Exploration of Spherical Geometry

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Abstract. We explore how geometry on a sphere compares to traditional plane geometry. We present formulas and theorems about the 2-gon and the 3-gon in spherical geometry. We end with an alternative proof of Euler's Formula using spherical geometry.

1. Introduction. Euclid's fifth postulate, the "parallel postulate," guarantees that there is one and only one line parallel to a given line l through a point not on l. In contrast to Euclidean geometry, here we treat a particular geometry where there are no parallel lines: spherical geometry.

There are many geometries without parallel lines. For example, consider the geometry consisting of exactly one point. In this geometry, there are no lines whatsoever, and so no parallel lines in particular. But this geometry is trivial and uninteresting. On the other hand, spherical geometry is interesting, because it is a non-Euclidean geometry, and because it is a model for studying shapes on nonplanar surfaces, such as the Earth.

In Section 2, we use Euclidean definitions to describe objects in spherical geometry. These definitions help to provide a set of abstract formal definitions for spherical geometry, and also to strengthen the analogies between spherical and plane geometry.

In Section 3, we use the definitions of Section 2 to see if spherical geometry satisfies Hilbert's axioms, so is, by definition, a Hilbert Plane. It isn't! However, were spherical geometry a Hilbert plane, then we would automatically have many results in spherical geometry because so much has already been proved about Hilbert planes.

In Section 4, we quit trying to fit spherical geometry into a planar hole. Instead, we develop new results specific to spherical geometry. The key is the absence of parallel lines. Its consequences are profound and surprising. For one, two-sided polygons exist! For another, angles are more important than sides in proving congruences and computing areas.

In Section 5, we use the framework of spherical geometry to support an alternative proof of Euler's Formula; thus spherical geometry can be used to prove results in other geometries.

2. Definition of Spherical Geometry. Spherical geometry is a geometry where all the points lie on the surface of a sphere. Nevertheless, we can use points off the sphere and results from Euclidean geometry to develop spherical geometry. For example, the center of the sphere is the fixed point from which the points in the geometry are equidistant. However, the center itself is not contained in the geometry.

In spherical geometry, we define a point, or *S-point*, to be a Euclidean point on the surface of the sphere. However, we cannot define an S-line to be a Euclidean line. Rather, consider a plane that passes through the center of the sphere. Its intersection

with the sphere is a circle—indeed, a *great circle*, that is, a circle whose diameter is equal to the sphere's. We define an *S-line* to be a great circle.

Consider two distinct planes that contain the center of the sphere. Because the two planes intersect, they must intersect in a Euclidean line. Because this line passes through the center of the sphere, it must also pass through the surface of the sphere in two S-points. As these two S-points are like the opposite poles on Earth, we call them *antipodal points*.

Antipodal points are often treated as a special case of a pair of S-points. Because any pair of S-lines is formed by distinct planes that contain the center of the sphere, any two distinct S-lines intersect in two antipodal points. Recall that *parallel* lines are defined as lines that do not intersect; thus, there are no parallel lines in spherical geometry.

Consider S-points A, B, and C that lie on less than half an S-line. Consider their respective projections from the center of the sphere to Euclidean points A', B', and C' in a plane p that does not contain the center. The points A', B', and C' are collinear in p. We say that the S-point B is *between* S-points A and C, and write $A * B * C$ or $C * B * A$, if B' is between A' and C' in p.

Admittedly, our notion of betweenness in spherical geometry differs from the intuitive notion. For example, even though we may expect that an S-point equidistant from two antipodal points is considered to be between them, it is not because the three S-points do not lie in less than half an S-line. In this sense, the definition of betweenness in spherical geometry is weak, but betweenness is a difficult concept to define in a geometry that mostly revolves around circles. Nevertheless, we need a definition of betweenness to make sense of Hilbert's third axiom about congruence of line segments.

An *S-segment* is an arc of a great circle whose length is less than half the circumference of the great circle. So an S-segment is determined by its endpoints. These endpoints cannot be antipodal because of the restriction on the length of the S-segment.

Consider an S-line l and two distinct S-points A , B on it, and another S-line m and two distinct S-points C, D on it. The S-lines l and m may be the same. If the chord connecting A and B is congruent to the chord connecting C and D, then the S-segments AB and CD are called *congruent*. Clearly, two S-segments are congruent exactly when their arcs are congruent.

An S-angle is analogous to a plane angle in Euclidean geometry in that it fits Euclid's definition, given in [**3**]: "[An angle is] the inclination to one another of two lines . . . which meet one another and do not lie in a straight line." More precisely, we define the *S-angle* formed by the intersection of two distinct S-lines as the angle in the plane formed by the tangent lines of the S-lines. Two S-angles are said to be *congruent* if the angles of their tangents are congruent.

3. Hilbert's Axioms. Unfortunately, spherical geometry does not satisfy Hilbert's axioms, so we cannot always apply the theory of the Hilbert plane to spherical geometry. In this section, we determine which axioms hold and why the others do not.

First, we recall Hilbert's axioms for a geometry from [**1**, pp. 66, 73–74, 82, 90–91]. Hilbert's axioms of incidence are as follows:

(I1) Any two distinct points lie on a unique line.
(I2) Every line contains at least two points.
(I3) There exist three noncollinear points.

Not all of Hilbert's axioms hold in spherical geometry. Notably, (I1) does not. Indeed, any two antipodal points lie on infinitely many distinct S-lines. However, any

other two points do lie on a unique S-line. Nevertheless, the shortest path between any two S-points is an S-line. Ironically, (I1) is one of the simplest axioms of Euclidean geometry.

The remaining two axioms of incidence, (I2) and (I3), do hold in spherical geometry. The second axiom holds because every S-line in spherical geometry is a circle, which contains infinitely many S-points.

The third axiom holds too. Indeed, consider three distinct S-lines where each pair is perpendicular. The three S-lines divide the sphere into eight equal triangular regions, where the corners of each region are three noncollinear S-points. But note that, given any two antipodal S-points, every third S-point is colinear with them.

Hilbert's axioms of betweenness are as follows:

(B1) If $A * B * C$, then A, B, C are three distinct points on a line, and also $C * B * A$.
(B2) For any two distinct points A and B, there exists a point C such that $A * B * C$.
(B3) Given three distinct points on a line, exactly one is between the other two.
(B4) Let A, B, C be three noncollinear points, and l a line not containing any of them. If l contains a point lying between A and B, then l must also contain either a point lying between A and C, or a point lying between B and C, but not both.

Although our definition of betweenness is weak, (B1) does hold. Indeed, three points that exhibit betweenness are collinear by definition. We also have that $A * B * C$ is equivalent to $C * B * A$ by definition.

However, (B2) fails under our definition of betweenness. Indeed, if A and B are antipodal points, then there is no S-point C such that $A * B * C$ because A and B do not lie on less than half an S-line.

Not surprisingly, (B3) fails as well owing to antipodal points. Indeed, given three distinct S-points where two are antipodal, the third cannot lie between the other two because the three do not lie on less than half an S-line.

The failure of (B3) yields some counterintuitive results. For example, consider four points A, B, C, D lying in order on an S-line with A and D antipodal. Then $A * B * C$ and $B * C * D$ hold, but $A * B * D$ and $A * C * D$ fail.

Strictly speaking, (B4) holds: if the S-line l intersects S-segment AB at D, then it intersects either S-segment AC or S-segment BC. However, let n be the S-segment not intersected by l. Then the S-line m that contains n has two points in common with l, but neither of these two points is on n.

Given the difficulties involving betweenness, it is a wonder why we try to define it at all, or at least in this manner. However, Axiom (C3), stated below, requires a definition of betweenness, so some definition of betweenness is necessary to discuss it. Our definition takes advantage of the Euclidean definition of betweenness to formalize a property that obviously is difficult to describe.

Hilbert's axioms of congruence ($\cong$) are as follows:

(C1) Given a line segment AB, and given a ray r originating at a point C, there exists a unique point D on r such that $AB \cong CD$.
(C2) If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.
(C3) Given three colinear points A, B, and C satisfying $A * B * C$, and three further colinear points D, E, and F satisfying $D * E * F$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

(C4) Given an angle $\angle BAC$ and a ray DF, there exists a unique ray DE on a given side of the line DF such that $\angle BAC \cong \angle EDF$.

(C5) For any three angles α, β, and γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

(C6) Given triangles ABC and DEF, suppose that $AB \cong DE$ and $AC \cong DF$, and that $\angle BAC \cong \angle EDF$. Then the two triangles are congruent, namely $BC \cong EF$, and also $\angle ABC \cong DEF$ and $\angle ACB \cong \angle DFE$.

Axioms (C1)–(C3) hold in spherical geometry. For (C1), consider the Euclidean line segment that connects A and B. Construct a sphere with center at C and the segment as its radius. The point where the sphere first intersects r is D. Indeed, D lies on r, and the chord from C to D is congruent to the chord from A to B. Because the chords are congruent, S-line AB is congruent to S-line CD. If D did not exist, then the sphere about C and the sphere containing C would not intersect, so the chord connecting A to B would have to be larger than the diameter of the sphere on which A, B, and C lie; this situation is clearly impossible.

For (C2), note that the chord connecting A to B is congruent to the chord connecting C to D. Similarly, the chord connecting A to B is congruent to the chord connecting E to F. From transitivity, it follows that the chord connecting C to D is congruent to the chord connecting E to F. So, by definition, $CD \cong EF$. Additionally, because the chord adjoining a segment is congruent to itself, every line segment is congruent to itself in spherical geometry.

As mentioned above, (C3) depends on the definition of betweenness. By hypotheses, $AB \cong DE$ and $BC \cong EF$. Because $A,\ B,\ C,\ D,\ E$, and F are points on some sphere whose center is O, say, and because all radii of a sphere are congruent, it follows that

$$OA \cong OB \cong OC \cong OD \cong OE \cong OF.$$

By the side-side-side congruence theorem (SSS), we have $\triangle OAB \cong \triangle ODE$ and $\triangle OBC \cong \triangle OEF$ where $\triangle XYZ$ denotes the Euclidean triangle determined by any three distinct points $X, Y,$ and Z. Hence,

$$\begin{aligned} \angle ABO + \angle CBO &= \angle ABC, \\ &\cong \angle DEO + \angle FEO, \\ &= \angle DEF. \end{aligned}$$

Because $\angle ABC \cong \angle DEF$, it follows that $\triangle ABC \cong \triangle DEF$ by the side-angle-side congruence theorem (SAS). It follows that $AC \cong DF$. So S-line AC is congruent to S-line DF.

As is the case for Hilbert's axioms on the congruence of segments, his axioms on the congruence of angles hold in spherical geometry. Once again, we use our definitions of congruence and Euclidean geometry to show that the axioms hold.

For (C4), take angle $\angle BAC$ and project it onto the tangent plane at A. Project points D and F onto the tangent plane at D, choose a side of DF, and draw the projected angle $\angle BAC$ along DF opening up in the chosen side of DF. Project the new angle onto the sphere. This angle is congruent to $\angle BAC$ because their projections on the tangent plane are congruent by construction.

As expected, transitivity holds similarly for congruence of angles. If $\alpha \cong \beta$, then the projected angles on the tangent planes to the sphere at their vertices are congruent.

Similarly, if $\alpha \cong \gamma$, then the projected angles on the tangent planes to the sphere at their vertices are also congruent. By transitivity, these angles are equal to one another. Therefore, $\beta \cong \gamma$. Additionally, because, for an angle on the sphere, the projected angle on the tangent plane is congruent to itself, every angle is congruent to itself.

The final congruence axiom (C6) is essential for proving theorems about the congruence of triangles, and is also sometimes referred to as the side-angle-side axiom (SAS). By hypothesis, the projected angles at A and D are congruent. Because S-line AB is congruent to S-line DE, and S-line AC is congruent to S-line DF, the projections of these lines are also congruent in the tangent plane. Hence, in the tangent plane, the projected triangles of ABC and DEF are congruent by SAS. Because the projected angles are congruent, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$. It also follows that the projections of S-lines BC and EF are congruent; so their chords are congruent. Therefore, the congruence of the S-lines themselves are congruent.

4. Polygons in Spherical Geometry. Polygons in spherical geometry are analogous to those in Euclidean geometry in that they are figures made of line segments; however, there are vast differences in the theorems about their congruence and their angle sums.

In Euclidean geometry, a polygon must have at least three sides, but not in spherical geometry, where we have the 2-sided lune. A *lune* is a polygon formed by two half-S-lines with common endpoints, its (antipodal) vertices.

A lune has two equal angles, which sit at its two vertices. And these angles determine the lune, up to congruence. The two sides have the same length, namely, πR on a sphere with radius R. So any two lunes have sides of the same length; in particular, congruent sides do not mean congruent lunes.

As expected, a spherical triangle has three sides. Consider the corresponding three S-lines—they form eight spherical triangles. Suppose the S-lines arise from three perpendicular planes that intersect at the center of the sphere. Then any two of the S-lines are also perpendicular. Thus, the eight congruent triangles have three right angles. Hence, in each triangle, the angles sum to 270°. Clearly, this situation is different from that in Euclidean geometry, where the sum of the angles of any triangle is 180°. But there is an analogous statement, which is given in Theorem 4-1.

Theorem 4-1 (Girard's Theorem). *Let T be a spherical triangle on a sphere of radius R. Then, in radians, the sum of the angles is equal to*

$$\pi + \text{area}(T)/R^2.$$

Proof: We present the proof in [**4**]. Figure 4-1 shows T and its opposite triangle T'.

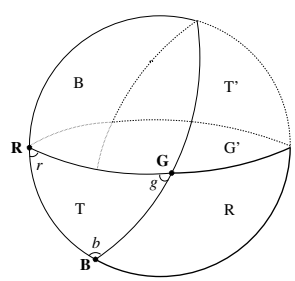

Figure 4-1. Spherical Triangle T.

It also shows the spherical triangles R, G', B, which come out of the page, and their opposites R, G', B. Let $\mathbf{R}$, $\mathbf{G}$, $\mathbf{B}$ be the vertices of T opposite the sides that border R, G, B. Let r, g, b be the angles at these vertices. Let L_r, L_g, L_b be the lunes composed of T and R, of T and G, and of T and B. Although this labelling may be confusing, the web site [**4**] has the figure in a color Java applet, which allows us to rotate the sphere. Accompanied by the applet, the proof may be easier to follow; so the same notation is used here.

For simplicity, use the same symbol to denote both a figure and its area. Then summing the areas of all six lunes, we get

$$\begin{aligned} L_r + L_g + L_b + L'_r + L'_g + L'_b &= (T+R) + (T+G) + (T+B) \\ &\quad + (T'+R') + (T'+G') + (T+B') \\ &= (T+R+G+B+T'+R'+G'+B') \\ &\quad + (T+T+T'+T') \\ &= \text{area(sphere)} + 2T + 2T'. \end{aligned}$$

The area of a sphere is $4\pi R^2$. Hence the area of a lune is $a/2\pi \cdot 4\pi R^2$, or $2R^2 a$, where a is the angle of the lune, as $a/2\pi$ is the fraction of the sphere that the lune occupies. Plugging into the first and last lines above, we get

$$\begin{aligned} 2R^2 r + 2R^2 g + 2R^2 b + 2R^2 r + 2R^2 g + 2R^2 b &= 4R^2(r+g+b) \\ &= 4\pi R^2 + 2T + 2T'. \end{aligned}$$

Now, T and T' are images of each other under the antipodal map. It is an isometry. So T and T' have the same area. Dviding by $4R^2$ now yields the assertion. □

Owing to Girard's Theorem, the sum of the angles of a triangle must always be greater than 180°, because area$(T)/R^2$ is always positive.

Rewriting the formula in Girard's Theorem, we obtain Girard's Formula for the area of a spherical triangle:

$$\text{area}(T) = R^2(r+g+b-\pi).$$

Note that the area depends on the angles, rather than on the length and height of the triangle as in Euclidean geometry. Also note that triangles with congruent angles have the same area.

In fact, angle-angle-angle (AAA) congruence implies congruence of the spherical triangles themselves; whereas, in Euclidean geometry, AAA only implies similarity. Indeed, consider the triangles ABC and $AB'C'$ in Figure 4-2. Assume $\angle BCA \cong \angle B'C'A$

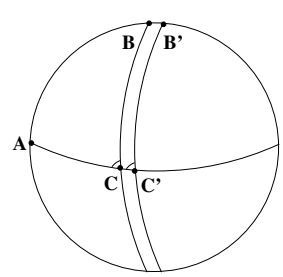

Figure 4-2. Proof of AAA.

and $\angle ABC \cong \angle AB'C'$. Then, by Girard's Theorem, the two spherical triangles have the same area. Hence they coincide.

The congruence theorems SAS, ASA, and SSS are also valid in spherical geometry; see [**2**] for proofs. In Euclidean geometry, because the sum of the angles is a constant, ASA implies AAS: if two triangles have two consecutive angles congruent and also a side not contained by the two angles congruent, then the third angles of the triangle must also be congruent, and enclose the congruent side; so the triangles are congruent by ASA. Yet the sum of the angles is not constant in spherical geometry; so it is not surprising that AAS does not hold.

For example, consider the spherical triangles PAB and PAB' in Figure 4-3. Angles

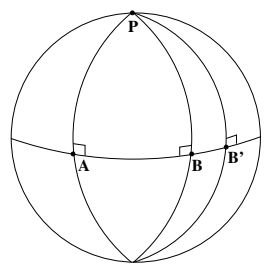

Figure 4-3. Why AAS and SSA Fail.

$\angle PAB$, $\angle PBA$, and $\angle PB'A$ are all right angles. The S-segments PB and PB' are both 1/4 of the circumference of the sphere, so they are congruent. Thus PAB and PAB' satisfy the hypotheses of AAS, yet they are clearly not congruent, because the area of PAB is strictly less than that of PAB', as one is contained within the other.

In addition, SSA also fails in spherical geometry. For example, again refer to Figure 4-3. Note that $PA \cong PA$ and $PB \cong PB'$ as before, and that triangles PAB and PAB' share right angle PAB. We have already shown that triangle PAB is not congruent to triangle PAB'.

5. Euler's Formula. Consider a convex polyhedron P with V vertices, E edges, and F faces. Euler proved that

$$V - E + F = 2.$$

It is possible to extend Girard's Theorem to spherical polygons and then to derive Euler's result in an alternative manner. We present such a proof, following Polking [**4**].

In Euclidean geometry, the sum of the angles in a convex n-gon is $(n-2)\pi$ radians. Indeed, any n-gon can be divided into $n-2$ triangles by drawing lines from one vertex of the polygon to the other nonadjacent vertices as shown in Figure 5-1. The sum of the

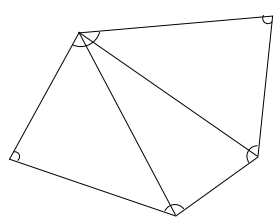

Figure 5-1. Dissecting a polygon into triangles.

angles of these triangles is the sum of the angles of the entire polygon. Since the sum of the angles in each triangle is π radians, the sum of all of the angles in the polygon is $(n-2)\pi$ radians. Applying the same idea to spherical polygons, we get

$$\sum\big(\text{angles of the } n\text{-gon}\big) = (n-2)\pi + \frac{1}{R^2}\cdot\big(\text{area of the } n\text{-gon}\big)n. \tag{5-1}$$

Let C be point in the interior of the given polyhedron P. Then we can construct a sphere centered at C that is large enough to contain P entirely, such as the one shown in Figure 5-2. Draw a line from C through each vertex of P, and find its intersection

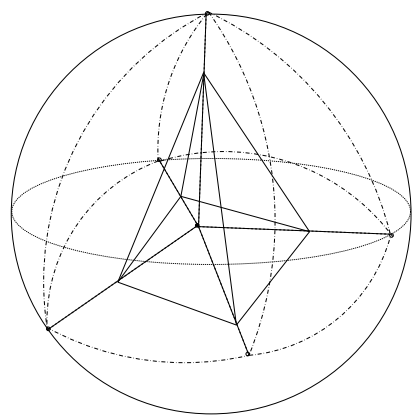

Figure 5-2. Polyhedron inside a sphere.

with the surface of the sphere. Connect these projected points on the sphere as they are connected in P, thus carving the sphere into various spherical polygons Q_i, each of which corresponds to a face of P. Say Q_i has e_i sides.

Using Formula (5-1), we get

$$\sum_{i=1}^{F}\big(\text{sum of angles of } Q_i\big) = \sum_{i=1}^{F}(e_i-2)\pi + \frac{1}{R^2}\sum_{i=1}^{F}\text{area } Q_i.$$

because the sum of the angles around every vertex is 2π, we have

$$2\pi V = \sum_{i=1}^{F}(e_i-2)\pi + \frac{1}{R^2}\sum_{i=1}^{F}\text{area } Q_i.$$

Since the spherical polygons cover the sphere, which has area $4\pi R^2$, we obtain

$$\begin{aligned} 2\pi V &= \sum_{i=1}^{F}(e_i-2)\pi + 4\pi \\ &= \sum_{i=1}^{F} e_i\pi - \sum_{i=1}^{F} 2\pi + 4\pi. \end{aligned}$$

Because every edge is counted twice in this sum, we have

$$\begin{aligned} 2\pi V &= 2\pi E - \sum_{i=1}^{F} 2\pi + 4\pi \\ &= 2\pi E - 2\pi F + 4\pi. \end{aligned}$$

Thus, we obtain Euler's famous formula:

$$V - E + F = 2.$$

References

[**1**] Hartshorne, R., "Geometry: Euclid and Beyond," Springer, 2000.

[**2**] Hollyer, L., http://www.uh.edu/~hollyer/Module6/m6ppt/sld083.htm.

[**3**] Joyce, D., "Euclid's Elements," http://aleph0.clarku.edu/∼djoyce/java/elements/elements.html.

[**4**] Polking, J. C., http://math.rice.edu/∼pcmi/sphere/gos4.html.