Math 420 - Homework 4

October 2, 2012

1 Coin Flip

We will often study hyperbolic geometry in the Poincaré disk. The following questions discuss properties of that object.

1. Often we will use the unit circle $|z| < 1$ for $z \in \mathbb{C}$ to define the Poincaré disk. Obviously this is not an infinite object, but it is meant to represent an infinite domain. Explain this contradiction.
2. Why is the boundary of the circle, $|z| = 1$, not included in the Poincaré disk?
3. Refer to the picture of the gentleman taking a stroll below. Explain what is happening in hyperbolic space, and contrast that with what is happening in Euclidean space.
4. What's going on at the center of the Poincaré disk? Why does that look different than elsewhere on the disk?

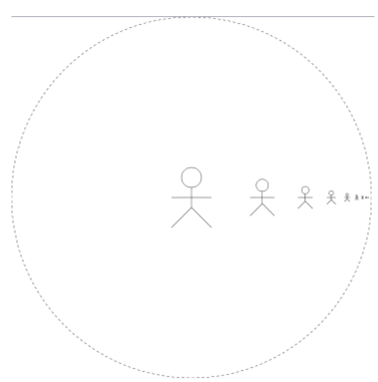

Figure 1: A hyperbolic man walks into a bar 10 miles east from his house in the middle of the disk, and the Euclidean bartender says "We don't serve minors."

2 20 yard line

1. When we consider Euclidean space, or any geometry, what is our definition of a straight line?
2. How did we represent straight lines in a spherical geometry?
3. How do we represent straight lines in the Poincaré disk?
4. Derive the equation for a hyperbolic line. The equation should be a circle in Euclidean space.
5. When will a line in hyperbolic space not be a circle in Euclidean space?
6. What does it mean for two lines to be parallel in hyperbolic space? Why can we allow two lines to intersect, but be parallel to the same line in hyperbolic space, but not in Euclidean space?
7. Why do the hyperbolic lines intersect the edge of the Poincaré disk at right angles?

3 Crossing Route

This section will require the use of the NonEuclid software.

1. How do we measure angles of hyperbolic lines? Explain it in terms of the Euclidean circles that have generated the lines.

2. Triangles in hyperbolic space can have any angle sum between $0 < \alpha + \beta + \gamma < \pi$, where α, β, and γ are the angles in the triangle. Explain what a triangle with very small angle sum looks like. Explain what a triangle with very large angle sum looks like.

3. The following NonEuclid activities for triangles have been filched from the website. Decide the following statements' validity, and construct an example, or a counter-example.

   (a) The longest side of a triangle is opposite the greatest angle.

   (b) The three altitudes of a triangle intersect at a single point. Apparently this is a difficult problem, so check out the website (Activity 3.03-3) for some hints.

   (c) The sum of any two sides of a triangle is always greater than the length of the third side.

   (d) The product "base times height" is the same regardless of which side is chosen as the base.

4. Follow the instructions for (Activity 3.03-9) to learn how to construct an isosceles triangle. Are the base angles of an isosceles triangle congruent, as they are in Euclidean space.

5. For an equilateral triangle in Euclidean space, all angles are $\pi/3$. What about in hyperbolic space? If a triangle is equilateral, does it have to have all angles equal?

6. If two triangles have all their angles congruent, are the two triangles also congruent?

4 4th and inches

Here again, it will be beneficial to use the NonEuclid software, although it won't always be necessary.

1. Define a rectangle.

   (a) Given that definition, can a rectangle exist in hyperbolic space?

2. Can you create a quadrilateral with 4 equal angles? What about a quadrilateral with 4 right angles?

3. Can you create a square in hyperbolic space, if we defined a square to be a quadrilateral with 4 equal sides?

4. Does the pythagoream theorem hold in hyperbolic space?

5 2 point conversion

1. Circles exist in hyperbolic space, much as they exist in Euclidean space. In Euclidean geometry, the ratio of circumference to diameter is a constant, π. Before mathematicians had a solid grasp on π, they would approximate it by using regular polygons inscribed within a circle. As the polygons got closer to the circle, the ratio of area to diameter reached a limit. Find that limit for hyperbolic space, if you can.