Math 333 Quiz 7 - April 3, 2013

Question 1

Compute the Laurent series for the function

$$f(z) = \frac{\mathrm{e}^{-z}}{z}$$

centered at $z_0 = 0$, and state the domain of convergence of that series. Recall the exponential series

$$\mathrm{e}^{z} = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \qquad z \in \mathbb{C}.$$

Answer 1

Using the series for e^z above, we know that

$$\mathrm{e}^{-z} = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} = 1 - z + \frac{1}{2}z^2 - \dots,$$

which is a series centered at $z_0 = 0$. The other part of $f(z)$, the $1/z$ term, is already a Laurent series centered at $z_0 = 0$. Combining these two series gives

$$\begin{aligned} \frac{\mathrm{e}^{-z}}{z} &= \frac{1}{z}\left(1 - z + \frac{1}{2}z^2 - \dots\right) \\ &= \frac{1}{z} - 1 + \frac{1}{2}z - ... \end{aligned}$$

To determine the convergence, we need the intersection of the domains of convergence of the two components. $1/z$ converges when $|z| > 0$ (for any nonzero z) and the series for e^{-z} is convergent everywhere, so the convergence for $f(z)$ is $|z| > 0$.

Question 2

Compute the Laurent series for the function

$$f(z) = \frac{1}{(z-1)z}$$

centered at $z_0 = 1$, and state the domain of convergence of that series. Recall the geometric series

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \qquad |z| < 1.$$

Answer 2

Again we need a series for a function in two pieces. The term

$$\frac{1}{z-1}$$

is already a series centered at $z_0 = 1$ (albeit a very short series), which is convergent for $|z-1| > 0$. We now need to expand the other half of $f(z)$ around $z_0 = 1$. To do so,

$$\begin{aligned}\frac{1}{z} &= \frac{1}{1+z-1} \\ &= \frac{1}{1-(-(z-1))} \\ &= \sum_{k=0}^{\infty}(-(z-1))^k = 1-(z-1)+(z-1)^2-\ldots\end{aligned}$$

This series is a *geometric* series, which is convergent for

$$|z-1| < 1.$$

Plugging this series into f gives

$$\begin{aligned}f(z) = \frac{1}{z-1}\frac{1}{z} &= \frac{1}{z-1}\left(1-(z-1)+(z-1)^2-\ldots\right) \\ &= \frac{1}{z-1} - 1 + (z-1) - \ldots.\end{aligned}$$

Again, to find the domain of convergence we take the intersection of the domains of convergence of the components of this series, which gives

$$0 < |z-1| < 1.$$

The left portion of this inequality comes from the $1/z$ term, and the right portion comes from the geometric series.