Applications of Guaranteed Adaptive Quasi-Monte Carlo Algorithms

Lluís Antoni Jiménez Rugama
Joint work with Da Li (IIT), Fred J. Hickernell (IIT), Clémentine Prieur (Univ. Grenoble-Alpes), Elise Arnaud (Univ. Grenoble-Alpes), and Laurent Gilquin (Univ. Grenoble-Alpes)

Room 120, Bldg E1, Department of Applied Mathematics
Illinois Institute of Technology, Chicago, 60616 IL
Email: ljimene1@hawk.iit.edu

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Outline

- Introduction
- Multivariate Normal Probability
- Sobol Indices
- Efficiency improvements
  - Importance Sampling
  - Control Variates
Outline

- Introduction—Overview of the GAIL quasi-Monte Carlo cubature algorithms
- Multivariate Normal Probability
- Sobol Indices
- Efficiency improvements
  - Importance Sampling
  - Control Variates
Numerical Integration Problem

Given \( \varepsilon_a \) and \( x \mapsto f(x) \), we want \( \hat{I} \) such that

\[
\left| \int_{[0,1]^d} f(x) \, dx - \hat{I}(x \mapsto f(x), \varepsilon_a) \right| \leq \varepsilon_a,
\]

where

\[
\hat{I}(x \mapsto f(x), \varepsilon_a) = \frac{1}{2m} \sum_{i=0}^{2m-1} f(z_i \oplus \Delta),
\]

for some automatic and adaptive choice of \( m \), \( \{z_i\}_{i=0}^{\infty} \in \{\text{Lattice, Digital}\} \)

sequence, and

\[
\text{cost} \left( \hat{I}(x \mapsto f(x), \varepsilon_a) \right) = O \left( (m + \$f)2^m \right).
\]
Examples of Sequences

Shifted rank-1 lattice sequence with generating vector \((1, 47)\).

Digitally shifted scrambled Sobol' sequence.
Adaptive Algorithm

The idea behind the results in Jiménez Rugama and Hickernell (2014) and Hickernell and Jiménez Rugama (2014) is that for all \( f \in \mathcal{C} \),

\[
\left| \int_{[0,1]^d} f(x) \, dx - \hat{I}(x \mapsto f(x), \varepsilon_a) \right| \leq a(r, m) \sum_{\kappa = [2^{m-r-1}]}^{2^{m-r}-1} |\tilde{f}_{m,\kappa}|.
\]

- \( \tilde{f}_{m,\kappa} \) = discrete Fourier \( \{\text{Exponential, Walsh}\} \) coefficients of \( f \).
- \( a(r, m) \) = inflation factor that depends on \( \mathcal{C} \).
In the Monte Carlo case, if $I(f) = \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x}$,

$$
\mathbb{E} \left[ \left( I(f) - \frac{1}{2^m} \sum_{i=0}^{2^m-1} f(z_i) \right)^2 \right] \leq \frac{1}{2^m} \sum_{\kappa \neq 0} \left| \hat{f}_\kappa \right|^2 \operatorname{Var}[f(Z)]
$$

- $z_i$ are IID samples with distribution $U[0,1]^d$.
- $f(z) = \sum_\kappa \hat{f}_\kappa \phi(z)$ is the Fourier decomposition of $f(z)$. 

 iid Monte Carlo Error Depends on All Fourier Coefficients
QMC Error Bounds Depend Only on Some Fourier Coefficients for $f \in C$

In the quasi-Monte Carlo case,

$$I(f) - \frac{1}{2^m} \sum_{i=0}^{2^m-1} f(z_i \oplus \Delta) \leq \sum \circ \leq a(r, m) \sum_{\kappa = [2^{m-r-1}]}^{2^{m-r}-1} |\tilde{f}_{m, \kappa}|$$

$$C = \left\{ \sum \diamond \text{ bounds} \sum \circ \right\}$$

$$|f_\kappa|, \quad 2^{12} \leq \kappa$$

$$|f_\kappa|, \quad 2^7 \leq \kappa < 2^8$$
Inside and Outside $\mathcal{C}$
Outline

- Introduction
- **Multivariate Normal Probability**—Improvements by using Genz (1992) transformation
- Sobol Indices
- Efficiency improvements
  - Importance Sampling
  - Control Variates
Multivariate Normal Probability

Given $\Sigma \in \mathbb{R}^{d \times d}$ semi-positive definite, we are interested in approximating

$$
P(a \leq X \leq b) = \int_{a}^{b} \frac{e^{x^t \Sigma^{-1} x}}{(2\pi)^{d/2} |\Sigma|^{1/2}} \, dx.
$$

This can be done by evaluating $P(a \leq X \leq b)$ as

$$
P(a \leq X \leq b) = \int_{[0,1)^d} f_{\text{affine}}(t) \, dt, \quad x_j = a_j + (b_j - a_j)t_j,
$$

$$
= \int_{[0,1)^{d-1}} f_{\text{Genz}}(t) \, dt, \quad \text{using Genz (1992) transformation.}
$$
Multivariate Normal Probability in Dimension 8

For

- \( a, b \in \mathbb{R}^8 \) chosen randomly,
- \( \Sigma = 0.4 I + 0.6 11^T \),
- \( \varepsilon_a = 10^{-4} \),

we found that \( P(a \leq X \leq b) \) takes

- 8,388,608 points and 13 seconds for \( f_{\text{affine}} \).
- 8,192 points and 0.05 seconds for \( f_{\text{Genz}} \).
Multivariate Normal Probability in Dimension 8

\[ |\hat{f}_{\text{affine,}\kappa}| \] decreases much slower than \[ |\hat{f}_{\text{Genz,}\kappa}| \], leading to a smaller sample size needed.

![Graph showing comparison between Affine transform and Alan Genz transform](image-url)
Outline

- Introduction
- Multivariate Normal Probability
- **Sobol Indices**—Computing how important each dimension is
- Efficiency improvements
  - Importance Sampling
  - Control Variates
Computing Sobol Indices

For $X \sim U[0, 1]^d$, we are interested in estimating the first-order and total-effect indices,

$$S_j = \frac{\text{Var} [\mathbb{E} (f(X)|X_j)]}{\text{Var} (f(X))} = 1 - \frac{\mathbb{E} [\text{Var} (f(X)|X_j)]}{\text{Var} (f(X))},$$

$$S_{j}^{\text{tot}} = 1 - \frac{\text{Var} [\mathbb{E} (f(X)|X_{-j})]}{\text{Var} (f(X))} = \frac{\mathbb{E} [\text{Var} (f(X)|X_{-j})]}{\text{Var} (f(X))}.$$

satisfying $0 \leq S_j \leq S_{j}^{\text{tot}} \leq 1$. 
Computing Sobol Indices

For $\mathbf{X} \sim U[0, 1]^d$, we are interested in estimating the *first-order* and *total-effect* indices,

\[
S_j = \frac{\text{Var} \left[ \mathbb{E} (f(\mathbf{X}) | X_j) \right]}{\text{Var} (f(\mathbf{X}))} = 1 - \frac{\mathbb{E} \left[ \text{Var} (f(\mathbf{X}) | X_j) \right]}{\text{Var} (f(\mathbf{X}))},
\]

\[
S_j^{\text{tot}} = 1 - \frac{\text{Var} \left[ \mathbb{E} (f(\mathbf{X}) | X_{-j}) \right]}{\text{Var} (f(\mathbf{X}))} = \frac{\mathbb{E} \left[ \text{Var} (f(\mathbf{X}) | X_{-j}) \right]}{\text{Var} (f(\mathbf{X}))},
\]

satisfying $0 \leq S_j \leq S_j^{\text{tot}} \leq 1$. The *first-order* index is composed by,

\[
S_j = 1 - \frac{I^{(1)}}{I^{(2)} - I^{(3)}},
\]

where \( I^{(1)} \) is a $2d - 1$ dim. integral, \( I^{(2)} \) is a $d$ dim. integral, and \( I^{(3)} \) is a $d$ dim. integral.

Error bounds for $S_j$ require more care than error bounds for $I^{(k)}$. 

ljimene1@hawk.iit.edu
Bratley et al. (1992)

\[ f(X) = \sum_{i=1}^{6} (-1)^i \prod_{j=1}^{i} X_j = -X_1 + X_1X_2 - X_1X_2X_3 + \ldots \]

<table>
<thead>
<tr>
<th>( j )</th>
<th>( S_j )</th>
<th>( S_j^{\text{tot}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = 1 )</td>
<td>0.653</td>
<td>0.740</td>
</tr>
<tr>
<td>( j = 2 )</td>
<td>0.179</td>
<td>0.266</td>
</tr>
<tr>
<td>( j = 3 )</td>
<td>0.037</td>
<td>0.077</td>
</tr>
<tr>
<td>( j = 4 )</td>
<td>0.013</td>
<td>0.034</td>
</tr>
<tr>
<td>( j = 5 )</td>
<td>0.001</td>
<td>0.007</td>
</tr>
<tr>
<td>( j = 6 )</td>
<td>0.001</td>
<td>0.007</td>
</tr>
</tbody>
</table>

Computation time was less than 2 seconds for \( \varepsilon_a = 10^{-3} \).
Multivariate Normal Probability

\[ P(a \leq X \leq b) = \int_a^b \frac{e^{x^t\Sigma^{-1}x}}{(2\pi)^{d/2} \mid \Sigma \mid^{1/2}} \, dx. \]

<table>
<thead>
<tr>
<th>( f_{\text{affine}} )</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( j = 4 )</th>
<th>( j = 5 )</th>
<th>( j = 6 )</th>
<th>( j = 7 )</th>
<th>( j = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_j )</td>
<td>0.003</td>
<td>0.020</td>
<td>0.008</td>
<td>0.000</td>
<td>0.025</td>
<td>0.021</td>
<td>0.014</td>
<td>0.028</td>
</tr>
<tr>
<td>( S_j^{\text{tot}} )</td>
<td>0.174</td>
<td>0.467</td>
<td>0.265</td>
<td>0.048</td>
<td>0.535</td>
<td>0.497</td>
<td>0.407</td>
<td>0.570</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( f_{\text{Genz}} )</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( j = 4 )</th>
<th>( j = 5 )</th>
<th>( j = 6 )</th>
<th>( j = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_j )</td>
<td>0.379</td>
<td>0.229</td>
<td>0.036</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>( S_j^{\text{tot}} )</td>
<td>0.713</td>
<td>0.550</td>
<td>0.097</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Arithmetic Mean Asian Call Option

For the payoff of an arithmetic mean Asian option with $S_0 = 100$, $K = 200$, $r = 2\%$, $\sigma = 50\%$, $T = 1$, generating the Brownian motion using PCA with $d = 6$ time steps, and $\varepsilon_a = 10^{-3}$,

\[
\begin{array}{ccccccc}
  j &=& 1 & j &=& 2 & j &=& 3 & j &=& 4 & j &=& 5 & j &=& 6 \\
  S_j &=& 0.992 & S_j &=& 0.000 & S_j &=& 0.000 & S_j &=& 0.000 & S_j &=& 0.000 & S_j &=& 0.000 \\
  S_{\text{tot}}^j &=& 1.000 & S_{\text{tot}}^j &=& 0.007 & S_{\text{tot}}^j &=& 0.002 & S_{\text{tot}}^j &=& 0.001 & S_{\text{tot}}^j &=& 0.001 & S_{\text{tot}}^j &=& 0.000 \\
\end{array}
\]
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- Multivariate Normal Probability
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- Efficiency improvements
  - Importance Sampling—How is importance sampling affecting the adaptive algorithm
  - Control Variates
Arithmetic Mean Asian Call Option

Suppose that the price of an option is \( \mathbb{E}[f_{\text{payoff}}(Z)] \) where \( Z \sim N(0, \Sigma) \). Choose \( h \) such that \( Z = h(X) \) with \( X \sim U[0, 1]^d \). Then,

\[
\text{Price} = \int_{[0,1]^d} f_{\text{payoff}}(h(x)) \, dx = \int_{[0,1]^d} \frac{f_{\text{payoff}}(\mu + h(x)) e^{-\mu^T \Sigma^{-1}(\mu/2 + h(x))}}{f(x, \mu)} \, dx.
\]

The price of the previous arithmetic mean Asian option, now considering \( d = 52 \) time steps, takes

- **131,072** points and **1.6** seconds for \( \mu = 0 \).
- **16,384** points and **0.2** seconds for \( \mu = 1.5(T/d, 2T/d, \ldots, T) \).
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  - Control Variates—Alternative control variates based on the adaptive algorithm
Control Variates Introduction

If \( I(g) = \int_{[0,1]^d} g(\mathbf{x}) \, d\mathbf{x} \) is known, then

\[
\int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} = \int_{[0,1]^d} f(\mathbf{x}) + \beta (I(g) - g(\mathbf{x})) \, d\mathbf{x},
\]

and, based on our cubatures, we choose \( \beta \) such that

\[
\sum_{\kappa=[2^m-1]}^{2^m-1} \left| \hat{f}_\kappa - \beta \hat{g}_\kappa \right| \xrightarrow{m \to \infty} 0 \quad \text{faster than} \quad \sum_{\kappa=[2^m-1]}^{2^m-1} \left| \hat{f}_\kappa \right| \xrightarrow{m \to \infty} 0.
\]

This choice is not necessarily

\[
\beta = \frac{\text{cov}(f(\mathbf{X}), g(\mathbf{X}))}{\text{Var}(g(\mathbf{X}))} = \arg\min_{b} \sum_{\kappa=0}^{\infty} \left| \hat{f}_\kappa - b \hat{g}_\kappa \right|^2,
\]

as noted by Hickernell et al. (2005).
Arithmetic Mean Asian Call Option

For the same configuration as in the importance sampling example and using the geometric mean Asian call option payoff as control variate, computing the price takes

- 131,072 points and 2.2 seconds without control variates.
- 32,768 points and 0.9 seconds with control variates.

\[ | \hat{f}_\kappa | \text{ are smaller than } | \hat{f}_\kappa - \beta \hat{g}_\kappa |, \text{ leading to a smaller sample size needed.} \]
Discussion

- These examples are a work in progress.
- We can handle error tolerances of the form $\max(\varepsilon_a, \varepsilon_r |I|)$.
- Examples can inform the default parameters that define $C$.

Future work includes

- Extending the Sobol Indices algorithm to any set of dimensions.
- Designing an adaptive and automatic multi-level quasi-Monte Carlo algorithm.
- Adapting the conditions that define the cone $C$ when the data-based necessary conditions for $f \in C$ are violated.


Mapping Properties for Embedded Sequences
Alan Genz’s Transform

Be $\Sigma = CC'$ the Cholesky decomposition. Furthermore,

$$a'_1 = a_1/c_{11}, \quad b'_1 = b_1/c_{11}, \quad q_1 = \phi(a'_1), \quad e_1 = \phi(b'_1)$$

where $\phi$ is the univariate standard normal probability distribution function. If for $i = 2, \ldots, d$ we recursively define,

$$y_{i-1} = \phi^{-1}(q_{i-1} + w_{i-1}(e_{i-1} - q_{i-1})),$$

$$a'_i = \frac{a_i - \sum_{j=1}^{i-1} c_{ij}y_j}{c_{ii}}, \quad b'_i = \frac{b_i - \sum_{j=1}^{i-1} c_{ij}y_j}{c_{ii}},$$

$$q_i = \phi(a'_i), \quad e_i = \phi(b'_i).$$
We identified $\hat{S}$ such that

$$\hat{S}_j(b_1, \ldots, b_4, c_1, \ldots, c_4) \leq S_j(a_1, \ldots, a_4) \leq \hat{S}_j(c_1, \ldots, c_4, b_1, \ldots, b_4)$$

for all $b_\ell \leq a_\ell \leq c_\ell$. Then, when one has

$$\hat{S}_j(c_1, \ldots, c_4, b_1, \ldots, b_4) - \hat{S}_j(b_1, \ldots, b_4, c_1, \ldots, c_4) \leq 2\varepsilon,$$

one may stop and choose

$$S_j(a_1, \ldots, a_4) \approx \frac{1}{2} \left( \hat{S}_j(b_1, \ldots, b_4, c_1, \ldots, c_4) + \hat{S}_j(c_1, \ldots, c_4, b_1, \ldots, b_4) \right)$$
European Call Option

Using as importance measure $\mathcal{N}(0, 4)$, the European Call with $S_0 = 100$, $K = 100$, $r = 2\%$, $\sigma = 5\%$, and $T = 2$ has discrete coefficients:
Alternative Optimal $\beta$

Based on our algorithm, we can find $\beta$ minimizing,

$$\left| \int_{[0,1)^d} f(x) \, dx - \frac{1}{2^m} \sum_{i=0}^{2^m-1} f(z_i \oplus \Delta) + \beta(I(g) - g(z_i \oplus \Delta)) \right|$$

$$\leq a(r, m) \sum_{\kappa = [2^{m-r-1}]}^{2^{m-r-1}} |\tilde{f}_{m,\kappa} - \beta \tilde{g}_\kappa|.$$  

Note that $\beta = 0$ corresponds to the initial problem and that this problem has an optimal solution.
Arithmetic Mean Asian Call Option

With $\mu = 0 \alpha = 1$

With $\mu = 0.5 \alpha = 1.1$

ljimene1@hawk.iit.edu

BIRS workshop
Research 2015