Math 333 — Exam 2 — Monday, April 29, 2012

Question 1

Use the residue theorem to evaluate the integral

$$\oint_C \frac{2-z}{(z+1)(z+\imath)^2}dz, \qquad C : |z| = 2.$$

Answer 1

This integrand f has two singularities, $z_1 = -1$ and $z_2 = -\imath$. Because f is a rational function, we know both of these are poles, of orders 1 and 2 respectively. The residual must be computed at each of these poles, which can be done using the formula you were provided. First we have

$$\begin{aligned}
\operatorname{Res}(f,-1) &= \lim_{z\to-1}(z+1)\frac{2-z}{(z+1)(z+\imath)^2} \\
&= \lim_{z\to-1}\frac{2-z}{(z+\imath)^2} \\
&= \frac{2-(-1)}{(-1+\imath)^2} = \frac{3}{2}\imath,
\end{aligned}$$

and we also have

$$\begin{aligned}
\operatorname{Res}(f,-\imath) &= \lim_{z\to-i}\frac{d}{dz}\left[(z+\imath)^2\frac{2-z}{(z+1)(z+\imath)^2}\right] \\
&= \lim_{z\to-1}\frac{d}{dz}\left[\frac{2-z}{z+1}\right] \\
&= \lim_{z\to-1}\frac{(z+1)(-1)-(2-z)(1)}{(z+1)^2} = -\frac{3}{2}\imath.
\end{aligned}$$

Now we use the reside theorem to find

$$\oint_C \frac{2-z}{(z+1)(z+\imath)^2}dz = 2\pi\imath\left(\frac{3}{2}\imath - \frac{3}{2}\imath\right) = 0.$$

Question 2

These questions deal with basic vector manipulations.

Question 2.a

Given a vector

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

find a unit vector pointing in the same direction.

Answer 2.a

Any multiple of that vector would still be pointing in the same direction, so we just need to find a multiple that would make its norm one. Its current norm is

$$\left\| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

We can divide by this to create a unit norm vector

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Question 2.b

Suppose you are given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Determine the value σ such that

$$\boldsymbol{c} = \boldsymbol{b} + \sigma \boldsymbol{a}$$

is orthogonal to $\boldsymbol{a}$.

Answer 2.b

Two vectors are orthogonal if their inner product is 0, so we require

$$\begin{aligned} \boldsymbol{a}^T \boldsymbol{c} &= 0 \\ \boldsymbol{a}^T (\boldsymbol{b} + \sigma \boldsymbol{a}) &= 0 \\ \boldsymbol{a}^T \boldsymbol{b} + \sigma \boldsymbol{a}^T \boldsymbol{a} &= 0 \\ \sigma &= \frac{\boldsymbol{a}^T \boldsymbol{b}}{\boldsymbol{a}^T \boldsymbol{a}}. \end{aligned}$$

Question 3

Solve the system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ where $\mathsf{A} = \mathsf{LU}$ and

$$\mathsf{L} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ 3 & 0 & 1 & \\ -1 & -1 & 2 & 1 \end{pmatrix}, \quad \mathsf{U} = \begin{pmatrix} 1 & -2 & -1 & 0 \\ & 1 & 0 & 2 \\ & & 1 & 2 \\ & & & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -2 \\ 5 \\ -3 \\ 6 \end{pmatrix}.$$

Answer 3

Given the LU decomposition, we first solve $\mathsf{L}\boldsymbol{y} = \boldsymbol{b}$ and then solve $\mathsf{U}\boldsymbol{x} = \boldsymbol{y}$. The lower triangular system can be written as

$$\begin{aligned} y_1 \quad & = -2 \\ -y_1 + y_2 \quad & = 5 \\ 3y_1 \qquad + y_3 \quad & = -3 \\ -y_1 - y_2 + 2y_3 + y_4 & = 6 \end{aligned}$$

which can be directly solved from top to bottom to find

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 3 \\ 1 \end{pmatrix}.$$

Let's now solve the upper triangular system

$$\begin{aligned} x_1 - 2x_2 - x_3 \qquad & = -2 \\ x_2 + \qquad 2x_4 & = 3 \\ x_3 + 2x_4 & = 3 \\ x_4 & = 1 \end{aligned}$$

which gives the answer

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Question 4

Suppose you were given a matrix of the form

$$\mathsf{A} = \mathsf{I}_n + \boldsymbol{u}\boldsymbol{v}^T, \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n.$$

You believe the inverse of this matrix is of the form

$$\mathsf{A}^{-1} = \mathsf{I}_n + \alpha \boldsymbol{u}\boldsymbol{v}^T.$$

Using the definition of the inverse, determine what value α must take in order for this to be true.

Answer 4

The definition of inverse tells us that

$$\begin{aligned} \mathsf{A}\mathsf{A}^{-1} &= \mathsf{I}_n \\ (\mathsf{I}_n + \boldsymbol{u}\boldsymbol{v}^T)(\mathsf{I}_n + \alpha \boldsymbol{u}\boldsymbol{v}^T) &= \mathsf{I}_n. \end{aligned}$$

We can expand this out, and absorb the I_n products to form

$$\begin{aligned} \mathsf{I}_n + \boldsymbol{u}\boldsymbol{v}^T + \alpha \boldsymbol{u}\boldsymbol{v}^T + \alpha \boldsymbol{u}\boldsymbol{v}^T\boldsymbol{u}\boldsymbol{v}^T &= \mathsf{I}_n \\ \boldsymbol{u}\boldsymbol{v}^T + \alpha \boldsymbol{u}\boldsymbol{v}^T + \alpha \boldsymbol{u}\boldsymbol{v}^T\boldsymbol{u}\boldsymbol{v}^T &= \mathsf{0} \end{aligned}$$

The right hand side $\mathsf{0}$ represents a matrix of all zeros. Now we can extract the inner product from the product of outer products to get

$$\begin{aligned} \boldsymbol{u}\boldsymbol{v}^T + \alpha \boldsymbol{u}\boldsymbol{v}^T + \alpha(\boldsymbol{v}^T\boldsymbol{u})\boldsymbol{u}\boldsymbol{v}^T &= \mathsf{0} \\ (1 + \alpha + \alpha(\boldsymbol{v}^T\boldsymbol{u}))\boldsymbol{u}\boldsymbol{v}^T &= \mathsf{0} \end{aligned}$$

This term $1 + \alpha + \alpha(\boldsymbol{v}^T\boldsymbol{u})$ is a scalar, which means that for this equation to be valid we either need $\boldsymbol{u}\boldsymbol{v}^T = \mathsf{0}$ (which is the trivial solution such that $\mathsf{A} = \mathsf{I}_n$) or

$$\begin{aligned} 1 + \alpha + \alpha(\boldsymbol{v}^T\boldsymbol{u}) &= 0 \\ \alpha &= -\frac{1}{1 + \boldsymbol{v}^T\boldsymbol{u}}. \end{aligned}$$

In an interesting twist, this matrix A does not have an inverse if $\boldsymbol{v}^T\boldsymbol{u} = -1$. This is easy to prove if you are comfortable with the SVD, but we didn't cover that in this class, so whatever.

Question 5

Consider the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Question 5.a

How many eigenvalues and eigenvectors does this matrix have? Why should the eigenvectors of A be orthogonal?

Answer 5.a

Every 2×2 matrix has 2 eigenvalues (though they may not be distinct). Technically, every matrix has infinitely many eigenvectors, because any multiple of an eigenvector is also an eigenvector. For this particular matrix, the eigenvectors are orthogonal because A is symmetric.

Question 5.b

Determine which of the following are eigenvectors of A. If it is an eigenvector, determine the associated eigenvalue.

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Answer 5.b

This is an easy one, just perform the matrix-vector products

$$\begin{aligned}
\mathsf{A}\boldsymbol{v}_1 &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\mathsf{A}\boldsymbol{v}_2 &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = (3) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathsf{A}\boldsymbol{v}_3 &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{aligned}$$

Clearly, $\boldsymbol{v}_1$ is an eigenvector with associated eigenvalue $\lambda_1 = -1$ and $\boldsymbol{v}_2$ is an eigenvector with associated eigenvalue $\lambda_2 = 3$, and $\boldsymbol{v}_3$ is not an eigenvector.

Question 5.c (BONUS)

Using the solution to (5.b), write the spectral decomposition of A.

Answer 5.c

This question was easy if you exploited the symmetric structure to produce the spectral decomposition $\mathsf{A} = \mathsf{V}\mathsf{D}\mathsf{V}^T$. Using the answer from (5.b) you must normalize the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to unit norm vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, and your job is done:

$$\begin{aligned} \mathsf{A} &= \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1 & \\ & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \end{aligned}$$

You could also have done this without the symmetric knowledge:

$$\begin{aligned} \mathsf{A} &= \mathsf{X}\Lambda\mathsf{X}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & \\ & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}, \end{aligned}$$

but that would require you to find the inverse of that matrix. It's not really difficult, but it is more difficult (I think) that normalizing vectors. The result of this computation is

$$\mathsf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & \\ & 3 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$