Math 333 — Exam 1 — Wednesday, March 13, 2013

Question 1

These questions deal with the concept of an analytic function in the complex plane.

Question 1.a

What does it mean for a function to be analytic?

Answer 1.a

A function is analytic if it is differentiable at a point and at all points in a neighborhood around that point. When a function is analytic, it has all its derivatives.

Question 1.b

How do we check if a function $f(z) = u(x, y) + \imath v(x, y)$ is analytic, if $z = x + \imath y$?

Answer 1.b

One test for analyticity is the Cauchy-Riemann equations. Technically you could also prove that the derivatives are valid in a neighborhood, but I don't know how you would do that in general.

Question 1.c

Why does the Independence of Path Theorem require f to be analytic?

Answer 1.c

We proved the independence of path theorem by creating a closed contour C_1 from z_0 to z_1 and then back along C_2 from z_1 to z_0. This closed contour integral must be equal to zero, meaning that the integral over C_1 is the negative of the integral along C_2. If the function were not analytic, then the closed contour $C_1 + C_2$ would not be 0, so independence of path would not exist.

Question 2

Use the Cauchy Integral Formula to evaluate the following integrals.

Question 2.a

$$\oint_C \frac{3}{(z-1)(z-4)} dz, \qquad C : |z-1| = 1$$

Answer 2.a

Only the singularity $z_0 = 1$ is inside C, so only it will contribute to the solution. That means that $f(z) = 3/(z-4)$, and plugging in to the formula gives

$$\begin{aligned} \oint_C \frac{1}{z(z-3)} dz &= \frac{2\pi\imath}{0!} f(1) \\ &= \frac{2\pi\imath(3)}{1-4} = -2\pi\imath. \end{aligned}$$

Question 2.b

$$\oint_C \frac{\sinh z}{(z-1)^3} dz, \qquad C : |z+1| = 5$$

Answer 2.b

Here we have a singularity at $z_0 = 1$, and in terms of the Cauchy Integral Formula, we have $n = 2$ and $f(z) = \sinh z$. Recall that $f''(z) = \sinh z$, meaning that our integral is

$$\oint_C \frac{\sinh z}{(z-1)^3} dz = \frac{2\pi\imath}{2!} \sinh(1) = \pi\imath \sinh(1).$$

Question 3

Evaluate the following quantities, and put the answer in $x + \imath y$ form.

Question 3.a

$$\mathrm{e}^{1-\pi\imath}$$

Answer 3.a

Using the formula that was given,

$$\mathrm{e}^{1-\pi\imath} = \mathrm{e}^{1}\mathrm{e}^{\pi\imath} = -\mathrm{e}$$

Question 3.b

$$\sin^{-1}(-2)$$

Answer 3.b

Using the formulas that were given,

$$\begin{aligned}
\sin^{-1}(-2) &= -\imath \ln[\imath(-2) + (1-(-2)^2)^{1/2}] \\
&= -\imath \ln[-2\imath \pm \imath\sqrt{3}] \\
&= -\imath \ln[\imath(-2 \pm \sqrt{3})] \\
&= -\imath \left[\log(2 \pm \sqrt{3}) - \frac{\pi}{2}\imath\right] \\
&= -\imath \log(2 \pm \sqrt{3}) - \frac{\pi}{2}
\end{aligned}$$

Question 4

Evaluate the integral

$$\int_C \bar{z}\, dz$$

where C is in two pieces: the straight line from $z = 0$ to $z = 1 + \imath$ and the straight line from $z = 1 + \imath$ to $z = \imath$. Use whatever method is appropriate for this integral.

Answer 4

Because $f(z) = \bar{z}$ is not analytic, we need to use parameterization. Many different choices are possible, but I will select

$$\begin{aligned}
C_1 : z(t) &= (1 + \imath)t, \qquad 0 \leq t \leq 1, \\
z'(t) &= 1 + \imath, \\
\bar{z}(t) &= (1 - \imath)t, \\
C_2 : z(t) &= 1 - t + \imath, \qquad 0 \leq t \leq 1, \\
z'(t) &= -1, \\
\bar{z}(t) &= 1 - t - \imath.
\end{aligned}$$

This converts our integral into two pieces. Using the parameterization rule

$$\int_C f(z) dz = \int_0^1 f(z(t)) z'(t) dt,$$

we compute

$$\begin{aligned}
\int_C \bar{z}\, dz &= \int_{C_1} \bar{z}\, dz + \int_{C_2} \bar{z}\, dz \\
&= \int_0^1 [(1 - \imath)t](1 + \imath) dt + \int_0^1 [1 - t - \imath](-1) dt \\
&= \int_0^1 2t dt + \int_0^1 [t + (\imath - 1)] dt \\
&= t^2 |_0^1 + \left[\frac{1}{2} t^2 + (\imath - 1)t \right]_0^1 \\
&= 1 + \frac{1}{2} + (\imath - 1) = \frac{1}{2} + \imath
\end{aligned}$$

Question 5

Choose only one of the following two questions to answer.

Question 5.a

Plot all the following sets on the same graph:

1. $\Re(z) = 1$
2. $\Im(z) = -1$
3. $|z + 2 + 2\imath| < 1$
4. $\text{Arg}(z) \geq 3\pi/4$

Try to label them well.

Answer 5.a

I'm not going to plot this here, I'll just explain the answer. $\Re(z) = 1$ is a vertical straight line through the value 1 on the real axis. $\Im(z) = -1$ is a horizontal straight line through the value -1 on the imaginary axis. $|z + 2 + 2\imath| < 1$ is a circle centered at $z_0 = -2 - 2\imath$ with radius 1 but not including the boundary of the circle. $\text{Arg}(z) \geq 3\pi/4$ is a wedge in the second quadrant including values less than or equal to the line $-\Re(z) = \Im(z)$. The boundary is included.

Question 5.b

Determine the region where the series

$$\sum_{k=1}^{\infty} k \frac{(-2z)^k}{(3-\imath)^{k-1}}$$

converges for $z \in \mathbb{C}$, and plot that region in a graph.

Answer 5.b

This takes a little care, but all that is required here is the limit ratio test from calculus II. The series will converge for any $z \in \mathbb{C}$ where the ratio of subsequent terms in the series is less than 1 in the limit as $k \to \infty$.

$$\lim_{k\to\infty} \left| \frac{(k+1)\frac{(-2z)^{k+1}}{(3-\imath)^k}}{k\frac{(-2z)^k}{(3-\imath)^{k-1}}} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{k+1}{k} \frac{2z}{3-\imath} \right| < 1$$

$$\left| \frac{2z}{3-\imath} \right| < 1$$

$$|z| < \frac{1}{2} |3 - \imath|$$

$$|z| < \frac{\sqrt{10}}{2}$$

This region in the plane is a circle centered at $z = 0$ with radius $\sqrt{10}/2$. The boundary of the circle is not included.