# Chromatic Polynomial and Counting DP Colorings of a Graph


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Joint work with

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# Graph Coloring

- Allocation of colors (limited resource) to vertices (entities) so that pairs of vertices with an edge (conflict) are given different colors.

- Color vertices so that any vertices with an edge between them must get different colors.
- Partition the set of all vertices into independent sets (edge-free sets/ “conflict-free” sets)

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# List Coloring

- List coloring was introduced independently by Vizing (1976) and Erdős, Rubin, and Taylor (1979), as a generalization of usual graph coloring.

- For graph $G$ suppose each $v \in V(G)$ is assigned a list, $L(v)$, of colors. We refer to $L$ as a list assignment. If all the lists associated with the list assignment $L$ have size $k$, we say that $L$ is a $k$-assignment.

- An L-coloring for $G$ is a proper coloring, $f$, of $G$ such that $f(v) \in L(v)$ for all $v \in V(G)$.

- When an $L$-coloring for $G$ exists, we say that $G$ is L-colorable or L-choosable.

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# List Chromatic Number

- The list chromatic number of a graph $G$, written $\chi_\ell(G)$, is the smallest $k$ such that $G$ is $L$-colorable whenever $|L(v)| \geq k$ for each $v \in V(G)$.

- When $\chi_\ell(G) = k$ we say that $G$ has list chromatic number $k$ or that $G$ is k-choosable.

- We immediately have that if $\chi(G)$ is the typical chromatic number of a graph $G$, then

$$\chi(G) \leq \chi_\ell(G).$$

But we know the gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large: $\chi_\ell(K_{k,t}) = k + 1$ when $t \geq k^k$.

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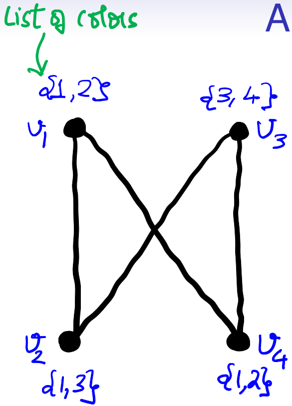
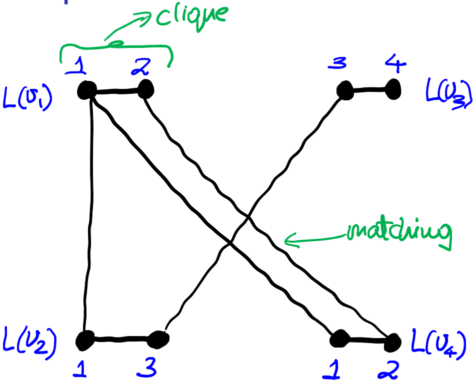
# A Different Perspective




**Proper L-coloring**

$v_1 \to 2$
$v_2 \to 1$
$v_3 \to 3$
$v_4 \to 1$

**corresponds to**

**Independent set of size 4 here.**

$2 \in L(v_1)$, $1 \in L(v_2)$, $3 \in L(v_3)$, $1 \in L(v_4)$

# DP-Coloring

- In 2015, Dvořák and Postle introduced DP-coloring (they called it correspondence coloring) of graphs.
- Intuitively, DP-coloring considers the worst-case scenario of how many colors we need in the lists if we no longer can identify the names of the colors. Each vertex still gets a list of colors but identification of which colors are different can vary from edge to edge.

- A cover of $G$ is a pair $\mathcal{H} = (L, H)$ consisting of a graph $H$ and a function $L : V(G) \to \mathcal{P}(V(H))$ satisfying:

  (1) the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$;
  (2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
  (3) if $E_H(L(u), L(v))$ is nonempty, then $u = v$ or $uv \in E(G)$;
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- Intuition:
  Blow up each vertex $u$ in $G$ into a clique of size $|L(u)|$;
  Add a matching (possibly empty) between any two such cliques for vertices $u$ and $v$ if $uv$ is an edge in $G$.

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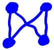
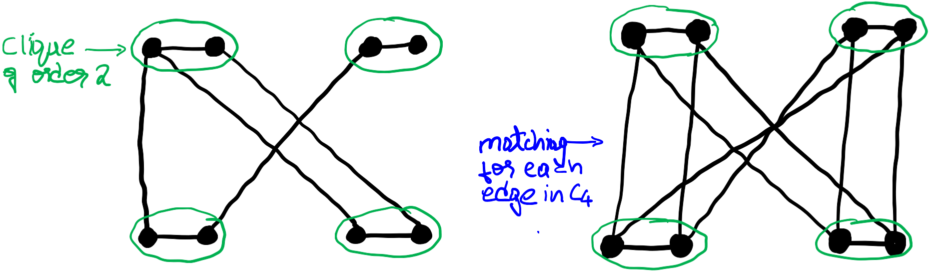
- A cover $\mathcal{H} = (L, H)$ is called $m$-fold if $|L(u)| = m$ for all $u$.
- Two 2-fold covers of $C_4$: 



# DP-Chromatic Number of a Graph

- Given $\mathcal{H} = (L, H)$, a cover of $G$, an $\mathcal{H}$-coloring of $G$ is an independent set in $H$ of size $|V(G)|$.
- The DP-chromatic number of a graph $G$, $\chi_{DP}(G)$, is the smallest $m$ such that $G$ admits an $\mathcal{H}$-coloring for every $m$-fold cover $\mathcal{H}$ of $G$.
- $\chi_{DP}(C_4) > 2 = \chi_{\ell}(C_4)$:

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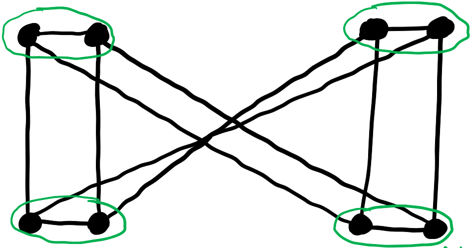
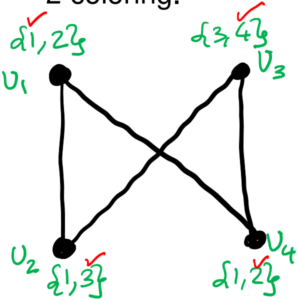


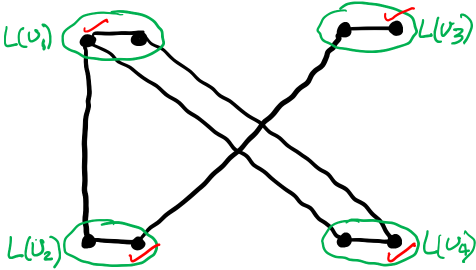
No independent set of size 4 in this 2-fold cover of $C_4$

# DP-Coloring and List Coloring

- Given an $m$-assignment, $L$, for a graph $G$, it is easy to construct an $m$-fold cover $\mathcal{H}$ of $G$ such that:
  $G$ has an $\mathcal{H}$-coloring if and only if $G$ has a proper $L$-coloring.




L- coloring




H- coloring

- $\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G)$.

# The List Color Function

- The chromatic polynomial of $G$, $P(G, m)$ equals the number of proper colorings of $G$ with colors $[m]$.

- $P(G, L)$ be the number of proper $L$-colorings of $G$. The list color function $P_\ell(G, m)$ is the minimum value of $P(G, L)$ over all possible $m$-assignments $L$ for $G$.

- In general, $P_\ell(G, m) \leq P(G, m)$.

- $P(K_{2,4}, 2) = 2$, and yet $P_\ell(K_{2,4}, 2) = 0$.
- $P_\ell(K_{3,26}, 3) \leq 3^8 2^{12} < 3^1 2^{26} \leq P(K_{3,26}, 3)$.

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- $P_\ell(G, m) \le P(G, m)$. And for some $G$, $P_\ell(G, m) < P(G, m)$

- $P_\ell(G, m)$ need not be a polynomial.

Theorem (Wang, Qian, Yan (2017); improving Thomassen (2009), Donner (1992))

*For any connected graph $G$ with $t$ edges,*
$P_\ell(G, m) = P(G, m)$ *for* $m > \frac{t-1}{ln(1+\sqrt{2})} \approx 1.135(t-1)$.

Theorem (Kostochka, Sidorenko (1990); K., Mudrock (2021))

*1)* $P_\ell(G, m) = P(G, m)$ *for all* $m$, *if* $G$ *is chordal.*
*2)* $P_\ell(C_n, m) = P(C_n, m) = (m-1)^n + (-1)^n(m-1)$ *for all* $m$.
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# A Natural Question

We know:

**Theorem (Wang, Qian, Yan (2017))**

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# DP Color Function is different

**Theorem (K., Mudrock (2021))**

*If $G$ is a graph with girth that is even, then there is an $N$ such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N$.*

*Furthermore, for any integer $g \geq 3$ there exists a graph $M$ with girth $g$ and an $N$ such that $P_{DP}(M, m) < P(M, m)$ whenever $m \geq N$.*

# Tools for DP Color Function - I

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**Proposition (K., Mudrock (2021))**

$P_{DP}(G, m) \leq \frac{m^n(m-1)^{|E(G)|}}{m^{|E(G)|}}$ *for all $m$.*

**Lemma (from Whitney's Broken Circuit Theorem (1932))**

*$G$ be a connected graph on $n$ vertices and $s$ edges with girth $g$. Suppose $P(G, m) = \sum_{i=0}^{n} (-1)^i a_i m^{n-i}$. Then, for $i = 0, 1, \ldots, g-2$*

*$a_i = \binom{s}{i}$ and $a_{g-1} = \binom{s}{g-1} - t$,*

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- This upper bound is the same as the lower bound on $P(G, m)$ when $G$ is bipartite, as claimed by the well-known Sidorenko's conjecture on counting homomorphisms from bipartite graphs.

**Corollary (K., Mudrock (2021))**
*For any connected graph $G$, $P_{DP}(G, m) = \frac{m^{|V(G)|}(m-1)^{|E(G)|}}{m^{|E(G)|}}$ for all $m$ if and only if $G$ is a tree.*

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- The expression above is the expected number of $(L,H)$-colorings of $G$.

# A Probabilistic Proof

**Proposition (K., Mudrock (2021))**
*$P_{DP}(G, m) \leq \frac{m^n(m-1)^{|E(G)|}}{m^{|E(G)|}}$ for all $m$.*

- Form an $m$-fold cover, $(L, H)$, of $G$ by the following (partially random) process.
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# Tools for DP Color Function - II

**Theorem (K., Mudrock (2021))**

*For any integer $g \geq 3$ there exists a graph $M$ with girth $g$ and an $N$ such that $P_{DP}(M,m) < P(M,m)$ whenever $m \geq N$.*

**Proposition (K., Mudrock (2021))**

*Let $G$ be a graph with $e \in E(G)$.
If $m \geq 2$ and $P(G - \{e\}, m) < \frac{m}{m-1} P(G,m)$,
then $P_{DP}(G,m) < P(G,m)$.*

**Theorem (K., Mudrock (2021))**

*Let $G_2$ be any graph and $G_1 = C_{2k+2}$ with exactly two vertices and one edge in common, and denote $G = G_1 \oplus G_2$. Then, $P_{DP}(G,m) < P(G,m)$ whenever $m \geq \max\{2, \chi(G_2)\}$.*

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# Second Natural Question

- For which graphs $G$ does $P_{DP}(G, m) = P(G, m)$ for all $m$?
- For which graphs $G$ does there exist $N$ such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$?

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*If $G$ is chordal, then $P_{DP}(G, m) = P(G, m)$ for every $m$.*

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- Let $\mathcal{H} = (L, H)$ be an $m$-fold cover of $G$. We say that $\mathcal{H}$ has a canonical labeling if it is possible to name the vertices of $H$ so that $L(u) = \{(u, j) : j \in [m]\}$ and $(u, j)(v, j) \in E(H)$ for each $j \in [m]$ whenever $uv \in E(G)$.
- When $\mathcal{H}$ has a canonical labeling, $G$ has an $\mathcal{H}$-coloring if and only if $G$ has a proper $m$-coloring.
- Trees have a canonical labeling.
- Using canonical labeling, we can develop tools to handle graphs that are close to being a forest.

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# Tools for DP Color Function - III

- A sharp bound when removing an edge gives us a canonical labeling.

**Lemma (K., Mudrock (2021))**

*Let $\mathcal{H} = (L, H)$ be an $m$-fold cover of $G$ with $m \geq 2$. Suppose $e = uv \in E(G)$. Let $H' = H - E_H(L(u), L(v))$ so that $\mathcal{H}' = (L, H')$ is an $m$-fold cover of $G - \{e\}$. If $\mathcal{H}'$ has a canonical labeling, then*

$$P_{DP}(G, \mathcal{H}) \geq P(G - e, m) - \max\left\{P(G - e, m) - P(G, m), \frac{P(G,m)}{m-1}\right\}$$

*Moreover, there exists an $m$-fold cover of $G$, $\mathcal{H}^* = (L, H^*)$, s.t.*

$$P_{DP}(G, \mathcal{H}^*) = P(G - e, m) - \max\left\{P(G - e, m) - P(G, m), \frac{P(G,m)}{m-1}\right\}.$$

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# Tools for DP Color Function - III

## Lemma (K., Mudrock (2021))

*Let $\mathcal{H} = (L, H)$ be an m-fold cover of G with $m \geq 3$. Let $e_1, e_2$ be the edges of an induced path P of length two.*
*Let $G_0 = G - \{e_1, e_2\}$, $G_1 = G - e_1$, $G_2 = G - e_2$, and $G^*$ be the graph obtained by making P into $K_3$. Suppose $\mathcal{H}'$, the m-fold cover of $G_0$ induced by $\mathcal{H}$, has a canonical labeling. Let*

$$A_1 = P(G_0, m) - P(G, m), A_2 = P(G_0, m) - P(G_2, m) + \frac{1}{m-1}P(G, m),$$

$$A_3 = P(G_0, m) - P(G_1, m) + \frac{1}{m-1}P(G, m),$$

$$A_4 = \frac{1}{m-1}\left(P(G_1, m) + P(G_2, m) + P(G^*, m) - P(G, m)\right), \text{ and}$$

$$A_5 = \frac{1}{m-1}\left(P(G_1, m) + P(G_2, m) - \frac{1}{m-2}P(G^*, m)\right).$$

*Then, $P_{DP}(G, \mathcal{H}) \geq P(G_0, m) - \max\{A_1, A_2, A_3, A_4, A_5\}$. Moreover, there exists an m-fold cover of G that achieves equality above.*

# Unicyclic Graphs

- A unicyclic graph is a connected graph containing exactly one cycle.
- If $G$ is a unicyclic graph on $n$ vertices that contains a cycle on $t$ vertices, then $P(G, m) = (m-1)^n + (-1)^t(m-1)^{n-t+1}$

**Theorem (K., Mudrock (2021))**

*Suppose $G$ is a unicyclic graph on $n$ vertices.*

*(1) If $G$ contains a cycle on $2k+1$ vertices, then $P_{DP}(G, m) = P(G, m)$ for all $m$.*

*(2) If $G$ contains a cycle on $2k+2$ vertices, then $P_{DP}(G, m) = (m-1)^n - (m-1)^{n-2k-2}$ for all $m \geq 2$.*

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# Theta Graphs

- A Generalized Theta graph $\Theta(l_1, \ldots, l_k)$ consists of a pair of end vertices joined by $k$ internally disjoint paths of lengths $l_1, \ldots, l_k$. $\Theta(l_1, l_2, l_3)$ is simply called a Theta graph.
- $P(\Theta(l_1, \ldots, l_k), m) = \frac{\prod_{i=1}^{k}((m-1)^{l_i+1}+(-1)^{l_i+1}(m-1))}{(m(m-1))^{k-1}} + \frac{\prod_{i=1}^{k}((m-1)^{l_i}+(-1)^{l_i}(m-1))}{m^{k-1}}.$
- Widely studied for many graph theoretic problems and are the main subject of two classical papers on the chromatic polynomial by Sokal, which include the celebrated result that the zeros of the chromatic polynomials of the Generalized Theta graphs are dense in the whole complex plane with the possible exception of the unit disc around the origin (by including the join of Generalized Theta graphs with $K_2$ this extends to all of the complex plane).

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# Theta Graphs

Extending results of K. and Mudrock (2021),

**Theorem** (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))

*Let $G = \Theta(l_1, l_2, l_3)$ and $2 \le l_1 \le l_2 \le l_3$.*

*(1) If the parity of $l_1$ is different from both $l_2$ and $l_3$, then $P_{DP}(G, m) = P(G, m)$ for all $m$.*

*(2) If the parity of $l_1$ is the same as $l_2$ and different from $l_3$, then for $m \ge 2$: $P_{DP}(G, m) =$*

$$\frac{1}{m}\left[(m-1)^{l_1+l_2+l_3} + (m-1)^{l_1} - (m-1)^{l_2+1} - (m-1)^{l_3} + (-1)^{l_3+1}(m-2)\right].$$

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# Two Fundamental Questions

- For which graphs $G$ does there exist $N$ such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$?

- Given a graph $G$ does there always exist an $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ whenever $m \geq N$?

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- Statement (i) does not answer the question of whether $P_{DP}(G, m)$ equals a polynomial for sufficiently large $m$. To answer that question, we study the DP color function of a class of graphs that contains all Generalized Theta graphs.

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# Graphs with a Feedback Vertex Set of Order One

- A feedback vertex set of a graph is a subset of vertices whose removal makes the resulting induced subgraph acyclic. Clearly, a Generalized Theta graph has a feedback vertex set of size one.

**Theorem** (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))

*Let $G$ be a graph with a feedback vertex set of order one. Then there exists $N$ and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ for all $m \geq N$.*

- We consider a decomposition $G$ into a star $G_1$ and a spanning forest $G_0$, and then carefully count the number of $\mathcal{H}_0$-colorings of $G_0$ that are not $\mathcal{H}$-colorings of $G$, where $\mathcal{H}_0$ is the $m$-fold cover of $G_0$ induced by a given $m$-fold cover $\mathcal{H}$ of $G$.

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## Questions?

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- Study the DP color function threshold of G, $\tau_{DP}(G)$, the smallest $N \geq \chi(G)$ such that $P_{DP}(G,m) = P(G,m)$ whenever $m \geq N$.

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