Math 420 - Homework 6

November 6, 2012

1 Heavy bag

Examine the image in Figure 1, which is constructed after being given triangle $\triangle ABC$.

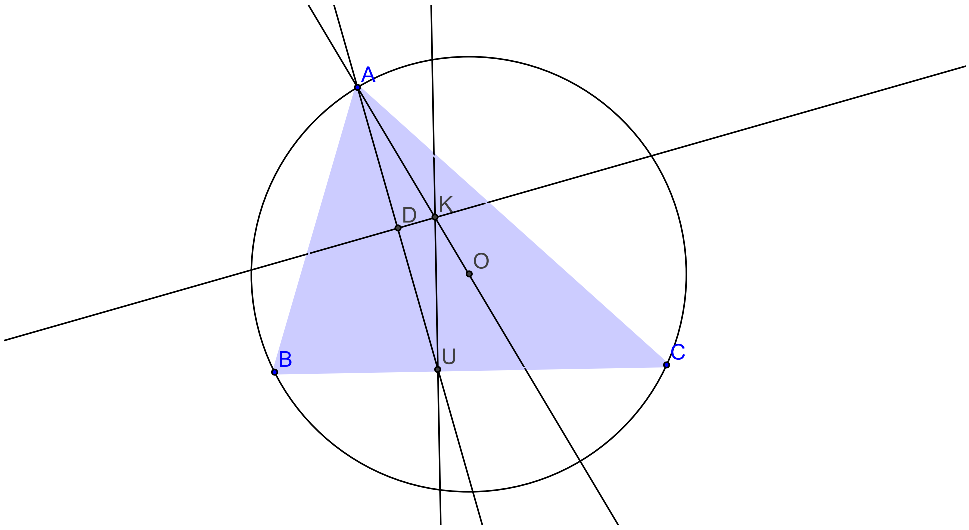

Figure 1: This triangle and associated constructs were created in GeoGebra.

1. Recreate this picture in GeoGebra using the following information.
   - The circle centered at O is the unique circle through A, B and C.
   - AU is the bisector of $\angle BAC$.
   - D is the midpoint of AU.
2. Prove that the point K is the intersection of the following three lines:
   - Line AO, which includes a diameter of the circle,
   - The line perpendicular to BC which includes U,
   - The perpendicular bisector of the segment AU.

2 The squared circle

This section deals with Saccheri quadrilaterals (S.Q.) and their properties. Recall that a S.Q. has two adjacent right angles on either side of the base, two congruent arms extending from the base, and a summit opposite the base. Sometimes, you may read that the summit is called the "upper base", though I don't use that terminology.

A brief introduction is posted on the class website, and that material may prove useful for these proofs. Specifically, here we will prove that all triangles (not in a spherical geometry) need to have angle sum less than π.

1. Prove that the summit angles of a S.Q. are congruent.
2. Prove that in any S.Q. $\square ABCD$ (with base AD) that $\angle BDC \geq \angle ABD$.
3. Prove that if $\triangle ABD$ has $\angle BAD = \pi/2$, then $\angle ABD + \angle BDA \leq \pi/2$.
4. Given $\triangle ABC$, let D be the foot of the perpendicular from B to AC. Prove that, if AC is the longest side of the triangle, then D is between A and C. *Hint: Proof by contradiction may be the best choice here.*
5. Using these results, prove that the angle sum of any triangle is less than or equal to π.

3 One-two punch

In our discussion on the area function, we frequently exploited the relationship between triangles and S.Q. Here I want you to prove some of the content that we glossed over in the lectures.

1. I often said that every triangle $\triangle ABC$ had an associated S.Q. $\square ABDE$. Draw this situation, correctly label the appropriate lines and angles, and make sure that it is acceptable for both a Euclidean and hyperbolic geometry.
2. What determines the length of the arms of the S.Q.?
3. Prove that $\triangle ABC$ and $\square ABDE$ are equivalent via finite dissection.
4. Why is it important that $\triangle ABC \equiv \square ABDE$?
5. Given $\triangle ABC$ and S.Q. $\square ABDE$, consider a new triangle $\triangle ABH$. Denote the intersection of AH and the line passing through DE as G. Prove that if $AG \cong GH$ then $\square ABDE$ is the S.Q. for $\triangle ABH$ as well.

4 Split decision

I'll be impressed if you can get all of this problem, part of which will be bonus.

At the end of the discussion on the relationship between hyperbolic area and defect, we suggested the following proposal

- Given $\triangle ABC$ and $\triangle DEF$ with $\delta(\triangle ABC) > \delta(\triangle DEF)$, we should be able to find a point P between A and C such that $\delta(\triangle BPC) = \delta(\triangle DEF)$.

Let's define the point P_r such that the measure of $\angle ABP_r = r$. Think of r as a variable and P_r is the resulting point on the segment AC. Define the function f as

$$f(r) = \delta(\triangle ABP_r).$$

1. What would be a logical definition of $f(0)$? Explain why.
2. On what domain should f be defined? Explain why.
3. Prove that f is a strictly increasing function.
4. Assuming that f is continuous, prove the proposal above.
5. Prove that f is continuous. *Hint: This is the bonus part.*