function $f: A \to B$. These observations are the basis of the following definition. In this definition, we merely are saying that the function *is* the sort of collection of ordered pairs that we have been discussing.

DEFINITION. Let A and B be sets. *A function with domain A and range B* is a collection f of ordered pairs (a, b), such that

(1) for each (a, b) in f, $a \in A$;

(2) each a in A is the first term of exactly one pair (a, b) in f; and

(3) for each (a, b) in f, $b \in B$.

When we write $b = f(a)$, we mean that (a, b) belongs to the collection f. From here on, we proceed to handle functions exactly as before.

A somewhat similar device enables us to give an explicit definition of the idea of a *relation defined on a set A*. We have been using this idea somewhat informally, writing $a < b$ to mean that a has the relation $<$ to b, and, more generally, $a * b$ is written to mean that a has the relation $*$ to b. Now, given a relation $*$, defined on the set A, we can form the collection

$$\{(a, b) | a * b\}.$$

Conversely, given any collection of ordered pairs of elements of A, we can define a relation $*$, by saying that $a * b$ if the pair (a, b) belongs to the collection. In the following definition, we are saying that the relation *is* the collection. Recall, of course, that $A \times A$ is the set of all ordered pairs of elements of A.

DEFINITION. A *relation defined on a set A* is a subset of $A \times A$.

For example, let

$$A = \{1, 2, 3\},$$

and let

$$* = \{(1, 2), (1, 3), (2, 3)\}.$$

Then $*$ is a relation. (It is, in fact, the usual relation $<$.)

It is not necessary, of course, to denote relations by peculiar symbols. For example, if $A = \{1, 2, 3\}$, as before, we may let

$$G = \{(2, 1), (3, 1), (3, 2)\}.$$

Thus $2G1$, $3G1$ and $3G2$, because $(2, 1)$, $(3, 1)$, and $(3, 2)$ belong to G. (In fact, G is the relation $>$.)

PROBLEM SET 3.2

1. Let $A = \{1, 2, 3, 4\}$. Let

$$G = \{(4, 2), (4, 1), (4, 3), (2, 1), (2, 3), (1, 3)\}.$$

Is G a relation? Is G an order relation?

2. Let A be as before, and let G be the set of all ordered pairs (a, b) such that a and b belong to A and $a \neq b$. Is G a relation? Is G an order relation?

3. Is the following collection a function? If so, what are its domain and image?

$$\{(0, 0), (1, 1), (2, 4), (3, 2), (4, 2), (5, 4), (6, 1)\}$$

Can you see a systematic way in which this collection might have been constructed?

4. Is the following collection a function?

$$\{(0, 1), (1, 0), (0, 0)\}$$

5. Let f be the set of all ordered pairs (x, y) such that x and y belong to R and $y = x^2$. Is this a function?

6. The same question is asked for the set of all ordered pairs (x, y) such that x and y belong to R and $x = y^2$.

7. Consider a rectangular coordinate system in the plane, in the usual sense of analytic geometry. Every point has a pair of coordinates (x, y). For the purposes of this question, let us regard points as indistinguishable from the ordered pairs (x, y) that describe them. Thus every figure, that is, every set of points, becomes a collection of ordered pairs of real numbers. Under what conditions, if any, do the following figures represent functions?

(a) a triangle
(b) a single point
(c) a line
(d) a circle
(e) a semicircle, including the end points
(f) an ellipse

What, in general, is the geometric condition that a figure in the coordinate plane must satisfy, to be a function?

3.3 THE DISTANCE FUNCTION

So far, the structure dealt with in our geometry has been the triplet

$$[S, \mathcal{L}, \mathcal{P}].$$

We shall now add to the structure by introducing the idea of distance. To each pair of points there will correspond a real number called the distance between them. Thus we want a distance function d, subject to the following postulates.

D-0. d is a function

$$d: S \times S \to \mathbf{R}.$$

D-1. For every P, Q, $d(P, Q) \geqq 0$.

D-2. $d(P, Q) = 0$ if and only if $P = Q$.

D-3. $d(P, Q) = d(Q, P)$ for every P and Q in S.

Here we have numbered our first postulate D-0 because it is never going to be cited in proofs; it merely explains what sort of object d is. Of course $d(P, Q)$ will be called the distance between P and Q, and, for the sake of brevity, we shall write $d(P, Q)$ simply as PQ. (We shall be using distances so often that we ought to reserve for them the simplest notation available, and "PQ" is absolute rock-bottom.)

Surely any reasonable notion of distance ought to satisfy D-1 through D-3. We might have required also that

$$PQ + QR \geqq PR,$$

which would say, approximately, that "a straight line is the shortest distance between two points." But as it happens, we don't need to make this statement a postulate, because it can be proved on the basis of other geometric postulates, to be stated later.

Henceforth, until further notice, the distance function d is going to be part of our structure. Thus the structure, at the present stage, is

$$[S, \mathcal{L}, \mathcal{P}, d].$$

The distance function is connected up with the rest of the geometry by the ruler postulate D-4, which we shall state presently.

We ordinarily think of the real numbers as being arranged on a line, like this:

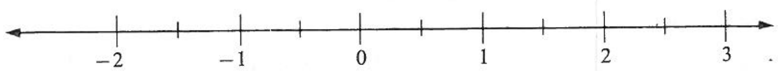

FIGURE 3.1

If the "lines" in our geometry, that is, the elements of $\mathcal{L}$, really "behave like lines," then we ought to be able to apply the same process in reverse and label the points of any line L with numbers in the way that we label the points of the x-axis in analytic geometry:

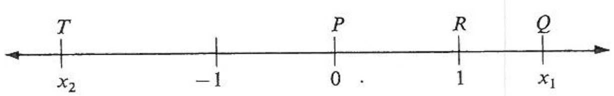

FIGURE 3.2

If this is done in the usual way, then we have a one-to-one correspondence,

$$f: L \leftrightarrow \mathbf{R},$$

between the points of L and the real numbers. This correspondence will turn out to be a *coordinate system,* in a sense which we shall soon define. Meanwhile, therefore, if $x = f(P)$, we shall refer to x as the *coordinate* of P. In the figure, the coordinates of P, Q, R, and T are $0, x_1, 1$, and x_2. If the coordinates are related to distance in the usual way, then

$$PQ = |x_1| \quad \text{and} \quad PT = |x_2|.$$

In fact, no matter where Q and T may lie on the line, we will always have

$$QT = |x_2 - x_1|.$$

(You can check this for the cases $x_2 < x_1 < 0$, $x_2 < 0 < x_1$, and $0 < x_2 < x_1$. There is no harm in assuming that $x_2 < x_1$, because when x_1 and x_2 are interchanged, both sides of our equation are unchanged.)

Obviously nothing can be proved by this discussion, because the postulates that we have stated so far do not describe any connection at all between the distance function and lines. All that we have been trying to do is to indicate why the following definition, and the following postulate, are reasonable.

DEFINITION. Let

$$f: L \leftrightarrow \mathbf{R}$$

be a one-to-one correspondence between a line L and the real numbers. If for all points P, Q of L, we have

$$PQ = |f(P) - f(Q)|,$$

then f is a *coordinate system* for L. For each point P of L, the number $x = f(P)$ is called the *coordinate* of P.

D-4. *The Ruler Postulate.* Every line has a coordinate system.

The postulate D-4 is called the ruler postulate because, in effect, it furnishes us with an infinite ruler which can be laid down on any line and used to measure distances along the line. This kind of ruler is not available in classical Euclidean geometry. When we speak of "ruler-and-compass constructions" in classical geometry, the first of these abstract drawing instruments is not really a ruler, because it has no marks on it. It is, properly speaking, merely a straight-edge. You can use it to draw the line containing two different points, but you can't use it to measure distances with numbers or even to tell whether two distances PQ, RT are the same.

As it stands, D-4 says merely that every line has at least one coordinate system. It is easy to show, however, that there are lots of others.

Theorem 1. If f is a coordinate system for L, and

$$g(P) = -f(P)$$

for each point P of L, then g is a coordinate system for L.

Proof. It is plain that the condition $g(P) = -f(P)$ defines a function $L \to \mathbf{R}$. And this function is one to one, because if $x = g(P)$, it follows that $-x = f(P)$, and $P = f^{-1}(-x)$, so that P is uniquely determined by x.

It remains to check the distance formula. Given that

$$x = g(P), \qquad y = g(Q),$$

we want to prove that

$$PQ = |x - y|.$$

We know that

$$-x = f(P), \qquad -y = f(Q).$$

Since f is a coordinate system, it follows that

$$PQ = |(-x) - (-y)|.$$

Therefore

$$PQ = |y - x|$$
$$= |x - y|.$$

which was to be proved.

Theorem 1 amounts to a statement that if we reverse the direction of the coordinate system, then we get another coordinate system. We can also shift the coordinates to left or right.

Theorem 2. Let f be a coordinate system for the line L. Let a be any real number; and for each $P \in L$, let

$$g(P) = f(P) + a.$$

Then $g: L \to \mathbf{R}$ is a coordinate system for L.

The proof is very similar to that of the preceding theorem. Combining the two, we get the following theorem.

Theorem 3. *The Ruler Placement Theorem.* Let L be a line, and let P and Q be any two points of L. Then L has a coordinate system in which the coordinate of P is 0 and the coordinate of Q is positive.

Proof. Let f be any coordinate system for L. Let $a = f(P)$; and for each point T of L, let $g(T) = f(T) - a$.

Then g is a coordinate system for L; and $g(P) = 0$. If $g(Q) > 0$, then g is the system that we were looking for. If $g(Q) < 0$, let $h(T) = -g(T)$ for every $T \in L$. Then h satisfies the conditions of the theorem.

PROBLEM SET 3.3

1. Show that D-1, D-2, and D-3 are consequences of the ruler postulate.

3.4 BETWEENNESS

One of the simplest ideas in geometry is that of betweenness for points on a line. In fact, Euclid seems to have regarded it as too simple to analyze at all, and he uses it, without comment, in proofs, but doesn't mention it at all in his postulates.

Roughly speaking, B is between A and C on the line L if the points are situated like this:

FIGURE 3.3

or like this:

FIGURE 3.4

(Logically speaking, of course, the second figure is superfluous, because on a line, there is no way to tell left from right or up from down.) What we need, to handle betweenness mathematically, is an exact definition which conveys our common-sense idea of what betweenness ought to mean. One such definition is as follows.

DEFINITION. Let A, B, and C be three collinear points. If

$$AB + BC = AC,$$

then B *is between* A *and* C. In this case we write A-B-C.

As we shall see, this definition is workable. It enables us to prove that betweenness has the properties that it ought to have.

Theorem B-1. If A-B-C, then C-B-A.

This is a triviality. If $AB + BC = AC$, then $CB + BA = CA$.

The rest of the basic theorems on betweenness are going to depend essentially on the ruler postulate.

Betweenness for real numbers is defined in the expected way; y is between x and z if either $x < y < z$ or $z < y < x$. In this case we write x-y-z. (Confusion with subtraction is unlikely to occur, because "x minus y minus z" would be ambiguous anyway.)

Lemma 1. Given a line L with a coordinate system f and three points A, B, C with coordinates x, y, z, respectively. If x-y-z, then A-B-C.

Proof of lemma. (1) If $x < y < z$, then

$$AB = |y - x| = y - x,$$

because $y - x > 0$. For the same reasons,

$$BC = |z - y| = z - y$$

and

$$AC = |z - x| = z - x.$$