Math 333 Quiz 3 - February 18, 2012

Question 1

Assuming $z = x + \imath y$, write the following functions $f(z)$ in $u + \imath v$ form, and state explicitly what $u(x, y)$ and $v(x, y)$ are.

Question 1.a

$$f(z) = z^2 + z$$

Answer 1.a

Substituting in $z = x + \imath y$, we get

$$\begin{aligned} f(z) &= (x + \imath y)^2 + (x + \imath y) \\ &= x^2 + 2\imath xy - y^2 + x + \imath y \\ &= x^2 - y^2 + x + \imath(2xy + y). \end{aligned}$$

Therefore

$$\begin{aligned} u(x, y) &= x^2 - y^2 + x, \\ v(x, y) &= 2xy + y. \end{aligned}$$

Question 1.b

$$f(z) = \frac{1}{1 + z^2}$$

Answer 1.b

Substituting in $z = x + \imath y$, we get

$$\begin{aligned} f(z) &= \frac{1}{1 + (x + \imath y)^2} \\ &= \frac{1}{1 + x^2 - y^2 + 2\imath xy} \\ &= \frac{1}{1 + x^2 - y^2 + 2\imath xy} \frac{1 + x^2 - y^2 - 2\imath xy}{1 + x^2 - y^2 - 2\imath xy} \\ &= \frac{1 + x^2 - y^2 - 2\imath xy}{(1 + x^2 - y^2)^2 + (2xy)^2}. \end{aligned}$$

Therefore

$$
\begin{aligned}
u(x,y) &= \frac{1+x^2-y^2}{(1+x^2-y^2)^2+(2xy)^2}, \\
v(x,y) &= \frac{-2xy}{(1+x^2-y^2)^2+(2xy)^2}.
\end{aligned}
$$

Question 2

Determine if the function

$$f(z) = z^3$$

is analytic by testing the Cauchy-Riemann equations.

Answer 2

The function f can be written in $u + \imath v$ form as

$$
\begin{aligned}
f(x+\imath y) = (x+\imath y)^3 &= (x+\imath y)^2(x+\imath y) \\
&= (x^2 - y^2 + 2\imath xy)(x+\imath y) \\
&= x^3 - xy^2 - 2xy^2 + \imath(2x^2y + x^2y - y^3),
\end{aligned}
$$

which yields

$$
\begin{aligned}
u(x,y) &= x^3 - 3xy^2, \\
v(x,y) &= 3x^2y - y^3.
\end{aligned}
$$

All first-order partial derivatives are

$$
\begin{aligned}
u_x &= 3x^2 - 3y^2, \qquad & u_y &= -6xy, \\
v_y &= 3x^2 - 3y^2, \qquad & v_x &= 6xy.
\end{aligned}
$$

The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and by plugging in our results above, we see that the Cauchy-Riemann equations are satisfied. That means that f is analytic.