Math 333 Quiz 4 - February 20, 2012

Question 1

For these problems, assume that $z = 1 + \imath$.

Question 1.a

Compute $\ln(z)$.

Answer 1.a

We need to write z in polar form $z = re^{\imath\theta}$:

$$r = |z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2},$$
$$\theta = \arg(z) = \tan^{-1}(1/1) = \pi/4 + 2\pi k, \qquad k \in \mathbb{Z}.$$

Plugging the polar form of z into $\ln(z)$ gives

$$\ln\left(\sqrt{2}\exp(\pi/4\imath + 2\imath\pi k)\right) = \log\sqrt{2} + \imath(\pi/4 + 2\pi k).$$

The value k can take infinitely many values, which means that ln also has infinitely many values.

Question 1.b

Using the value above, determine what $\mathrm{Ln}(z)$ is.

Answer 1.b

When we compute $\mathrm{Ln}(z)$, we use the principal value of the angle θ, meaning that the polar form of z must include an angle in $(-\pi, \pi]$. In this case, that angle is $\mathrm{Arg}(z) = \pi/4$, meaning that

$$\mathrm{Ln}(z) = \log\sqrt{2} + \imath\pi/4,$$

which has only one value.

Question 2

Using the information

$$\cosh(z) = \frac{1}{2}(\mathrm{e}^z + \mathrm{e}^{-z}),$$

derive the formula for $\cosh^{-1}(z)$. *Hint: Start by defining* $\cosh(w) = z$*, and use the substitution* $u = \mathrm{e}^w$*.*

Answer 2

Let's follow the hint:

$$\cosh(w) = z \qquad \Leftrightarrow \qquad w = \cosh^{-1}(z).$$

Using the definition of cosh gives

$$\frac{1}{2}(\mathrm{e}^w + \mathrm{e}^{-w}) = z.$$

Our goal now is to solve for w. Applying the substitution I suggested yields

$$\begin{aligned} \frac{1}{2}\left(u + \frac{1}{u}\right) &= z, \\ u^2 - 2zu &= -1. \end{aligned}$$

This is a quadratic equation in u, so we can complete the square,

$$\begin{aligned} u^2 - 2zu + z^2 &= z^2 - 1, \\ (u - z)^2 &= z^2 - 1, \\ u &= z + (z^2 - 1)^{1/2}. \end{aligned}$$

Notice the use of the complex square root notation $(z^2 - 1)^{1/2}$; this should remind you that there are 2 values to that expression. Returning back from our original substitution gives

$$\begin{aligned} \mathrm{e}^w &= z + (z^2 - 1)^{1/2}, \\ w &= \ln\left(z + (z^2 - 1)^{1/2}\right). \end{aligned}$$

Finally we can write the definition of $\cosh^{-1}$ as

$$\cosh^{-1}(z) = \ln\left(z + (z^2 - 1)^{1/2}\right).$$