Math 333 Quiz 5 - March 4, 2013

Question 1

Evaluate the integral

$$\int_C \bar{z} dz,$$

by parameterizing C, defined by

$$y = x^2, \qquad 0 \leq x \leq 1.$$

Answer 1

We need to write our parameterization of $z(t)$:

$$z(t) = t + \imath t^2, \qquad 0 \leq t \leq 1.$$

and compute its derivative

$$z'(t) = 1 + 2\imath t.$$

Plugging this into the integral gives

$$\begin{aligned}
\int_C \bar{z} dz &= \int_0^1 \overline{(t + \imath t^2)}(1 + 2\imath t) dt \\
&= \int_0^1 (t - \imath t^2)(1 + 2\imath t) dt \\
&= \int_0^1 (t + \imath t^2 + 2t^3) dt \\
&= \frac{1}{2}t + \frac{1}{3}\imath t^3 + \frac{1}{2}t^4 \Big|_0^1 \\
&= 1 + \frac{1}{3}\imath.
\end{aligned}$$

Question 2

Question 2a

What is the value of

$$\int_C f(z)dz$$

where C is a closed contour and f is analytic inside C?

Answer 2a

That integral is always 0.

Question 2b

Evaluate the integral

$$\int_C \frac{2z}{z^2+4}dz, \qquad C : |z+2\imath| = 2$$

using partial fraction decomposition, properties of integrals, and the theorem

$$\int_C \frac{1}{(z-z_0)^n}dz = \begin{cases} 2\pi\imath & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

from chapter 18.2, where z_0 is **inside** C.

Answer 2b

First, we need to do partial fraction decomposition on the integrand,

$$\frac{2z}{z^2+4} = \frac{1}{z+2\imath} + \frac{1}{z-2\imath}.$$

Now we can write

$$\int_C \frac{2z}{z^2+4}dz = \int_C \frac{1}{z+2\imath}dz + \underbrace{\int_C \frac{1}{z-2\imath}dz}_{=0}$$

The second integral is equal to 0, because it is analytic inside C. The first integral can be evaluated with the theorem above, because the point where it is not analytic $z_0 = -2\imath$ is inside C. This means that

$$\int_C \frac{2z}{z^2+4}dz = 2\pi\imath + 0 = 2\pi\imath.$$