A Polynomial Method for Counting Colorings of Sparse Graphs

Hemanshu Kaul

Illinois Institute of Technology

www.math.iit.edu/∼kaul

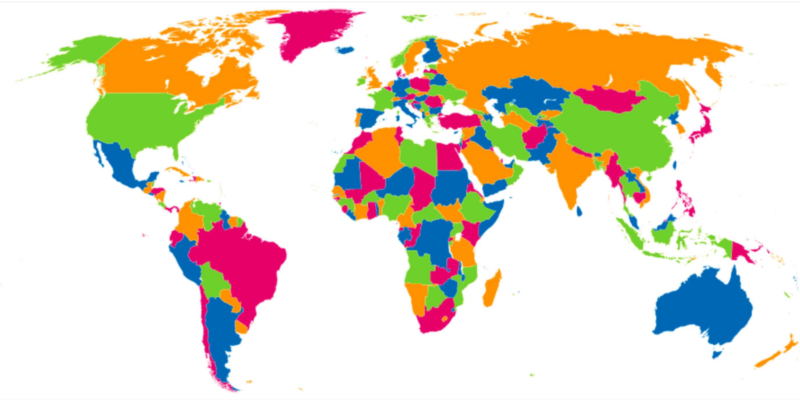
kaul@iit.edu

Joint work with
Samantha Dahlberg (Illinois Tech)
Jeffrey Mudrock (U. South Alabama)

Exponentially Many Colorings of Planar Graphs!

The history of coloring of planar graphs and its subfamilies, is intertwined with the related conjectures and results on existence of exponentially many such colorings (exponential in n, the number of vertices) going back at the least to Birkhoff's and Whitney's works in 1930s.

Exponentially Many Colorings of Planar Graphs!



Four Colors for the World Map

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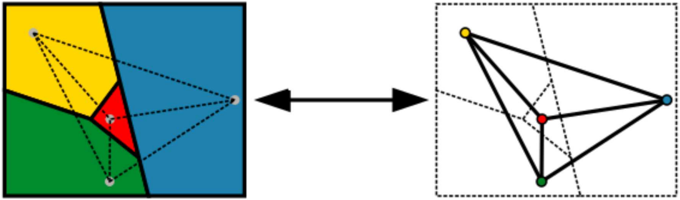


Coloring a Planar Graph

Exponentially Many Colorings of Planar Graphs!

- Francis Guthrie (October 23, 1852): Do Four colors suffice for any planar graph?

- Kempe (1879) published a proof claiming to solve it. Honored as Fellow of the Royal Society and elected President of LMS.
- Heawood (1890) found an error in Kempe's proof. The fixed proof showed: Five colors suffice for any planar graph.

Exponentially Many Colorings of Planar Graphs!

- Birkhoff (1912): Chromatic Polynomial, $P(G, k)$, the number of colorings of G using k colors.

- Birkhoff and Lewis (1946) conjectured:
  For any planar graph G,
  $P(G, k) \geq k(k-1)(k-2)(k-3)^{n-3}$ for all real numbers $k \geq 4$.

  They (essentially, Birkhoff (1930)) proved this is true for $k \geq 5$, thus giving exponentially many 5-colorings of planar graphs: $P(G, 5) > 2^n$.

Exponentially Many Colorings of Planar Graphs!

- Grötzsch (1959): $P(G, 3) > 0$, for any triangle-free planar graph.
- Appel and Haken (1976): Four Color Theorem! $P(G, 4) > 0$ for every planar graph G.

Exponentially Many Colorings of Planar Graphs!

- Vizing (1975), Erdös, Rubin, and Taylor (1979): Introduced List Coloring. Instead of same colors for each vertex, vertices are assigned lists of (possibly different) colors. Kostochka and Sidorenko (1990): List Color Function, $P_\ell(G, k)$, the guaranteed number of list colorings of G, no matter which lists of k-colors are assigned to each vertex.
- $P_\ell(G, k) \leq P(G, k)$.

- Thomassen (1995): $P_\ell(G, 5) > 0$ for any planar graph G.
- Thomassen (2007): $P_\ell(G, 5) > 2^{n/9}$ for any planar graph G.

Exponentially Many Colorings of Planar Graphs!

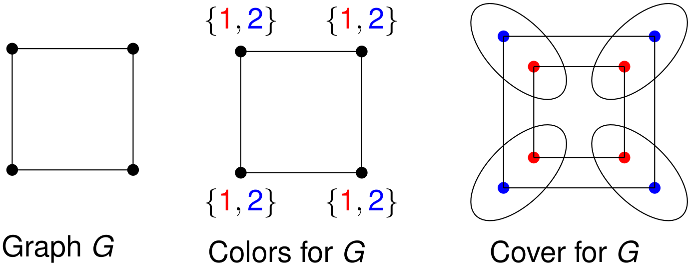
- Thomassen (2007): $P_\ell(G, 5) > 2^{n/9}$ for any planar graph G.
- Since 2000s, there has been much work done on showing that planar graphs and their subfamilies have exponentially many list k-colorings for appropriate $k \in \{3, 4, 5\}$.
- These proofs are typically intricate topological arguments specialized to the family of planar graphs under consideration.

Can we unify these results and arguments in a systematic way?

Classic Graph Coloring

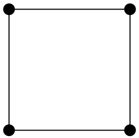
- Color vertices of a graph so that any vertices with an edge between them must get different colors.
- A proper k-coloring of a graph G is a labeling $c : V(G) \to \{1, \ldots, k\}$, such that $c(u) \neq c(v)$ whenever u and v are adjacent in G.
- Each vertex has the same list of colors $[k]$ available to it.

A More General Perspective

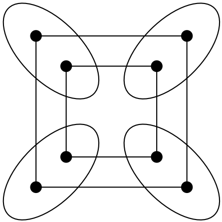


In the cover of G, vertices correspond to the available colors for G, and edges correspond to conflicts between those colors based on edges of G. Picking a coloring of G corresponds to choosing an independent set of order n in the cover.

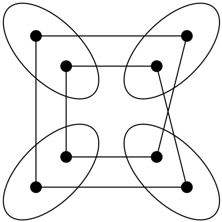
A More General Perspective



Graph G



A Cover for G



Another Cover for G

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A More General Perspective

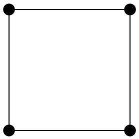
A topological aside:

What we are informally calling cover of a graph, can be formally defined in the language of covering map. A graph is a topological space, a one-dimensional simplicial complex, and covering maps can be defined and studied for graphs.

A surjective map $\phi : V(H) \to V(G)$ where G, H are graphs is a covering map if for every $x \in V(H)$, the neighbor set $N_H(x)$ is mapped bijectively to $N_G(\phi(x))$. When such a mapping exists and is k-to-1, we say that H is a k-lift, or k-fold cover of G.

Lifts of graphs have been studied in algebraic/ topological graph theory since 1980s (see Godsil & Royle, Algebraic Graph Thry); and in random graph theory since 2000 (see seminal papers of Linial).

A More General Perspective



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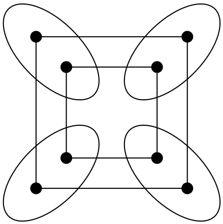

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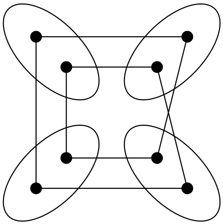

Another Cover for G

A cover of G can be expressed with a permutation on each edge of G. The permutation models the conflict betwwen those colors.

S-labeling and coloring

Jin, Wang, Zhu (2019):

- Let A be a finite set, $|A| = k$, and $S \subseteq S_A$ be a subset of the permutations of A. Think of A as "colors".
- An S-labeling of G is a pair (D, σ) consisting of an orientation D of G and an edge labeling $\sigma : E(D) \rightarrow S$.

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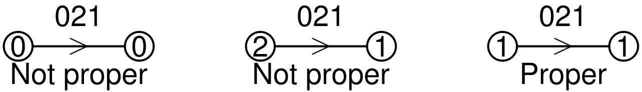
- Let $A = \{0, 1, 2\}$ and $S = S_A$.



021 denotes the permutation $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$.

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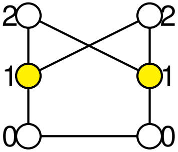

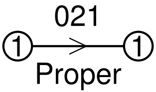

- An S-k-coloring of (D, σ) is $\kappa : V(G) \to A$ such that for each edge $(u, v) \in E(D)$ if $\pi = \sigma(u, v)$ then $\pi(\kappa(u)) \neq \kappa(v)$.

S-labeling and coloring

- We call G S-k-colorable if there exists an S-k-coloring for every S-labeling of G.
- Coloring of S-labeled graphs is a common generalization of many well studied notions of colorings.

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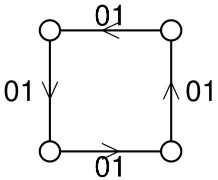
- $S = \{\mathrm{id}_A\}$ gives classical coloring.
- $S \subseteq \mathcal{L}$, linear permutations, gives Signed-coloring. Introduced in 1930s (formally, the seminal paper of Zaslavsky (1982))with many applications in context of psychological models, root systems, Ising model, etc.
- Signed $\mathbb{Z}_k$-coloring.
- Group $\mathbb{Z}_k$-coloring; Field coloring.
- Coloring of Gain graphs.
- $S = S_A$ gives DP-coloring. Introduced in 2015 by Dvořák and Postle and widely studied since then.

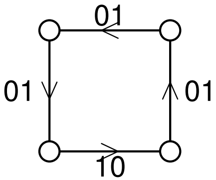
Counting Colorings of S-labeled Graphs

$$P_S(G,k) = \min_{S\text{-labelings}} \#\{S\text{-}k\text{-colorings of } G\}.$$

Let $A = \{0,1\}$ and $S = S_A$.



Two $\{\mathrm{id}_A\}$-k-colorings



No S-2-colorings

$$P_S(C_4,2) = 0$$

$P_{S_A}(G,k) = P_{DP}(G,k)$, DP Color Function.

Counting Colorings

Let A be a finite set of "colors", $|A| = k$, and $S \subseteq S_A$ be a subset of the permutations of A.

$$P_S(G, k) = \min_{S\text{-labelings}} \#\{S\text{-}k\text{-colorings of } G\}.$$

- Chromatic Polynomial, $P(G, k) = P_{\{\mathrm{id}_A\}}(G, k)$.

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- Chromatic Polynomial, $P(G,k) = P_{\{\mathrm{id}_A\}}(G,k)$.
- (Dahlberg, K., Mudrock, (2023+)) k-colorings of Signed Graphs are k-colorings of S-labeled graphs for some $S \subseteq \mathcal{L}_k$, where $\mathcal{L}_k$ is the set of all linear permutations on A.
- Signed chromatic function, the guaranteed number of signed colorings of G, $P_{\pm}(G,k) \geq P_{\mathcal{L}_k}(G,k)$.

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- Signed chromatic function, the guaranteed number of signed colorings of G, $P_{\pm}(G,k) \geq P_{\mathcal{L}_k}(G,k)$.
- DP Color Function, $P_{DP}(G,k) = P_{S_A}(G,k)$.

A Poset of Graph Coloring notions

- Coloring of S-labeled graphs is a common generalization of many well studied notions of colorings.
- Any choice of subset of permutations $S \subseteq S_A$ leads to a notion of coloring.

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- $P_{DP}(G, k) \leq \ldots \leq P_S(G, k) \leq \cdots \leq P(G, k)$

A Poset of Graph Coloring notions

- Coloring of S-labeled graphs is a common generalization of many well studied notions of colorings.
- Any choice of subset of permutations $S \subseteq S_A$ leads to a notion of coloring.
- $P_{DP}(G, k) \leq \ldots \leq P_S(G, k) \leq \cdots \leq P(G, k)$
- The subset relation over the symmetric group, induces a partial order on all these notions of coloring with the DP coloring as the unique maximal element, and the classical coloring as a minimal element.
  In fact, it's a distributive lattice of notions of colorings.

List Colorings

What about list colorings and the list color function $P_\ell(G, k)$?

Even though list coloring does not fit into the framework of S-labeling, we still have DP color function as a lower bound on the list color function.

List Colorings

- $P_{DP}(G, k) \leq P_\ell(G, k) \leq P(G, k)$.

- In addition to, $P_{DP}(G, k) \leq \ldots \leq P_S(G, k) \leq \cdots \leq P(G, k)$, and in particular, $P_{DP}(G, k) \leq P_{\mathcal{L}}(G, k) \leq P(G, k)$.

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- In addition to, $P_{DP}(G,k) \leq \ldots \leq P_{S}(G,k) \leq \cdots \leq P(G,k)$, and in particular, $P_{DP}(G,k) \leq P_{\mathcal{L}}(G,k) \leq P(G,k)$.
- Exponential lower bound on $P_{DP}(G,k)$ would give an exponential lower bound on all these (and more) colorings of G.

Polynomial Method

In a survey article, Terrence Tao describes the polynomial method as:

"strategy is to capture the arbitrary set of objects in the zero set of a polynomial whose degree is in control; for instance the degree may be bounded by a function of the number of the objects."

Then we use algebraic tools to understand this zero set.

This paradigm has been used for breakthrough results in arithmetic combinatorics, additive combinatorics, number theory, graph theory, discrete geometry, and more.

Number of non-zeros of a polynomial

Theorem (Alon, Füredi (1993))

Let $\mathbb{F}$ be an arbitrary field, let A_1, A_2, ..., A_n be any non-empty subsets of $\mathbb{F}$, and let $B = \prod_{i=1}^{n} A_i$. Suppose that $P \in \mathbb{F}[x_1, \ldots, x_n]$ is a polynomial of degree d that does not vanish on all of B. Then, the number of points in B for which P has a non-zero value is at least $\min \prod_{i=1}^{n} q_i$ where the minimum is taken over all integers q_i such that $1 \leq q_i \leq |A_i|$ and $\sum_{i=1}^{n} q_i \geq -d + \sum_{i=1}^{n} |A_i|$.

Corollary (B. Bosek, J. Grytczuk, G. Gutowski, O. Serra, M. Zajac (2022))

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To apply this to count colorings of a graph G, we need to design a polynomial that has the property that its non-zeros correspond to proper colorings of G.

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- $\mathbb{F} = \mathbb{F}_k$ where $k = p^r$ is a power of a prime.
- $B = \mathbb{F}_k^n$, k color choices for each vertex. Each A_i is $\mathbb{F}_k$.
- The graph polynomial of G is $f_G = \displaystyle\prod_{ij \in E(G)} (x_i - x_j)$.
- $f_G(\mathbf{x}) \neq 0$ implies $\mathbf{x} = (x_1, x_2, \ldots x_n)$ gives a k-coloring.
- $\chi(G) \leq k$ & $nk \geq n + m \implies P(G, k) \geq k^{(kn-n-m)/(k-1)}$.

Number of non-zeros of a polynomial

- Using the graph polynomial over the field of reals, we easily get:

Proposition (Dahlberg, K., Mudrock (2023+))

Suppose G is an n-vertex graph with m edges, and k is a positive integer greater than 1 satisfying $\chi_\ell(G) \leq k$. If $m \leq (k-1)n$, then

$$P_\ell(G,k) \geq k^{n-\frac{m}{k-1}}.$$

(We recently generalized this to counting "packings of list colorings".)

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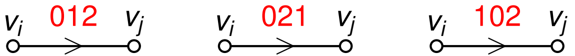
$$P_\ell(G, k) \ge k^{n - \frac{m}{k-1}}.$$

But this doesn't work with DP Color Function. Need to work in $\mathbb{F}_k$, a finite field, with a different polynomial.

A Polynomial for $k = 3$

- We want to define for a S-labeling of G, (D, σ), $f_{(D,\sigma)} = \prod_{ij \in E(D)} f_{\sigma(ij)}$, such that $f_{(D,\sigma)}(\mathbf{x}) \neq 0 \implies \mathbf{x}$ is an S-k-coloring.

Let $A = \mathbb{F}_3$ and $S = S_A$.



$x_i - x_j$	$x_i + x_j$	$x_i + x_j - 1$
$0 - 0 = 0$	$0 + 0 = 0$	$0 + 1 - 1 = 0$
$1 - 1 = 0$	$1 + 2 = 0$	$1 + 0 - 1 = 0$
$2 - 2 = 0$	$2 + 1 = 0$	$2 + 2 - 1 = 0$

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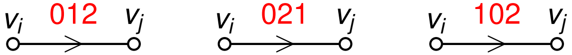

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- Dahlberg, K., Mudrock (2023): For $\mathbb{F}_3$, $f_{\sigma(ij)} = (x_i + (-1)^{c_{ij}} x_j - \beta_{ij})$ works, and gives $f_{(D,\sigma)}$ of degree m, the same as the graph polynomial.

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- This is based on the observation that
  Suppose σ is a permutation of $\mathbb{F}_3$. Then, either $z - \sigma(z)$ is the same for all $z \in \mathbb{F}_3$, or $z + \sigma(z)$ is the same for all $z \in \mathbb{F}_3$.

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- For $k > 3$, we are forced to use non-linear polynomials for each edge, and the degree of our polynomial consequently goes up.

A Polynomial for $k \geq 3$

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- Our building blocks will be L-polynomials.
  An L-polynomial is a polynomial in $\mathbb{F}_k[x, y]$ constructed from $i, j \in \mathbb{F}_k$ and $\pi \in S_{\mathbb{F}_k}$ given by

$$L^{\pi}_{i,j}(x, y) := (j - i)(y - \pi(i)) - (\pi(j) - \pi(i))(x - i)$$

  Essentially, $L^{\pi}_{i,j}(x, y)$ will be zero on all points that lie on the line between the points $(i, \pi(i))$ and $(j, \pi(j))$.

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- The degree of the polynomial $f_{\sigma(ij)}$ build this way using the L-polynomials can be as high as $k - 2$. We believe this is the optimal degree (computationally verified for small values of k).

A Polynomial for $k \geq 3$

- We want to define for a S-labeling of G, (D, σ), $f_{(D,\sigma)} = \prod_{ij \in E(D)} f_{\sigma(ij)}$, such that $f_{(D,\sigma)}(\mathbf{x}) \neq 0 \implies \mathbf{x}$ is an S-k-coloring.

- Our building blocks will be L-polynomials.
  An L-polynomial is a polynomial in $\mathbb{F}_k[x, y]$ constructed from $i, j \in \mathbb{F}_k$ and $\pi \in S_{\mathbb{F}_k}$ given by

$$L^{\pi}_{i,j}(x, y) := (j - i)(y - \pi(i)) - (\pi(j) - \pi(i))(x - i)$$

  Essentially, $L^{\pi}_{i,j}(x, y)$ will be zero on all points that lie on the line between the points $(i, \pi(i))$ and $(j, \pi(j))$.

- The degree of the polynomial $f_{\sigma(ij)}$ can be lowered to $\lfloor k/2 \rfloor$ but under additional assumptions about the DP-chromatic number of an associated multigraph.

Main Theorem

Theorem (Dahlberg, K., Mudrock (2023+))

Let $k = p^r$ where p is prime, $r \in \mathbb{N}$, and $k > 2$. Suppose G is a connected n-vertex simple graph with m edges. Then the following statements hold.

(i) If $\chi_{DP}(G) \leq k$ and $m \leq 2n - \frac{k-3}{k-2}$, then

$$P_{DP}(G,k) \geq k^{((2n-m)(k-2)-(k-3))/(k-1)}.$$

(ii) Let $q = \lfloor k/2 \rfloor$. Let G' be the multigraph obtained from G by adding $(q-1)$ parallel edges to each edge $e \in E(G) - E(T)$, where T is a spanning tree of G.
If $\chi_{DP}(G') \leq k$ and $m \leq n(1 + (k-2)/q) - 1 + 1/q$, then

$$P_{DP}(G,k) \geq k^{(n(q+k-2)-qm+1-q)/(k-1)}.$$

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Corollary

Let $c \geq 2$ and $k = p^r$ be a power of prime with $c \leq k$. Suppose G is a connected n-vertex simple graph with m edges. If $\chi_{DP}(G) \leq c$ and $m \leq \frac{n(c+k-4)}{k-2} - \frac{k-3}{k-2}$, then

$$P_{DP}(G, c) \geq c^{(n(c+k-4)-(k-2)m-(k-3))/(c-1)}.$$

Linear S-labelings and Signed Colorings

Theorem (Dahlberg, K., Mudrock (2023+))

Let $k = p^r$ where p is prime, $r \in \mathbb{N}$, and $k > 2$. If an n-vertex graph G with m edges is S-k-colorable for some $S \subseteq \mathcal{L}_k$ and $m \leq (k-1)n$, then

$$P_S(G,k) \geq k^{(kn-n-m)/(k-1)} = k^{n-\frac{m}{k-1}}$$

and particularly $P_{\mathcal{L}}(G,k) \geq k^{n-\frac{m}{k-1}}$.

Corollary

Let G be an n-vertex signed graph with m edges. Let k be a power of a prime. If $\chi_{\pm}(G) \leq k$ and $m \leq (k-1)n$, then

$$P_{\pm}(G,k) \geq k^{n-\frac{m}{k-1}}.$$

Exponentially Many Colorings of Sparse Graphs

Theorem (Dahlberg, K., Mudrock (2023+))

Let $k = p^r$ where p is prime, $r \in \mathbb{N}$, and $k > 2$. Suppose G is a connected n-vertex simple graph with m edges.

If $\chi_{DP}(G) \leq k$ and $m \leq 2n - \frac{k-3}{k-2}$, then

$$P_{DP}(G, k) \geq k^{((2n-m)(k-2)-(k-3))/(k-1)}.$$

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and particularly $P_{\mathcal{L}}(G, k) \geq k^{n - \frac{m}{k-1}}$.

Exponentially Many Colorings of Planar Graphs

Giving an enumerative extension of the four color conjecture, Birkhoff and Lewis (1946) conjectured:
For any planar graph G,
$P(G, k) \geq k(k-1)(k-2)(k-3)^{n-3}$ for all real numbers $k \geq 4$.

They proved this is true for $k \geq 5$, thus giving exponentially many 5-colorings of planar graphs

Exponentially Many Colorings of Planar Graphs

After Thomassen (1994) proved all planar graphs are 5-choosable, it was asked whether there are exponentially many 5-list-colorings of planar graphs.

Since then, there has been much work done on showing that planar graphs and their subfamilies have exponentially many list k-colorings for appropriate $k \in \{3, 4, 5\}$.

Exponentially Many Colorings of Planar Graphs

After Thomassen (1994) proved all planar graphs are 5-choosable, it was asked whether there are exponentially many 5-list-colorings of planar graphs.

Thomassen(2007) proved $P_\ell(G,5) \geq 2^{n/9}$ where G is a planar graph.
Extended by Postle and Smith-Roberge (2023+) to $P_{DP}(G,5) \geq 2^{n/67}$.

Dahlberg, K., Mudrock (2023+): $P_\pm(G,5) \geq 5^{n/4}$.

Exponentially Many Colorings of Planar Graphs

The question of colorings of sparse planar graphs, where sparsity is controlled by forbidding short cycles, also has a long history.

For planar graph G of girth 5
Thomassen: G is 3-choosable (1995), and
$P_\ell(G,3) \geq 2^{n/1000}$ (2007).
Improved by Postle and Smith-Roberge (2023+) to
$P_{DP}(G,3) \geq 2^{n/282}$.

Dahlberg, K., Mudrock (2024): $P_{DP}(G,3) \geq 3^{n/6}$.

Our Applications - 1

Theorem (Dahlberg, K., Mudrock (2023+))

Let G be an n-vertex graph of girth at least 5 embedded on a surface of Euler genus g. Suppose G is DP-k-colorable for some k, power of a prime. If $n \geq 5g$, then $P_{DP}(G, k) \geq k^{(((n-5g)(k-2)/3)-(k-3))/(k-1)}$.

This Theorem generalizes Thomassen(2007) that such graphs have $P(G, 3) \geq 2^{(n-5g)/9}$.

Our Applications - 2

Theorem (Dahlberg, K., Mudrock (2023+))

Let G be an n-vertex planar graph, and k be a power of prime.

1. *If G has no cycle of length in $\{4,5,6,7,8\}$, then $P_{DP}(G,k) \geq k^{\frac{n}{5}\frac{k-2}{k-1}-1}$ for $k \geq \chi_{DP}(G)$.*
2. *If G has no cycle of length in $\{4,5,6,9\}$, then $P_{DP}(G,k) \geq k^{\frac{n}{11}\frac{k-2}{k-1}-1}$ for $k \geq 3$. In particular, $P_{DP}(G,3) \geq 3^{\frac{n}{22}-1}$.*
3. *If G has no intersecting triangles and no cycle of length in $\{4,5,6,7\}$, then $P_{DP}(G,k) \geq k^{\frac{2n}{13}\frac{k-2}{k-1}-1}$ for $k \geq 3$. In particular, $P_{DP}(G,3) \geq 3^{\frac{n}{13}-1}$.*
4. *If G has no cycle of length in $\{4,5,6\}$, then $P_{DP}(G,k) \geq k^{\frac{n}{11}\frac{k-2}{k-1}-1}$ for $k \geq 4$. In particular, $P_{DP}(G,4) \geq 3^{\frac{n}{33}-1}$.*
5. *If G has no cycle of length in $\{4,5,7,9\}$, then $P_{DP}(G,k) \geq k^{\frac{2n}{13}\frac{k-2}{k-1}-1}$ for $k \geq 3$. In particular, $P_{DP}(G,3) \geq 3^{\frac{n}{22}-1}$.*

Our Applications - 3

Theorem (Dahlberg, K., Mudrock (2023+))

Let G be an n-vertex planar graph, and k be a power of prime.

1. $P_{\pm}(G,k) \geq k^{\frac{n(k-4)}{k-1}}$ *for* $k \geq 5$, *and in particular* $P_{\pm}(G,5) \geq 5^{\frac{n}{4}}$.
2. *If G is triangle free, then* $P_{\pm}(G,k) \geq k^{\frac{n(k-3)}{k-1}}$ *for* $k \geq 4$. *In particular,* $P_{\pm}(G,4) \geq 4^{\frac{n}{3}}$.
3. *If the girth of G is* $g \geq 5$, *then* $P_{\pm}(G,k) \geq k^{\frac{n(3k-8)}{3(k-1)}}$ *for* $k \geq 3$. *In particular,* $P_{\pm}(G,3) \geq 3^{\frac{n}{6}}$.
4. *If G doesn't have any cycles of length in* $\{4,5,6,7,8\}$, *then* $P_{\pm}(G,k) \geq k^{\frac{n(5k-14)}{5(k-1)}}$ *for* $k \geq 3$. *In particular,* $P_{\pm}(G,3) \geq 3^{\frac{n}{10}}$.

Our Applications - 4

Let G be a triangle-free planar n-vertex graph.

Grötzsch (1959): $\chi(G) \leq 3$.

However there exist triangle-free planar graphs that are not 3-choosable and hence not DP-3-colorable.

$\chi_\ell(G), \chi_{DP}(G) \leq 4$, by degeneracy.

Our Applications - 4

Let G be a triangle-free planar n-vertex graph.

Thomassen (2007) Conjecture: G has exponentially many 3-colorings.

Asadi, Dvořák, Postle, Thomas (2013): $P(G, 3) \geq 2^{\sqrt{n/212}}$.

Our Applications - 4

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Thomassen (2007) Conjecture: G has exponentially many 3-colorings.

Asadi, Dvořák, Postle, Thomas (2013): $P(G,3) \geq 2^{\sqrt{n/212}}$.

Dvořák, Postle (2022): The best we can hope for is $64^{n^{0.731}}$ 3-colorings.
The Thomassen conjecture is false!

Our Applications - 4

Conjecture (Dahlberg, K., Mudrock (2023+))

There exists a constant $c > 1$*, such that for any triangle-free planar n-vertex graph G,* $P_{DP}(G, 4) \geq c^n$.

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Theorem (Dahlberg, K., Mudrock (2023+))

Let G be an n-vertex triangle-free planar graph with m edges.

1. $P_\ell(G, 4) \geq 4^{\frac{n+4}{3}}$.
2. $P_\pm(G, 4) \geq 4^{\frac{n+4}{3}}$.
3. *Suppose there exists $c > 0$ such that $m \leq (2 - c)n$ and $n > 1/3c$, then $P_{DP}(G, 4) \geq 4^{(4cn-1)/3}$.*

Our Applications - 5

Question (K., Mudrock (2021))

If $P_{DP}(G, k) = P(G, k)$ for some $k \geq \chi(G)$, does it follow that $P_{DP}(G, k+1) = P(G, k+1)$?

NO!!!!!!

Corollary (Dahlberg, K., Mudrock (2024))

There are infinitely many graphs G with the property that $\chi_{DP}(G) = 3$, $P_{DP}(G, 3) = P(G, 3)$, and $P_{DP}(G, m) < P(G, m)$ for sufficiently large m.

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- The corresponding question for the list color function remains open.

Thank You!

Questions?

- Is it true that for k, a power of a prime, $P_{\mathcal{L}}(G,k) = P_{DP}(G,k)$?
- Does there exist $c > 1$, such that for any triangle-free planar n-vertex graph G, $P_{DP}(G,4) \geq c^n$?
- Given a graph G does there always exist an $N \in \mathbb{N}$ and a polynomial $p(k)$ such that $P_{DP}(G,k) = p(k)$ whenever $k \geq N$?
- For which graphs G does $\exists N$ such that $P_{DP}(G,k) = P(G,k)$ for all $k \geq N$? That is, when is $\tau_{DP}(G) := N$ finite?
- Given a graph G and $p \in \mathbb{N}$, what is the value of $\tau_{DP}(K_p \vee G)$?
- For fixed n what is the asymptotic behavior of $\tau_{\ell}(K_{n,l})$ as $l \to \infty$?
- Kirov and Naimi 2016: If $P_{\ell}(G,k) = P(G,k)$ for some $k \geq \chi(G)$, does it follow that $P_{\ell}(G,k+1) = P(G,k+1)$?

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