MATH 380

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Multivariable non-linear Optimization models

We make 40" & 50" TVs.
Manufacturing costs: \$195 per 40", \$225 per 50", plus \$400000 fixed costs.

Suggested retail price: \$339 for 40" & \$399 for 50".

To sell all produced TVs in a competitive market:
we drop prices by 1¢ per unit of that type sold, and additionally
price of 40" drops 0.3¢ per unit of 50" sold,
price of 50" drops 0.4¢ per unit of 40" sold.

How many units of each type of TV should we manufacture?

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How many units of each type of TV should we manufacture?

Let x_1 = # 40" TVs , x_2 = # 50" TVs

Maximize Profit [Profit = Revenue − Costs]

Costs, $C(x_1, x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R(x_1, x_2) = (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2$

Maximize $P(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$ over $x_1 \geqslant 0, x_2 \geqslant 0$

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Method 1 Set $\nabla P = \left(\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2} \right) = \vec{0}$

$\boxed{\text{Gradient}}$

$$\frac{\partial P}{\partial x_1} = 339 - 0.02x_1 - 0.003x_2 - 0.004x_2 - 195 = 0$$

$$\frac{\partial P}{\partial x_2} = -0.003x_1 + 399 - 0.004x_1 - 0.02x_2 - 225 = 0$$

Solve : $x_1 \approx 4735$, $x_2 \approx 7043$

Verify it's really the maximum – 2nd deriv. test or sketch graph.

Profit $\approx \$553641$

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Maximize $P(x_1,x_2) = R(x_1,x_2) - C(x_1,x_2)$ over $x_1 \geq 0, x_2 \geq 0$

<u>Method 2</u> <u>Gradient Method of Steepest ascent/descent</u>

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Method 2 Gradient Method of Steepest ascent / descent

Idea ∇P at a point in domain of a differentiable function $P(x,y)$ always points in the direction of max rate of increase of P

Iterative process Need (x_0, y_0) initial approximation and a sequence of step-sizes λ_k.

See Table 13.1 in textbook

For $k = 0, 1, \ldots\ldots, N$

$$x_{k+1} = x_k + \lambda_k \frac{\partial P}{\partial x}(x_k, y_k)$$

$$y_{k+1} = y_k + \lambda_k \frac{\partial P}{\partial y}(x_k, y_k)$$

How to choose λ_k? As $(x_k, y_k) \to$ extreme pt., $\nabla P \to (0,0)$, so we will need λ_k to be larger... e.g. $\lambda_k = \lambda_0 \delta^k$ for fixed $\delta > 1$

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Maximize $P(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$ over $x_1 \geq 0, x_2 \geq 0$

Sensitivity? e.g. "Price elasticity" of 40" TV

set $a = 0.01$

price of 40" TV $= 339 - ax_1 - 0.003x_2$

Costs, $C(x_1,x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R_a(x_1,x_2) = \underline{(339 - \cancel{0.01}^{\,a} x_1 - 0.003x_2)x_1} + (399 - 0.004x_1 - 0.01x_2)x_2$

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<u>Sensitivity?</u> e.g. "Price elasticity" of 40" TV

set $a = 0.01$

price of 40" TV $= 339 - ax_1 - 0.003x_2$

Setting $\nabla P_a = \vec{0}$ gives $x_1 = \dfrac{1662000}{400000a - 49}$, $x_2 = 8700 - \dfrac{581700}{400000a - 49}$

At $a = 0.01$, Sensitivity, $S(x_1,a) = \left(\dfrac{dx_1}{da}\right)\left(\dfrac{a}{x_1}\right) = \cdots = \dfrac{-400}{351} \approx -1.1$

$S(x_2,a) = \left(\dfrac{dx_2}{da}\right)\left(\dfrac{a}{x_2}\right) = \cdots = \dfrac{9695}{36123} \approx 0.27$

e.g. 10% increase in price elasticity of 40" TVs means we should make ~11% fewer 40" TVs and 2.7% more 50" TVs.

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Maximize $P_a(x_1,x_2) = R_a(x_1,x_2) - C(x_1,x_2)$ over $x_1 \geq 0, x_2 \geq 0$

Sensitivity of Profit to price elasticity of 40" TVs

$$\frac{dP_a}{da} = \frac{\partial P_a}{\partial x_1}\frac{\partial x_1}{\partial a} + \frac{\partial P_a}{\partial x_2}\frac{\partial x_2}{\partial a} + \frac{\partial P_a}{\partial a}$$

$$= 0 + 0 + -x_1^2 = -x_1^2 \quad \text{at } a = 0.01, \text{ since } \nabla P_a = 0$$

So, sensitivity $S(P_a, a) = \left(\frac{dP_a}{da}\right)\left(\frac{a}{P_a}\right)$ at $a = 0.01$

$$\approx -(4735)^2 \frac{0.01}{553641} \approx -0.40$$

A 10% increase in price elasticity of 40" TVs will cause about 4% drop in profits.

Constrained Non-linear Optimization Models

An Oil Transfer company needs to minimize costs associated with dispensing and holding the oil to maintain sufficient oil to satisfy demand while meeting the restricted tank storage space constraint.

x_i = amount of type i oil available

a_i = cost of oil type i

b_i = withdrawal rate per unit time of oil type i

h_i = storage cost per unit time of oil type i

t_i = space in cubic feet to store one unit of oil type i

T = total space available for storage.

minimize $f(x_1, x_2) = \left(\frac{a_1 b_1}{x_1} + \frac{h_1 x_1}{2}\right) + \left(\frac{a_2 b_2}{x_2} + \frac{h_2 x_2}{2}\right)$ ← total cost

s.t. $g(x_1, x_2) = t_1 x_1 + t_2 x_2 = T$ ← space constraint

Given data

OU	a_i (\$)	b_i	h_i (\$)	t_i (ft^3)
1	9	3	0.50	2
2	4	5	0.20	4

& $T = 24\, ft^3$

leads to

$$\min f(x_1, x_2) = \frac{27}{x_1} + 0.25 x_1 + \frac{20}{x_2} + 0.10 x_2$$

s.t. $\quad 2x_1 + 4x_2 = 24$

Given data

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s.t. $2x_1 + 4x_2 = 24$

Method of Lagrange Multipliers

Extreme points of $f(x_1, \ldots, x_n)$ over constraints $g_1(x_1, \ldots, x_n) = c_1$,
$\vdots$
$g_k(x_1, \ldots, x_n) = c_k$

must satisfy $\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k$

i.e., $\dfrac{\partial f}{\partial x_1} = \lambda_1 \dfrac{\partial g_1}{\partial x_1} + \cdots + \lambda_k \dfrac{\partial g_k}{\partial x_1}$

$\vdots$

$\dfrac{\partial f}{\partial x_n} = \lambda_1 \dfrac{\partial g_1}{\partial x_n} + \cdots + \lambda_k \dfrac{\partial g_k}{\partial x_n}$

if $\nabla g_1, \ldots, \nabla g_k$ are linearly independent.

Given data

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leads to $\min f(x_1, x_2) = \frac{27}{x_1} + 0.25x_1 + \frac{20}{x_2} + 0.10x_2$

s.t. $g(x_1, x_2) = 2x_1 + 4x_2 = 24$

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda(g(x_1, x_2) - T)$$
$$= \frac{27}{x_1} + 0.25x_1 + \frac{20}{x_2} + 0.10x_2 + \lambda(2x_1 + 4x_2 - 24)$$

Solve for x_1, x_2, λ using

$\frac{\partial L}{\partial x_1} = 0$, i.e., $\frac{-27}{x_1^2} + 0.25 + 2\lambda = 0$

$\frac{\partial L}{\partial x_2} = 0$, i.e., $\frac{-20}{x_2^2} + 0.10 + 4\lambda = 0$

$\frac{\partial L}{\partial \lambda} = 0$, i.e., $-2x_1 + 4x_2 - 24 = 0$

Using a solver (based on Newton's m.??) we get $x_1 = 5.097$, $x_2 = 3.452$, $\lambda = 0.3947$, and $f(x_1, x_2) = \$12.71$

λ has a special meaning – Shadow price!

$\lambda \equiv$ amount f changes per unit change in constraint corresp to λ.

If capacity increases $24 \to 25 \Rightarrow$ cost increases by \$0.3947

Interpretation of Lagrange Multipliers

It represents the sensitivity of objective function to the corresponding constraint.

min/max $f(x_1,x_2)$ s.t. $g(x_1,x_2)=t$

then sensitivity $S(f,t) = \left(\frac{df}{dt}\right)\left(\frac{t}{f}\right)$

& we know $\nabla f = \lambda \nabla g$ at optimum

which gives $\frac{df}{dt} = \lambda$ at an optimum.

Theorem Suppose M is the max/min value of $f(x_1,\ldots,x_n)$ subject to constraints $g_i(x_1,\ldots,x_n)=k_i$, $i=1,\ldots,p$.
Then the Lagrange Multiplier λ_i corresp. to $g_i(\bar{x})=k_i$ is the rate of change of f w.r.t. g_i at M

i.e. $\lambda_i \approx$ change in M resulting from 1-unit increase in k_i

example An editor has been allocated \$60000 to spend on development and promotion of a new book.
It is estimated if x thousand dollars is spent on development and y on promotion, then approximately $f(x,y) = 20x^{3/2}y$ copies of the book are expected to be sold.

How much money should the editor allocate to development and how much to promotion, in order to maximize sales?

How can the editor make an argument for increasing the budget from \$60000?

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we want to maximize $f(x,y) = 20x^{3/2}y$ s.t. $g(x,y) = 60$

where $g(x,y) = x+y$

The corresponding Lagrange equations are:

$30x^{1/2}y = \lambda$, $20x^{3/2} = \lambda$, $x+y = 60$

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$$\left.\begin{array}{l} \underbrace{30x^{1/2}y = \lambda \quad , \quad 20x^{3/2} = \lambda}, \quad x+y=60 \\ \Downarrow \\ 30x^{1/2}y = 20x^{3/2} \underset{x\neq 0}{\Longrightarrow} y = \frac{2}{3}x \end{array}\right\} \Longrightarrow x + \frac{2}{3}x = 60$$

i.e. $x = 36$ & $y = 24$

$\therefore$ to maximize sales, editor should spend \$36000 on development and \$24000 on promotion. Then $f(36,24) = 103680$ copies of the book are predicted to be sold

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How can the editor make an argument for increasing the budget from \$60000?

We found $\lambda = 30x^{1/2}y = 30(36)^{1/2}(24) = 4320$
which means that book sales will increase by 4320 copies (from 103680 to 108000) if the budget is increased from \$60000 to \$61000

Theorem Let $\vec{x} = (x_1, \ldots, x_n)$ be variables

Consider $\max f(\vec{x})$ s.t. $g(\vec{x}) = \omega$

Let $\vec{x}^*(\omega)$ be the $\vec{x}$ that maximizes problem above for ω.

Corresponding to $\vec{x}^*(\omega)$, there is a value $\lambda = \lambda^*(\omega)$ for the Lagrange multipliers

Then $\lambda^*(\omega) = \frac{d}{d\omega} f(\vec{x}^*(\omega))$, the rate of change in the optimal output resulting from change of the constant ω.

Relevant & important in economics where f is often the utility function of the inputs, ω denotes the value of these inputs, and the Lagrange multiplier is the "marginal utility of money".

e.g. Cobb-Douglas utility function $U(x, y) = K x^{\lambda_1} y^{\lambda_2}$ where $K > 0$
$\lambda_1, \lambda_2 \in [0, 1]$
and $\lambda_1 + \lambda_2 = 1$

Managing Renewable Resources: Fishing Industry

Artic Baleen Whales : peak catch of 2.8 million tons in 1937
but only 50000 Tons in 1978

Peruvian Anchoveta : 12.3 mil. Tons in 1970 but 500000 Tons in 1978

Overharvesting of a renewable resource.

What should be done?

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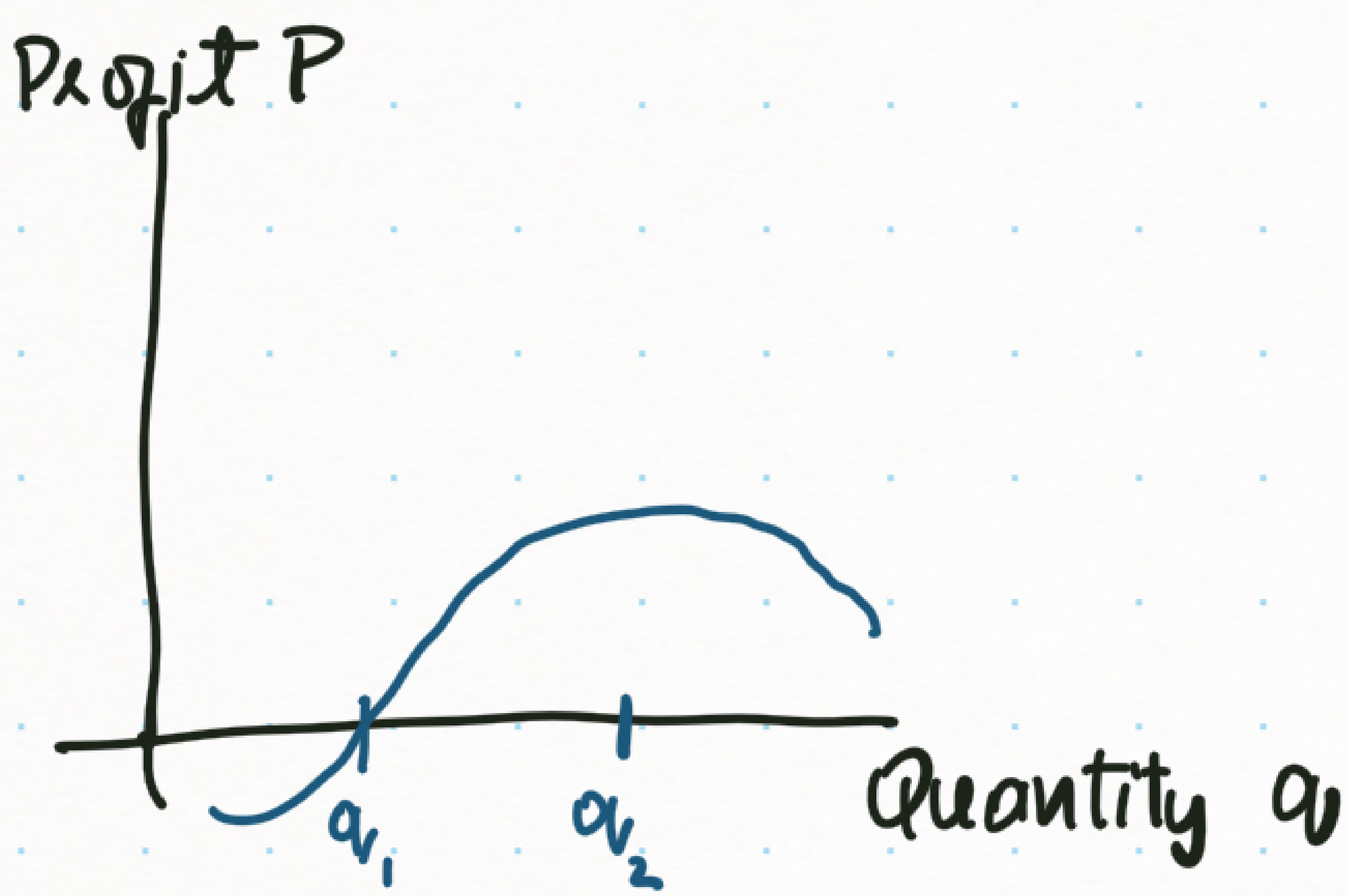
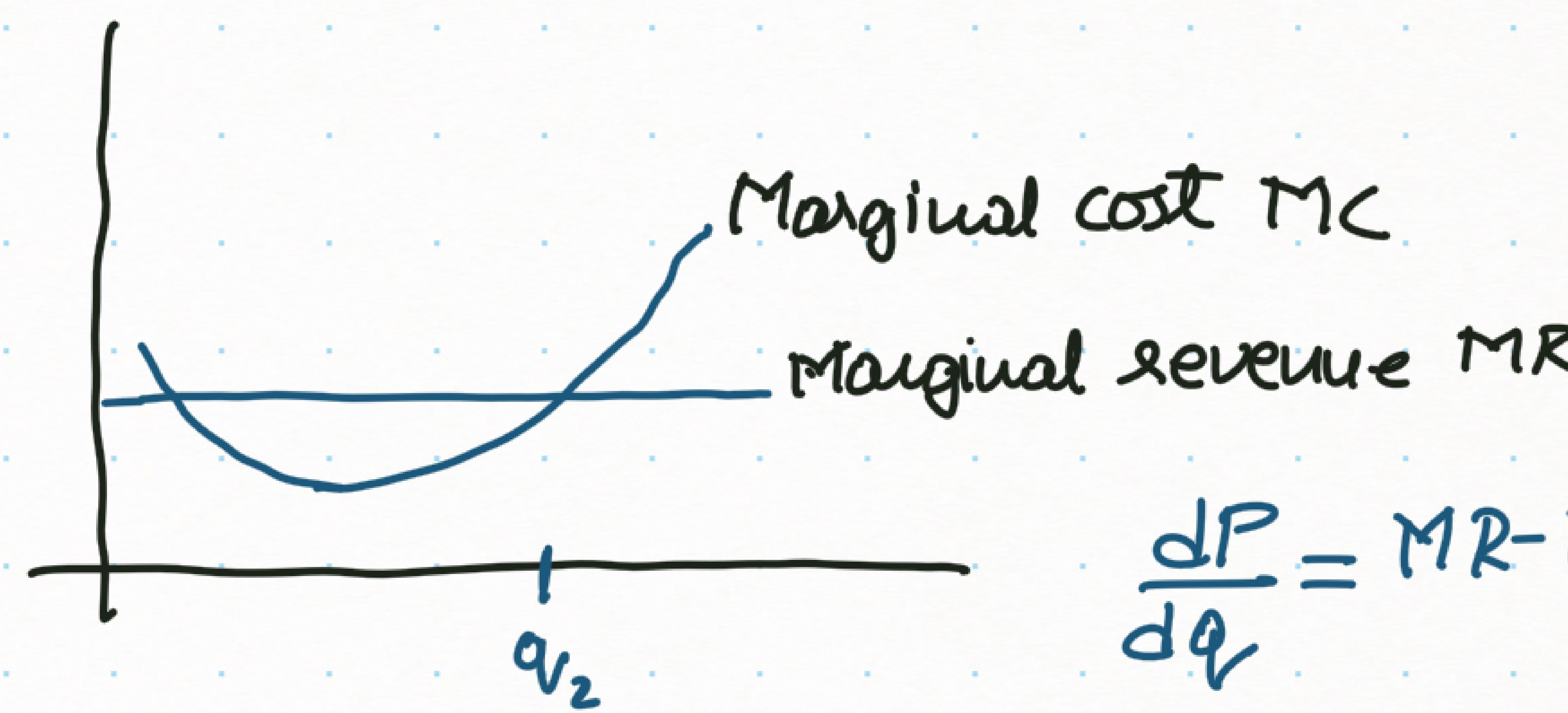
Basic economic model of a firm





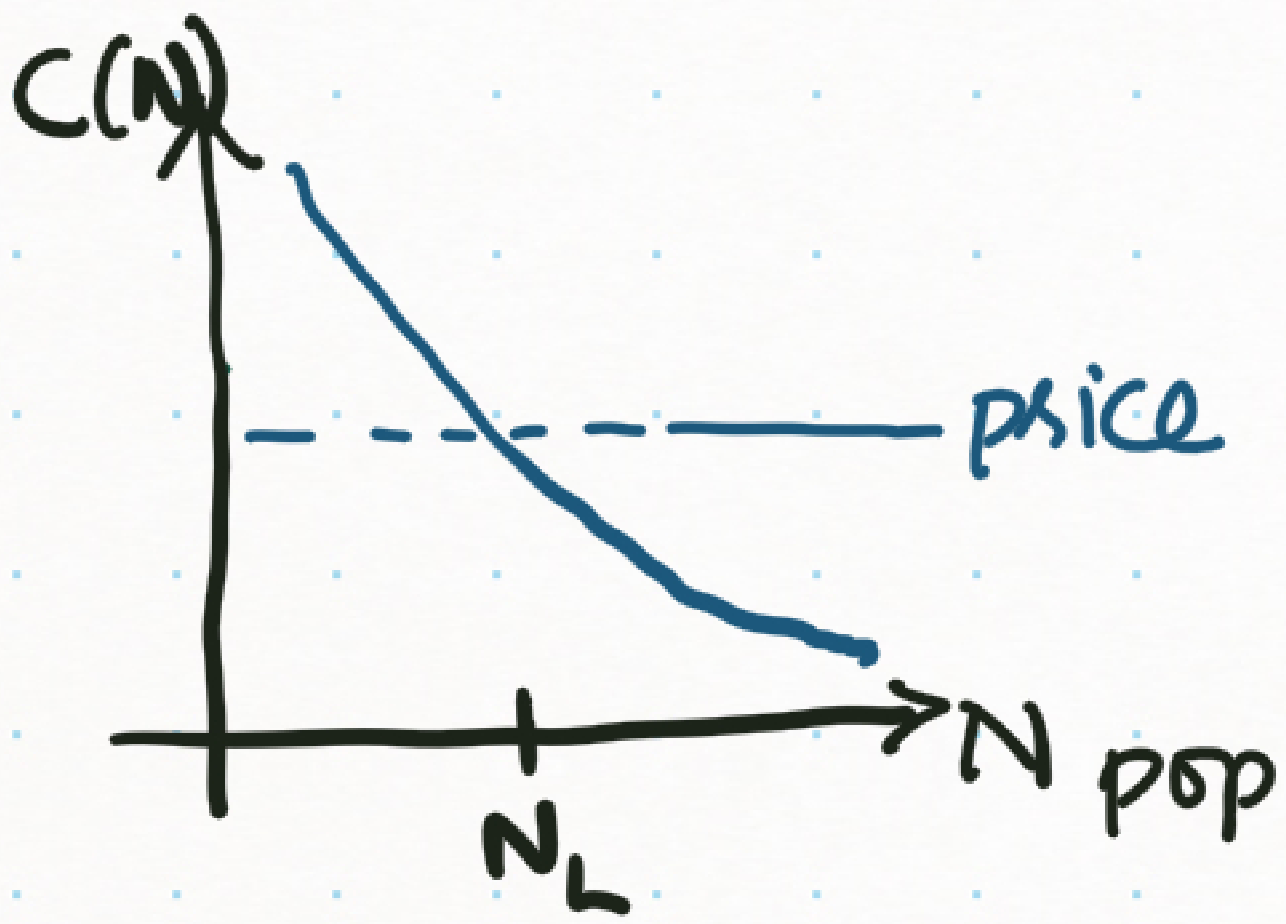
$$\frac{dP}{dq} = MR - MC$$

How to create an economic model for harvesting common fish?

→ Assume price/fish is constant (interchangeable with others) (similar common species)

→ easier to harvest when population N is higher

i.e., lower cost $c(N)$ / fish.



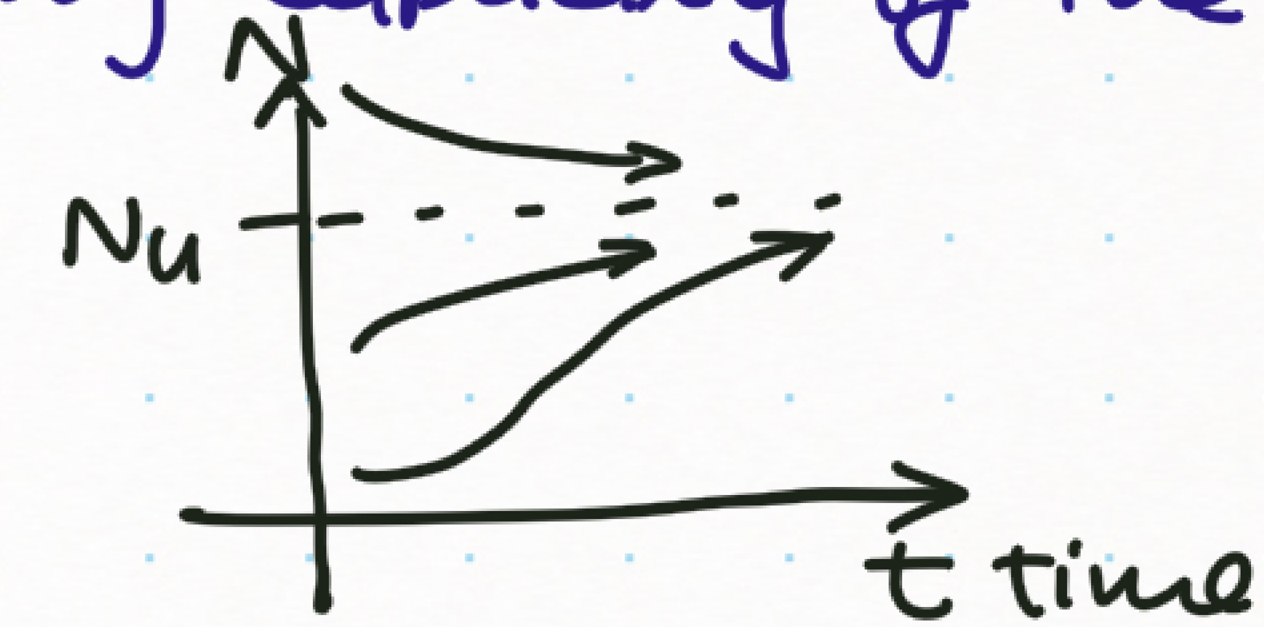
N_L is the population below which fishing is unprofitable.

Will fishing "naturally" increase until N drops to N_L?

Since profit = yield × profit per fish depends on N,
we need a submodel for population.

$g(N)$ = growth rate of fish population (with no harvesting)

$\propto N(N_u - N)$ where N_u is the carrying capacity of the environment.



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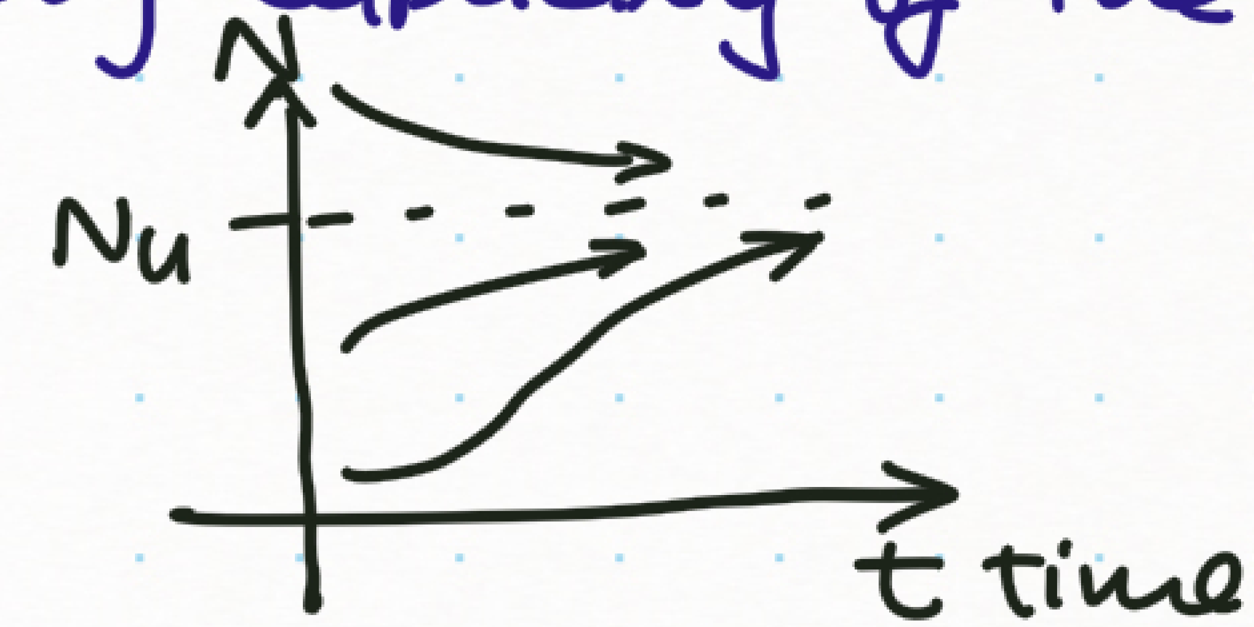
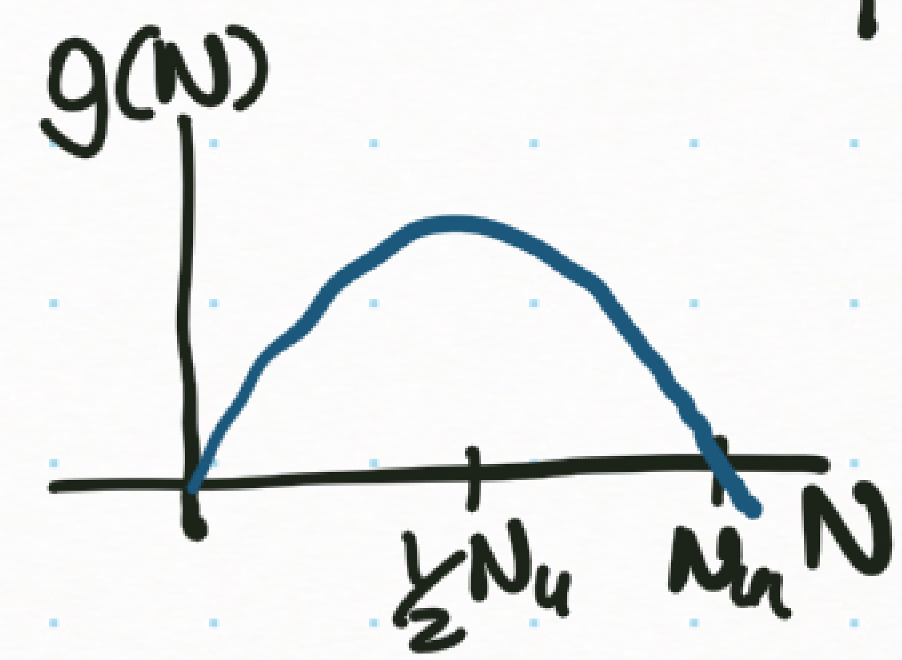


Biologically Optimal Level

Population with max growth rate

$g'(N) = 0 \Rightarrow k(N_u - N) + kN(0-1) = 0$

$\Rightarrow k(N_u - 2N) = 0 \Rightarrow N = \frac{1}{2}N_u$



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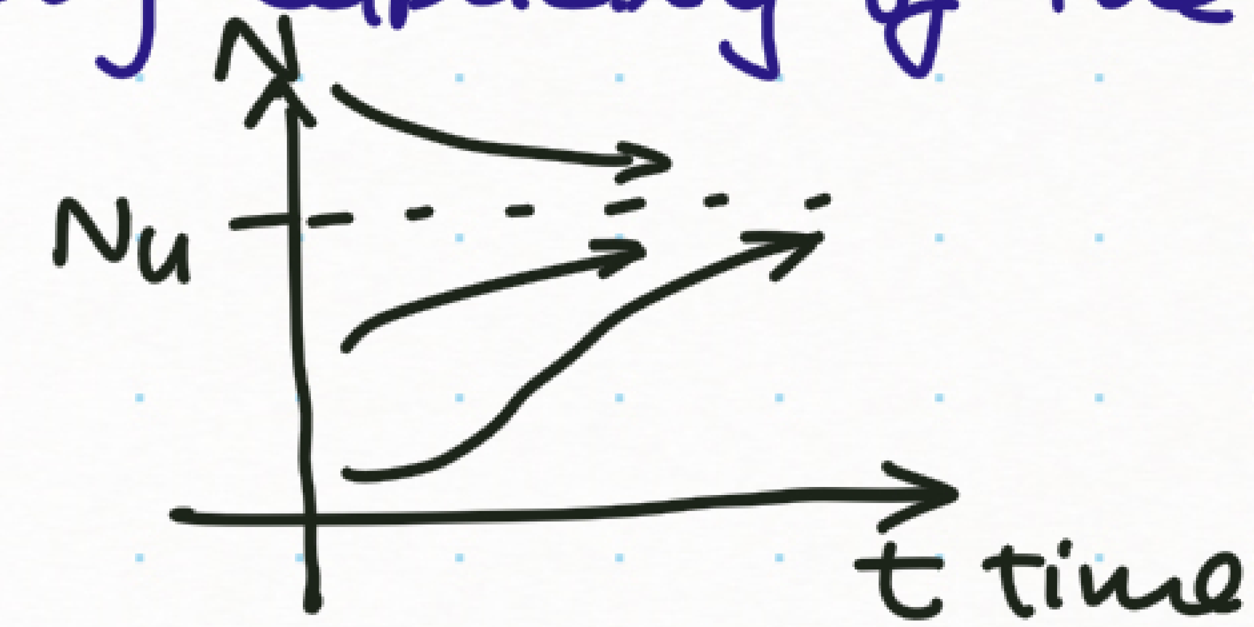

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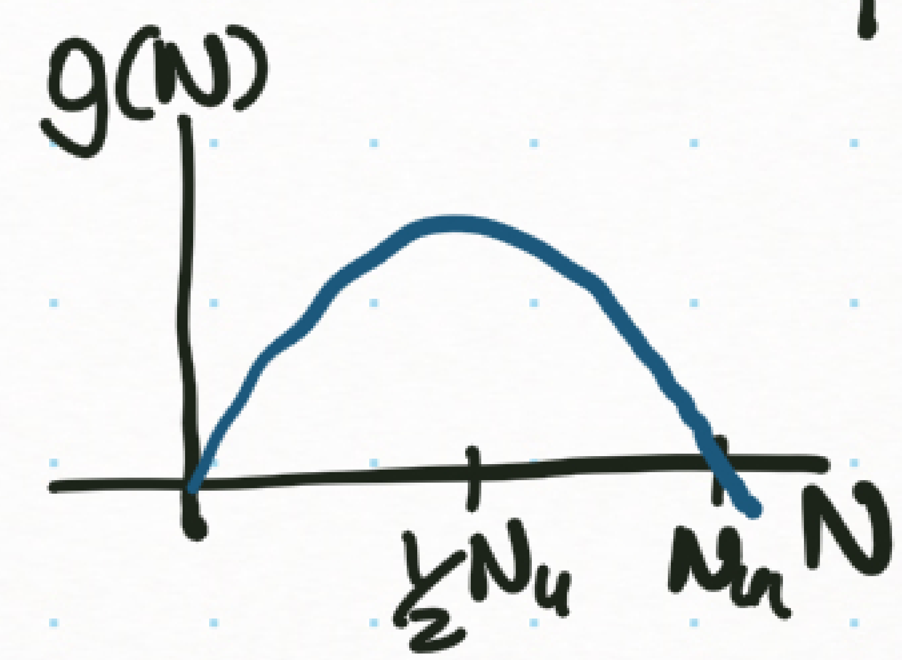

Socially Optimal Level

Population allowing the largest stable harvest

so, let harvest rate = growth rate $g(N)$

. Its also $N = \frac{1}{2}N_u$.

Economically optimal level

Population N_p that maximizes total profit. [as an industry overall]

Total profit = (yield) × (profit per fish)

say, we prefer a stable yield, i.e., constant harvest so to maximize, let rate of harvest = $g(N)$

$$\max \; TP = g(N)\left(\text{constant price} - c(N)\right)$$

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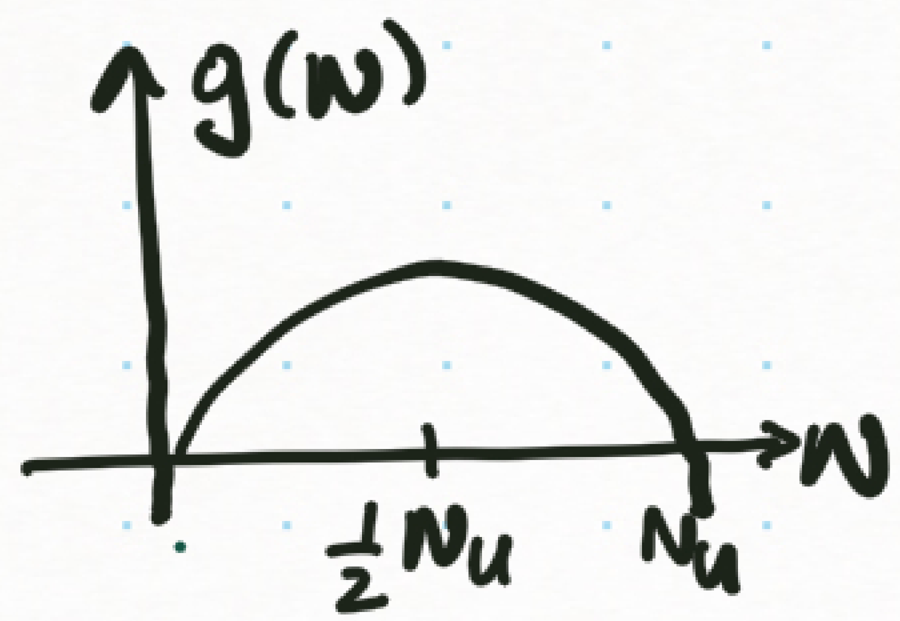

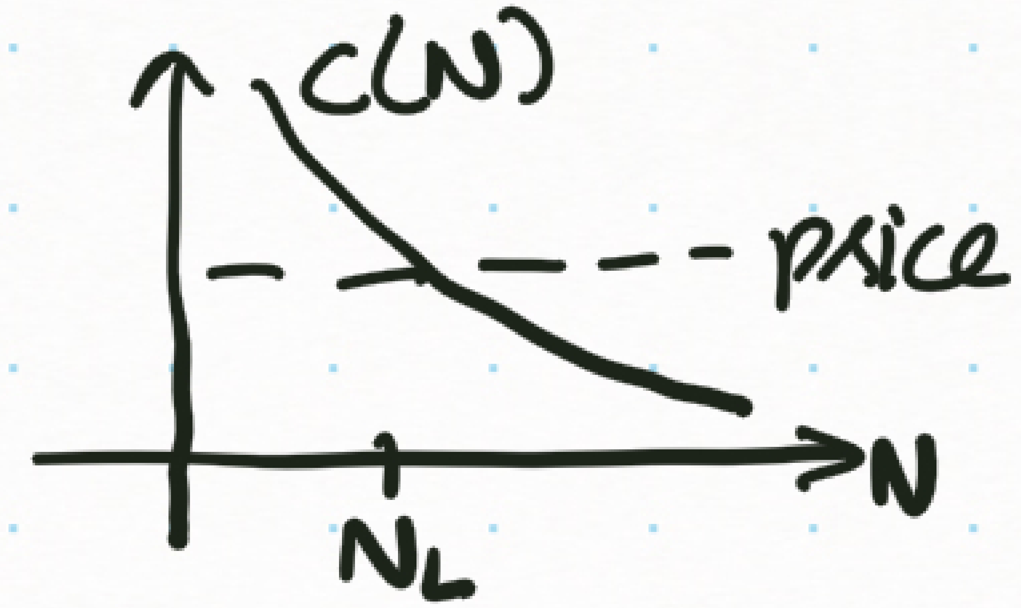

TP is increasing from $N=0$ to $N=\frac{1}{2}N_u$, so $N_p \geq \frac{1}{2}N_u$

$$TP'(N) = \underbrace{g'(N)}_{\text{positive for } N \leq \frac{1}{2}N_u} \underbrace{(P - c(N))}_{\text{positive for } N > N_L} + \underbrace{g(N)}_{\text{positive for } N < N_u} \underbrace{(-c'(N))}_{\text{negative}}$$

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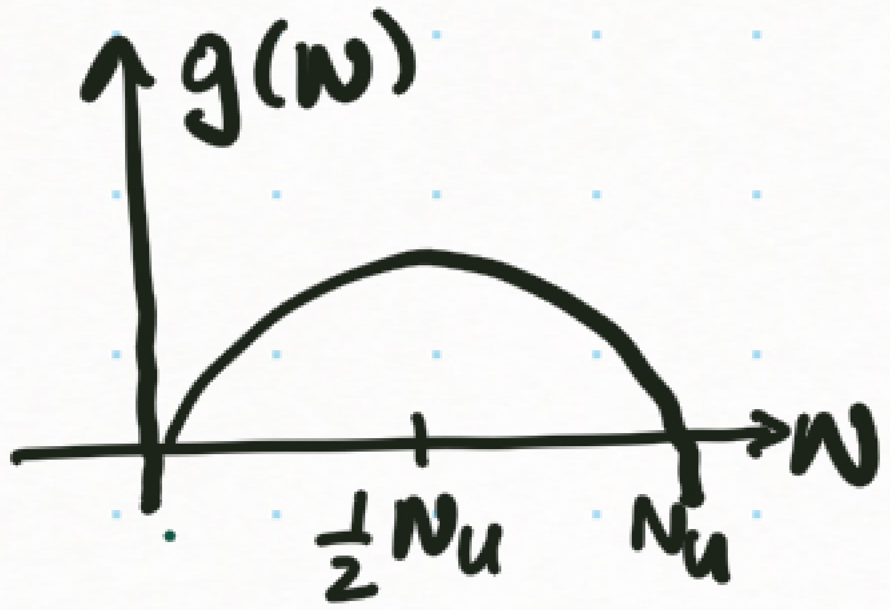

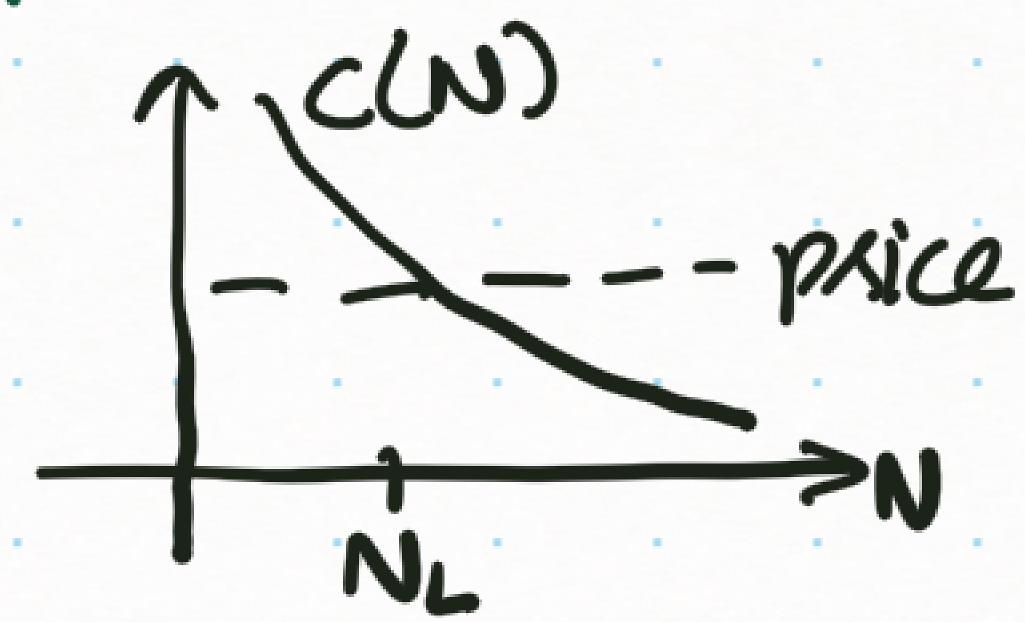

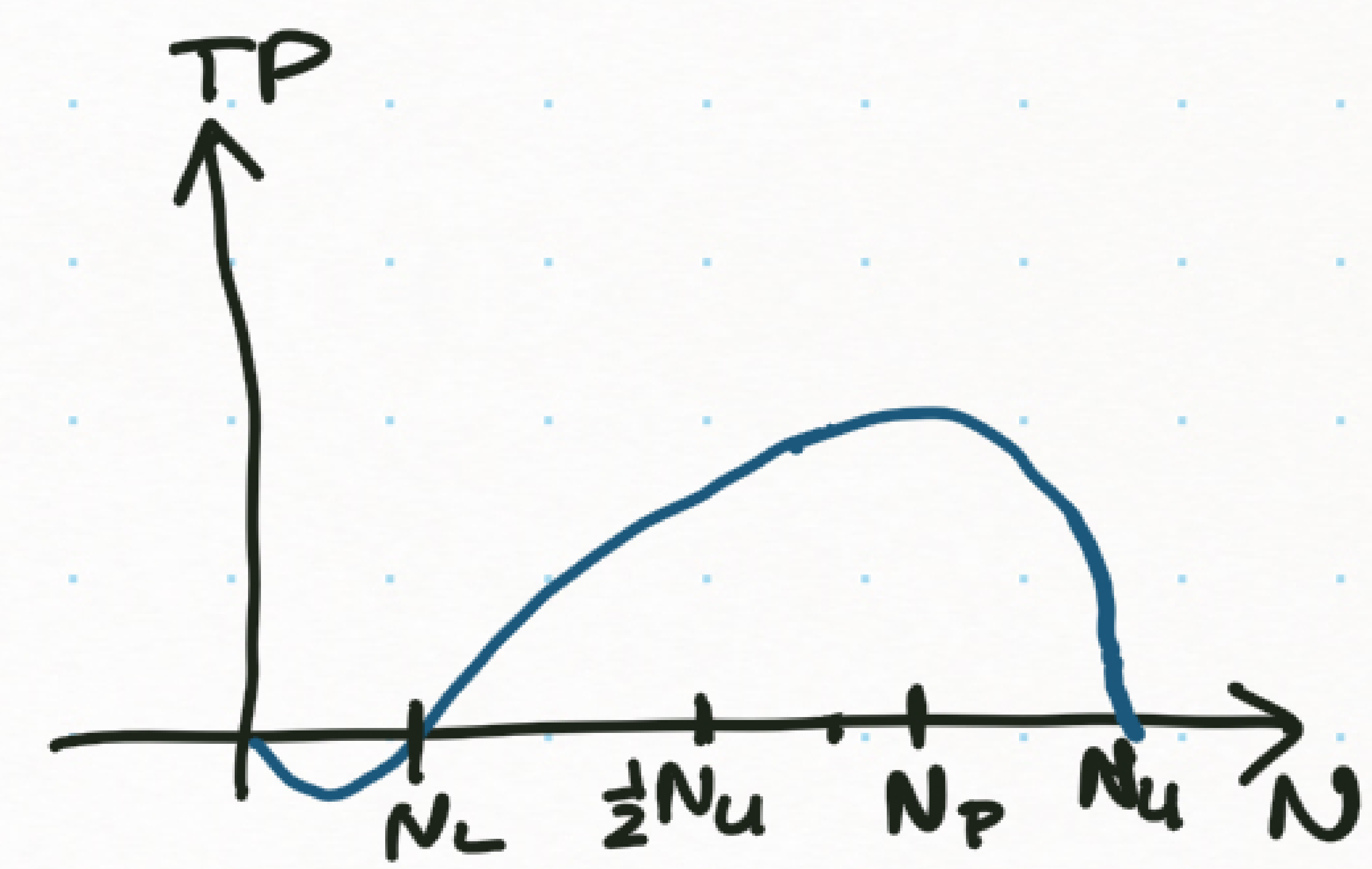

TP is increasing from $N=0$ to $N=\frac{1}{2}N_u$, so $N_P \geq \frac{1}{2}N_u$

$$TP'(N) = \underbrace{g'(N)}_{\text{positive for } N \leq \frac{1}{2}N_u}\,\underbrace{(P - c(N))}_{\text{positive for } N > N_L} + \underbrace{g(N)}_{\text{positive for } N < N_u}\,\underbrace{(-c'(N))}_{\text{negative}}$$

$TP'(N) > 0$ for $N_L < N \leq \frac{1}{2}N_u$

$TP'(N) < 0$ for $N < N_L$

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Population N_p that maximizes total profit. [as an industry overall]

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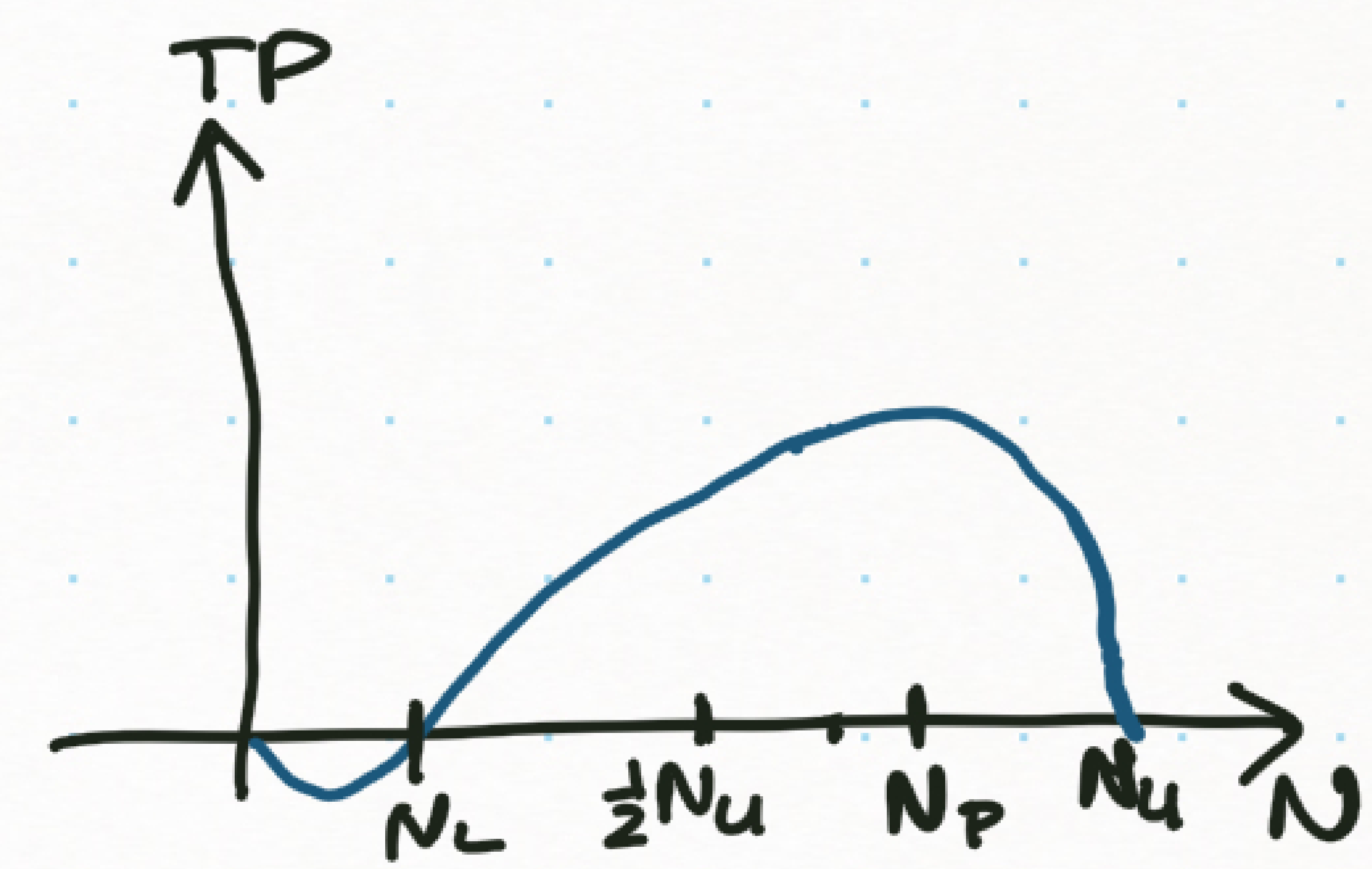

$N_p > \frac{1}{2}N_u$

More money is made if the fishing industry takes fewer fish than the maximum they could sustainably harvest (so population N is higher than the biologically optimum level $\frac{1}{2}N_u$)