Math 333 Quiz 6 - March 6, 2013

Question 1

A path independent integral can be evaluated by

$$\int_C f(z)dz = F(z_1) - F(z_0), \qquad F'(z) = f(z),$$

where the contour C begins at the point z_0 and ends at the point z_1.

Question 1.a

What condition must f satisfy in order for this theorem to apply?

Answer 1.a

f must be analytic to invoke path independence.

Question 1.b

Using this theorem, evaluate the integral

$$\int_C \mathrm{e}^z dz$$

along the curve

$$C : y = \frac{1}{x}, \qquad 1 \leq x \leq 2.$$

To save time, the answer does not need to be in $x + \imath y$ form.

Answer 1.b

We know that e^z is analytic, because it was proven earlier. That means that we do not need to parameterize, but rather just use path independence. This integral moves along the curve $z(t) = t + \imath/t$, which means that

$$\int_C \mathrm{e}^z dz = \int_{1+\imath}^{2+\imath/2} \mathrm{e}^z dz.$$

This integral is pretty easy, leaving the answer

$$\int_C \mathrm{e}^z dz = \mathrm{e}^{2+\imath/2} - \mathrm{e}^{1+\imath},$$

which for this problem is simplified enough.

Question 2

The Cauchy Integral Formula states that

$$\int_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi\imath}{(n-1)!}f^{(n-1)}(z_0),$$

where C is a closed contour and z_0 lies inside C.

Question 2.a

What condition must f satisfy in order for this formula to apply?

Answer 2.a

f must be analytic inside the closed contour C to invoke the Cauchy Integral Formula.

Question 2.b

Evaluate the integral

$$\int_C \frac{z^2-1}{z^2+1}dz, \qquad C : |z+2\imath| = 2$$

using the Cauchy Integral Formula.

Answer 2.b

We need to write this integral in two components: a numerator which is analytic, and a denominator with an exposed singularity. We know that

$$z^2+1 = (z-\imath)(z+\imath)$$

so we can write

$$\int_C \frac{z^2-1}{z^2+1}dz = \int_C \frac{z^2-1}{(z+\imath)(z-\imath)}dz = \int_C \frac{\left(\frac{z^2-1}{z-\imath}\right)}{z+\imath}dz.$$

Because $f = \frac{z^2-1}{z-\imath}$ is analytic inside C, this form is acceptable. For this integral, $n=1$ and $z_0 = -\imath$, so

$$\begin{aligned}\int_C \frac{z^2-1}{z^2+1}dz &= \frac{2\pi\imath}{1}\frac{(-\imath)^2-1}{-\imath-\imath} \\ &= 2\pi\imath\frac{-1-1}{-\imath-\imath} \\ &= 2\pi\imath\frac{1}{\imath} \\ &= 2\pi\end{aligned}$$