Math 333 Quiz 8 - April 8, 2013

Question 1

Consider the function

$$f(z) = \frac{1}{z(z-2)}.$$

Question 1.a

Identify and classify the singularities of this function.

Answer 1.a

The function f is a quotient of two polynomials, so we automatically know that any singularities will either be removable or poles, but not essential. The numerator will never be 0, so we can never have removable singularities; therefore, any singularities are poles.

The denominator is equal to 0 when $z = 0$ or $z = 2$, therefore both those points are poles. The order of the poles can be determined by looking at the power of the factor - in this case, both poles are order 1.

Question 1.b

Expand this function in a series which is convergent in the domain $0 < |z| < 2$.

Answer 1.b

This question calls for a series centered at $z_0 = 0$. $f(z)$ has two components, one of which is already centered at $z_0 = 0$: $1/z$. We need to convert the other part to a series centered at $z_0 = 0$:

$$\begin{aligned} \frac{1}{z-2} &= -\frac{1}{2}\frac{1}{1-z/2} \\ &= -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right) = -\frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \dots \end{aligned}$$

This is a geometric series, valid when $|z/2| < 1$, or equivalently, $|z| < 2$. Substituting into our original $f(z)$ gives

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{z-2} \\ &= \frac{1}{z}\left(-\frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \dots\right) \\ &= -\frac{1}{2}\frac{1}{z} - \frac{1}{4} - \frac{1}{8}z - \dots \end{aligned}$$

Question 2

This question deals with residues.

Question 2.a

Given a generic Laurent expansion for a function f:

$$f(z) = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

identify the center of the series and the residue of f at that center.

Answer 2.a

The center of the series is z_0 and the residue is a_{-1}.

Question 2.b

Cauchy's Residue Theorem states that

$$\oint_C f(z)dz = 2\pi\imath \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

for C a closed contour, and f analytic except at z_k *within* C. $\mathrm{Res}(f, z_k)$ is the residue of f at z_k.

Use this theorem to evaluate the integral

$$\oint_C \frac{2z+5}{z^2+4}dz, \qquad C : |z - \imath| = 2.$$

You can evaluate the residue of an order 1 pole of f at $z = z_k$ with

$$\mathrm{Res}(f, z_k) = \lim_{z\to z_k} (z - z_k) f(z).$$

Answer 2.b

For this problem, $f(z)$ is a rational function. Its poles can be found by solving $z^2 + 4 = 0$, which can be factored as $(z + 2\imath)(z - 2\imath) = 0$ and has solutions $z = \pm 2\imath$. Neither of those values causes the numerator to be zero, so their are both poles of order 1.

Only the pole $z_0 = 2\imath$ is inside the closed contour C, so only it contributes to the value of the integral. The residue at that point can be computed with the formula provided:

$$\mathrm{Res}(f, 2\imath) = \lim_{z\to 2\imath} (z - 2\imath)\frac{2z+5}{z^2+4} = \lim_{z\to 2\imath} \frac{2z+5}{z+2\imath} = \frac{4\imath+5}{4\imath}.$$

Plugging that in to the residue theorem gives

$$\oint_C \frac{2z+5}{z^2+4}dz = 2\pi\imath\,\mathrm{Res}(f, 2\imath) = 2\pi\imath\left(\frac{4\imath+5}{4\imath}\right) = \frac{\pi}{2}(4\imath+5).$$