plane geometry that model is always the familiar geometry of the plane with the familiar notion of point and line. But it is not be the only model of Euclidean plane geometry we could consider! To illustrate the variety of forms that geometries can take consider the following example.

3.1.5 Example. Denote by $\mathbf{P}^2$ the geometry in which the 'points' (here called P-points) consist of all the Euclidean lines through the origin in 3-space and the P-lines consist of all Euclidean planes through the origin in 3-space.

Since exactly one plane can contain two given lines through the origin, there exists exactly one P-line through each pair of P-points in $\mathbf{P}^2$ just as in Euclidean plane geometry. But what about parallel P-lines? For an abstract geometry $\mathcal{G}$ we shall say that two lines m, and l in $\mathcal{G}$ are *parallel* when l and m contain no common points. This makes good sense and is consistent with our usual idea of what parallel means. Since any two planes through the origin in 3-space must always intersect in a line in 3-space we obtain the following result.

3.1.6 Theorem. In $\mathbf{P}^2$ there are no parallel P-lines.

Actually, $\mathbf{P}^2$ is a model of Projective plane geometry. The following figure illustrates some of the basic ideas about $\mathbf{P}^2$.



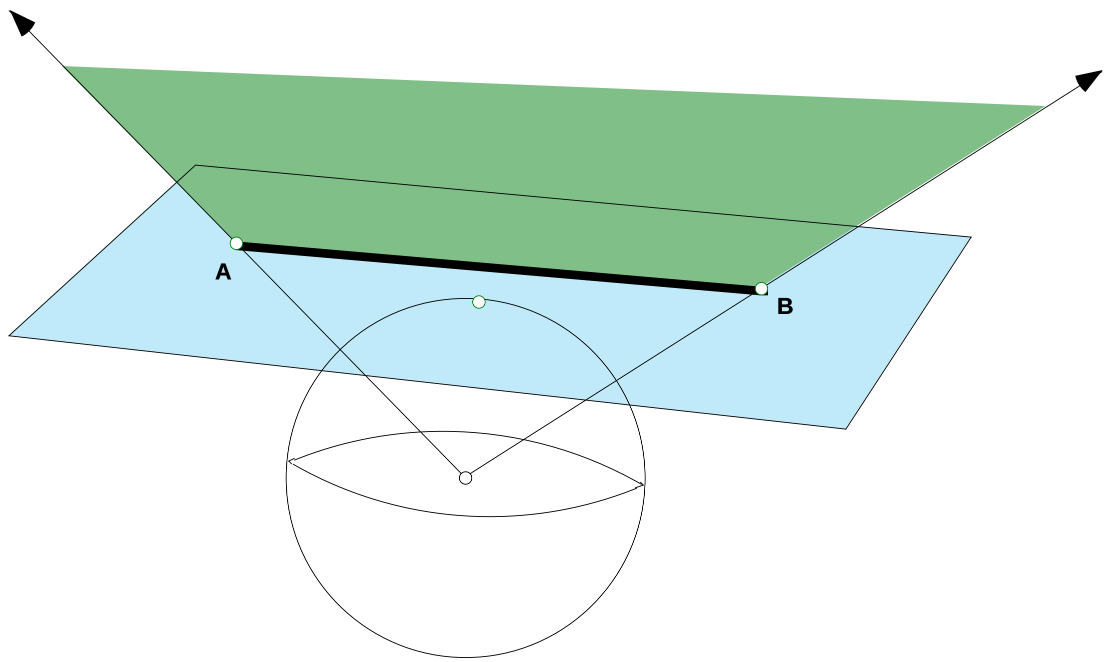
The two Euclidean lines passing through A and the origin and through B and the origin specify two P-points in $\mathbf{P^2}$, while the indicated portion of the plane containing these lines through A and B specify the 'P-line segment' $\overline{AB}$.

Because of Theorem 3.1.6, the geometry $\mathbf{P^2}$ cannot be a model for Euclidean plane geometry, but it comes very 'close'. Fix a plane passing through the origin in 3-space and call it the *Equatorial Plane* by analogy with the plane through the equator on the earth.

3.1.7 Example. Denote by $\mathbf{E^2}$ the geometry in which the E-points consist of all lines through the origin in 3-space that are not contained in the equatorial plane and the E-lines consist of all planes through the origin save for the equatorial plane. In other words, $\mathbf{E^2}$ is what is left of $\mathbf{P^2}$ after one P-line and all the P-points on that P-line in $\mathbf{P^2}$ are removed.

The claim is that $\mathbf{E^2}$ can be identified with the Euclidean plane. Thus there must be parallel E-lines in this new geometry $\mathbf{E^2}$. Do you see why? Furthermore, $\mathbf{E^2}$ satisfies Euclid's Fifth Postulate.

The figure below indicates how $\mathbf{E^2}$ can be identified with the Euclidean plane. Look at a fixed sphere in Euclidean 3-Space centered at the origin whose equator is the circle of intersection with the fixed equatorial plane. Now look at the plane which is tangent to this sphere at the North Pole of this sphere.



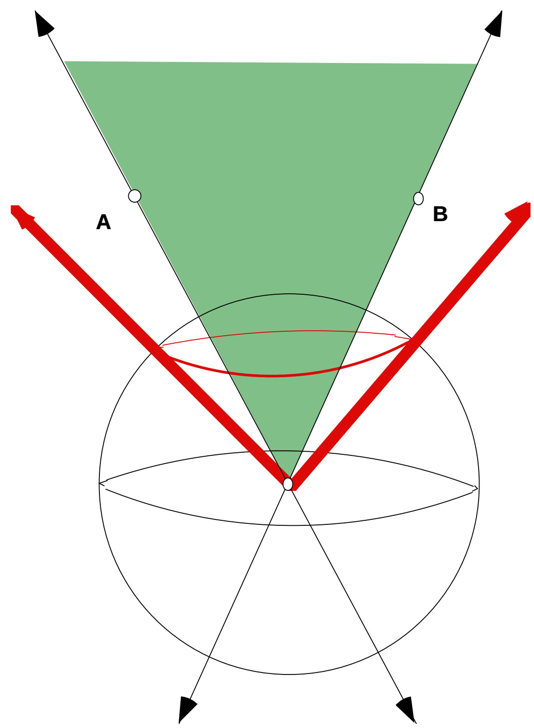
Every line through the origin in 3-space will intersect this tangent plane in exactly one point unless the line is parallel in the usual 3-dimensional Euclidean sense to the tangent plane at the North Pole. But these parallel lines are precisely the lines through the origin that lie in the equatorial plane. On the other hand, for each point A in the tangent plane at the North Pole there is exactly one line in 3-space passing through both the origin and the given point A in the tangent plane. Thus there is a 1-1 correspondence between the E-points in $\mathbf{E}^2$ and the points in the tangent plane at the North Pole. In the same way we see that there is a 1-1 correspondence between E-lines in $\mathbf{E}^2$ and the usual Euclidean lines in the tangent plane. The figure above illustrates the 1-1 correspondence between E-line segment $\overline{AB}$ in $\mathbf{E}^2$ and the line segment $\overline{AB}$ in Euclidean plane geometry.

For reasons, which will become very important later in connection with transformations, this 1-1 correspondence can be made explicit through the use of coordinate geometry and ideas from linear algebra. Let the fixed sphere centered at the origin having radius 1. Then the point (x, y) in the Euclidean plane is identified with the point $(x, y, 1)$ in the tangent plane at the North Pole, and this point is then identified with the line $\{\alpha(x, y, 1): -\infty < \alpha < \infty\}$ through the origin in 3-space.

Since there are no parallel lines in $\mathbf{P}^2$ it is clear that the removal from $\mathbf{P}^2$ of that one P-line and all P-points on that P-line must be very significant.

3.1.8 Exercise. What points do we need to add to the Euclidean plane so that under the identification of the Euclidean plane with $\mathbf{E}^2$ the Euclidean plane together with these additional points are in 1-1 correspondence with the points in $\mathbf{P^2}$? What line do we need to add to the Euclidean plane so that we get a 1-1 correspondence with all the lines in $\mathbf{P^2}$?

Note first that by restricting further the points and lines in $\mathbf{P^2}$ we get a model of a different geometry. The set of all lines passing through the origin in 3-space and through the 45th parallel in the Northern Hemisphere of the fixed sphere model determines a *cone* in 3-space to be denoted by **L**.

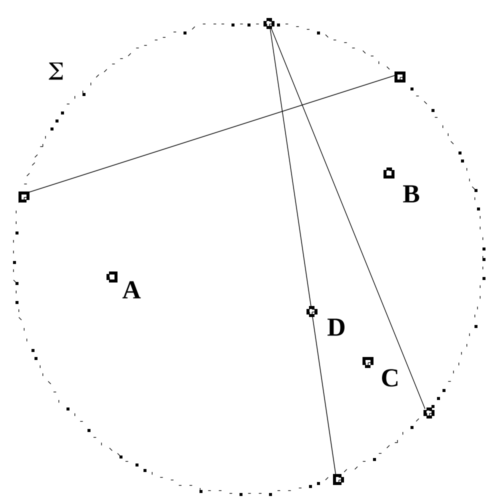


3.1.9 Definition. Denote by $\mathbf{H^2}$ the geometry whose h-points consists of Euclidean lines through the origin in 3-space that lie in the inside the cone **L** and whose h-lines consist of the intersections of the **interior** of **L** and planes through the origin in 3-space.

Again the Euclidean lines through *A* and *B* represent h-points *A* and *B* in $\mathbf{H^2}$ and the 'h-line segment' $\overline{AB}$ is (as indicated in the above figure by the shaded region) the sector of a plane containing the Euclidean lines through the origin which are passing through points on the line segment connecting *A* and *B*. $\mathbf{H^2}$ is a model of *Hyperbolic* plane geometry. The reason why it's a model of a 'plane' geometry is clear because we have only defined points and lines, but what is not at all obvious is why the name 'hyperbolic' is used. To understand that let's try to use $\mathbf{H^2}$ to create other models. For instance, our intuition about 'plane' geometries suggests that we should try to find models in which h-points really are points, not lines through the origin! One way of doing this is by looking at surfaces in 3-space, which intersect the lines inside the cone **L** exactly once. There are two natural candidates, both presented here. The second one presented realizes Hyperbolic plane geometry as the points on a hyperboloid, - hence the name 'Hyperbolic' geometry. The first one presented realizes Hyperbolic plane geometry as the points inside a disk. This first one, known as the *Klein Model,* is very useful for solving the following exercise because its h-lines are realized as open Euclidean line segments. In the next section we study a third model known as the *Poincaré Disk.*

3.1.10 Exercise. Given an h-line *l* in Hyperbolic plane geometry and an h-point *P* not on the h-line, how many h-lines parallel to *l* through *P* are there?

3.1.11 Klein Model. Consider the tangent plane *M*, tangent to the unit sphere at its North Pole, and let the origin in *M* be the point of tangency of *M* with the North Pole. Then *M* intersects the cone **L** in a circle, call it Σ, and it intersects each line inside **L** in exactly one point inside Σ. In fact, there is a 1-1 correspondence between the lines inside **L** and the points inside Σ. On the other hand, the intersection of *M* with planes is a Euclidean line, so the lines in $\mathbf{H}^2$ are in 1-1 correspondence with the chords of Σ, except that we must remember that points *on* circle Σ correspond to lines *on* **L**. So the lines in the Klein model of Hyperbolic plane geometry are exactly the chords of Σ, omitting the endpoints of a chord. In other words, the hyperbolic h-lines in this model are *open* line segments. The following picture contains some points and lines in the Klein model,



the dotted line on the circumference indicating that these points are omitted.

3.1.11a Exercise. Solve Exercise 3.1.10 using the Klein model.

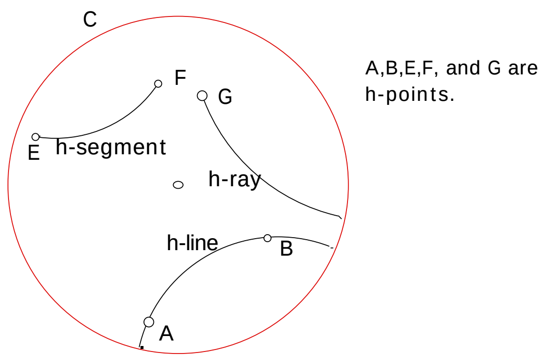
3.1.12 Hyperboloid model. Consider the hyperbola $z^2 - x^2 = 1$ in the x,z-plane. Its asymptotes are the lines $z = \pm x$. Now rotate the hyperbola and its asymptotes about the z-axis. The asymptotes generate the cone **L**, and the hyperbola generates a two-sheeted hyperboloid lying inside **L**; denote the upper hyperboloid by **B**. Then every line through the origin in 3-space intersects **B** exactly once – see **Exercise 3.1.13**; in fact, there is a 1-1 correspondence between the points on **B** and the points in $\mathbf{H}^2$. The lines in $\mathbf{H}^2$ correspond to the curves on **B** obtained by intersecting the planes though the origin in 3-space. With this model, the hyperboloid **B** is a realization of Hyperbolic plane geometry.

3.1.13 Exercise. Prove that every line through the origin in 3-space intersects **B** (in the Hyperbolic model above) exactly once.

3.2 POINCARÉ DISK. Although the line geometries of the previous section provide a very convenient, coherent, and illuminating way of introducing models of non-Euclidean geometries, they are not convenient ones in which to use Sketchpad. More to the point, they are not easy to visualize or to work with. The Klein and Hyperboloid models are more satisfactory ones that conform more closely to our intuition of what a 'plane geometry' should be, but the definition of distance between points and that of angle measure conform less so. We instead focus on the Poincaré Model **D**, introduced by Henri Poincaré in 1882, where 'h-points' are points as we usually think of them - points in the plane - while 'h-lines' are arcs of particular Euclidean circles. This too fits in with our usual experience of Euclidean plane geometry if one thinks of a straight line through point **A** as the limiting case of a circle through point **A** whose radius approaches ∞ as the center moves out along a perpendicular line through **A**. The Poincaré Disk Model allows the use of standard Euclidean geometric ideas in the development of the geometric properties of the models and hence of Hyperbolic plane geometry. We will see later that **D** is actually a model of the "same" geometry as $\mathbf{H}^2$ by constructing a 1-1 transformation from $\mathbf{H}^2$ onto **D**.

Let **C** be a circle in the Euclidean plane. Then **D** is the geometry in which the 'h-points' are the points inside **C** and the 'h-lines' are the arcs *inside* **C** of any circle intersecting **C** at right angles. This means that we omit the points of intersection of these circles with **C**. In addition, any diameter of the bounding circle will also be an h-line, since any straight line through the center of the bounding circle intersects the bounding circle at right angles and (as before) can be regarded as the limiting case of a circle whose radius approaches infinity.

As in the Klein model, points on the circle are omitted and hyperbolic h-lines are *open* --- in this case, open arcs of circles. As we are referring to points inside **C** as *h-points* and the hyperbolic lines inside **C** as *h-lines*; it will also be convenient to call **C** the *bounding circle*. The following figure illustrates these definitions:



More technically, we say that a circle intersecting **C** at right angles is *orthogonal* to **C**. Just as for Euclidean geometry, it can be shown that through each pair of h-points there passes exactly one h-line. A coordinate geometry proof of this fact is included in Exercise 3.6.2. We suggest a synthetic proof of this in Section 3.5. Thus the notion of *h-line segment* between h-points A and B makes good sense: it is the portion between A and B of the unique h-line through A and B. In view of the definition of h-lines, the h-line segment between A and B can also be described as the arc between A and B of the unique circle through A and B that is orthogonal to **C**. Similarly, an *h-ray* starting at an h-point A in **D** is either one of the two portions, between A and the bounding circle, of an h-line passing through A.

Having defined **D**, the first two things to do are to introduce the *distance*, $d_h(A, B)$, between h-points A and B as well as the *angle measure* of an *angle* between h-rays starting at some h-point A. The distance function should have the same properties as the usual Euclidean distance, namely:

- (Positive-definiteness): For all points A and B ($A \neq B$),

$$d_h(A, B) > 0 \quad \text{and} \quad d_h(A, A) = 0;$$

- (Symmetry): For all points A and B,

$$d_h(A, B) = d_h(B, A);$$

- (Triangle inequality): For all points A, B and C,

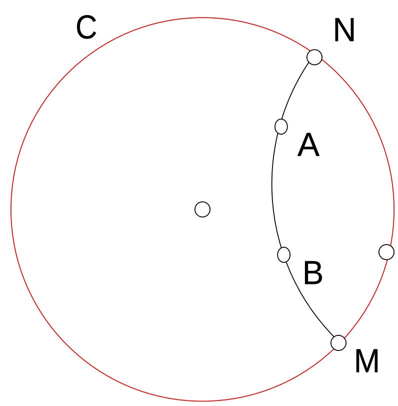
$$d_h(A, B) \leq d_h(A, C) + d_h(C, B).$$

Furthermore the distance function should satisfy the **Ruler Postulate.**

3.2.0. Ruler Postulate: The points in each line can be placed in 1-1 correspondence to the real numbers in such a way that:

- each point on the line has been assigned a unique real number (its *coordinate*);
- each real number is assigned to a unique point on the line;
- for each pair of points A, B on the line, $d_h(A, B) = |a - b|$, where a and b are the respective coordinates of A and B.

The function we adopt for the distance looks very arbitrary and bizarre at first, but good sense will be made of it later, both from a geometric and transformational point of view. Consider two h-points A, B in **D** and let M, N be the points of intersection with the bounding circle of the h-line through A, B as in the figure:



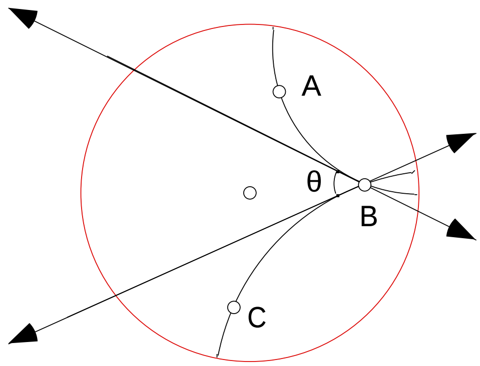
We set

$$d_h(A,B) = \left| \ln\left(\frac{d(A,M)d(B,N)}{d(A,N)d(B,M)} \right) \right|$$

where $d(A,M)$ is the usual Euclidean distance between points A and M. Using properties of logarithms, one can check that the role of M and N can be reversed in the above formula (see Exercise 3.3.7).

3.2.1 Exercise. Show that $d_h(A, B)$ satisfies the positive-definiteness and symmetry conditions above.

We now introduce angles and angle measure in **D.** Just as in the Euclidean plane, two *h*-rays starting at the same point form an angle. In the figure below we see two intersecting h-lines forming BAC.



To find the hyperbolic measure $m_h \; BAC$ of BAC we appeal to angle measure in Euclidean geometry. To do that we need the tangents to the arcs at the point *A*. The hyperbolic measure of the angle BAC is then defined to be the Euclidean measure of the Euclidean angle between these two tangents, i.e. $m_h \; BAC = m \; \theta$.

Just as the notions of points, lines, distance and angle measure are defined in Euclidean plane geometry, these notions are all defined in **D**. And, we can exploit the hyperbolic tools for Sketchpad, which correspond to the standard Euclidean tools, to discover facts and theorems about the Poincaré Disk and hyperbolic plane geometry in general.

- Load the “Poincaré” folder of scripts by moving the sketch “Poincare Disk.gsp” into the Tool Folder. To access this sketch, first open the folder “Samples”, then “Sketches”, then “Investigations”. Once Sketchpad has been restarted, the following scripts will be available:

    - **Hyperbolic Segment** - Given two points, constructs the h-segment joining them
    - **Hyperbolic Line** - Constructs an h-line through two h-points
    - **Hyperbolic P. Bisector** - Constructs the perpendicular bisector between two h-points
    - **Hyperbolic Perpendicular** - Constructs the perpendicular of an h-line through a third point not on the h-line.
    - **Hyperbolic A. Bisector –** Constructs an h-angle bisector.
    - **Hyperbolic Circle By CP** - Constructs an h-circle by center and point.

- **Hyperbolic Circle By CR –** Constructs an h-circle by center and radius.
- **Hyperbolic Angle –** Gives the hyperbolic angle measure of an h-angle.
- **Hyperbolic Distance** - Gives the measure of the hyperbolic distance between two h-points which do not both lie on a diameter of the Poincare disk.

- The sketch "Poincare Disk.gsp" contains a circle with a specially labeled center called, 'P. Disk Center', and point on the disk called, 'P. Disk Radius'. The tools listed above work by using Auto-Matching to these two labels, so if you use these tools in another sketch, you must either label the center and radius of your Poincare Disk accordingly, or match the disk center and radius before matching the other givens for the tool**.** We are now ready to investigate properties of the Poincaré Disk. Use the line tool to investigate how the curvature of *h-lines* changes as the line moves from one passing close to the center of the Poincaré disk to one lying close to the bounding circle. Notice that this line tool never produces h-lines passing through the center of the bounding circle for reasons that will be brought out in the next section. In fact, if you experiment with the tools, you will find that the center of the Poincare Disk and the h-lines which pass through the center are problematic in general. Special tools need to be created to deal with these cases.

(There is another very good software simulation of the Poincaré disk available on the web at

http://math.rice.edu/~joel/NonEuclid.

You can download the program or run it online. The site also contains some background material that you may find interesting.)

3.2.2 Demonstration: Parallel Lines. As in Euclidean geometry, two *h-lines* in **D** are said to be *parallel* when they have no *h-points* in common.

- In the Poincaré disk construct an *h-line* l and an *h-point* P not on l. Use the *h-line* script to investigate if an *h-line* through P parallel to l can be drawn. Can more than one be drawn? How many can be drawn? **End of Demonstration 3.2.2.**

3.2.3 Shortest Distance. In Euclidean plane geometry the line segment joining points P and Q is the path of shortest distance; in other words, a line segment can be described both in *metric* terms and in *geometric* terms. More precisely, there are two natural definitions of a line segment $\overline{PQ}$, one as the shortest path between P and Q, a metric property, the other as all points between P, Q on the unique line l passing through P and Q - a geometric property. But what do we mean by *between*? That is easy to answer in terms of the metric: the line segment $\overline{PQ}$ consists of all points R on l such that $d_h(P, R) + d_h(R, Q) = d_h(P, Q)$. This last definition makes good sense also in **D** since there we have defined a notion of distance.

3.2.3a Demonstration: Shortest Distance.

- In the Poincaré disk select two points A and B. Use the "Hyperbolic Distance" tool to investigate which points C minimize the sum

$$d_h(C, A) + d_h(C, B).$$

What does your answer say about an *h-line segment* between A and B?

End of Demonstration 3.2.3a.

3.2.4 Demonstration: Hyperbolic Versus Euclidean Distance. Since Sketchpad can measure both Euclidean and hyperbolic distances we can investigate hyperbolic distance and compare it with Euclidean distance.

- Draw two h-line segments, one near the center of the Poincaré disk, the other near the boundary. Adjust the segments until both have the same hyperbolic length. What do you notice about the Euclidean lengths of these arcs?
- Compute the ratio

$$d_h(A, B) \big/ d(A, B)$$

of the hyperbolic and Euclidean lengths of the respective hyperbolic and Euclidean line segments between points A, B in the Poincaré disk. What is the largest value you can obtain? **End of Demonstration 3.2.4.**

3.2.5 Demonstration: Investigating d_h further.

- Does this definition of d_h depend on where the boundary circle lies in the plane?
- What is the effect on d_h if we change the center of the circle?
- What is the effect on d_h of doubling the radius of the circle?

By changing the size of the disk, but keeping the points in the same proportion we can answer these questions. Draw an h-line segment $\overline{AB}$ and measure its length.

Over on the toolbar change the select arrow to the **Dilate tool**. Select "P. Disk Center", then **Transform** "Mark Center." Under the **Edit** menu "Select All," then deselect the “Distance =”. Now, without deselecting these objects, drag the P. Disk Radius to vary the size of the P-Disk and of all the Euclidean distances between objects inside proportionally. What effect does changing the size of the P-Disk proportionally (relative to the P-Disk Center) have on the hyperbolic distance between the two endpoints of the hyperbolic segment?

Over on the toolbar change the select arrow to the **Rotate tool**. Select “P-Disk Center”, then **Transform** “Mark Center.” Under the **Edit** menu “Select All,” then deselect the “Distance =”. Now, without deselecting these objects, drag the P-Disk Radius to rotate the orientation of the P-Disk. What effect does changing the orientation of the P-Disk uniformly have on the hyperbolic distance between the two endpoints of the hyperbolic segment?

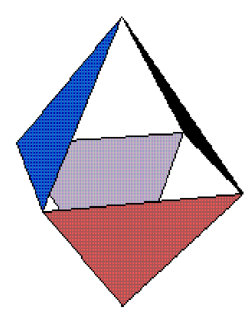
Over on the toolbar, change the Rotate tool back to the **select arrow**. Under the **Edit** menu “Select All,” then deselect the “Distance =”. Grab the P-Disk Center, and drag the Disk around the screen. What effect does changing the location of the P-Disk have on the hyperbolic distance between the two endpoints of the h-line segment?

End of Demonstration 3.2.5.

3.3 Exercises. This Exercise set contains questions related to Abstract Geometries and properties of the Poincaré Disk.

Exercise 3.3.1. Prove that in a 4-Point geometry there passes exactly 3 lines through each point.

Exercise 3.3.2. The figure to the right is an octahedron. Use this to exhibit a model of a 4-Point geometry that is very different from the tetrahedron model we used in class. Four of the faces have been picked out. Use these as the 4 points. What must the lines be if the octahedron is to be a model of a 4-Point geometry? Make sure you check that all the axioms of a 4-Point geometry are satisfied.



Exercise 3.3.3. We have stated that our definition for the hyperbolic distance between two points satisfies the ruler postulate, but it is not easy to construct very long *h-line segments*, say ones of length 10. The source of this difficulty is the rapid growth of the exponential

function. Suppose that the radius of the bounding circle is 1 and let *A* be an *h-point* that has Euclidean distance *r* from the origin ($r < 1$, of course). The diameter of the bounding circle passing through *A* is an *h-line*. Show the hyperbolic distance from the center of the bounding circle to *A* is

$$\left|\ln\frac{(1+r)}{(1-r)}\right|;$$

Find *r* when the hyperbolic distance from *A* to the center of the bounding circle is 10.

Exercise 3.3.4. Use Exercise 3.3.3 to prove that the second statement of the ruler postulate holds when the hyperbolic line is a diameter of the bounding circle and if to each point we assign the hyperbolic distance between it and the center of the bounding circle. That is, why are we guaranteed that each real number is assigned to a unique point on the line? Hint: Show your function for *r* from Exercise 3.3.3 is 1-1 and onto the interval *(-1,1)*.

Exercise 3.3.5. Explain why the ruler postulate disallows the use of the Euclidean distance formula to compute the distance between two points in the Poincaré Disk.

Exercise 3.3.6. Using Sketchpad open the Poincaré Disk Starter and find a counterexample within the Poincaré Disk to each of the following.

(a) If a line intersects one of two parallel lines, then it intersects the other.

(b) If two lines are parallel to a third line then the two lines are parallel to each other.

Exercise 3.3.7. Using properties of logarithms and properties of absolute value, show that, with the definition of hyperbolic distance,

$$d_h(A,B) = \left|\ln\left(\frac{d(A,M)d(B,N)}{d(A,N)d(B,M)}\right)\right| = \left|\ln\left(\frac{d(A,N)d(B,M)}{d(A,M)d(B,N)}\right)\right|,$$

i.e., the roles of *M* and *N* can be reversed and the same distance value results.

3.4 CLASSIFYING THEOREMS. For many years mathematicians attempted to deduce Euclid's fifth postulate from the first four postulates and five common notions. Progress came in the nineteenth century when mathematicians abandoned the effort to find a contradiction in the denial of the fifth postulate and instead worked out carefully and completely the consequences of such a denial. It was found that a coherent theory arises if one assumes the Hyperbolic Parallel Postulate instead of Euclid's fifth Postulate.

Hyperbolic Parallel Postulate: Through a point *P* not on a given line *l* there exists at least two lines parallel to *l*.

The axioms for hyperbolic plane geometry are Euclid's 5 common notions, the first four postulates and the Hyperbolic Parallel Postulate. Three professional mathematicians are credited with the discovery of Hyperbolic geometry. They were Carl Friedrich Gauss (1777-1855), Nikolai Ivanovich Lobachevskii (1793-1856) and Johann Bolyai (1802-1860). All three developed non-Euclidean geometry axiomatically or on a synthetic basis. They had neither an analytic understanding nor an analytic model of non-Euclidean geometry. Fortunately, we have a model now; the Poincaré disk **D** is a model of hyperbolic plane geometry, meaning that the five axioms, consisting of Euclid's first four postulates and the Hyperbolic Parallel Postulate, are true statements about **D**, and so any theorem that we deduce from these axioms must hold true for **D.** In particular, there are several lines though a given point parallel to a given line not containing that point.

Now, an abstract geometry (in fact, any axiomatic system) is said to be **categorical** if any two models of the system are equivalent. When a geometry is categorical, any statement which is true about one model of the geometry is true about all models of the geometry and will be true about the abstract geometry itself. Euclidean geometry and the geometries that result from replacing Euclid's fifth postulate with Alternative A or Alternative B are both categorical geometries.

In particular, Hyperbolic plane geometry is categorical and the Poincaré disk **D** is a model of hyperbolic plane geometry. So any theorem valid in **D** must be true of Hyperbolic plane geometry. To prove theorems about Hyperbolic plane geometry one can either deduce them from the axioms (*i.e.*, give a synthetic proof) or prove them from the model **D** (*i.e.*, give an analytic proof).

Since both the model **D** and Hyperbolic plane geometry satisfy Euclid's first four postulates, any theorems for Euclidean plane geometry that do not require the fifth postulate will also be true for hyperbolic geometry. For example, we noted in Section 1.5 that the proof that the angle bisectors of a triangle are concurrent is independent of the fifth postulate. By comparison, any theorem in Euclidean plane geometry whose proof used the Euclidean fifth postulate might not be valid in hyperbolic geometry, though it is not automatically ruled out, as there may be a proof that does not use the fifth postulate. For example, the proof we gave of the existence of the centroid used the fifth postulate, but other proofs, independent of the fifth postulate, do exist. On the other hand, all proofs of the existence of the circumcenter must rely in some way on the fifth postulate, as this result is false in hyperbolic geometry.

Exercise 3.4.0 After the proof of Theorem 1.5.5, which proves the existence of the circumcenter of a triangle in Euclidean geometry, you were asked to find where the fifth postulate was used in the proof. To answer this question, open a sketch containing a Poincare Disk with the center and radius appropriately labeled (P. Disk Center and P. Disk Radius). Draw a hyperbolic triangle and construct the perpendicular bisectors of two of the sides. Drag the vertices of the triangle and see what happens. Do the perpendicular bisectors always intersect? Now review the proof of Theorem 1.5.5 and identify where the Parallel postulate was needed.

We could spend a whole semester developing hyperbolic geometry axiomatically! Our approach in this chapter is going to be either analytic or visual, however, and in chapter 5 we will begin to develop some transformation techniques once the idea of Inversion has been adequately studied. For the remainder of this section, therefore, various objects in the Poincaré disk **D** will be studied and compared to their Euclidean counterparts.

3.4.1 Demonstration: Circles. A circle is the set of points equidistant from a given point (the center).

- Open a Poincaré Disk, construct two points, and label them by A and O.
- Measure the hyperbolic distance between A and O, $d_h(A, O)$, Select the point A and under the **Display** menu select Trace Points. Now drag A while keeping $d_h(A, O)$ constant.
- Can you describe what a hyperbolic circle in the Poincaré Disk should look like?
- To confirm your results, use the circle script to investigate hyperbolic circles in the Poincaré Disk. What do you notice about the center? **End of Demonstration 3.4.1.**

3.4.2 Demonstration: Triangles. A triangle is a three-sided polygon; two hyperbolic triangles are said to be congruent when they have congruent sides and congruent interior angles. Investigate hyperbolic triangles in the Poincaré Disk.

- Construct a hyperbolic triangle ΔABC and use the “Hyperbolic Angle” tool to measure the hyperbolic angles of ΔABC (keep in mind that three points are necessary to name the angle, the vertex should be the second point clicked).
- Calculate the sum of the three angle measures. Drag the vertices of the triangle around. What is a lower bound for the sum of the hyperbolic angles of a triangle? What is an

upper bound for the sum of the hyperbolic angles of a triangle? What is an appropriate conclusion about hyperbolic triangles? How does the sum of the angles change as the triangle is dragged around **D**?

The proofs of SSS, SAS, ASA, and HL as valid shortcuts for showing congruent triangles did not require the use of Euclid's Fifth postulate. Thus they are all valid shortcuts for showing triangles are congruent in hyperbolic plane geometry. Use SSS to produce two congruent hyperbolic triangles in **D.** Drag one triangle near the boundary and one triangle near the center of **D.** What happens?

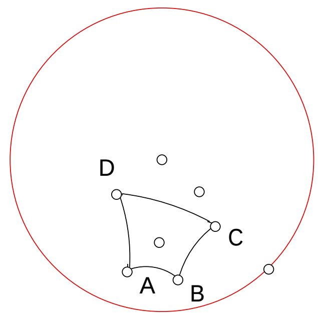
We also had AA, SSS, and SAS shortcuts for similarity in Euclidean plane geometry. Is it possible to find two hyperbolic triangles that are similar but not congruent? Your answer should convince you that it is impossible to magnify or shrink a triangle without distortion! **End of Demonstration 3.4.2.**

3.4.3 Demonstration: Special Triangles. An equilateral triangle is a triangle with 3 sides of equal length. An isosceles triangle has two sides of equal length.

- Create a tool that constructs hyperbolic equilateral triangles in the Poincaré disk. Is an equilateral triangle equiangular? Are the angles always $60°$ as in Euclidean plane geometry?
- Can you construct a hyperbolic isosceles triangle? Are angles opposite the congruent sides congruent? Does the ray bisecting the angle included by the congruent sides bisect the side opposite? Is it also perpendicular? How do your results compare to Theorem 1.4.6 and Corollary 1.4.7? **End of Demonstration 3.4.3.**

3.4.4 Demonstration: Polygons.

- A rectangle is a quadrilateral with four right angles. Is it possible to construct a rectangle in **D**?
- A regular polygon has congruent sides and congruent interior angles.
- To construct a regular quadrilateral in the Poincaré Disk start by constructing an h-circle and any diameter of the circle. Label the intersection points of the diameter and the circle as *A* and *C*. Next construct the perpendicular bisector of the diameter and label the intersection points with the circle as *B* and *D*. Construct the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, .



ABCD is a regular quadrilateral.

Then *ABCD* is a regular quadrilateral. Why does this work? Create a tool from your sketch.

- The following theorems are true for hyperbolic plane geometry as well as Euclidean plane geometry: Any regular polygon can be inscribed in a circle. Any regular polygon can be circumscribed about a circle. Consequently, any regular *n*-gon can be divided into *n* congruent isosceles triangles just as in Euclidean plane geometry.
- Modify the construction to produce a regular octagon and regular 12-gon. Create tools from your sketches.

End of Demonstration 3.4.4.

By now you may have started to wonder how one could define area within hyperbolic geometry. In Euclidean plane geometry there are two natural ways of doing this, one geometric, the other analytic. In the geometric definition we begin with the area of a fixed shape, a square, and then build up the area of more complicated figures as sums of squares so that we could say that the area of a figure is *n* square inches, say. Since squares don't exist in hyperbolic plane geometry, however, we cannot proceed in this way.

Now any definition of area should have the following properties:

- Every polygonal region has one and only one area, (a positive real number).
- Congruent triangles have equal area.
- If a polygonal region is partitioned into a pair of sub regions, the area of the region will equal the sum of the areas of the two sub regions.

Recall, that in hyperbolic geometry we found that the sum of the measures of the angles of any triangle is less than 180. Thus we will define the defect of a triangle as the amount by which the angle sum of a triangle misses the value 180.

3.4.5 Definition. The defect of triangle ΔABC is the number

$$\delta(\Delta ABC) = 180 - m\angle A - m\angle B - m\angle C$$

More generally, the defect can be defined for polygons.

3.4.6 Definition. The defect of polygon $P_1P_2\ldots P_n$ is the number

$$\delta(\Delta P_1P_2\ldots P_n) = 180(n-2) - m\;\; P_1 - m\;\; P_2 - \ldots - m\;\; P_n$$

It may perhaps be surprising, but this will allow us to define a perfectly legitimate area function where the area of a polygon $P_1P_2\ldots P_n$ is *k* times its defect. The value of *k* can be specified once a unit for angle measure is agreed upon. For example if our unit of angular measurement is degrees, and we wish to express angles in terms of radians then we use the constant $k = \pi/180^\circ$. It can be shown that this area function defined below will satisfy all of the desired properties listed above.

3.4.7 Definition. The area $Area_h(P_1P_2\ldots P)$ of a polygon $P_1P_2\ldots P_n$ is defined by

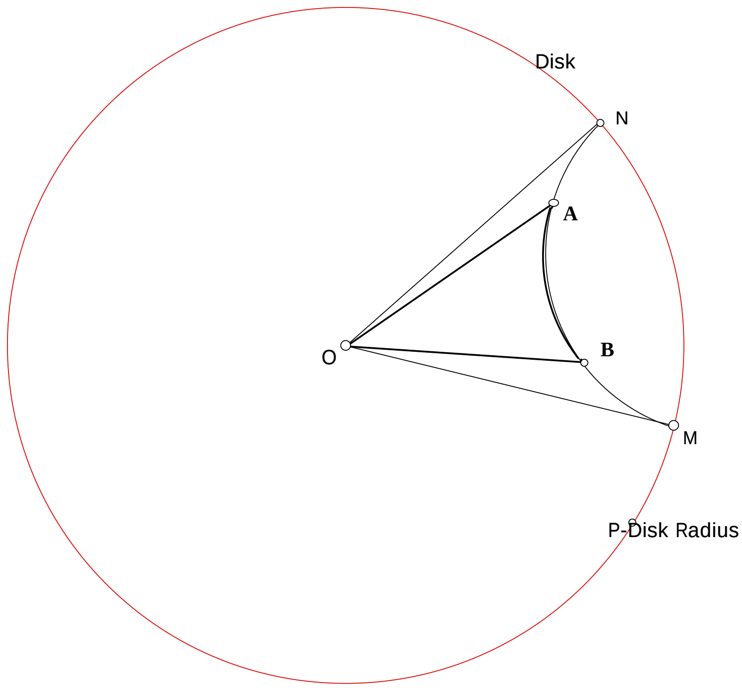
$$Area_h(P_1P_2\ldots P_n) = k\delta(P_1P_2\ldots P_n)$$

where *k* is a positive constant.

Note, that this puts an upper bound on the area of all triangles, namely $180\;\; k$. (More generally, $180\;\; (n-2)\;\; k$ for *n*-gons.) This definition becomes even stranger when we look at particular examples.

3.4.7a Demonstration: Areas of Triangles.

- Open a Poincare disk. Construct a hyperbolic 'triangle' $\Delta_h OMN$ having one vertex *O* at the origin and the remaining two vertices *M, N* on the bounding circle. This is not a triangle in the strict sense because points on the bounding circle are not points in the Poincare disk. Nonetheless, it is the limit of a hyperbolic triangle $\Delta_h OAB$ as *A, B* approach the bounding circle.



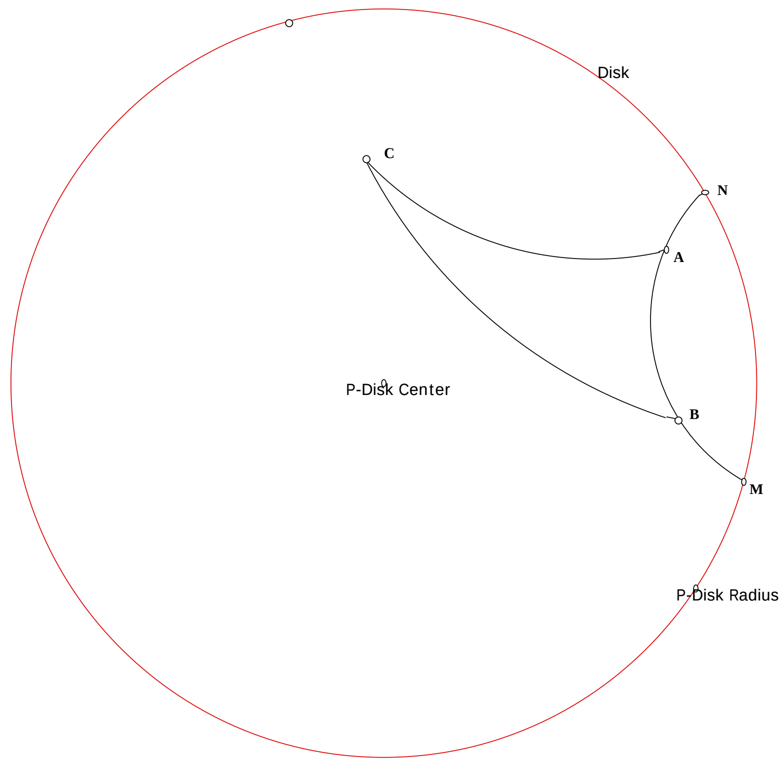
The 'triangle' $\Delta_h OMN$ is called a *Doubly-Asymptotic* triangle.

- Determine the length of the hyperbolic line segment $\overline{AB}$ using the length script. Then measure each of the interior angles of the triangle and compute the area of $\Delta_h OAB$ (use $k=1$). What happens to these values as A, B approach M, N along the hyperbolic line through A, B? Set

$$Area_h(\Delta_h OMN) = \lim Area_h(\Delta_h OAB)$$

Explain this value by relating it to properties of $\Delta_h OMN$.

- Repeat this construction, replacing the center O by any point C in the Poincaré disk.



What value do you obtain for $Area_h(\Delta_h CAB)$? Now let A, B approach M, N along the hyperbolic line through A, B and set

$$Area_h(\Delta_h CMN) = \lim Area_h(\Delta_h CAB);$$

again we say that $\Delta_h(CMN)$ is a doubly-asymptotic triangle. Relate the value of $Area_h(\Delta_h CMN)$ to properties of $\Delta_h CMN$.

- Select an arbitrary point L on the bounding circle and let C approach L. We call $\Delta_h LMN$ a *triply-asymptotic* triangle. Now set

$$Area_h(\Delta_h LMN) = \lim Area_h(\Delta_h CMN).$$

Explain your value for $Area_h(\Delta_h LMN)$ in terms of the properties of $\Delta_h LMN$.

End of Demonstration 3.4.7a.

Your investigations may lead you to conjecture the following result.

3.4.8 Theorem.

(a) The area of a hyperbolic triangle is at most 180*k* even though the lengths of its sides can be arbitrarily large.

(b) The area of a triply-asymptotic triangle is always 180*k* irrespective of the location of its vertices on the bounding circle.

By contrast, in Euclidean geometry the area of a triangle can become unboundedly large as the lengths of its sides become arbitrarily large. In fact, it can be shown that Euclid's Fifth Postulate is equivalent to the statement: *there is no upper bound for the areas of triangles.*

3.4.9 Summary. The following results are true in both Euclidean and Hyperbolic geometries:

- SAS, ASA, SSS, HL congruence conditions for triangles.
- Isosceles triangle theorem (Theorem 1.4.6 and Corollary 1.4.7)
- Any regular polygon can be inscribed in a circle.

The following results are strictly Euclidean

- Sum of the interior angles of a triangle is $180°$.
- Rectangles exist.

The following results are strictly Hyperbolic

- The sum of the interior angles of a triangle is less than $180°$.
- Parallel lines are not everywhere equidistant.
- Any two similar triangles are congruent.

Further entries to this list are discussed in Exercise set 3.6.

As calculus showed, there is also an analytic way introducing the area of a set *A* in the Euclidean plane as a double integral

$$\iint_A dxdy.$$

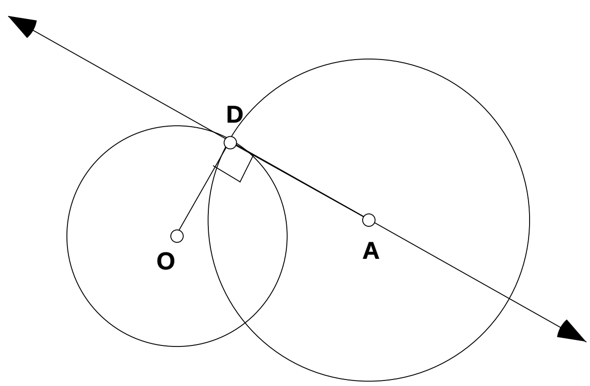
An entirely analogous analytic definition can be made for the Poincaré disk. What is needed is a substitute for *dxdy*. If we use standard polar coordinates (r,θ) for the Poincaré disk, then the hyperbolic area of a set *A* is defined by

$$Area_h(A) = \iint_A \frac{4r\,dr\,d\theta}{1-r^2}.$$

Of course, when *A* is an *n*-gon, it has to be shown that this integral definition of area coincides with the value defined by the defect of *A* up to a fixed constant *k* independent of *A*. Calculating areas with this integral formula often requires a high degree of algebraic ingenuity, however.

3.5 ORTHOGONAL CIRCLES. Orthogonal circles, *i.e.* circles intersecting at right angles, arise on many different occasions in plane geometry including the Poincaré disk model **D** of hyperbolic plane geometry introduced in the previous section. In fact, their study constitutes a very important part of Euclidean plane geometry known as Inversion Theory. This will be studied in some detail in Chapter 5, but here we shall develop enough of the underlying ideas to be able to explain exactly how the tools constructing h-lines and h-segments are obtained.

Note first that two circles intersect at right angles when the tangents to both circles at their point of intersection are perpendicular. Another way of expressing this is say that the tangent to one of the circles at their point of intersection *D* passes through the center of the other circle as in the figure below.

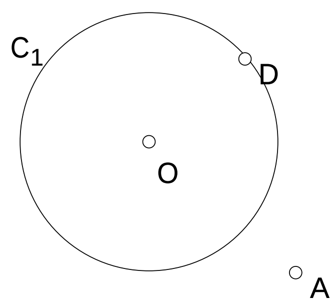


Does this suggest how orthogonal circles might be constructed?

3.5.1 Exercise. Given a circle C_1 centered at O and a point D on this circle, construct a circle C_2 intersecting C_1 orthogonally at D. How many such circles C_2 can be drawn?

It should be easily seen that there are many possibilities for circle C_2. By requiring extra properties of C_2 there will be only one possible choice of C_2. In this way we see how to construct the unique h-line through two points P, Q in **D**.

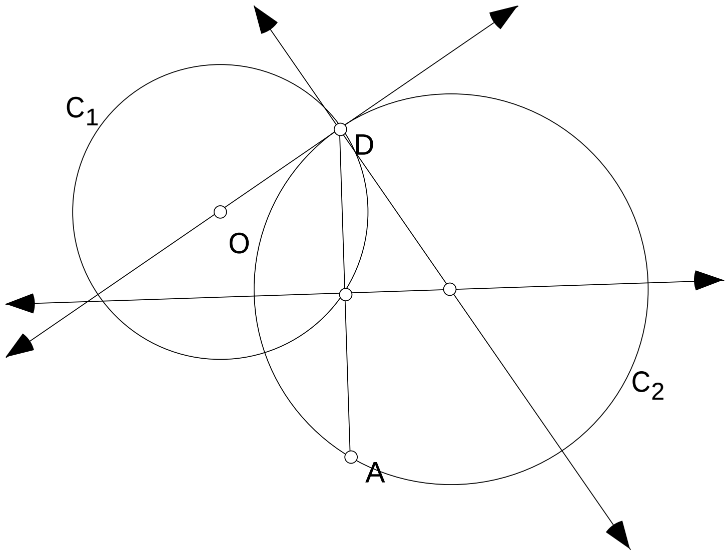
3.5.2 Demonstration. Given a circle C_1 centered at O, a point A not on C_1 , as well as a point D on C_1, construct a circle C_2 passing through A and intersecting C_1 at D orthogonally. How many such circles C_2 can be drawn?



Sketchpad provides a very illuminating solution to this problem.

- Open a new sketch. Draw circle C_1, labeling its center O, and construct point A not on the circle as well as a point D on the circle.
- Construct the tangent line to the circle C_1 at D and then the segment $\overline{AD}$.
- Construct the perpendicular bisector of $\overline{AD}$. The intersection of this perpendicular bisector with the tangent line to the circle at D will be the center of a circle passing through both A and D and intersecting the circle C_1 orthogonally at D. Why?

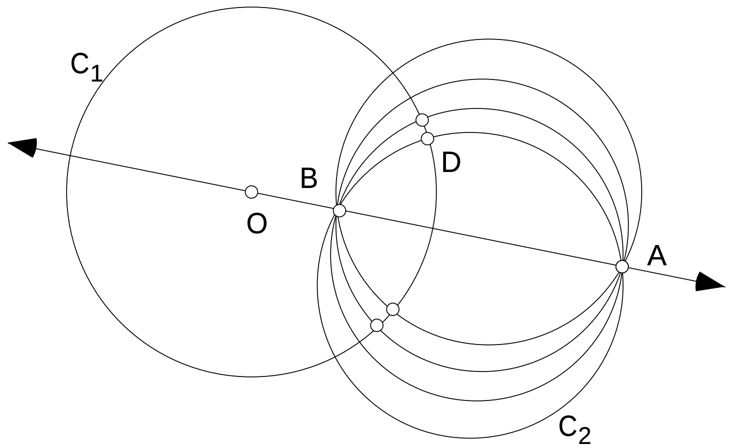
The figure below illustrates the construction when A is outside circle $C_{1.}$



What turns out to be of critical importance is the locus of circle C_2 passing through *A* and *D* and intersecting the given circle C_1 orthogonally at *D*, as *D* moves. Use Sketchpad to explore the locus.

- Select the circle C_2, and under the **Display menu** select trace circle. Drag *D*.
- Alternatively you can select the circle C_2, then select the point *D* and under the **Construct menu** select locus.

The following figure was obtained by choosing different *D* on the circle C_1 and using a script to construct the circle through *A* (outside C_1) and *D* orthogonal to C_2. The figure you obtain should look similar to this one, but perhaps more cluttered if you have traced the circle.



Your figure should suggest that all the circles orthogonal to the given circle C_1 that pass through *A* have a second common point on the line through *O* (the center of C_1) and *A*. In the figure above this second common point is labeled by *B*. [Does the figure remind you of anything in Physics - the lines of magnetic force in which the points *A* and *B* are the poles of the magnet. say?] Repeat the previous construction when *A* is inside C_1 and you should see the same result.

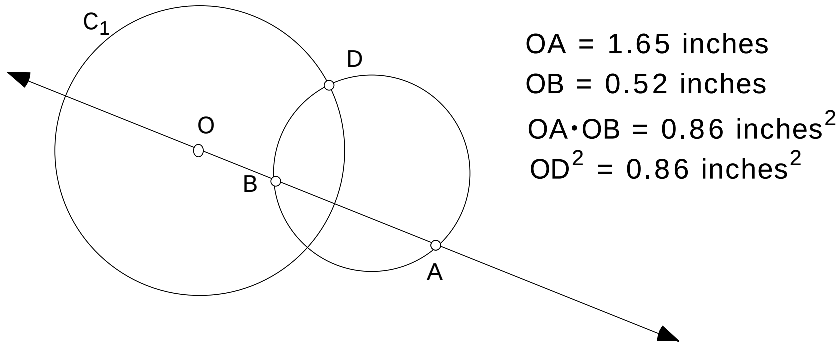
End of Demonstration 3.5.2.

At this moment, Theorem 2.9.2 and its converse 2.9.4 will come into play.

3.5.3 Theorem. Fix a circle C_1 with center O, a point *A* not on the circle, and point *D* on the circle, Now let *B* be the point of intersection of the line through *O* with the circle through *A* and *D* that is orthogonal to C_1. Then B satisfies

$$OA.OB = OD^2.$$

In particular, the point *B* is independent of the choice of point *D*. The figure below illustrates the theorem when *A* is outside C_1.



Proof. By construction the segment $\overline{OD}$ is tangential to the orthogonal circle. Hence $OA.OB = OD^2$ by Theorem 2.9.2. **QED**

Theorem 3.5.3 has an important converse.

3.5.4 Theorem. Let C_1 be a circle of radius r centered at O. Let A and B be points on a line through O (neither A or B on C_1). If $OA.OB = r^2$ then any circle through A and B will intersect the circle C_1 orthogonally.

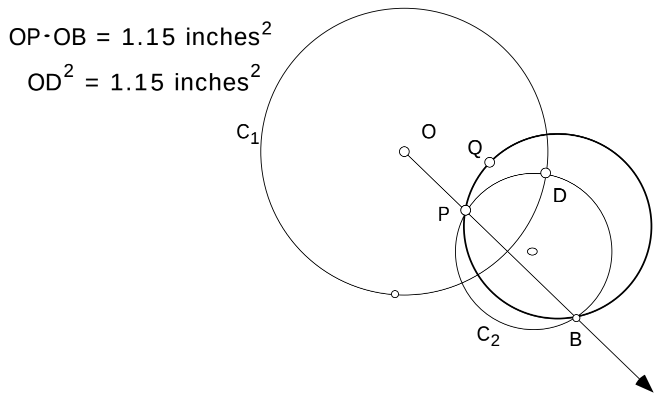
Proof. Let D denote a point of intersection of the circle C_1 with any circle passing through A and B. Then $OA.OB = OD^2$. So by Theorem 2.9.4, the line segment $\overline{OD}$ will be tangential to the circle passing through A, B, and D. Thus the circle centered at O will be orthogonal to the circle passing through A, B, and D. **QED**

Theorems 3.5.3 and 3.5.4 can be used to construct a circle orthogonal to a given circle C_1 and passing through two given points P, Q inside C_1. In other words, we can show how to construct the unique h-line through two given points P, Q in the hyperbolic plane **D.**

3.5.5 Demonstration.

- Open a new sketch and draw the circle C_1, labeling its center by O. Now select arbitrary points P and Q inside C_1.
- Choose any point D on C_1.
- Construct the circle C_2 passing through P and D that is orthogonal to C_1. Draw the ray starting at O and passing through P. Let B be the other point of intersection of this ray with C_2. By Theorem 3.5.3 $OP.OB = OD^2$. Confirm this by measuring OP, OB, and OD in your figure.

· Construct the circumcircle passing through the vertices P, Q and B of ΔPQB. By Theorem 3.3.4 this circumcircle will be orthogonal to the given circle.



If P and Q lie on a diameter of C_1 then the construction described above will fail. Why? This explains why there had to be separate scripts in Sketchpad for constructing h-lines passing through the center of the bounding circle of the Poincaré model **D** of hyperbolic plane geometry.

The points A, B described in Theorem 3.5.3 are said to be *Inverse Points*. The mapping taking A to B is said to be *Inversion*. The properties of inversion will be studied in detail in Chapter 5 in connection with tilings of the Poincaré model **D**. Before then in Chapter 4, we will study transformations. **End of Demonstration 3.5.5.**

3.6 Exercises. In this set of exercises, we'll look at orthogonal circles, as well as other results about the Poincaré Disk.

Exercise 3.6.1. To link up with what we are doing in class on orthogonal circles, recall first that the equation of a circle **C** in the Euclidean plane with radius r and center (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

which on expanding becomes

$$x^2 - 2hx + y^2 - 2ky + h^2 + k^2 = r^2.$$

Now consider the special case when **C** has center at the origin (0, 0) and radius 1. Show that the equation of the circle orthogonal to **C** and having center (h, k) is given by

$$x^2 - 2hx + y^2 - 2ky + 1 = 0.$$

Exercise 3.6.2. One very important use of the previous problem occurs when **C** is the bounding circle of the Poincaré disk. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two points inside the circle **C**, *i.e.*, two h-points. Show that there is one and only one choice of (h, k) for which the circle centered at (h, k) is orthogonal to **C** and passes through A, B. This gives a coordinate geometry proof of the basic Incidence Property of hyperbolic geometry saying that there is one and only one h-line through any given pair of points in the Poincaré Disk. Assume that A and B do not lie on a diagonal of C.

Exercise 3.6.3. Open a Poincaré Disk and construct a hyperbolic right triangle. (A right triangle has one $90°$ angle.) Show that the Pythagorean theorem does not hold for the Poincaré disk **D**. Where does the proof of Theorem 2.3.4 seem to go wrong?

Exercise 3.6.4. Open a Poincaré Disk. Find a triangle in which the perpendicular bisectors for the sides do not intersect. In Hyperbolic plane geometry, can any triangle be circumscribed by a circle? Can any triangle be inscribed by a circle? Why or why not?

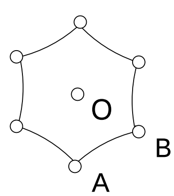
Exercise 3.6.5. Find a counterexample in the Poincaré Disk model for each of the following theorems. That is show each theorem is strictly Euclidean.

(a) The opposite sides of a parallelogram are congruent. (A parallelogram is a quadrilateral where opposite sides are parallel.)

(b) The measure of an exterior angle of a triangle is equal to the sum of the measure of the remote interior angles.

Exercise 3.6.6. Using Sketchpad open a Poincaré Disk. Construct a point P and any diameter of the disk not through P. Devise a script for producing the h-line through P perpendicular to the given diameter (also an h-line).

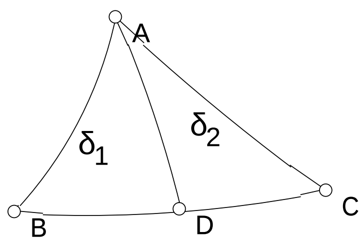
Exercise 3.6.7. The defect of a certain regular hexagon in hyperbolic geometry is 12. ($k=1$)



- Find the measure of each angle of the hexagon.
- If O is the center of the hexagon, find the measure of each interior angle of each sub-triangle making up the hexagon, such as ΔABO as shown in the figure.

- Are each of these sub-triangles equilateral triangles, as they would be if the geometry were Euclidean?

Exercise 3.6.8. Given ΔABC as shown with δ_1 and δ_2 as defects of the sub triangles ΔABD and ΔADC



prove $\delta(\Delta ABC) = \delta_1 + \delta_2$.