Math 420 - Homework 2

September 4, 2012

This homework will constitute most of our coverage of Chapter 2 in the book. You will be expected to read through the book to answer these questions. This homework, unlike the previous one, will be graded and I expect it to either be typed or very neatly written.

1 Proofs

This section deals with proofs involving geometric objects we have described in class. Some of them we have glossed over, so you will be expected to go through the text to learn about them. You may also use the internet, as I find this a useful source of information.

Question 1

From page 89: Prove that the shortest of all chords, passing through a point A taken in the interior of a given circle, is the one which is perpendicular to the diameter drawn through A.

Answer 1

Begin by assuming that there is a circle with radius r and center O and some point A inside the circle. Define the chord BC to be the chord perpendicular to the segment OA. Also define DE to be some other chord which is not perpendicular to OA.

Theorem 105 from the book says that the diameter perpendicular to a chord bisects that chord. Therefore, we can define H to be the midpoint of DE, and notice that $OH \perp DE$. This also tells us that A is the midpoint of BC, because $OA \perp BC$.

Let's now define some equations that we know to be true, because A and H are the midpoints of the chords BC and DE respectively

- $DH = HE = \frac{1}{2}DE$
- $AB = BC = \frac{1}{2}BC$

Our goal for this proof is to show that $|BC| < |DE|$, which would say that any chord other than BC is shorter than it.

We can exploit the fact that $OH \perp DE$ to realize that the triangle $\triangle OHE$ is a right triangle. Applying the pythagorean theorem, we know that the sides satisfy

$$|HE|^2 + |OH|^2 = |OE|^2.$$

We know that OE is a radius of the circle, which, combined with our earlier statements, allows us to write

$$\begin{aligned}
|HE|^2 + |OH|^2 &= |OE|^2, \\
|HE|^2 + |OH|^2 &= r^2, \\
\frac{1}{4}|DE|^2 + |OH|^2 &= r^2, \\
|DE| &= 2\sqrt{r^2 - |OH|^2}\ .
\end{aligned}$$

We are also aware that the the triangle $\triangle OAB$ is a right triangle, for the same reason given above, $OA \perp BC$. Following the same pattern just described we can end up with the equally useful statement

$$|BC| = 2\sqrt{r^2 - |OA|^2}\,. \tag{1}$$

Now, we can again leverage the fact that $OH \perp DE$ to notice that $\triangle OHA$ is a right triangle. Like all right triangles, it satisfies the Pythagorean theorem

$$\begin{aligned} |OH|^2 + |HA|^2 &= |OA|^2 \\ |OH|^2 &< |OA|^2 \end{aligned}$$

Using this inequality, we can substitute back in to (1) to get

$$\begin{aligned} &|BC| < 2\sqrt{r^2 - |OH|^2}\,, \\ &|BC| < |DE|, \end{aligned}$$

which is the desired result.

Question 2

From page 96: Prove that the shortest segment joining two non-intersecting circles lies on the line of centers.

Answer 2

This problem could be approached a few different ways. I think the easiest approach is to define circles with centers O and P and radii r_O and r_P respectively. We should assume that they do not intersect, since the solution to that problem is obvious. Without loss of generality, assume that $r_P < r_O$, which means that $C_{r_P}(P)$ either lies entirely inside $C_{r_O}(O)$ or entirely outside. The discussion below covers both cases.

Define the intersection of $C_{r_O}(O)$ with OP as A and define the intersection of $C_{r_P}(P)$ with OP as B. We are asked to prove that $|AB|$ is the shortest distance between $C_{r_O}(O)$ and $C_{r_P}(P)$.

We will begin by constructing the tangents to $C_{r_O}(O)$ at A and $C_{r_P}(P)$ at B, which we will call AC and BD respectively. Because tangent lines of a circle are perpendicular to the radius at that point, we know that $AC \perp OC$ and $BD \perp PB$. Furthermore, since AB is on the same line as OP, we can conclude that $AC \perp AB$ and $BD \perp AB$.

Since AC and BD are perpendicular to the same segment, we can conclude that $AC \parallel BC$, which is to say that the tangent lines we created from the two circles are parallel. All we need to do now is prove that the segment of shortest length connecting parallel lines is perpendicular to them. Now we have a lemma that we need to prove:

- **Lemma** - The shortest distance between two parallel lines is covered by a segment perpendicular to them.

- **Proof** - Call the two lines AB and CD, and require $AB \parallel CD$. We can assume, without loss of generality that $AC \perp CD$ and $AD \not\perp CD$. By the parallel postulate, we can use the fact that $AC \perp CD$ to conclude that $AC \perp AB$. Therefore AC is the perpendicular segment between AB and CD. We can prove that $|AD| > |AC|$ by noticing that $\angle ACD$ is a right angle, and therefore $\triangle ACD$ is a right triangle. The hypotenuse is AD and one leg is AC, which allows us to conclude that $|AD| > |AC|$.

Now that we have proved that, we can safely say that the shortest distance between AC and BD (from our problem, not the lemma) is a perpendicular segment. We can conclude that it is AB because AC and BD are tangent lines, meaning they only intersect their respective circles at one point, and thus that is the only point which can connect AC and BD with an orthogonal segment. Thus we know that the shortest distance between two circles passes through AB, which is on the line OP.

Question 3

From page 102: Let PA and PB be two tangents to a circle with center O, drawn from the same point P, and let BC be a diameter. Prove that CA and OP are parallel.

Answer 3

We already have all the necessary terms defined for us, so let's dive in. Start by noticing that $\triangle PBO$ and $\triangle PAO$ are congruent by SSA: they have a shared side OP, $OA \cong OB$ because they are both radii, and $\angle OAB$ and $\angle OBP$ are both right angles. This allows us to conclude that $\angle AOP \cong \angle BOP$.

Now we need to use some of our arc knowledge from the book. We can start by recalling **Theorem 123**, which says that *An inscribed angle measure half of the subtended arc.* We are going to use this on the inscribed angle $\angle ACB$ to say that $\angle ACB = \frac{1}{2}\overset{\frown}{AB}$. We can also conclude, from the definition of arc, that $\angle AOB = \overset{\frown}{AB}$.

Because we have already established that $AOP \cong \angle BOP$, we know that OP bisects $\angle AOB$. This in turn lets us conclude that $\angle POB = \frac{1}{2}\angle AOB$. Following this through, we see

$$\angle POB = \frac{1}{2}\angle AOB$$
$$\angle POB = \frac{1}{2}\overset{\frown}{AB}$$
$$\angle POB = \angle ACB$$

This lets us conclude that $\angle POB \cong \angle ACB$. Because these angles are formed by the lines OP and AC intersecting the same line CB, we can therefore conclude that $AC \parallel OP$. This last point was discussed in chapter 11 of the book.

2 Constructions

This section deals with Geometric constructions that I would like you to work through. Once again, you can find this material in the book at the pages I have listed. Please use GeoGebra, or some other software, to complete these constructions, because it is the 21st century and I would be embarrassed to see my students using rulers and compasses.

Construction 1

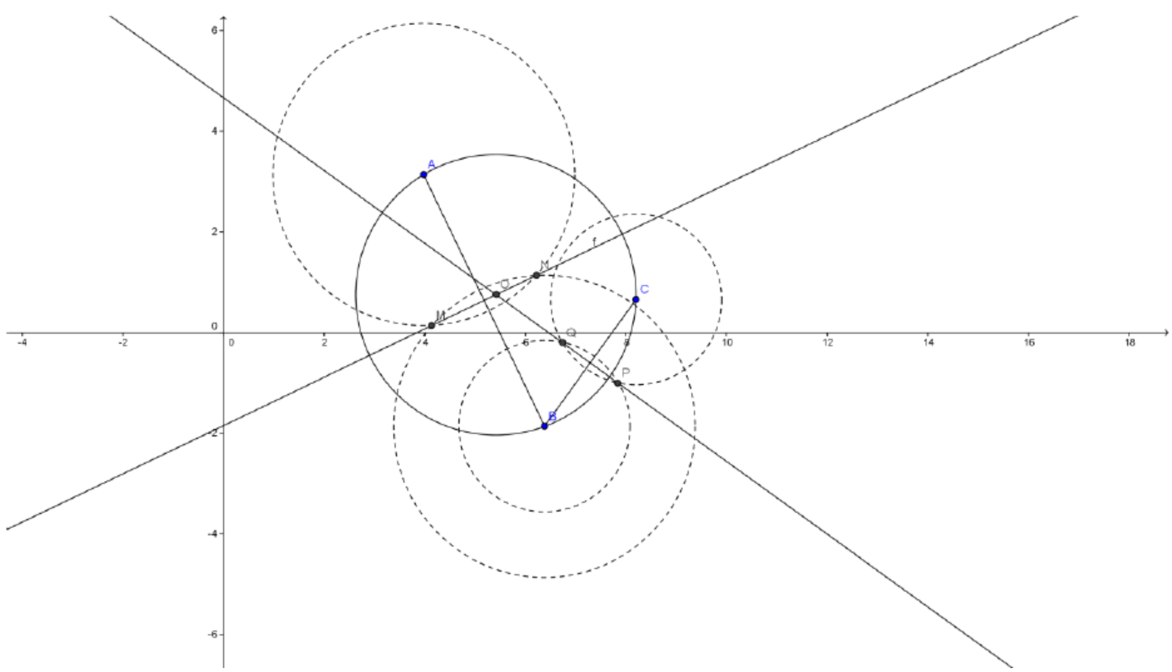
From page 83: Given three points not on a line (possibly $(1, 2)$, $(-1, 2)$, $(0, 1)$) construct the circle that passes uniquely through them.

Answer 1

Construction #1

Procedure

1. Construct segments AB and BC
2. Construct perpendicular bisectors MN and PQ of segments AB and BC, respectively
3. Label intersection point of MN and PQ as O
4. Take AO as radius



Construction 2

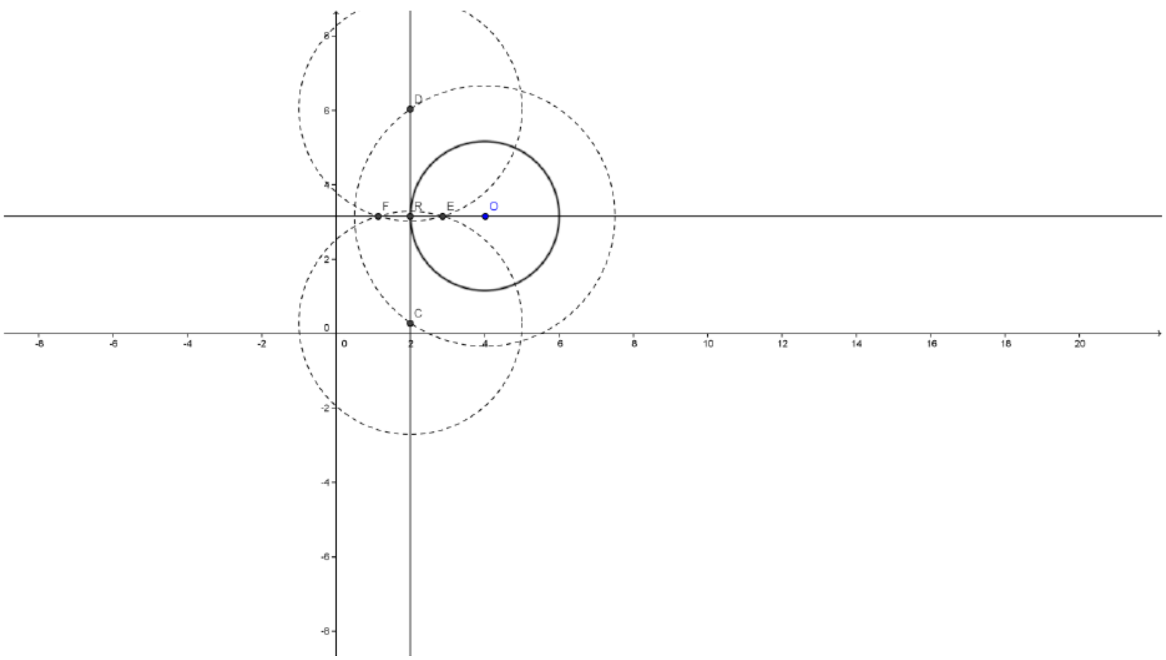
From page 92: Construct a circle which has a given radius $r = 2$ and is tangent to the vertical line $x = 2$.

Answer 2

Construction #2

<u>Procedure</u>

1. Construct line x = 2
2. Identify arbitrary point O on the line x = 4
3. Drop a perpendicular from point O to line x = 2
4. Label the intersection of perpendicular and x = 2 as R (tangency point)
5. Construct a circle with radius OR, passing through point R



Construction 3

From page 103: Given two circles (circle O has radius 2 and center $(3, 4)$ and circle O' has radius 1 and center $(0, 0)$) construct a common tangent line.

Answer 3

Construction #3

Procedure

1. **Construct an auxiliary circle centered at O with radius = 2 -1 = 1.**
2. **Construct a tangent O'C to auxiliary circle.**
3. **Through point C, draw radius OC and extend to intersect circle O at point A**
4. **Draw parallel line to O'C, intersecting circle at tangency point B.**