Math 420 - Homework 5

October 16, 2012

1 Batting Practice

This section deals with a couple proofs from some of our earlier material. It is assigned here to provide additional practice with proofs. As such, I expect these proofs to be complete, well thought out, and well presented. You will probably benefit from using methods of trigonometry (and maybe even calculus) in trying to prove these, so feel free to do so.

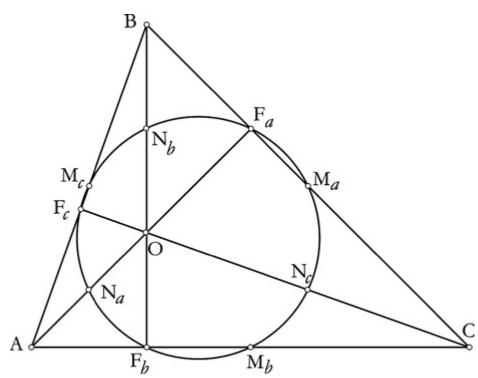
1. Given a triangle $\triangle ABC$, prove that the following nine points all lie on the same circle:

   - The three midpoints of the sides (M_a, M_b, M_c),
   - The three bases of the altitudes (intersection of the altitude with the side opposite the originating angle) (F_a, F_b, F_c), and
   - The midpoints of the three segments connecting the orthocenter (intersection of the three altitudes) and the originating angle (N_a, N_b, N_c).

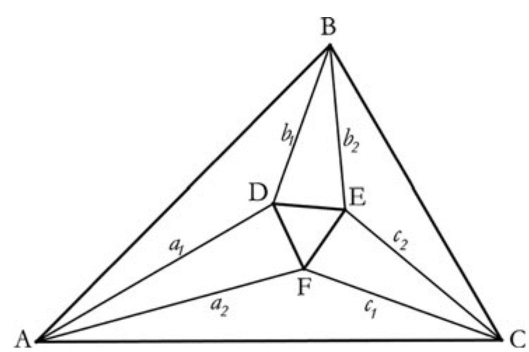
   A picture of this situation is provided in Figure 1a.

   Also determine the center and radius of this circle. Although I should make you do this with generic coordinates, I'll let you work with the coordinates $A = (0, 0)$, $B = (2, 4)$ and $C = (6, 0)$.

2. Consider a triangle, and draw in the lines that trisect all the angles: this situation is demonstrated in Figure 1b. Prove that the smaller triangle created by the intersection of these trisectors is an equilateral triangle.



(a) Prove that the nine points listed all lie on the same circle.



(b) Prove that this inner triangle is always equilateral.

Figure 1: Example triangles for this section.

2 Who's on first?

This section deals with the Ruler Postulate. Some material for this topic has been placed on the class website, although I still haven't found a great reference. If you come across one while working, please let me know about it.

1. Assume that P and Q are points on a line which satisfies the Ruler Postulate, i.e., $d(P,Q) = |f(P) - f(Q)|$ for some bijection f. Prove the following **Basic Properties**
   (a) $d(P,Q) \geq 0$,
   (b) $d(P,Q) = 0$ if and only if $P = Q$, and
   (c) $d(P,Q) = d(Q,P)$.

   Explain in normal parlance why each of these ideas makes sense from a physical standpoint.

2. Assume that $k > 0$. Prove that if d satisfies the Ruler Postulate, then so does
$$d_k(P,Q) = kd(P,Q).$$

3. Assume that $k > 0$ and ℓ is some fixed line. Define
$$d_{k,\ell}(P,Q) = \begin{cases} d_k(P,Q), & \text{if } P,Q \in \ell \\ d(P,Q), & \text{else} \end{cases}.$$
   Prove that $d_{k,\ell}$ satisfies the Ruler Postulate.

4. Explain why the outcome of this previous proof means that even though a distance function satisfies the Ruler Postulate, it need not satisfy the **Triangle Inequality**
$$d(P,Q) + d(Q,R) \geq d(P,R).$$

3 Infield fly rule

The Incidence Axiom often comes in three pieces:

- For every pair of distinct points P and Q there is exactly one line ℓ such that P and Q lie on ℓ.
- For every line ℓ there exist at least two distinct points P and Q such that both P and Q lie on ℓ.
- There exist three points that do not all lie on any one line.

Basically, it means that two points (in general) make a line, and three points (in general) make three lines.

1. I want you to explain why we call it the Ruler Postulate, and not the Ruler Theorem. Explain why, even with the Incidence Axiom and some mapping d which satisfies the basic properties and triangle inequality from earlier, we still can't prove that the Ruler Postulate must be satisfied.

2. Recall our work deriving the equation for the orthogonal circles relative to the Poincaré Disk, specifically from HW 4.
   (a) Prove that a circle with center (h,k) which is orthogonal to the Poincaré Disk has equation
$$x^2 - 2hx + y^2 - 2ky + 1 = 0.$$

(b) Demonstrate (and explain under what conditions) there is only one choice for (h, k) so that the circle passes through points (a_1, a_2) and (b_1, b_2) inside the Poincaré Disk.

(c) Explain what happens when such a circle does not exist.

(d) Explain how this gives a "proof" of the Incidence Axiom in hyperbolic space, if you have designed an appropriate coordinate geometry system. Of course this isn't a proof, because we have used the Incidence Axiom in Euclidean space to create all these structures, but it is still nice to see that things are consistent.

4 Big league chew

1. Prove that $f = \log(x/(a-x))$ is a bijection from $(0, a)$ to $\mathbb{R}$.

2. Suppose S is a nonempty set, and $d : S \times S \to \mathbb{R}$ is defined by

$$d(P,Q) = \begin{cases} 0, & P = Q \\ 1, & P \neq Q \end{cases}.$$

   Prove this d satisfies the basic properties and triangle inequality.

3. Suppose S is a nonempty set, $k > 0$ and d satisfies the basic properties and triangle inequality. Define $\hat{d} : S \times S \to \mathbb{R}$ as

$$\hat{d}(P,Q) = \begin{cases} 0, & P = Q \\ k + d(P,Q), & P \neq Q \end{cases}.$$

   Prove that $\hat{d}$ satisfies the basic properties and triangle inequality.

5 Extra innings

1. We have discussed in the past the idea that the angles of a hyperbolic triangle do not add up to π, but rather some value always less than π. This difference is often called the defect

$$\delta(\triangle ABC) = \pi - \angle A - \angle B - \angle C.$$

   Consider the situation in Figure 2

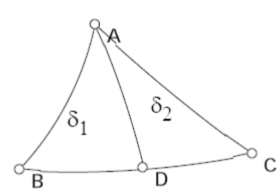


Figure 2: One hyperbolic triangle cut in pieces.

   Prove that $\delta(\triangle ABC) = \delta(\triangle ABD) + \delta(\triangle ACD)$.

2. Explain, using this result, why the triangles which seem the most Euclidean are the smallest.