Sometimes, in order to apply the basic integration formulas, it is necessary first to perform algebraic manipulations on the integrand, as Example 8 shows.

EXAMPLE 8 Using Algebraic Manipulation to Find an Indefinite Integral

Find $\int y^2\left(y+\frac{2}{3}\right)dy.$

Solution: The integrand does not fit a familiar integration form. However, by multiplying the integrand we get

$$\int y^2\left(y+\frac{2}{3}\right)dy = \int\left(y^3+\frac{2}{3}y^2\right)dy$$

$$= \frac{y^4}{4}+\left(\frac{2}{3}\right)\frac{y^3}{3}+C = \frac{y^4}{4}+\frac{2y^3}{9}+C$$

NOW WORK PROBLEM 41

CAUTION

In Example 8, we first multiplied the factors in the integrand. The answer could not have been found simply in terms of $\int y^2\,dy$ and $\int(y+\frac{2}{3})\,dy$. There is not a formula for the integral of a general product of functions.

EXAMPLE 9 Using Algebraic Manipulation to Find an Indefinite Integral

a. *Find* $\int \frac{(2x-1)(x+3)}{6}\,dx.$

Solution: By factoring out the constant $\frac{1}{6}$ and multiplying the binomials, we get

$$\int \frac{(2x-1)(x+3)}{6}\,dx = \frac{1}{6}\int(2x^2+5x-3)\,dx$$

$$= \frac{1}{6}\left((2)\frac{x^3}{3}+(5)\frac{x^2}{2}-3x\right)+C$$

$$= \frac{x^3}{9}+\frac{5x^2}{12}-\frac{x}{2}+C$$

b. *Find* $\int \frac{x^3-1}{x^2}\,dx.$

Solution: We can break up the integrand into fractions by dividing each term in the numerator by the denominator:

$$\int \frac{x^3-1}{x^2}\,dx = \int\left(\frac{x^3}{x^2}-\frac{1}{x^2}\right)dx = \int(x-x^{-2})\,dx$$

$$= \frac{x^2}{2}-\frac{x^{-1}}{-1}+C = \frac{x^2}{2}+\frac{1}{x}+C$$

Another algebraic approach to part (b) is

$$\int \frac{x^3-1}{x^2}\,dx = \int(x^3-1)x^{-2}\,dx$$

$$= \int(x-x^{-2})\,dx$$

and so on.

NOW WORK PROBLEM 49

Problems 14.2

In Problems 1–52, find the indefinite integrals.

*1. $\int 7\,dx$

2. $\int \frac{1}{2x}\,dx$

*3. $\int x^8\,dx$

4. $\int 5x^{24}\,dx$

*5. $\int 5x^{-7}\,dx$

6. $\int \frac{z^{-3}}{3}\,dz$

7. $\int \frac{2}{x^{10}}\,dx$

8. $\int \frac{7}{x^4\,dx}$

*9. $\int \frac{1}{t^{7/4}}\,dt$

10. $\int \frac{7}{2x^{9/4}}\,dx$

*11. $\int(4+t)\,dt$

12. $\int(r^3+2r)\,dr$

13. $\int(y^5-5y)\,dy$

14. $\int(5-2w-6w^2)\,dw$

*15. $\int(3t^2-4t+5)\,dt$

16. $\int(1+t^2+t^4+t^6)\,dt$

17. $\int(7+e)\,dx$

18. $\int(5-2^{-1})\,dx$

19. $\int\left(\frac{x}{7}-\frac{3}{4}x^4\right)dx$

20. $\int\left(\frac{2x^2}{7}-\frac{8}{3}x^4\right)dx$

*21. $\int \pi e^x\,dx$

22. $\int\left(\frac{e^x}{3}+2x\right)dx$

23. $\int(x^{8.3}-9x^6+3x^{-4}+x^{-3})\,dx$

24. $\int(0.7y^3+10+2y^{-3})\,dy$

25. $\int \frac{-2\sqrt{x}}{3}\,dx$

26. $\int dz$

27. $\int \frac{1}{4\sqrt[8]{x^2}}\,dx$

28. $\int \frac{-4}{(3x)^3}\,dx$

29. $\int\left(\frac{x^3}{3}-\frac{3}{x^3}\right)dx$

30. $\int\left(\frac{1}{2x^3}-\frac{1}{x^4}\right)dx$

31. $\int\left(\frac{3w^2}{2}-\frac{2}{3w^2}\right)dw$

32. $\int \frac{4}{e^{-s}}\,ds$

33. $\int \frac{3u-4}{5}\,du$

34. $\int \frac{1}{12}\left(\frac{1}{3}e^x\right)dx$

35. $\int(u^e+e^u)\,du$

36. $\int\left(3y^3-2y^2+\frac{e^y}{6}\right)dy$

37. $\int(2\sqrt{x}-3\sqrt[4]{x})\,dx$

38. $\int 0\,dt$

39. $\int\left(-\frac{\sqrt[3]{x^2}}{5}-\frac{7}{2\sqrt{x}}+6x\right)dx$

40. $\int\left(\sqrt[3]{u}+\frac{1}{\sqrt{u}}\right)du$

*41. $\int(x^2+5)(x-3)\,dx$

42. $\int x^4(x^3+8x^2+7)\,dx$

43. $\int\sqrt{x}(x+3)\,dx$

44. $\int(z+2)^2\,dz$

45. $\int(3u+2)^3\,du$

46. $\int\left(\frac{2}{\sqrt[5]{x}}-1\right)^2 dx$

47. $\int v^{-2}(2v^4+3v^2-2v^{-3})\,dv$

48. $\int(6e^u-u^3(\sqrt{u}+1))\,du$

*49. $\int \frac{z^4+10z^3}{2z^2}\,dz$

50. $\int \frac{x^4-5x^2+2x}{5x^2}\,dx$

51. $\int \frac{e^x+e^{2x}}{e^x}\,dx$

52. $\int \frac{(x^3+1)^2}{x^2}\,dx$

53. If $F(x)$ and $G(x)$ are such that $F'(x)=G'(x)$, is it true that $F(x)-G(x)$ must be zero?

54. **(a)** Find a function F such that $\int F(x)\,dx = xe^x+C$.
(b) Is there only one function F satisfying the equation given in part (a), or are there many such functions?

55. Find $\int \frac{d}{dx}\left(\frac{1}{\sqrt{x^2+1}}\right)dx.$

14.3 Integration with Initial Conditions

OBJECTIVE

To find a particular antiderivative of a function that satisfies certain conditions. This involves evaluating constants of integration.

If we know the rate of change, f', of the function f, then the function f itself is an antiderivative of f' (since the derivative of f is f'). Of course, there are many antiderivatives of f', and the most general one is denoted by the indefinite integral. For example, if

$$f'(x) = 2x$$

then

$$f(x) = \int f'(x)\,dx = \int 2x\,dx = x^2+C. \tag{1}$$

That is, *any* function of the form $f(x) = x^2+C$ has its derivative equal to $2x$. Because of the constant of integration, notice that we do not know $f(x)$ specifically. However, if f must assume a certain function value for a particular value of x, then we can determine the value of C and thus determine $f(x)$ specifically. For instance, if $f(1)=4$, then from Equation (1),

$$f(1) = 1^2+C$$

$$4 = 1+C$$

$$C = 3$$

Thus,

$$f(x) = x^2+3$$

That is, we now know the particular function $f(x)$ for which $f'(x)=2x$ and $f(1)=4$. The condition $f(1)=4$, which gives a function value of f for a specific value of x, is called an *initial condition*.

Since dr/dq is the derivative of total revenue r,

$$r = \int (2000 - 20q - 3q^2)\,dq$$

$$= 2000q - (20)\frac{q^2}{2} - (3)\frac{q^3}{3} + C$$

so that

$$r = 2000q - 10q^2 - q^3 + C \tag{7}$$

Revenue is 0 when q is 0.

We assume that **when no units are sold, there is no revenue;** that is, $r = 0$ when $q = 0$. This is our initial condition. Putting these values into Equation (7) gives

$$0 = 2000(0) - 10(0)^2 - 0^3 + C$$

Although $q = 0$ gives $C = 0$, this is not true in general. It occurs in this section because the revenue functions are polynomials. In later sections, evaluating at $q = 0$ may produce a nonzero value for C.

Hence, $C = 0$, and

$$r = 2000q - 10q^2 - q^3$$

To find the demand function, we use the fact that $p = r/q$ and substitute for r:

$$p = \frac{r}{q} = \frac{2000q - 10q^2 - q^3}{q}$$

$$p = 2000 - 10q - q^2$$

NOW WORK PROBLEM 11

EXAMPLE 5 Finding Cost from Marginal Cost

In the manufacture of a product, fixed costs per week are \$4000. (Fixed costs are costs, such as rent and insurance, that remain constant at all levels of production during a given time period.) If the marginal-cost function is

$$\frac{dc}{dq} = 0.000001(0.002q^2 - 25q) + 0.2$$

where c is the total cost (in dollars) of producing q pounds of product per week, find the cost of producing 10,000 lb in 1 week.

Solution: Since dc/dq is the derivative of the total cost c,

$$c(q) = \int [0.000001(0.002q^2 - 25q) + 0.2]\,dq$$

$$= 0.000001 \int (0.002q^2 - 25q)\,dq + \int 0.2\,dq$$

$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + C$$

When q is 0, total cost is equal to fixed cost.

Fixed costs are constant regardless of output. Therefore, when $q = 0$, $c = 4000$, which is our initial condition. Putting $c(0) = 4000$ in the last equation, we find that $C = 4000$, so

$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + 4000 \tag{8}$$

Although $q = 0$ gives C a value equal to fixed costs, this is not true in general. It occurs in this section because the cost functions are polynomials. In later sections, evaluating at $q = 0$ may produce a value for C that is different from fixed cost.

From Equation (8), we have $c(10{,}000) = 5416\frac{2}{3}$. Thus, the total cost for producing 10,000 pounds of product in 1 week is \$5416.67.

NOW WORK PROBLEM 15

Problems 14.3

In Problems 1 and 2, find y subject to the given conditions.

*1. $dy/dx = 3x - 4$; $y(-1) = \frac{13}{2}$

2. $dy/dx = x^2 - x$; $y(3) = \frac{19}{2}$

In Problems 3 and 4, if y satisfies the given conditions, find y(x) for the given value of x.

3. $y' = 5/\sqrt{x}$, $y(9) = 50$; $x = 16$

4. $y' = -x^2 + 2x$, $y(2) = 1$; $x = 1$

In Problems 5–8, find y subject to the given conditions.

*5. $y'' = -3x^2 + 4x$; $y'(1) = 2$, $y(1) = 3$

6. $y'' = x + 1$; $y'(0) = 0$, $y(0) = 5$

7. $y''' = 2x$; $y''(-1) = 3$, $y'(3) = 10$, $y(0) = 13$

8. $y''' = e^x + 1$; $y''(0) = 1$, $y'(0) = 2$, $y(0) = 3$

In Problems 9–12, dr/dq is a marginal-revenue function. Find the demand function.

9. $dr/dq = 0.7$

10. $dr/dq = 10 - \frac{1}{16}q$

*11. $dr/dq = 275 - q - 0.3q^2$

12. $dr/dq = 5{,}000 - 3(2q + 2q^3)$

In Problems 13–16, dc/dq is a marginal-cost function and fixed costs are indicated in braces. For Problems 13 and 14, find the total-cost function. For Problems 15 and 16, find the total cost for the indicated value of q.

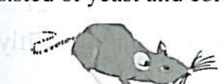
13. $dc/dq = 1.35$; {200}

14. $dc/dq = 2q + 75$; {2000}

*15. $dc/dq = 0.08q^2 - 1.6q + 6.5$; {8000}; $q = 25$

16. $dc/dq = 0.000204q^2 - 0.046q + 6$; {15,000}; $q = 200$

*17. **Diet for Rats** A group of biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.[2] The protein consisted of yeast and corn flour.



Over a period of time, the group found that the (approximate) rate of change of the average weight gain G (in grams) of a rat with respect to the percentage P of yeast in the protein mix was

$$\frac{dG}{dP} = -\frac{P}{25} + 2 \qquad 0 \le P \le 100$$

If $G = 38$ when $P = 10$, find G.

18. **Winter Moth** A study of the winter moth was made in Nova Scotia.[3] The prepupae of the moth fall onto the ground from host trees. It was found that the (approximate) rate at which prepupal density y (the number of prepupae per square foot of soil) changes with respect to distance x (in feet) from the base of a host tree is

$$\frac{dy}{dx} = -1.5 - x \qquad 1 \le x \le 9$$

If $y = 57.3$ when $x = 1$, find y.

19. **Fluid Flow** In the study of the flow of fluid in a tube of constant radius R, such as blood flow in portions of the body, one can think of the tube as consisting of concentric tubes of radius r, where $0 \le r \le R$. The velocity v of the fluid is a function of r and is given by[4]

$$v = \int -\frac{(P_1 - P_2)r}{2l\eta}\,dr$$

where P_1 and P_2 are pressures at the ends of the tube, η (a Greek letter read "eta") is fluid viscosity, and l is the length of the tube. If $v = 0$ when $r = R$, show that

$$v = \frac{(P_1 - P_2)(R^2 - r^2)}{4l\eta}$$

20. **Elasticity of Demand** The sole producer of a product has determined that the marginal-revenue function is

$$\frac{dr}{dq} = 100 - 3q^2$$

Determine the point elasticity of demand for the product when $q = 5$. (*Hint:* First find the demand function.)

21. **Average Cost** A manufacturer has determined that the marginal-cost function is

$$\frac{dc}{dq} = 0.003q^2 - 0.4q + 40$$

where q is the number of units produced. If marginal cost is \$27.50 when $q = 50$ and fixed costs are \$5000, what is the *average* cost of producing 100 units?

22. If $f''(x) = 30x^4 + 12x$ and $f'(1) = 10$, evaluate

$$f(965.335245) - f(-965.335245)$$

OBJECTIVE

To learn and apply the formulas for $\int u^n\,du$, $\int e^u\,du$, and $\int \frac{1}{u}\,du$.

14.4 More Integration Formulas

Power Rule for Integration

The formula

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \ne -1$$

[2]Adapted from R. Bressani, "The Use of Yeast in Human Foods," in *Single-Cell Protein*, ed. R. I. Mateles and S. R. Tannenbaum (Cambridge, MA: MIT Press, 1968).

[3]Adapted from D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada*, no. 46 (1965).

[4]R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

EXAMPLE 7 An Integral Involving $\frac{1}{u}\,du$

Find $\int \frac{(2x^3 + 3x)\,dx}{x^4 + 3x^2 + 7}$.

Solution: If $u = x^4 + 3x^2 + 7$, then $du = (4x^3 + 6x)\,dx$, which is two times the numerator giving $(2x^3 + 3x)\,dx = \frac{du}{2}$. To apply Equation (3), we write

$$\begin{aligned}\int \frac{2x^3 + 3x}{x^4 + 3x^2 + 7}\,dx &= \frac{1}{2}\int \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|x^4 + 3x^2 + 7| + C \qquad \text{(Rewrite } u \text{ in terms of } x.) \\ &= \frac{1}{2}\ln(x^4 + 3x^2 + 7) + C \qquad (x^4 + 3x^2 + 7 > 0 \text{ for all } x)\end{aligned}$$

NOW WORK PROBLEM 51

EXAMPLE 8 An Integral Involving Two Forms

Find $\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw$.

Solution:

$$\begin{aligned}\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw &= \int (1-w)^{-2}dw + \int \frac{1}{w-1}dw \\ &= -1\int (1-w)^{-2}[-dw] + \int \frac{1}{w-1}dw\end{aligned}$$

The first integral has the form $\int u^{-2}\,du$, and the second has the form $\int \frac{1}{v}\,dv$. Thus,

$$\begin{aligned}\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw &= -\frac{(1-w)^{-1}}{-1} + \ln|w-1| + C \\ &= \frac{1}{1-w} + \ln|w-1| + C\end{aligned}$$

For your convenience, we list in Table 14.2 the basic integration formulas so far discussed. We assume that u is a function of x.

TABLE 14.2 Basic Integration Formulas

1.	$\int k\,du = ku + C$	k a constant
2.	$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$	$n \neq -1$
3.	$\int \frac{1}{u}\,du = \ln|u| + C$	$u \neq 0$
4.	$\int e^u\,du = e^u + C$	
5.	$\int kf(x)\,dx = k\int f(x)\,dx$	k a constant
6.	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	

Problems 14.4

In Problems 1–80, find the indefinite integrals.

1. $\int (x+5)^7\,dx$
2. $\int 15(x+2)^4\,dx$
*3. $\int 2x(x^2+3)^5\,dx$
4. $\int (3x^2+10x)(x^3+5x^2+6)\,dx$
*5. $\int (3y^2+6y)(y^3+3y^2+1)^{2/3}\,dy$
6. $\int (15t^2-6t+1)(5t^3-3t^2+t)^{17}\,dt$
7. $\int \frac{5}{(3x-1)^3}\,dx$
8. $\int \frac{4x}{(2x^2-7)^{10}}\,dx$
9. $\int \sqrt{2x-1}\,dx$
10. $\int \frac{1}{\sqrt{x-5}}\,dx$
11. $\int (7x-6)^4\,dx$
12. $\int x^2(3x^3+7)^3\,dx$
13. $\int u(5u^2-9)^{14}\,du$
14. $\int 9x\sqrt{1+2x^2}\,dx$
*15. $\int 4x^4(27+x^5)^{1/3}\,dx$
16. $\int (4-5x)^9\,dx$
17. $\int 3e^{3x}\,dx$
18. $\int 5e^{3t+7}\,dt$
19. $\int (2t+1)e^{t^2+t}\,dt$
20. $\int -3w^2e^{-w^3}\,dw$
21. $\int xe^{7x^2}\,dx$
22. $\int x^3e^{4x^4}\,dx$
23. $\int 4e^{-3x}\,dx$
24. $\int x^4e^{-6x^5}\,dx$
25. $\int \frac{1}{x+5}\,dx$
26. $\int \frac{12x^2+4x+2}{x+x^2+2x^3}\,dx$
27. $\int \frac{3x^2+4x^3}{x^3+x^4}\,dx$
28. $\int \frac{6x^2-6x}{1-3x^2+2x^3}\,dx$
29. $\int \frac{6z}{(z^2-6)^5}\,dx$
30. $\int \frac{3}{(5v-1)^4}\,dv$
*31. $\int \frac{4}{x}\,dx$
32. $\int \frac{3}{1+2y}\,dy$
33. $\int \frac{s^2}{s^3+5}\,ds$
34. $\int \frac{2x^2}{3-4x^3}\,dx$
35. $\int \frac{5}{4-2x}\,dx$
36. $\int \frac{7t}{5t^2-6}\,dt$
37. $\int \sqrt{5x}\,dx$
38. $\int \frac{1}{(3x)^6}\,dx$
39. $\int \frac{x}{\sqrt{x^2-4}}\,dx$
40. $\int \frac{9}{1-3x}\,dx$
*41. $\int 2y^3e^{y^4+1}\,dx$
42. $\int 2\sqrt{2x-1}\,dx$
43. $\int v^2e^{-2v^3+1}\,dv$
44. $\int \frac{x^2}{\sqrt[3]{2x^3+9}}\,dx$
45. $\int (e^{-5x}+2e^x)\,dx$
46. $\int 4\sqrt[3]{y+1}\,dy$
47. $\int (8x+10)(7-2x^2-5x)^3\,dx$
48. $\int 2ye^{3y^2}\,dy$
49. $\int \frac{x^2+2}{x^3+6x}\,dx$
50. $\int (e^x+2e^{-3x}-e^{5x})\,dx$
*51. $\int \frac{16s-4}{3-2s+4s^2}\,ds$
52. $\int (6t^2+4t)(t^3+t^2+1)^6\,dt$
53. $\int x(2x^2+1)^{-1}\,dx$
54. $\int (8w^5+w^2-2)(6w-w^3-4w^6)^{-4}\,dw$
55. $\int -(x^2-2x^5)(x^3-x^6)^{-10}\,dx$
56. $\int \frac{3}{5}(v-2)e^{2-4v+v^2}\,dv$
57. $\int (2x^3+x)(x^4+x^2)\,dx$
58. $\int (e^{3.1})^2\,dx$
59. $\int \frac{7+14x}{(4-x-x^2)^5}\,dx$
60. $\int (e^x-e^{-x})^2\,dx$
61. $\int x(2x+1)e^{4x^3+3x^2-4}\,dx$
62. $\int (u^3-ue^{6-3u^2})\,du$
63. $\int x\sqrt{(8-5x^2)^3}\,dx$
64. $\int e^{-x/7}\,dx$
65. $\int \left(\sqrt{2x}-\frac{1}{\sqrt{2x}}\right)dx$
66. $\int 3\frac{x^4}{e^{x^5}}\,dx$
*67. $\int (x^2+1)^2\,dx$
68. $\int \left[x(x^2-16)^2-\frac{1}{2x+5}\right]dx$
69. $\int \left[\frac{x}{x^2+1}+\frac{x^5}{(x^6+1)^2}\right]dx$
70. $\int \left[\frac{3}{x-1}+\frac{1}{(x-1)^2}\right]dx$
71. $\int \left[\frac{2}{4x+1}-(4x^2-8x^5)(x^3-x^6)^{-8}\right]dx$
72. $\int (r^3+5)^2\,dr$
73. $\int \left[\sqrt{3x+1}-\frac{x}{x^2+3}\right]dx$
74. $\int \left[\frac{x}{3x^2+5}-\frac{x^2}{(x^3+1)^3}\right]dx$
75. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
76. $\int (e^5-3^e)\,dx$
77. $\int \frac{1+e^{2x}}{4e^x}\,dx$
78. $\int \frac{2}{t^2}\sqrt{\frac{1}{t}+9}\,dt$
79. $\int \frac{x+1}{x^2+2x}\ln(x^2+2x)\,dx$
80. $\int \sqrt[3]{x}e^{\sqrt[3]{8x^4}}\,dx$

In Problems 81–84, find y subject to the given conditions.

81. $y' = (3-2x)^2; \quad y(0) = 1$
82. $y' = \frac{x}{x^2+6}; \quad y(1) = 0$