DP-Coloring of Graphs from Random Covers

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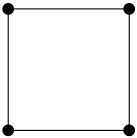
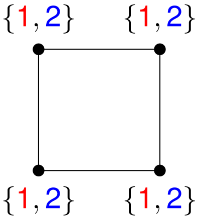
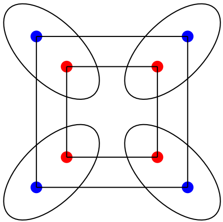
A General Perspective on Graph Coloring



Graph G



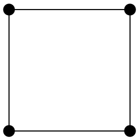
Colors for G



Cover for G

In the cover of G, vertices correspond to the available colors for G, and edges correspond to conflicts between those colors based on edges of G.
Picking a coloring of G corresponds to choosing an independent set of order n in the cover.

A General Perspective on Graph Coloring



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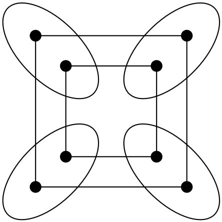
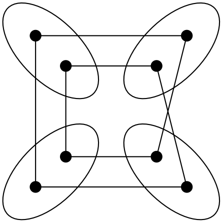


A Cover for G



Another Cover for G

In the cover of G, vertices correspond to the available colors for G, and edges correspond to conflicts between those colors based on edges of G.

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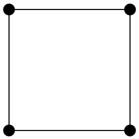
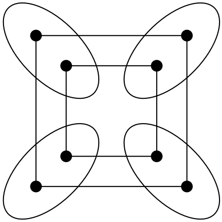
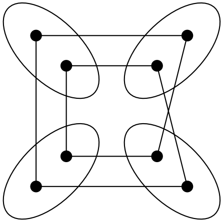
A General Perspective on Graph Coloring



Graph *G*



A Cover for *G*



Another Cover for *G*

A cover of *G* can be expressed with a <u>permutation</u> on each edge of *G*. The permutation models the conflict betwwen those colors.

A General Perspective on Graph Coloring

A topological aside:

What we are informally calling cover of a graph, can be formally defined in the language of covering map. A graph is a topological space, a one-dimensional simplicial complex, and covering maps can be defined and studied for graphs.

> A surjective map $\phi : V(H) \to V(G)$ where G, H are graphs is a covering map if for every $x \in V(H)$, the neighbor set $N_H(x)$ is mapped bijectively to $N_G(\phi(x))$. When such a mapping exists and is k-to-1, we say that H is a k-lift, or k-fold cover of G.

A General Perspective on Graph Coloring

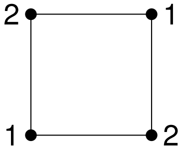
A topological aside:

What we are informally calling cover of a graph, can be formally defined in the language of covering map. A graph is a topological space, a one-dimensional simplicial complex, and covering maps can be defined and studied for graphs.

> Lifts of graphs have been studied in
> - algebraic/ topological graph theory since 1980s (see Negami's Planar Cover Conjecture (1988); Godsil & Royle, Algebraic Graph Theory (2001));
> - random graph theory since 2000 (see seminal papers of Linial).

Classical Coloring

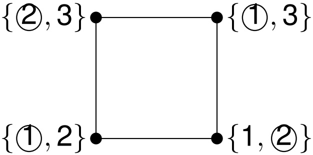
- Classical graph coloring assigns to each vertex in a graph some color, which we will represent as a natural number.
- A k-coloring of a graph G is a function $\phi : V(G) \to [k]$, where $[k] = \{1, 2, \ldots, k\}$.
- A proper k-coloring is a k-coloring ϕ such that every pair of adjacent vertices in G are assigned different colors, i.e. $\phi(u) \neq \phi(v)$ for all $uv \in E(G)$.
- The chromatic number of G, $\chi(G)$ is the smallest k such that G is proper k-colorable.



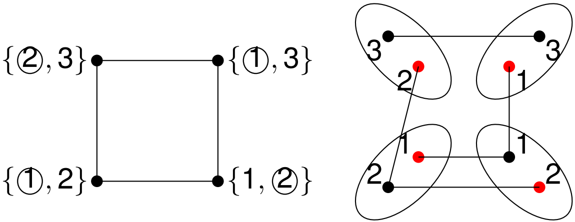
$\chi(C_4) = 2$.

List Coloring

- Introduced by Vizing (1976) and Erdös, Rubin and Taylor (1979).
- For graph G, suppose each $v \in V(G)$ is assigned a list, $L(v)$, of colors. We refer to L as a list assignment. L is an k-assignment if $|L(v)| = k$ for all $v \in V(G)$.
- An L-coloring for G is a proper coloring, ϕ, of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$ and $\phi(u) \neq \phi(v)\ \forall\ uv \in E(G)$.
- The list chromatic number of G, $\chi_\ell(G)$ is the smallest k such that G is L-colorable for all k-assignments L.



DP Coloring

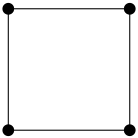


Introduced by Dvořák and Postle (2015).

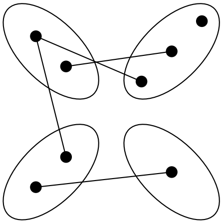
- A DP cover is a tuple $\mathcal{H} = (L, H)$ where H is a graph and $L : V(G) \to 2^{V(H)}$ satisfying:
  (i) $V(H) = \cup_{v \in V(G)} L(v)$,
  (ii) For adjacent $u, v \in V(G)$, $E_H(L(u), L(v))$ forms a matching,
  (ii) There are no other edges in H.

DP Cover

For $G = C_4$



A DP cover of C_4



DP-cover Intuition:

Blow up each vertex u in G into an independent set of size $|L(u)|$;

Add a matching (possibly empty) between any two such independent sets for vertices u and v if uv is an edge in G.

DP Coloring

- A transversal of $\mathcal{H} = (L, H)$ is a set of vertices $T \subseteq V(H)$ such that $|T \cap L(v)| = 1$ for all $v \in V(G)$.
- T is an independent transversal if it is an independent set in H.
- If $\mathcal{H}$ has an independent transversal, we say that G is $\mathcal{H}$-colorable.

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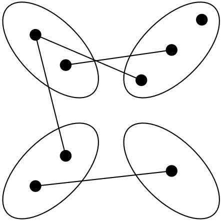

For $\mathcal{H}$ from the previous slide, C_4 is $\mathcal{H}$-colorable.

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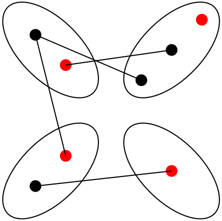
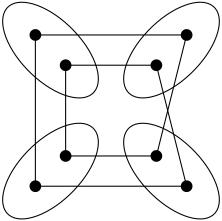


For $\mathcal{H}$ from the previous slide, C_4 is $\mathcal{H}$-colorable.

DP Coloring

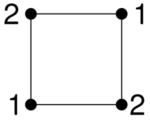
- $\mathcal{H} = (L, H)$ is a k-fold cover of G if $|L(v)| = k$ for all $v \in V(G)$. $\mathcal{H}$ is full if every matching in it is perfect.
- The DP chromatic number $\chi_{DP}(G)$ is the smallest k such that G is $\mathcal{H}$-colorable for every k-fold cover $\mathcal{H}$.
- $\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G)$.



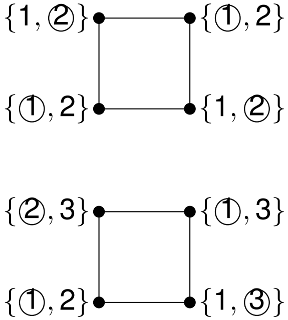
$\chi_{DP}(C_4) > 2$.

Comparing Classical, List, and DP Coloring

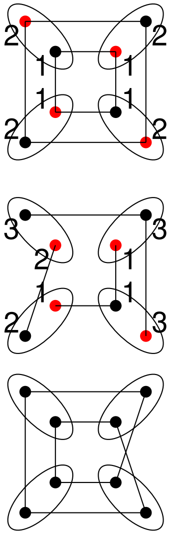
Classical Coloring



List Coloring



DP Coloring



Our Question

- The DP chromatic number offers a guarantee that we will always be able to find an independent transversal.
- If there is even a single k-fold cover of G that does not have an independent transversal, then $\chi_{DP}(G) > k$.
- The question we ask is: "is there a threshold on the value of k such that almost all k-fold covers of a graph have an independent transversal above the threshold, and almost none below the threshold?"
- We initiate this study by considering full DP-covers generated uniformly at random, and asking the natural probabilistic questions that arise from that context.

Historical Notes

- Random lists and Palette Sparsification.
  The list assignments of a given graph G are generated uniformly at random from a palette of given colors. Is there a threshold size of the assignments that shows a transition in the list colorability of G (parameterized by either the order or the chromatic number of the graph)?

  Introduced in 2004 by Krivelevich and Nachmias ("*The problem originated in the chemical industry and it is related to scheduling problems occurring in the production of colorants.*"). Studied for powers of cycles, complete graphs, complete multipartite graphs, graphs with bounded degree, etc.

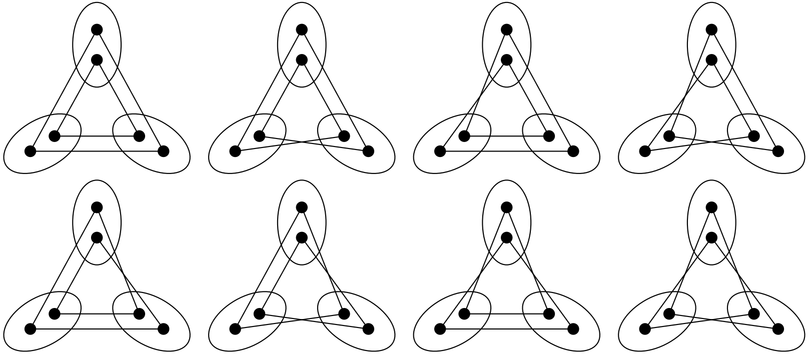
  Colorings from random list assignments – under the name palette sparsification – has recently found applications in the design of sublinear coloring algorithms starting from the work of Assadi, Chen, and Khanna (2019).

Historical Notes

- Random Lifts.
  A full DP-cover of G is equivalent to the previously studied notion of a lift (or a covering graph) of G. The notion of random k-lifts was introduced in 2000 by Amit and Linial. This work, and the large body of research following it studies random k-lifts as a random graph model. Their purpose was the study of the properties of random k-lifts, such as chromatic number, connectivity, expansion properties, etc., of a fixed graph G as $k \to \infty$.

Random Cover Example

Select one of the full 2-fold covers of K_3 uniformly at random.



Random Covers

- The random k-fold cover of G, $\mathcal{H}(G,k)$, is one of the $k!^{|E(G)|}$ full k-fold covers chosen uniformly at random.
  - We can think of this as creating a sample space of all full k-fold covers of G, then selecting one uniformly at random,
  - Or, we can think of this as creating our lists of size k, and selecting each perfect matching (or permutation) uniformly at random from the $k!$ possibilities.

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- We study the probability that a random cover of a graph has an independent transversal. G is k-DP-colorable with probability p when $\mathbb{P}(G \text{ is } \mathcal{H}(G, k)\text{-colorable}) = p$.

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- We study the probability that a random cover of a graph has an independent transversal. G is k-DP-colorable with probability p when $\mathbb{P}(G \text{ is } \mathcal{H}(G,k)\text{-colorable}) = p$.
- The density of graph G, $d(G)$, is $|E(G)|/|V(G)|$. The maximum density of G, $\rho(G)$, is $\max_{G'} d(G')$, where the maximum is taken over all nonempty subgraphs G' of G.
- A graph G is d-degenerate if there exists some ordering of the vertices in $V(G)$ such that each vertex has at most d neighbors among the preceding vertices.
  The degeneracy of a graph G is the smallest $d \in \mathbb{N}$ such that G is d-degenerate.
  Note that $\rho(G) \leq d \leq 2\rho(G)$.

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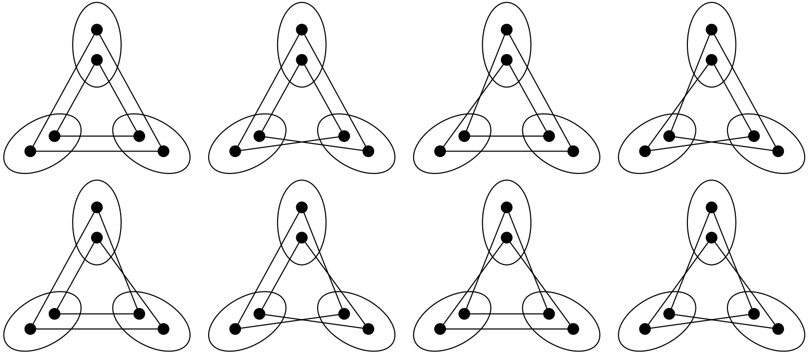
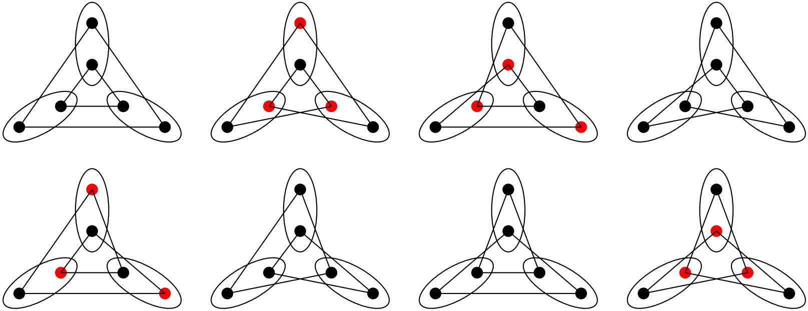


Random Cover Example

K_3 is 2-DP-colorable with probability 0.5



Threshold Behavior

Given a sequence of graphs $\mathcal{G} = (G_\lambda)_{\lambda \in \mathbb{N}}$ and a sequence of integers $\kappa = (k_\lambda)_{\lambda \in \mathbb{N}}$. We say that $\mathcal{G}$ is κ-DP-colorable with high probability if

$$\lim_{\lambda \to \infty} \mathbb{P}(G_\lambda \text{ is } \mathcal{H}(G_\lambda, k_\lambda)\text{-colorable}) = 1.$$

Similarly, we say that $\mathcal{G}$ is non-κ-DP-colorable w.h.p. if

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- A function $t_{\mathcal{G}} : \mathbb{N} \to \mathbb{R}$ is called a DP-threshold function for $\mathcal{G}$:
  if $k_\lambda = o(t_{\mathcal{G}}(\lambda))$, then $\mathcal{G}$ is non-κ-DP-colorable w.h.p.,
  while if $t_{\mathcal{G}}(\lambda) = o(k_\lambda)$, then $\mathcal{G}$ is κ-DP-colorable w.h.p.

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$$\lim_{\lambda \to \infty} \mathbb{P}(G_\lambda \text{ is } \mathcal{H}(G_\lambda, k_\lambda)\text{-colorable}) = 0.$$

- A function $t_{\mathcal{G}}$ is said to be a sharp DP-threshold function for $\mathcal{G}$ when for any $\epsilon > 0$,
  $\mathcal{G}$ is non-κ-DP-colorable w.h.p. when $k_\lambda \leq (1-\epsilon)t_{\mathcal{G}}(\lambda)$ for all large enough λ,
  and it is κ-DP-colorable w.h.p. when $k_\lambda \geq (1+\epsilon)t_{\mathcal{G}}(\lambda)$ for all large enough λ.

Threshold Results

Theorem (Bernshteyn, Dominik, K., Mudrock (2025))

Let $\mathcal{G} = (G_\lambda)_{\lambda \in \mathbb{N}}$ *be a sequence of graphs with* $|V(G_\lambda)|$, $\rho(G_\lambda) \to \infty$ *as* $\lambda \to \infty$.
Define a function $t_{\mathcal{G}}(\lambda) = \rho(G_\lambda) / \ln \rho(G_\lambda)$.

$$\text{If } \lim_{\lambda \to \infty} \frac{\ln \rho(G_\lambda)}{\ln \ln |V(G_\lambda)|} = \infty,$$

then $t_{\mathcal{G}}(\lambda)$ *is a DP-threshold function for* $\mathcal{G}$.

$$\text{If } \lim_{\lambda \to \infty} \frac{\ln \rho(G_\lambda)}{\ln |V(G_\lambda)|} = 1,$$

then $t_{\mathcal{G}}(\lambda)$ *is a sharp DP-threshold function for* $\mathcal{G}$.

Threshold Results

Corollary (Bernshteyn, Dominik, K., Mudrock (2025))

For $\mathcal{G} = (K_n)_{n \in \mathbb{N}}$, the sequence of complete graphs, $t_{\mathcal{G}}(n) = n/(2 \ln n)$ is a sharp DP-threshold function.

The existence of a (not necessarily sharp) DP-threshold function of order $\Theta(n/\ln n)$ for the sequence of complete graphs was recently proved by Dvořák and Yepremyan using different methods.

Corollary (Bernshteyn, Dominik, K., Mudrock (2025))

For $\mathcal{G} = (K_{m \times n})_{n \in \mathbb{N}}$ with constant $m \geq 2$, the sequence of complete m-partite graphs with n vertices in each part, $t_{\mathcal{G}}(n) = (m-1)n/(2 \ln n)$ is a sharp DP-threshold function.

DP-colorability with Low Probability

Theorem (Bernshteyn, Dominik, K., Mudrock (2025))

Let $\epsilon > 0$ and let G be a nonempty graph with $\rho(G) \geq \exp(e/\epsilon)$. If $1 \leq k \leq \rho(G)/\ln \rho(G)$, then G is k-DP-colorable with probability at most ϵ.

In fact, we prove a stronger result in context of fractional DP-coloring. Let $p^*(G, k) = \sup\{p : \exists a, b \in \mathbb{N} \text{ s.t. } a/b \leq k \text{ and } G \text{ is } (a, b)\text{-DP-colorable with probability } p\}$.

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Let $\epsilon > 0$ and let G be a nonempty graph with $\rho(G) \geq \exp(e/\epsilon)$. If $1 \leq k \leq \rho(G)/\ln \rho(G)$, then $p^*(G, k) \leq \epsilon$.

DP-colorability with High Probability for Dense Graphs

Theorem (Bernshteyn, Dominik, K., Mudrock (2025))

For all $\epsilon > 0$ and $s \in [0, 1/3)$, there is $n_0 \in \mathbb{N}$ such that the following holds. Suppose G is a graph with $n \geq n_0$ vertices such that $\rho(G) \geq n^{1-s}$, and

$$k \geq (1+\epsilon)\left(1 + \frac{s}{1-2s}\right)\frac{\rho(G)}{\ln \rho(G)}$$

Then G is k-DP-colorable with probability at least $1 - \epsilon$.

Notice how the lower bound on k increases from $\frac{\rho(G)}{\ln \rho(G)}$ to $\frac{2\rho(G)}{\ln \rho(G)}$ as $\rho(G)$ decreases from $n^{1-o(1)}$ to $n^{2/3}$.

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Then G is k-DP-colorable with probability at least $1 - \epsilon$.

This is proved through a long second-moment argument.

Can we lower the bound on ρ below $n^{2/3}$ if we aim to keep the bound on k at $\frac{2\rho(G)}{\ln \rho(G)}$ (off by a factor of two from the first moment bound of $\frac{\rho(G)}{\ln \rho(G)}$)?

DP-colorability with High Probability for Sparse Graphs

We use degeneracy to push the bound on density down to *polylog*(n).

Theorem (Bernshteyn, Dominik, K., Mudrock (2025))

For all $\epsilon \in (0, 1/2)$, there is $n_0 \in \mathbb{N}$ such that the following holds. Let G be a graph with $n \geq n_0$ vertices and degeneracy d such that $d \geq \ln^{2/\epsilon} n$. If $k \geq (1+\epsilon)d/\ln d$, then G is k-DP-colorable with probability at least $1 - \epsilon$.

DP-colorability with High Probability for Sparse Graphs

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Since $\rho(G) \leq d \leq 2\rho(G)$, we can compare this result to the earlier ones in terms of density.

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This is proved by analyzing a greedy algorithm for constructing an independent (b-fold) transversal in a random k-fold cover.

The random variables for each vertex being unavailable to be picked in the greedy transversal are negatively correlated.

And, use a form of the Chernoff–Hoeffding bound for negatively correlated Bernoulli random variables due to Panconesi and Srinivasan (1997).

What about very sparse graphs?

Proposition (Bernshteyn, Dominik, K., Mudrock (2025))

For any $\epsilon > 0$ and $n_0 \in \mathbb{N}$, there is a graph G with $n \geq n_0$ vertices such that $\rho(G) \geq (\ln n / \ln \ln n)^{1/3}$ but, for every $k \leq 2\rho(G)$, G is k-DP-colorable with probability less than ϵ.

We take $G = tK_q$, the disjoint union of t copies of K_q, where $t = \ln(1/\epsilon)\,(q-1)!^{\binom{q}{2}}$ and q is large enough.

A result of Bernshteyn (2019) shows: for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that every triangle-free regular graph G with $\rho(G) \geq C_\epsilon$ satisfies $\chi_{DP}(G) \leq (1+\epsilon)2\rho(G)/\ln \rho(G)$, and hence it is k-DP-colorable (with probability 1) for all $k \geq (1+\epsilon)2\rho(G)/\ln \rho(G)$.

What about very sparse graphs?

Proposition (Bernshteyn, Dominik, K., Mudrock (2025))

For any $\epsilon > 0$ and $n_0 \in \mathbb{N}$, there is a graph G with $n \geq n_0$ vertices such that $\rho(G) \geq (\ln n / \ln \ln n)^{1/3}$ but, for every $k \leq 2\rho(G)$, G is k-DP-colorable with probability less than ϵ.

A result of Bernshteyn (2019) shows: for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that every triangle-free regular graph G with $\rho(G) \geq C_\epsilon$ satisfies $\chi_{DP}(G) \leq (1+\epsilon)2\rho(G)/\ln \rho(G)$, and hence it is k-DP-colorable (with probability 1) for all $k \geq (1+\epsilon)2\rho(G)/\ln \rho(G)$.

So, for graphs with density below *polylog*(n), density alone is not enough to determine probability of DP-colorability.

A Conjecture

We conjecture that for density above *polylog*(n), we should get a sharp bound on k.

Conjecture (Bernshteyn, Dominik, K., Mudrock (2025))

For all $\epsilon > 0$, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose G is a graph with $n \geq n_0$ vertices such that $\rho(G) \geq \ln^C n$, and

$$k \geq (1+\epsilon)\frac{\rho(G)}{\ln \rho(G)}.$$

Then G is k-DP-colorable with probability at least $1-\epsilon$.

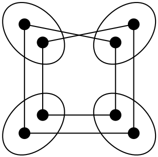
Summary of Results

Density lower bound	Cover size	$\mathbb{P}(G,k)$
$\exp(e/\epsilon)$	$k \leq \frac{\rho(G)}{\ln \rho(G)}$	$\leq \epsilon$
n^{1-s} for $s \in [0, 1/3)$	$k \geq (1+\epsilon)\left(1 + \frac{s}{1-2s}\right)\frac{\rho(G)}{\ln \rho(G)}$	$\geq 1 - \epsilon$
$\ln^{2/\epsilon} n$	$k \geq (1+\epsilon)\frac{2\rho(G)}{\ln \rho(G)}$	$\geq 1 - \epsilon$
No lower bound	$k > 2\rho(G)$	1

$\mathbb{P}(G,k)$ is the probability that G is $\mathcal{H}(G,k)$-colorable.

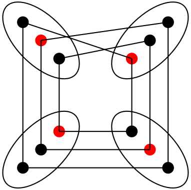
Fractional DP-Coloring

- We saw $\chi_{DP}(C_4) > 2$.



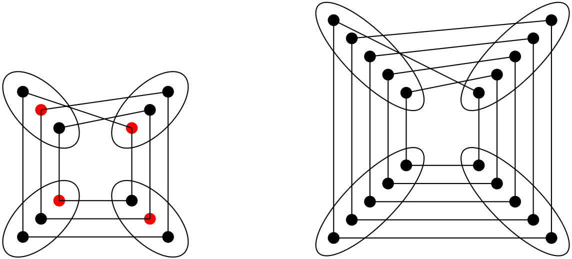
Fractional DP-Coloring

- In fact, $\chi_{DP}(C_4) = 3$.



Fractional DP-Coloring

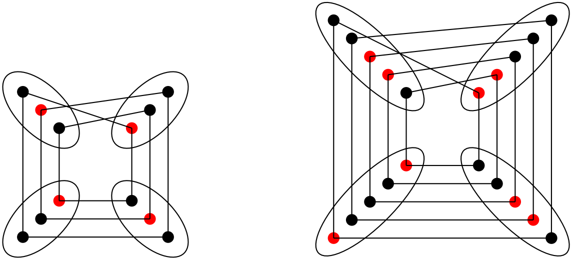
- Instead, let us look at a 5-fold cover of C_4



- $\chi_{DP}(C_4) = 3$.

Fractional DP-Coloring

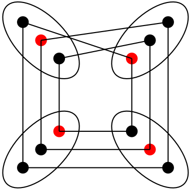
- Instead, let us look at a 5-fold cover of C_4 and find an independent 2-fold transversal in it.

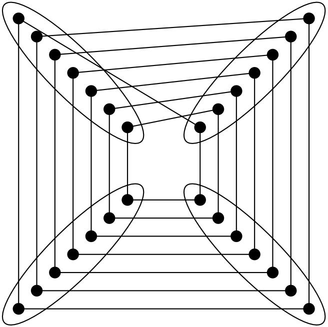


- $\chi_{DP}(C_4) = 3$.
- We can see that C_4 is $(5, 2)$-DP-colorable and $\chi^*_{DP}(C_4) \leq 5/2$.

Fractional DP-Coloring

- Instead, let us look at a 7-fold cover of C_4

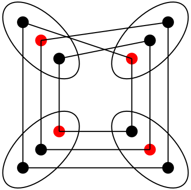


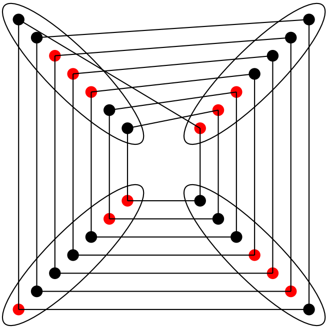


- $\chi_{DP}(C_4) = 3$.

Fractional DP-Coloring

- Instead, let us look at a 7-fold cover of C_4 and find an independent 3-fold transversal in it.

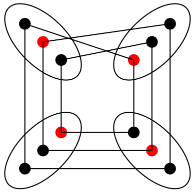


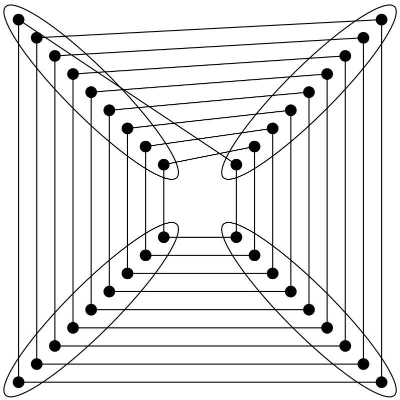


- $\chi_{DP}(C_4) = 3$.
- We can see that C_4 is $(7, 3)$-DP-colorable and $\chi^*_{DP}(C_4) \leq 7/3$.

Fractional DP-Coloring

- Instead, let us look at a 9-fold cover of C_4

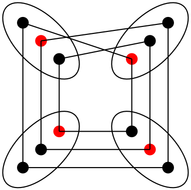


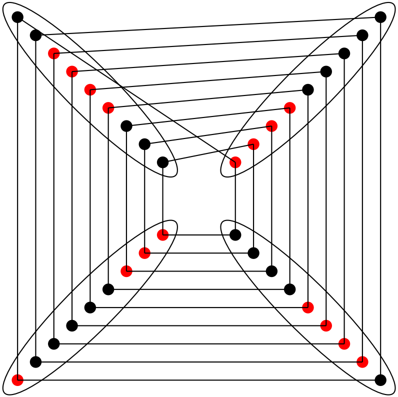


- $\chi_{DP}(C_4) = 3$.

Fractional DP-Coloring

- Instead, let us look at a 9-fold cover of C_4 and find an independent 4-fold transversal in it.





- $\chi_{DP}(C_4) = 3$.
- We can see that C_4 is $(9,4)$-DP-colorable and $\chi^*_{DP}(C_4) \le 9/4$.

Fractional DP-Coloring

- $\chi_{DP}(C_4) = 3$.
- In the limit we can see $\chi^*_{DP}(C_4) = 2$.

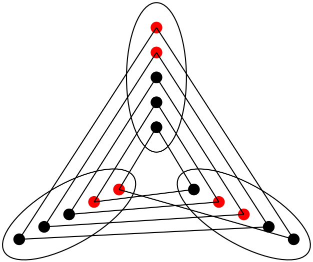
Fractional DP-Coloring Defined

- Given a graph G and $\mathcal{H}$, a cover of G, then G is $(\mathcal{H}, b)$-colorable if $\mathcal{H}$ contains an independent b-fold transversal.
- A graph G is (a, b)-DP-colorable if G is $(\mathcal{H}, b)$-colorable for all a-fold covers $\mathcal{H}$.
- The fractional DP-chromatic number is

$$\chi^*_{DP}(G) = \inf\left\{\frac{a}{b} : G \text{ is } (a, b)\text{-DP-colorable}\right\}.$$

- Introduced by Bernshteyn, Kostochka, and Zhu (2020).
- $\chi^*(G) = \chi^*_\ell(G) \leq \chi^*_{DP}(G) \leq \chi_{DP}(G)$.

Probability of Fractional-DP-Coloring



- If G is κ-DP-colorable, then G is fractional-k-DP-colorable.
- If G is non-k-DP-colorable, there may be some large a and b that still allows G to be fractional-k-DP-colorable.
- What is the probability of fractional-DP-colorability of G over $\mathcal{H}(G, k)$?

Probability of Fractional-DP-Coloring

Let $p^*(G,k) = \sup\{p : \exists a, b \in \mathbb{N} \text{ s.t. } a/b \le k$ and G is (a,b)-DP-colorable with probability $p\}$.

Theorem (Bernshteyn, Dominik, K., Mudrock (2025))

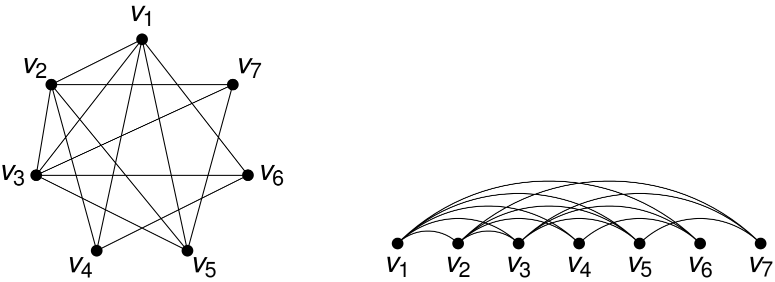
Let $\epsilon > 0$ and let G be a graph with $\rho(G) \ge \exp(e/\epsilon)$. If $1 \le k \le \rho(G)/\ln \rho(G)$, then $p^*(G,k) \le \epsilon$.

Theorem (Bernshteyn, Dominik, K., Mudrock (2025))

For all $\epsilon > 0$, there is $d_0 \in \mathbb{N}$ such that the following holds. Let G be a graph with degeneracy $d \ge d_0$ and let $k \ge (1+\epsilon)d/\ln d$. Then $p^*(G,k) \ge 1 - \epsilon$.

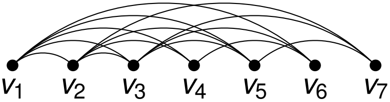
This extends the earlier result, where we required degeneracy $d \ge \ln^{2/\epsilon} n$, to any graph whose degeneracy is high enough as a function of ϵ (regardless of how small it is when compared to the number of vertices in the graph), at the cost of replacing DP-coloring with fractional DP-coloring.

Degeneracy



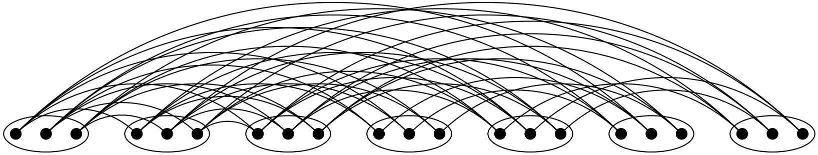
- A graph is *d*-degenerate if there is an ordering of its vertices such that no vertex has more than *d* neighbors preceding itself in the list.

Greedy Transversal Procedure



- Consider a random 3 fold cover of the graph from the previous slide.

Greedy Transversal Procedure



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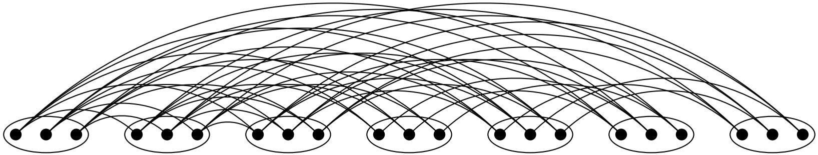
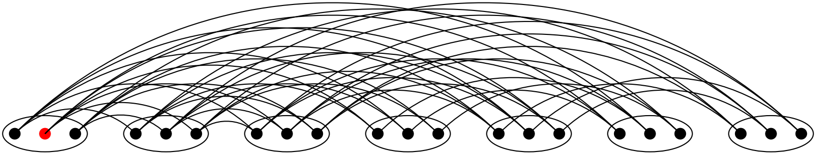


- Consider a random 3 fold cover of the graph from the previous slide.
- Select one available vertex from each list, starting with $L(v_1)$ and ending with $L(v_7)$.

Greedy Transversal Procedure



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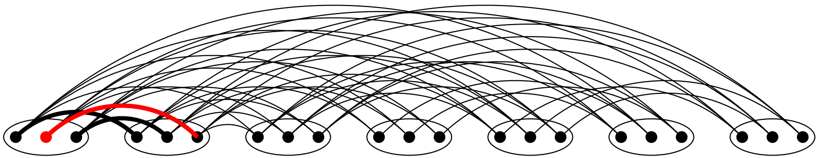

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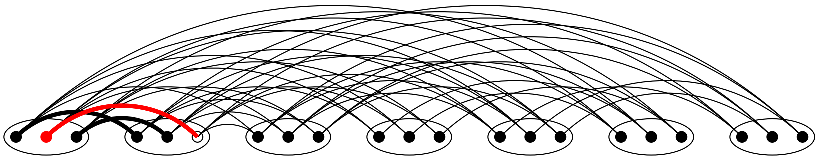

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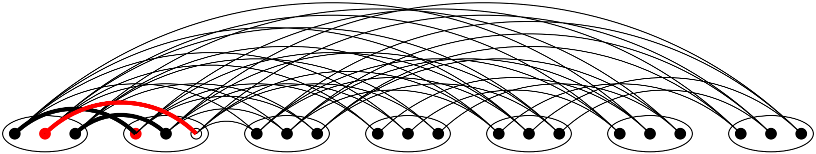

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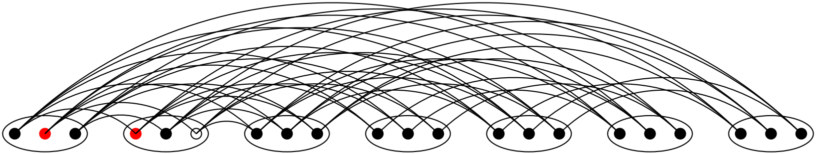

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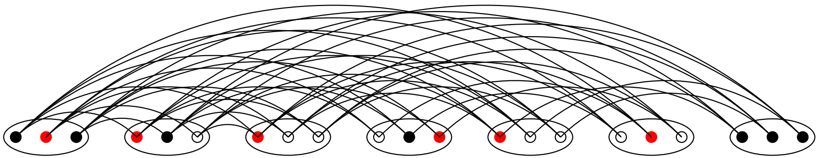

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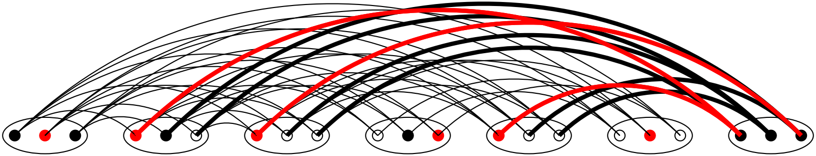

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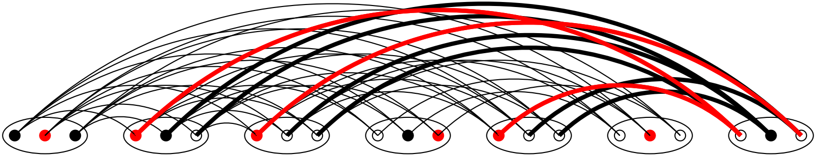

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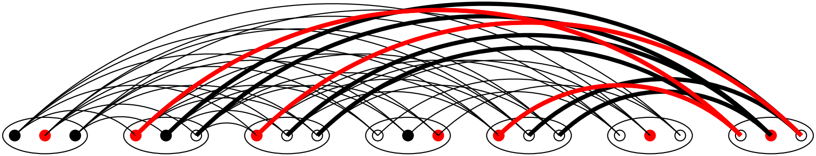

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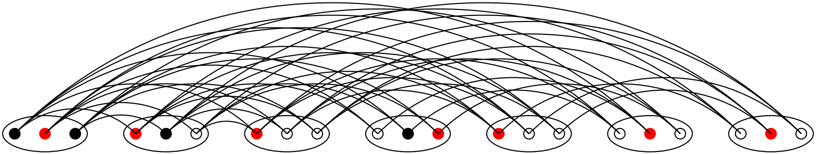

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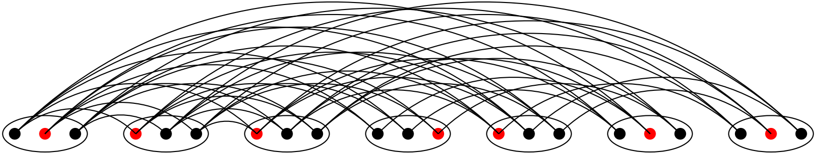

- Consider a random 3 fold cover of the graph from the previous slide.
- Select one available vertex from each list, starting with $L(v_1)$ and ending with $L(v_7)$.
- In the fractional setting, we pick b available vertices sequentially from each list, in a random a-fold cover.

Analyzing the Greedy Transversal Procedure

- Consider a random a-fold cover of a n-vertex graph G.
- For each list in the cover. Pick b available vertices sequentially, if possible. If not, then just pick any b vertices.
- Output is a b-fold transversal which is independent if at least b vertices are available at each step.

- For each $i \in [n]$ (one for each vertex of G) and $j \in [a]$ (one for each "color" in the lists of the cover), let $X_{i,j}$ be the indicator random variable of the event that the vertex $v_{i,j}$ is available in the list $L(v_i)$.
  Let $Y_{i,j} = 1 - X_{i,j}$.

Analyzing the Greedy Transversal Procedure

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  Let $Y_{i,j} = 1 - X_{i,j}$.

Lemma

Consider the set of random variables $X_{i,j}$ as defined above.

(i) For all $i \in [n]$ and $j \in [a]$, we have $\mathbb{E}(X_{i,j}) \geq \left(1 - \frac{b}{a}\right)^d$.

(ii) For each $i \in [n]$, the variables $(Y_{i,j})_{j \in [a]}$ are negatively correlated.

Analyzing the Greedy Transversal Procedure

- A collection $(Y_i)_{i\in[k]}$ of $\{0,1\}$-valued random variables is negatively correlated if for every subset $I \subseteq [k]$, we have $\mathbb{P}\left(\bigcap_{i\in I}\{Y_i = 1\}\right) \le \prod_{i\in I}\mathbb{P}(Y_i = 1)$.

- Sums of negatively correlated random variables satisfy Chernoff–Hoeffding style bounds, as discovered by Panconesi and Srinivasan (1997).

Lemma

Let $(X_i)_{i\in[k]}$ be $\{0,1\}$-valued random variables. Set $Y_i = 1 - X_i$ and $X = \sum_{i\in[k]} X_i$. If $(Y_i)_{i\in[k]}$ are negatively correlated, then

$$\mathbb{P}\left(X < \mathbb{E}(X) - t\right) < \exp\left(-\frac{t^2}{2\mathbb{E}(X)}\right) \quad \text{for all } 0 < t \le \mathbb{E}(X).$$

Analyzing the Greedy Transversal Procedure

- For each $i \in [n]$ (one for each vertex of G) and $j \in [a]$ (one for each "color" in the lists of the cover), let $X_{i,j}$ be the indicator random variable of the event that the vertex $v_{i,j}$ in the cover is available in the list $L(v_i)$.
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Consider the set of random variables $X_{i,j}$ as defined above.

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  Let $Y_{i,j} = 1 - X_{i,j}$.

- Let $X_i = \sum_{j \in [a]} X_{i,j}$ the number of available vertices in $L(v_i)$.

Analyzing the Greedy Transversal Procedure

- Let $X_i = \sum_{j \in [a]} X_{i,j}$ the number of available vertices in $L(v_i)$.
- We can show for degeneracy d,
  $\mathbb{E}(X_i) \geq a\left(1 - \frac{b}{a}\right)^d \geq b \cdot (1+\epsilon/2)\frac{d}{\ln d} \cdot \left(1 - \frac{\ln d}{(1+\epsilon/2)d}\right)^d \geq$
  $b \cdot (1+\epsilon/2)\frac{d}{\ln d} \cdot d^{-1/(1+\epsilon/2)} > b\, d^{\epsilon/3}$
  where the last step uses d is large as a function of ϵ.

Analyzing the Greedy Transversal Procedure

- Let $X_i = \sum_{j \in [a]} X_{i,j}$ the number of available vertices in $L(v_i)$.
- We showed $\mathbb{E}(X_i) > b\, d^{\epsilon/3}$.

  Using Chernoff-Hoeffding for negatively correlated r.v.s, we can show at least b vertices are available at each step pf the GT Procedure with high probability,

  $\mathbb{P}(X_i < b) \leq \mathbb{P}\left(X_i < \frac{\mathbb{E}(X_i)}{2}\right) < \exp\left(-\frac{\mathbb{E}(X_i)}{8}\right) \leq e^{-b/4} < \frac{\epsilon}{n}$
  where the last inequality uses b is large enough as a function of n.

- By the union bound, it follows that
  $\mathbb{P}(X_i < b \text{ for some } i \in [n]) < \epsilon$.

Thank You!

Any Questions?

Question

Under what conditions on $\mathcal{G} = (G_\lambda)_{\lambda \in \mathbb{N}}$ will $t_{\mathcal{G}}(\lambda) = \rho(G_\lambda)/\ln \rho(G_\lambda)$ be a DP-threshold function or a sharp DP-threshold function for $\mathcal{G}$?

Conjecture

For all $\epsilon > 0$, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose G is a graph with $n \geq n_0$ vertices such that $\rho(G) \geq \ln^C n$, and

$$k \geq (1+\epsilon)\frac{\rho(G)}{\ln \rho(G)}.$$

Then G is k-DP-colorable with probability at least $1-\epsilon$.

Question

What about fractional DP-coloring?

Thank You!

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Question

What about fractional DP-coloring?

Negative Correlation by Coupling

- A collection $(Y_i)_{i\in[k]}$ of $\{0,1\}$-valued random variables is negatively correlated if for every subset $I \subseteq [k]$, we have $\mathbb{P}\left(\bigcap_{i\in I}\{Y_i = 1\}\right) \leq \prod_{i\in I}\mathbb{P}(Y_i = 1)$.

- To prove that for each $i \in [n]$, the variables $(Y_{i,j})_{j\in[a]}$ are negatively correlated, we use a coupling argument.
  - Create two new probability spaces:
    - One finds the probability of getting certain matchings from all matchings that leave the j^{th} vertex available.
    - The other finds the probability of getting certain matchings after "fixing" the set of matchings so that the j^{th} vertex is available.
  - Show that these probability measures are equivalent and that we don't lose any events.