MATH 380

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Competitive Hunter Model

We want to stock a small pond with trout & bass fish. Can they coexist in the pond? Or, will one species dominate?

Let $x(t)$ = population of trout at time t

$y(t)$ = population of bass at time t

each population depends on –

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In isolation, each population is assumed to grow as $\frac{dx}{dt} \propto x$, $\frac{dy}{dt} \propto y$

(What other natural model can we consider here?)

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predators,
environmental factors, etc.

In isolation, each population is assumed to grow as $\frac{dx}{dt} \propto x$, $\frac{dy}{dt} \propto y$

so, $\frac{dx}{dt} = ax$ for some $a > 0$; $\frac{dy}{dt} = my$ for some $m > 0$

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Let $x(t)$ = population of trout at time t
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We modify the basic models to take into account the competition between trout & bass for the same limited resources like space & food.
Each species has a negative effect on the other species.

This negative effect is assumed to be proportional to the number of possible interactions between the two species:

↳ Remember this?

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We modify the basic models to take into account the competition between trout & bass for the same limited resources like space & food.

Each species has a negative effect on the other species.

This negative effect is assumed to be proportional to the number of possible interactions between the two species:

$\frac{dx}{dt} = ax - bxy$ where $a>0$ & $b>0$. Similarly for $\frac{dy}{dt}$.

Competitive Hunter Model

We want to stock a small pond with trout & bass fish. Can they coexist in the pond? Or, will one species dominate?

Let $x(t)$ = population of trout at time t
$y(t)$ = population of bass at time t

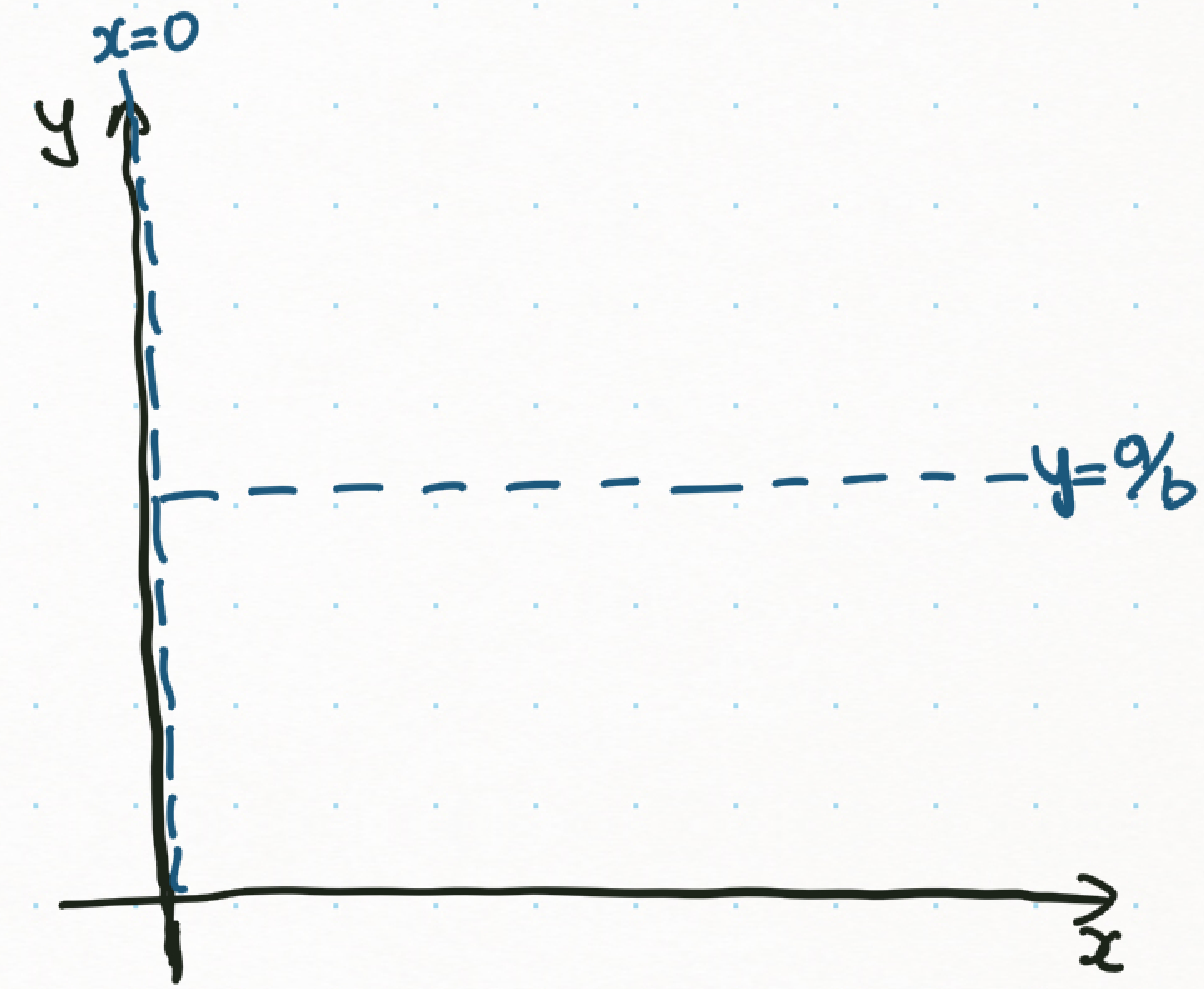
Under the previously stated assumptions,

$$\frac{dx}{dt} = \underbrace{(a-by)}_{\text{Intrinsic growth rate}}x \ , \quad \frac{dy}{dt} = \underbrace{(m-nx)}_{}y \ , \quad x(0)=x_0,\ y(0)=y_0,$$

where $a,b,m,n \in \mathbb{R}^+$ constants.

Equilibria? Behavior of population near & away from equil?
What is the overall behavior of the possible "solutions"?

$$\frac{dx}{dt} = 0 \iff (a-by)x = 0 \iff x=0 \text{ or } y = a/b$$



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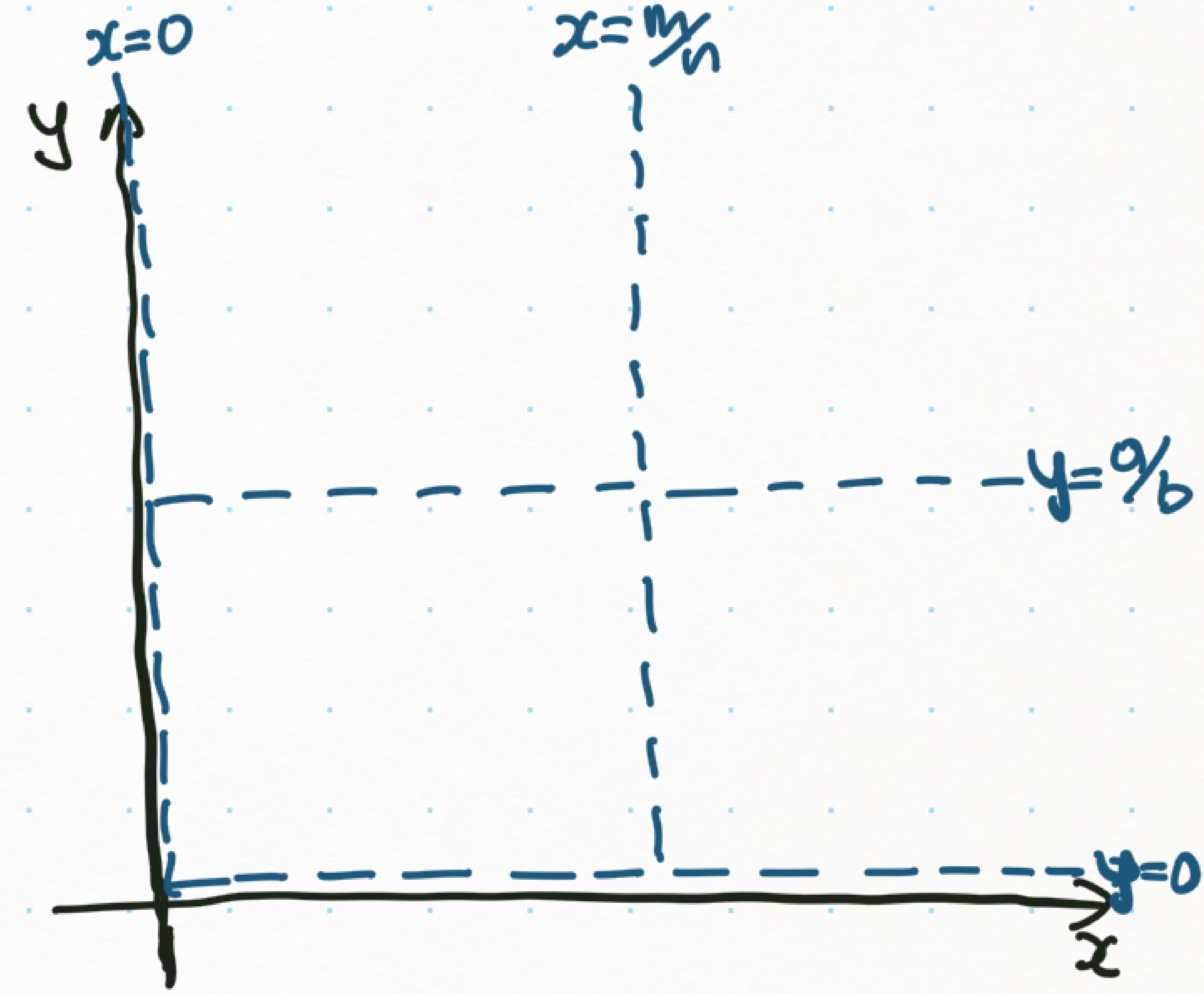

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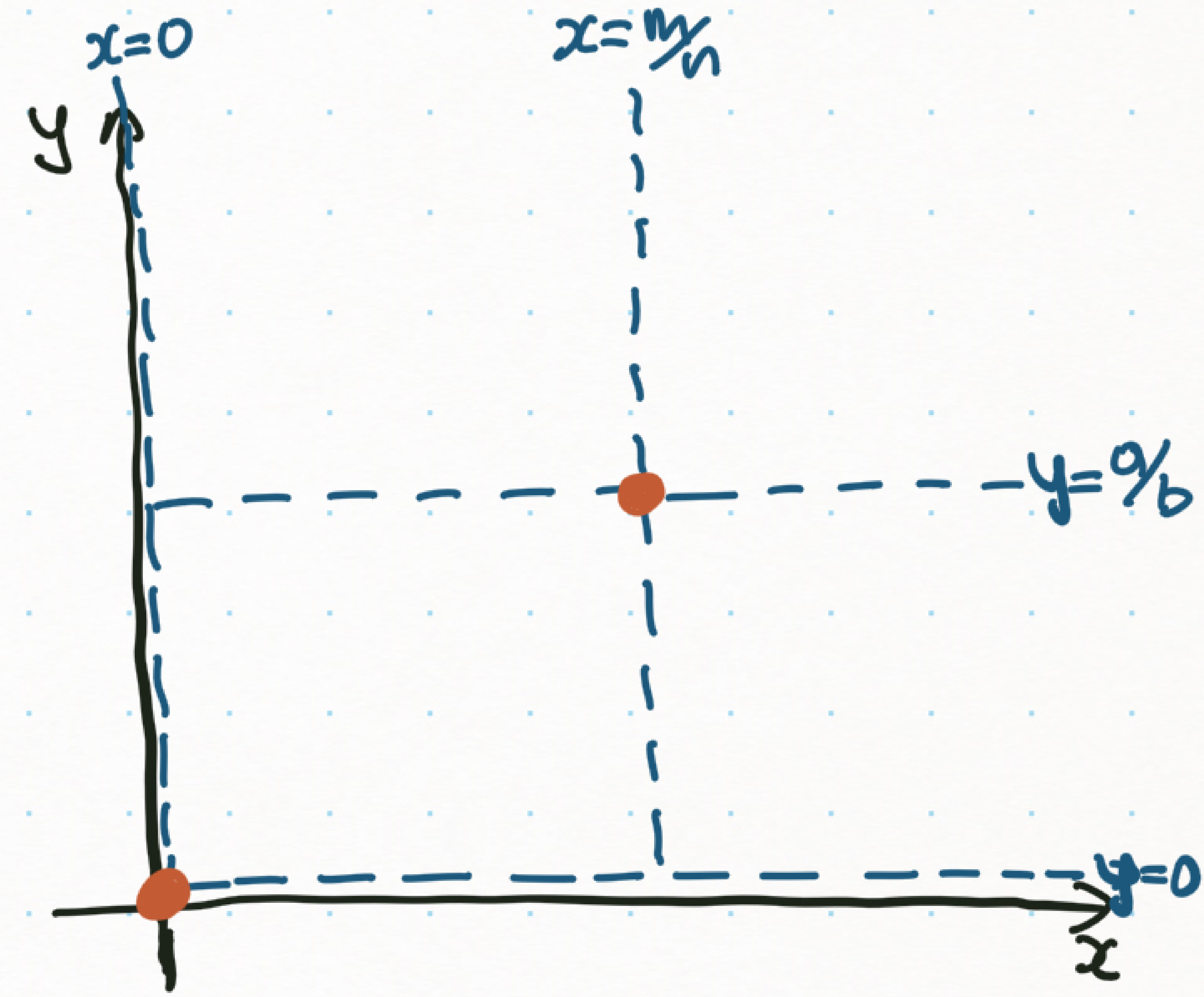
Equilibrium points:

$$\left.\begin{array}{l} \frac{dx}{dt} = 0 \\ \text{and } \frac{dy}{dt} = 0 \end{array}\right\} \iff \begin{array}{l} (a-by)x = 0 \\ (m-nx)y = 0 \end{array} \iff \begin{array}{l} (x,y) = (0,0) \\ (x,y) = (m/n, a/b) \end{array}$$



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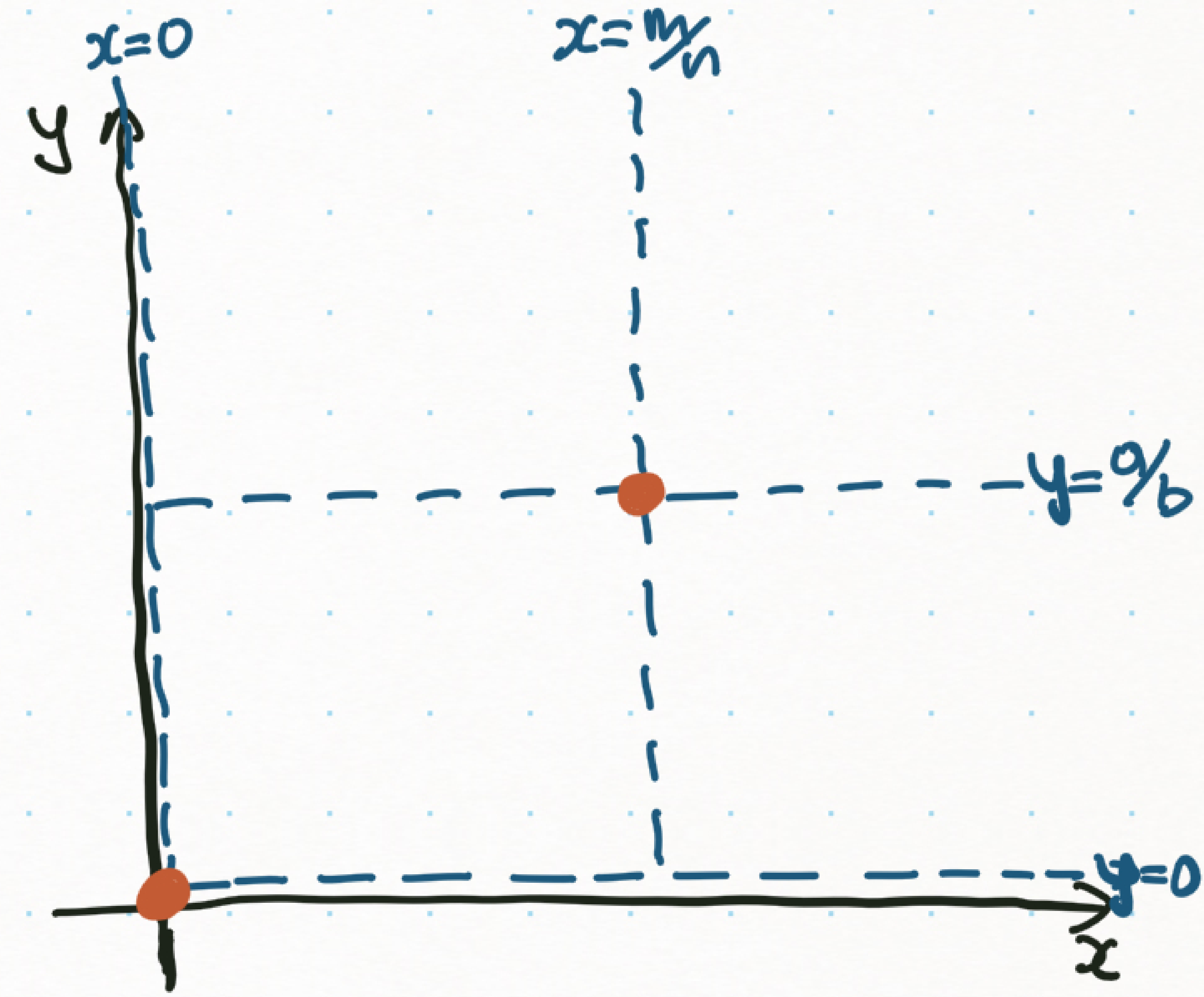
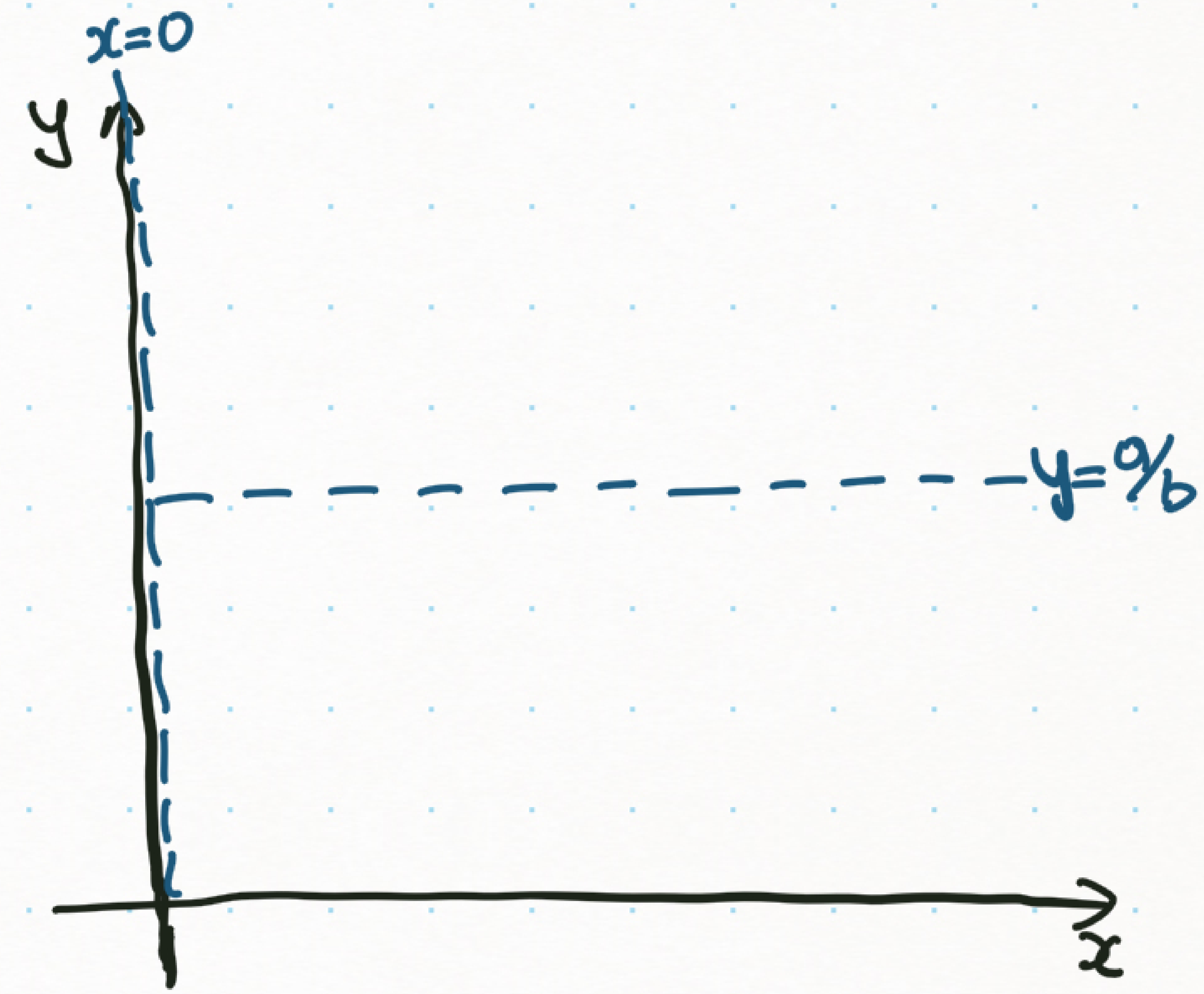


Since its unlikely we will know precise enough values of a, b, m, n, we want to first understand the behavior of the populations i.e., the solution curves around each equilibrium point.

$$\frac{dx}{dt} = 0 \iff (a-by)x = 0 \iff x=0 \text{ or } y = a/b$$

What is the behavior of $\frac{dx}{dt}$ in the phase plane?



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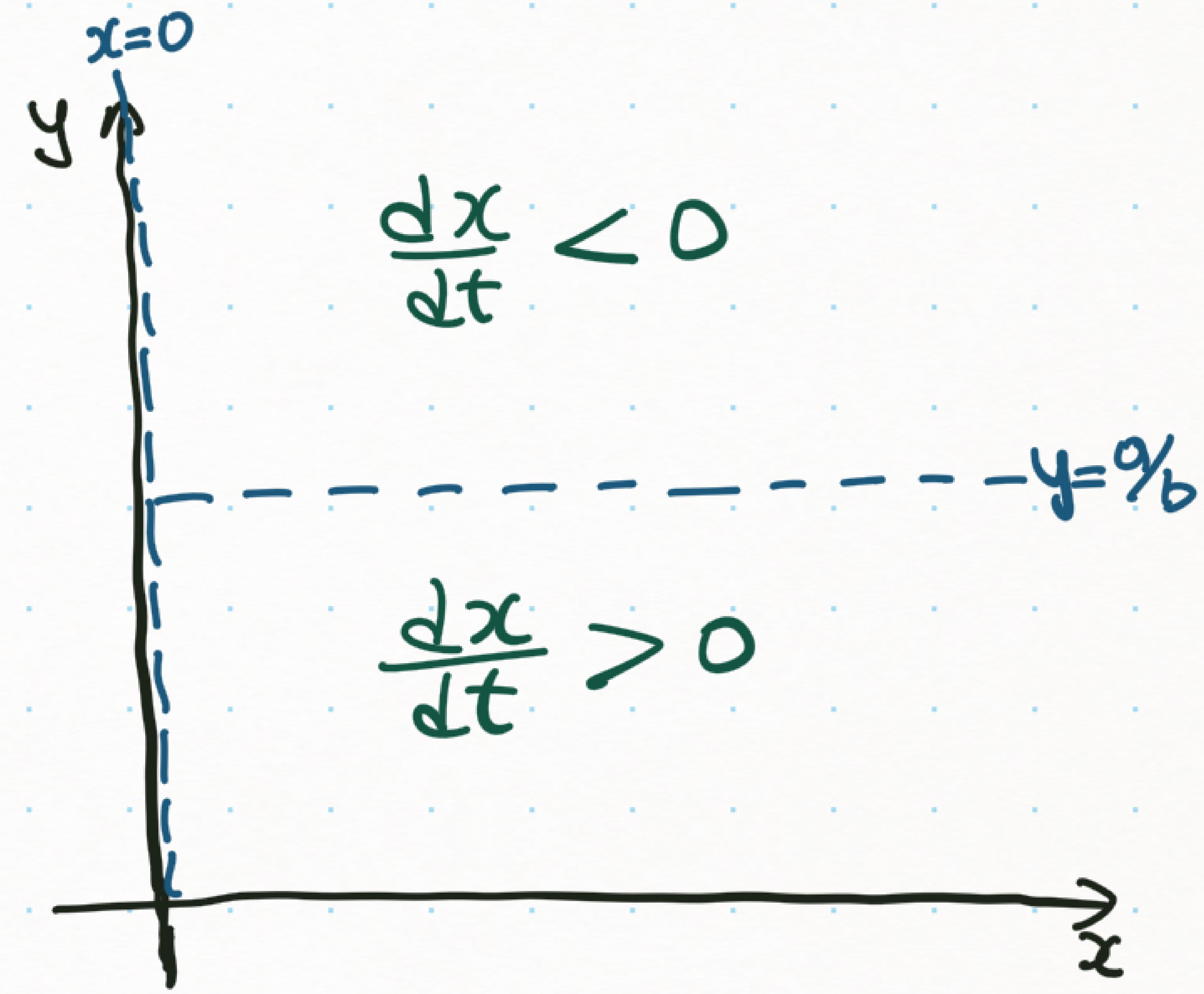

$$\frac{dy}{dt} = 0 \iff (m - nx)y = 0$$
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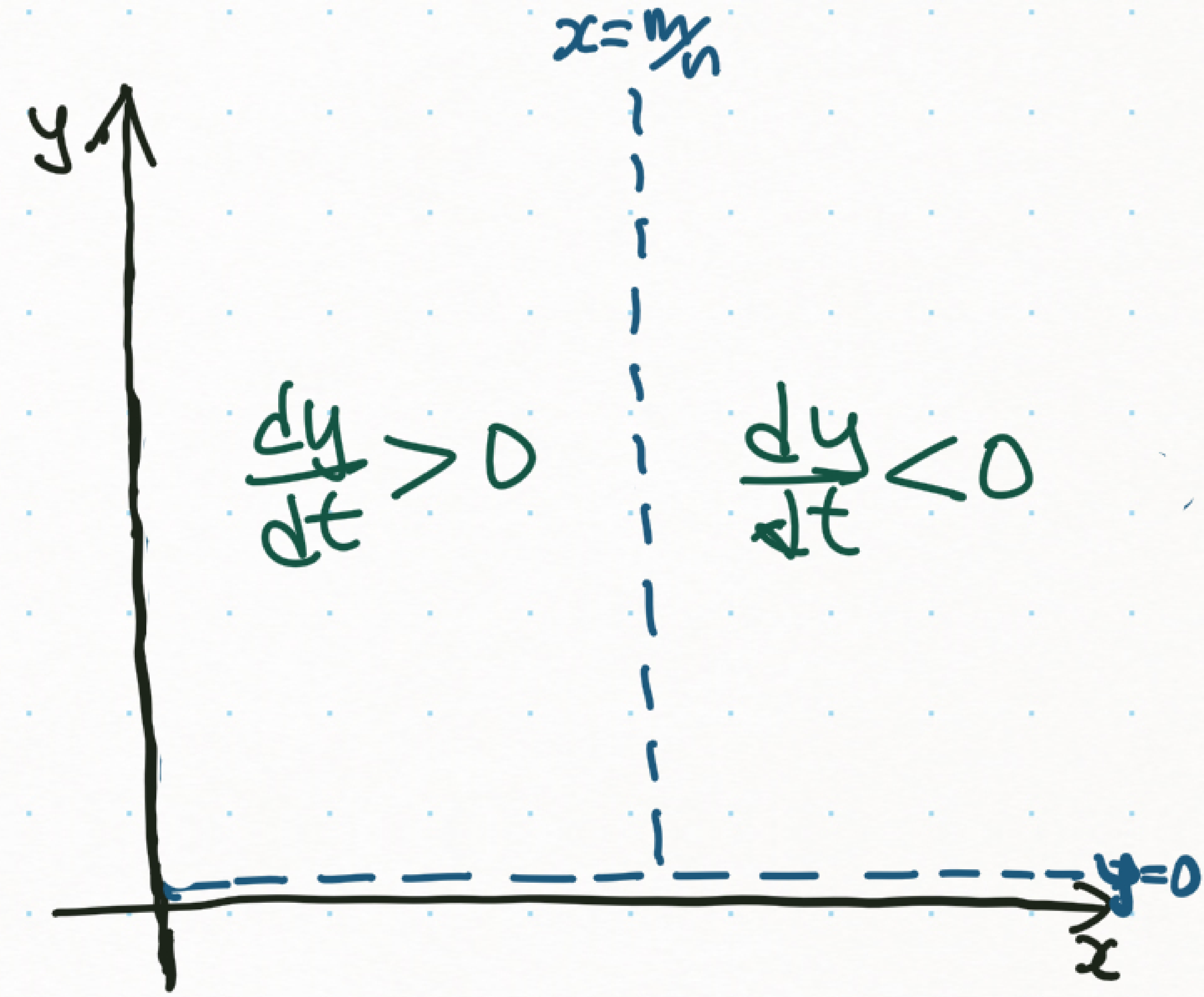
What is behavior of $\frac{dy}{dt}$ in the phase plane ?



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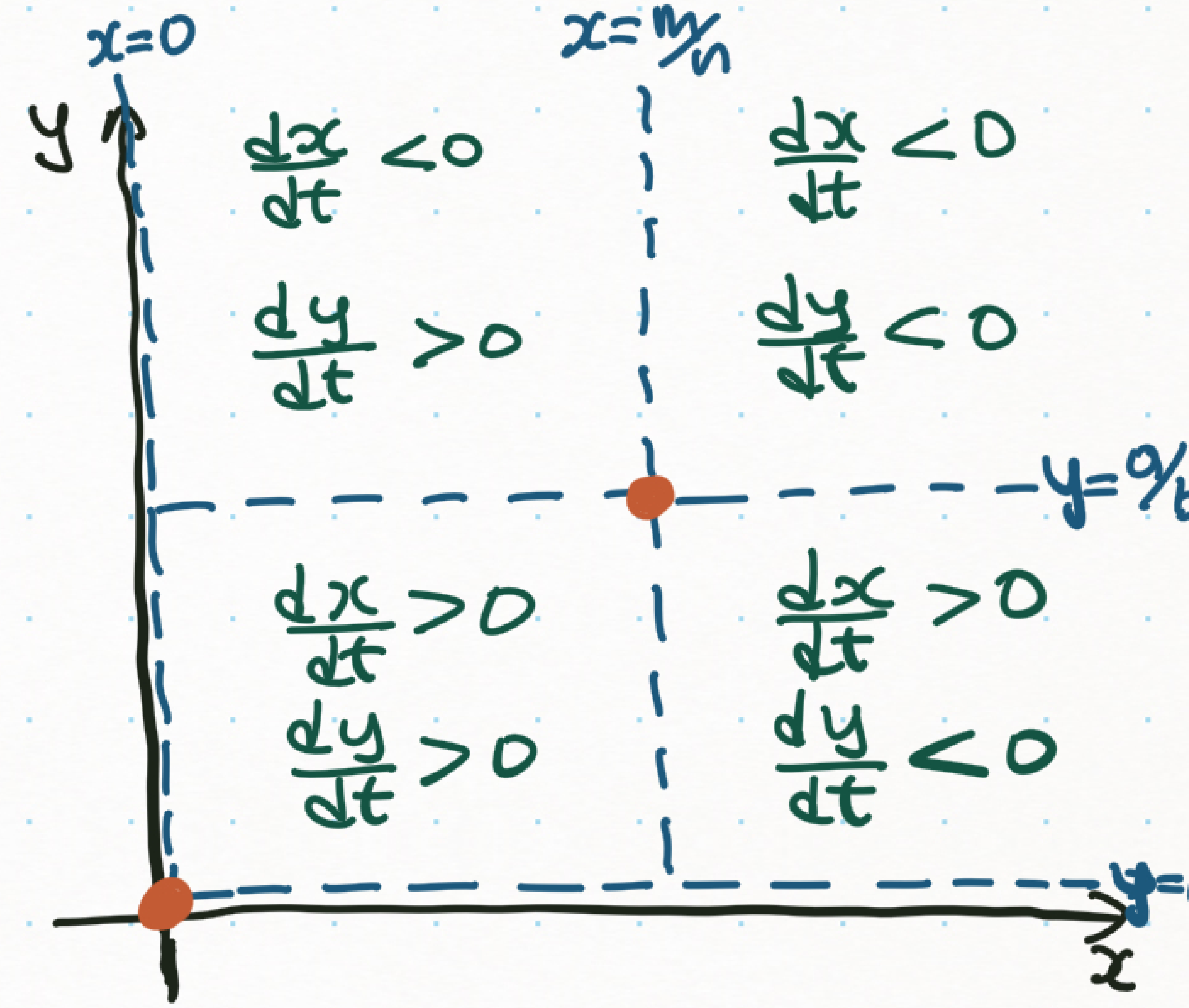

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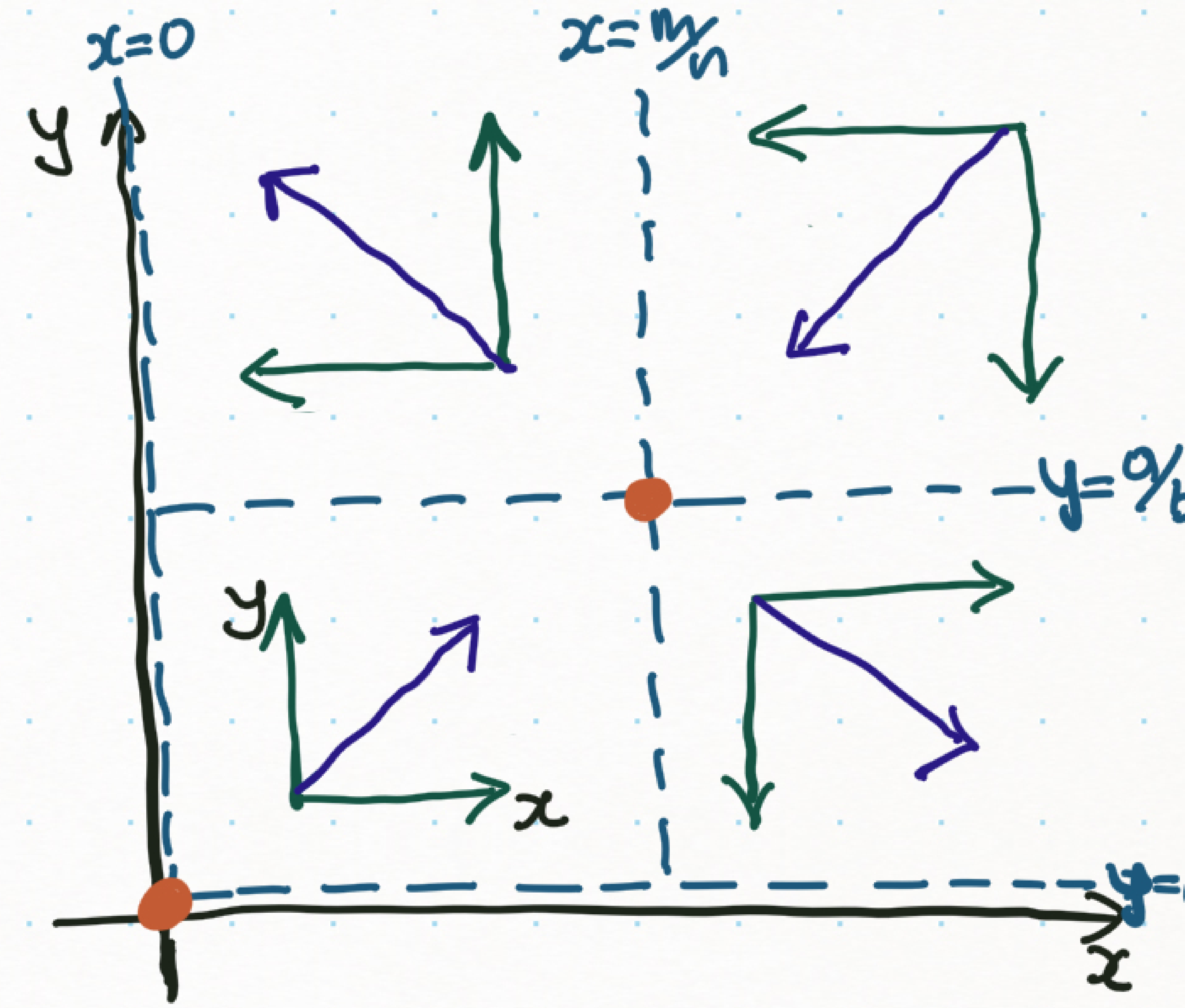

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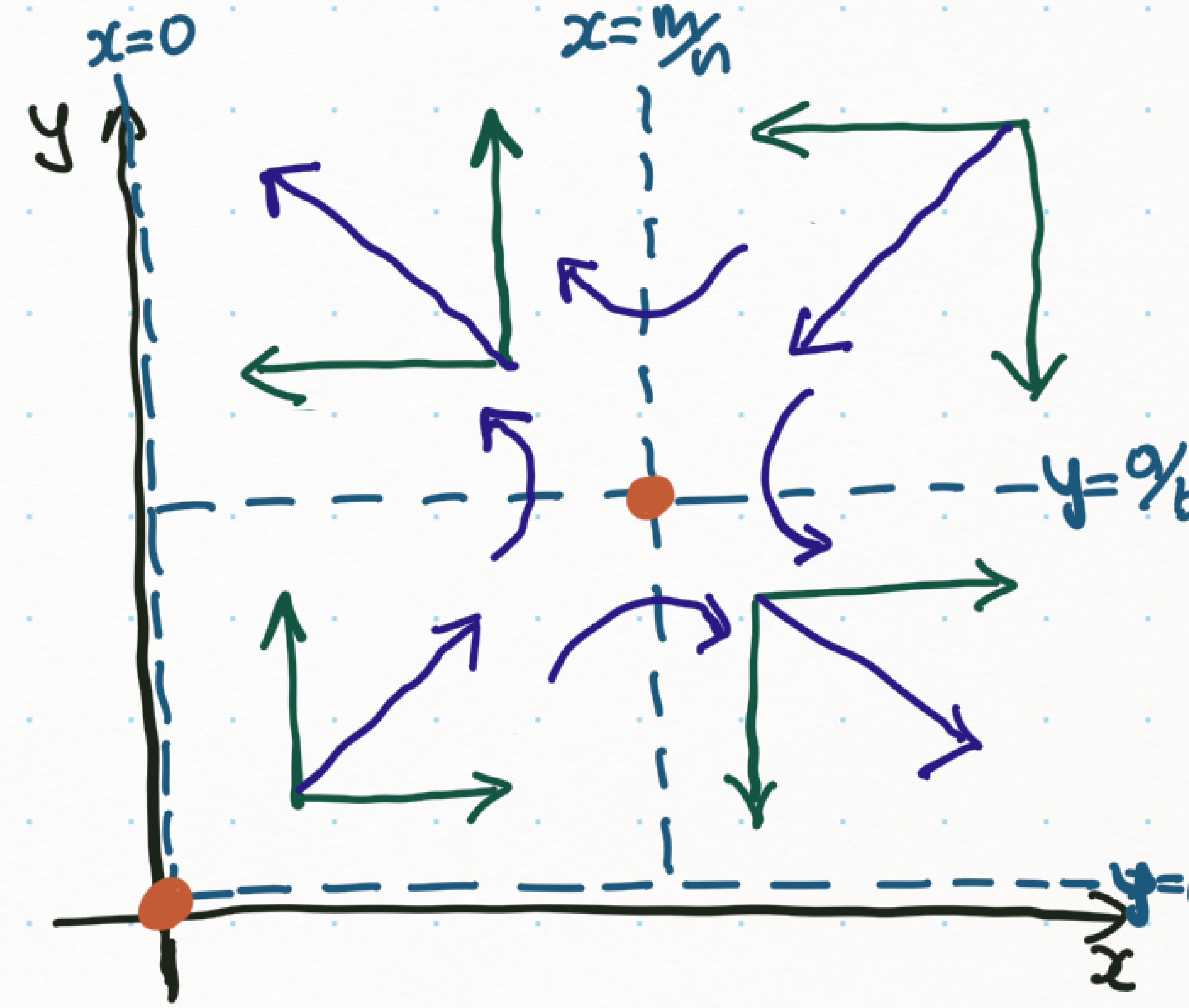

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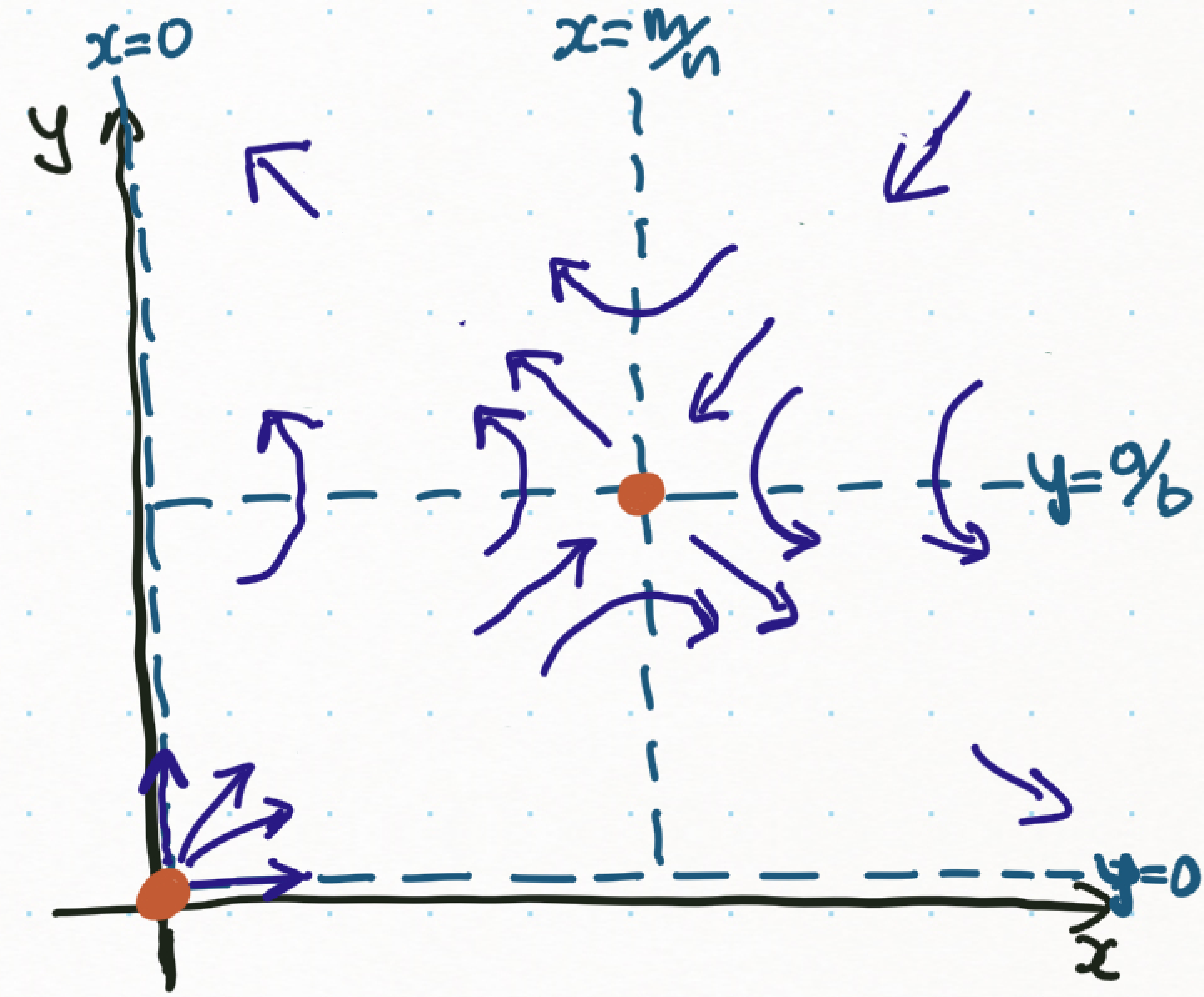

$\longrightarrow$ solution curves
arrow indicates time t inc.

$$\frac{dx}{dt}=0 \iff (a-by)x=0 \iff x=0 \text{ or } y=a/b$$

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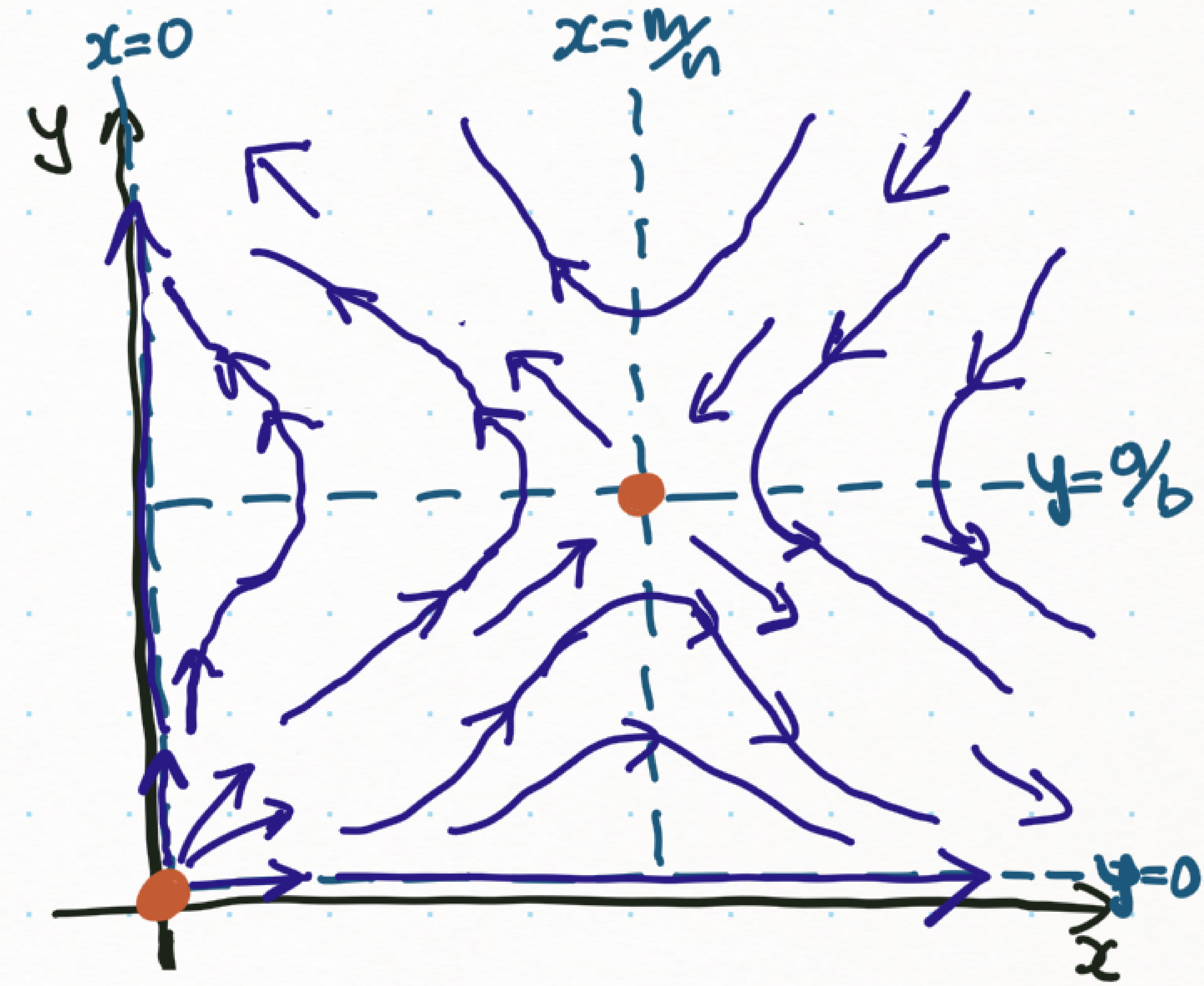

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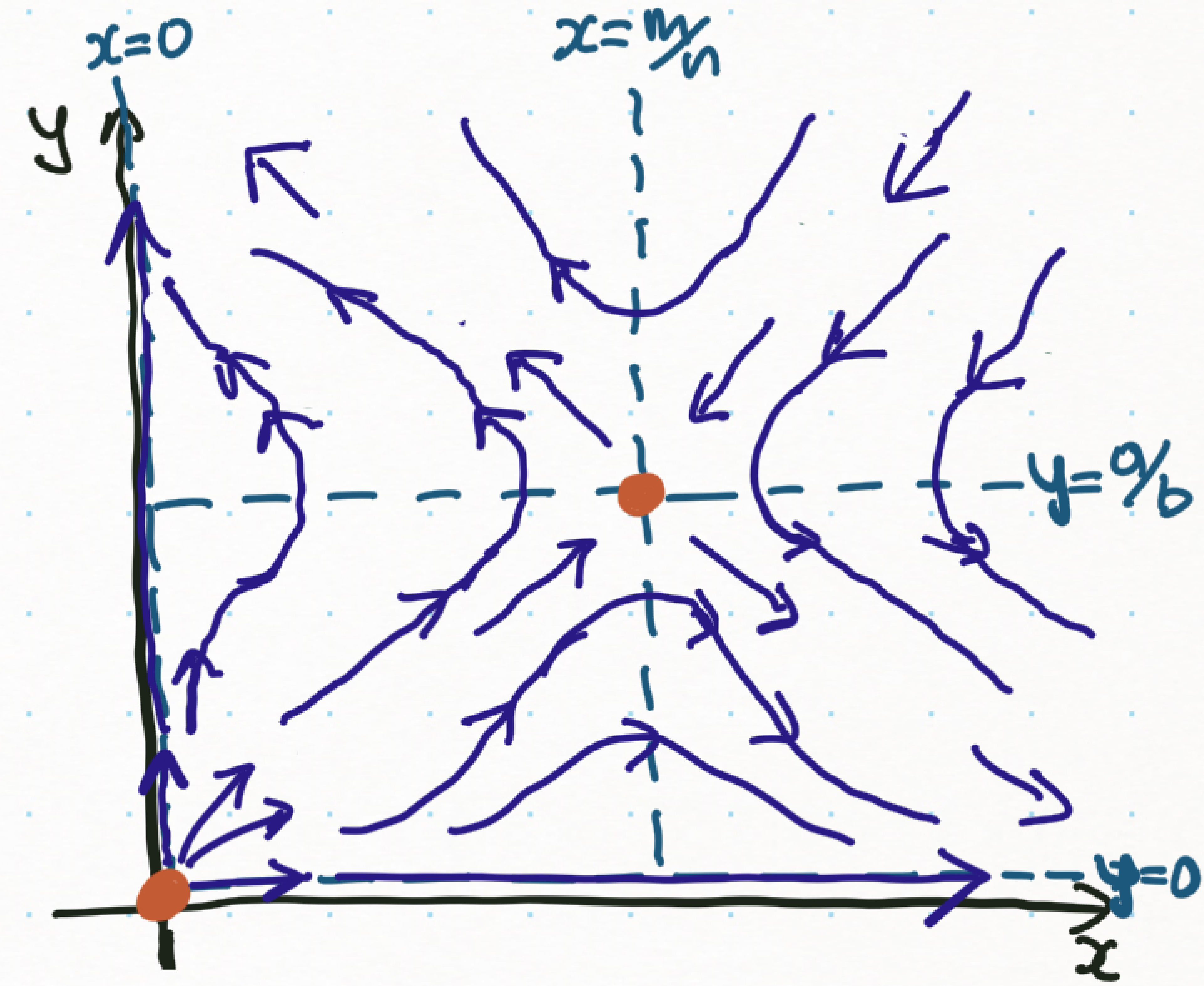

Long-term behavior?

$\frac{dx}{dt} = 0 \iff (a-by)x = 0 \iff x=0$ or $y = a/b$

$\frac{dy}{dt} = 0 \iff (m-nx)y = 0$
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Equilibrium points:

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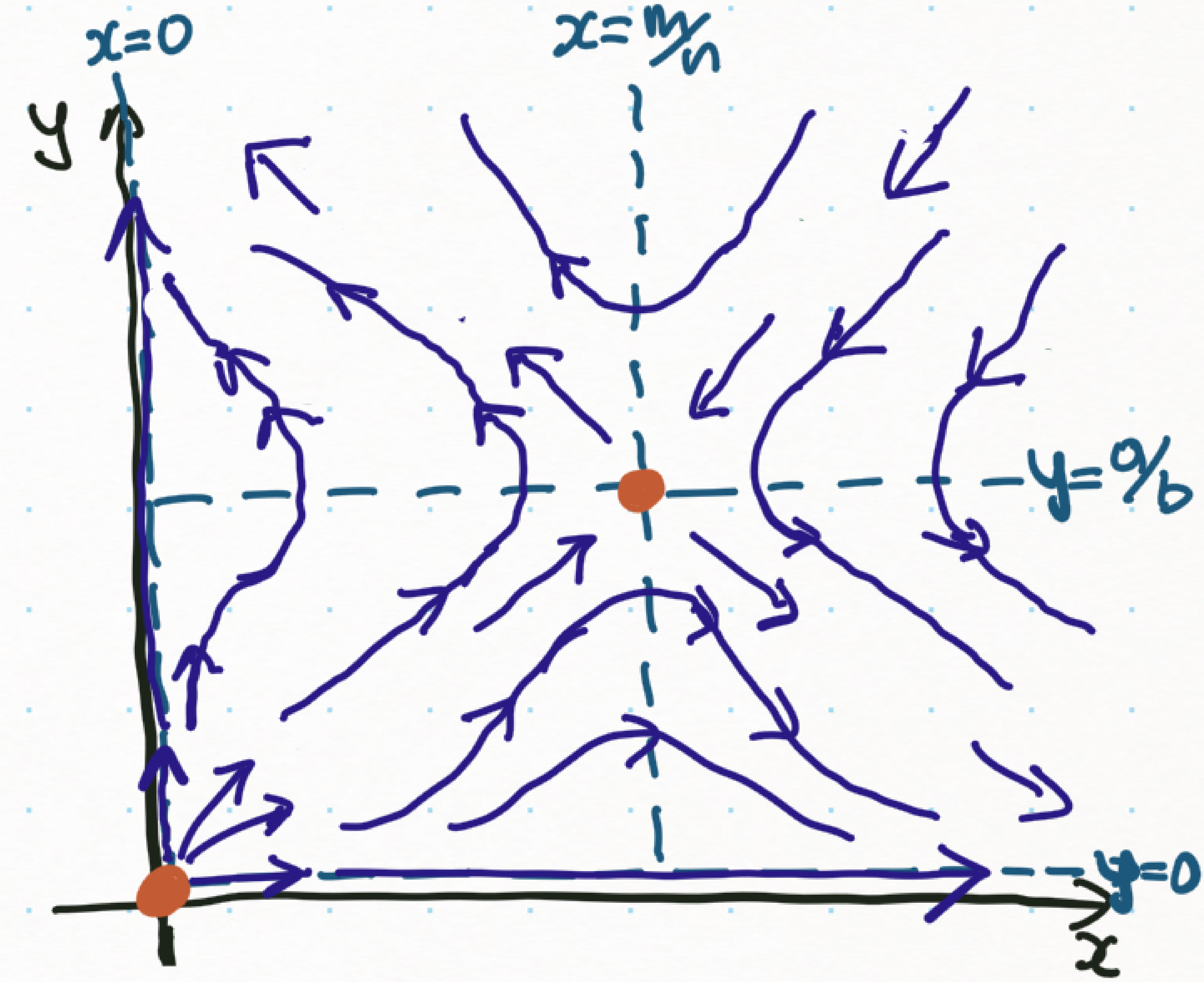

Long-term behavior?

(0,0) is unstable, so both bass & trout will not be extinct.

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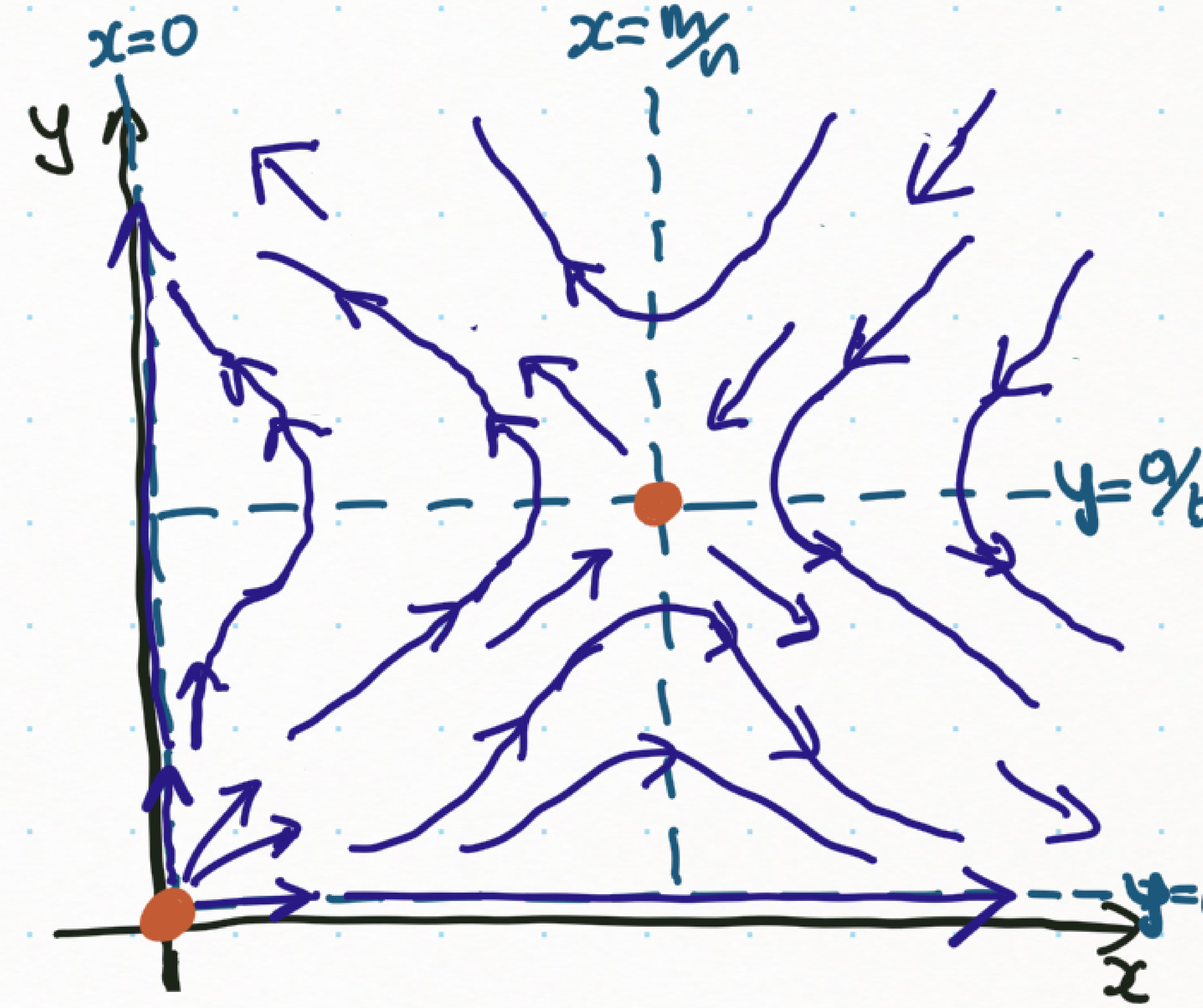

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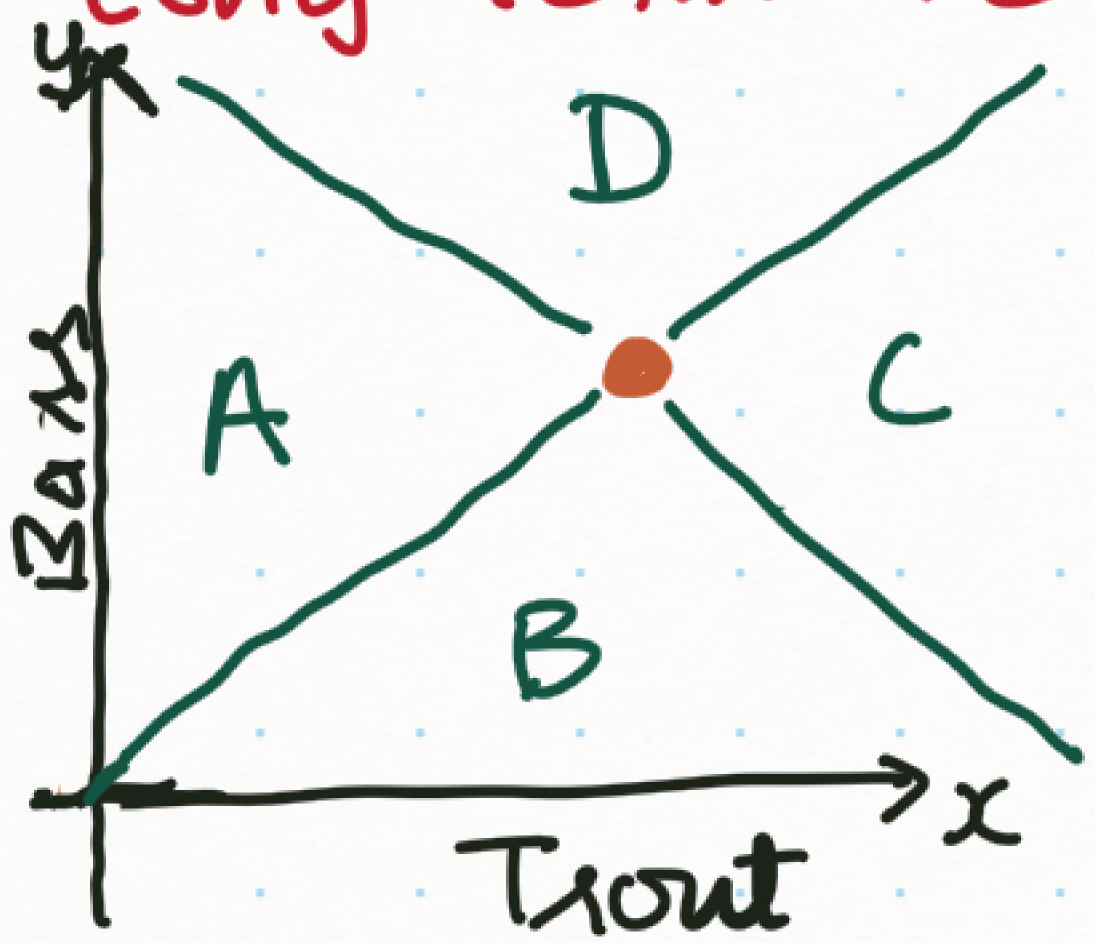


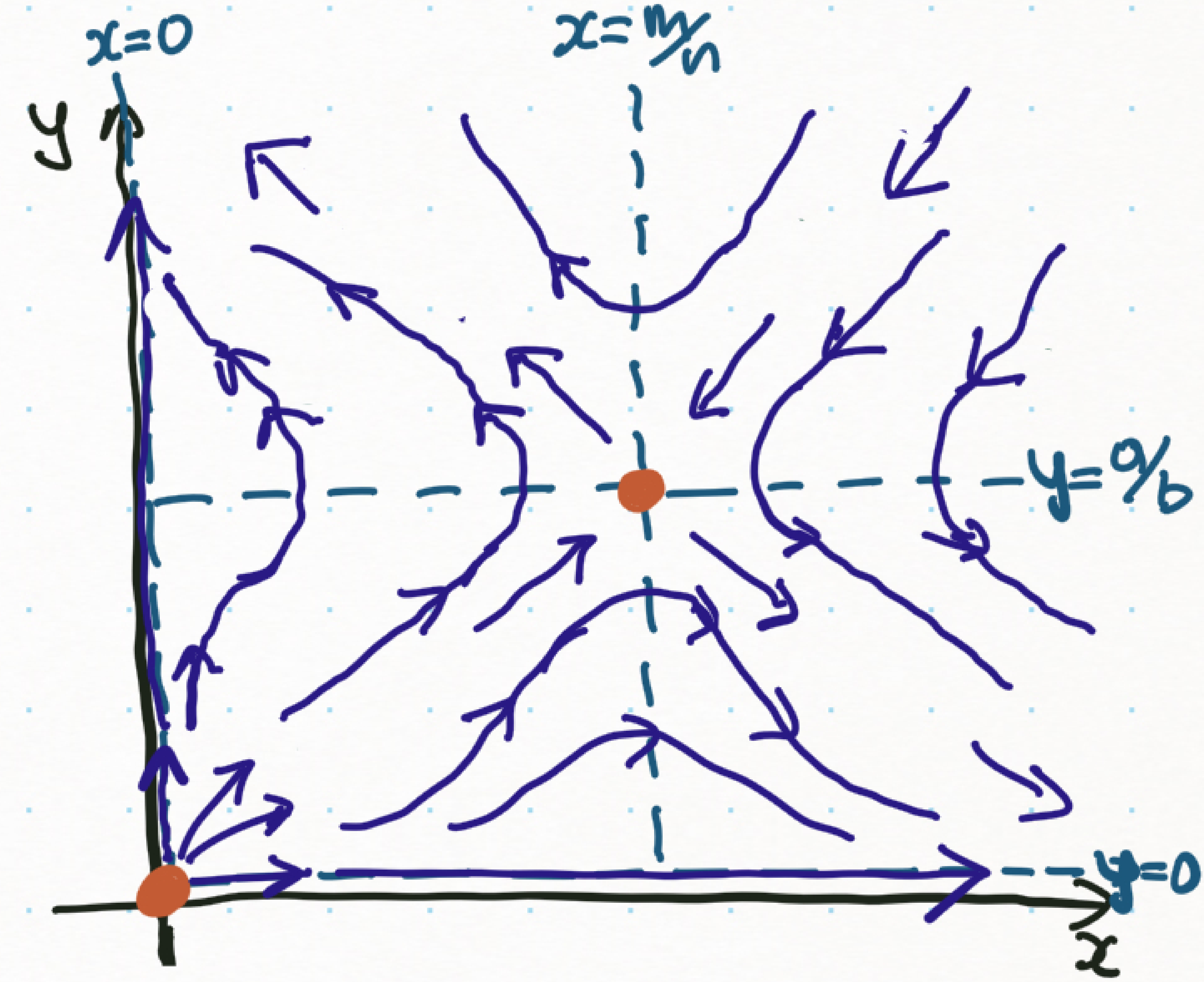
(0,0) is unstable, so both bass & trout will not be extinct.

$(\frac{m}{n}, \frac{a}{b})$ is also unstable but with peculiar behavior

$\frac{dx}{dt} = 0 \iff (a-by)x = 0 \iff x=0$ or $y = a/b$

$\frac{dy}{dt} = 0 \iff (m-nx)y = 0$

$\iff y=0$ or $x = m/n$



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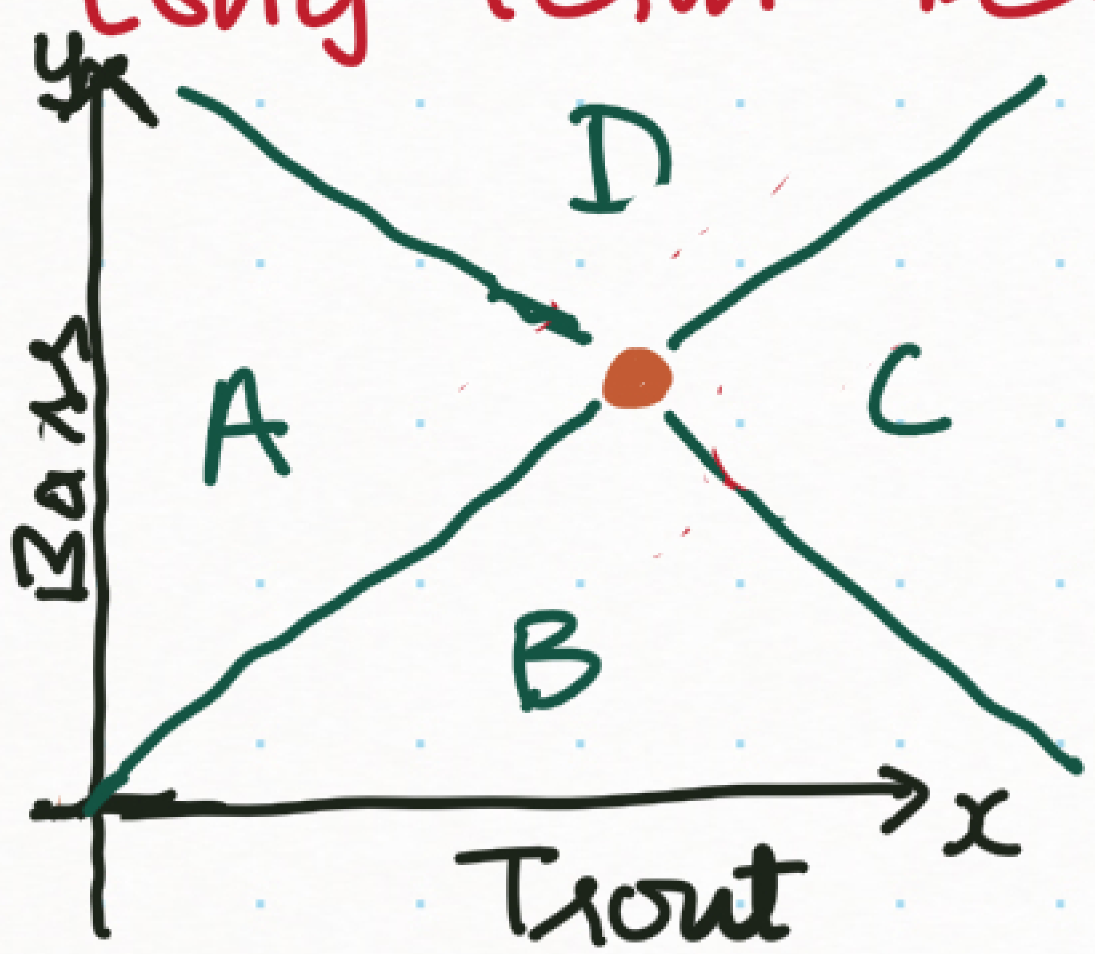

$(0,0)$ is unstable, so both bass & trout will not be extinct.
$(\frac{m}{n}, \frac{a}{b})$ is also unstable but with peculiar behavior
If initial population (x_0, y_0) is in region A then
initially both x & y increase but then y dominates & $x \to 0$

$$\frac{dx}{dt} = 0 \iff (a-by)x = 0 \iff x=0 \text{ or } y = a/b$$

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Equilibrium points:

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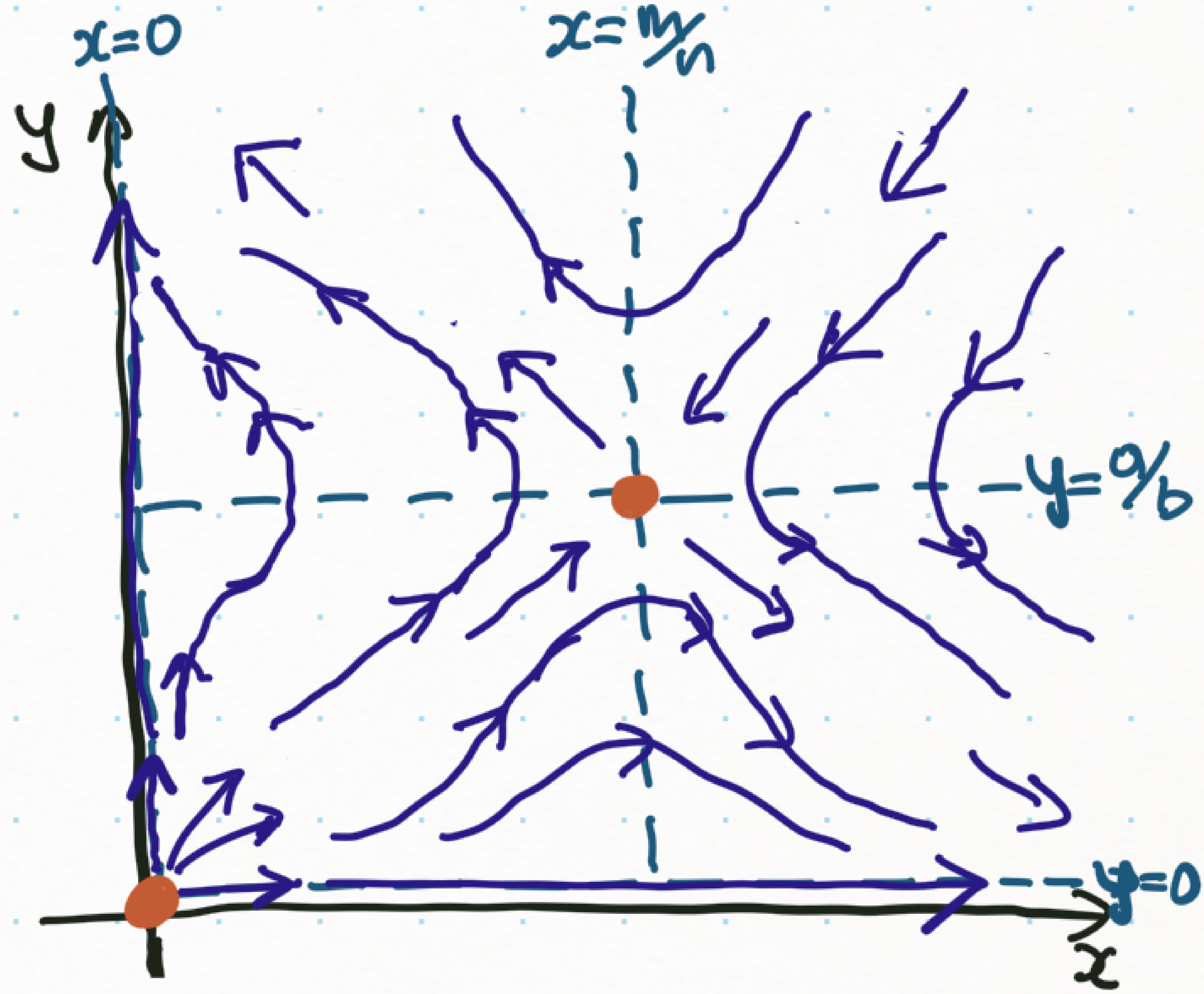

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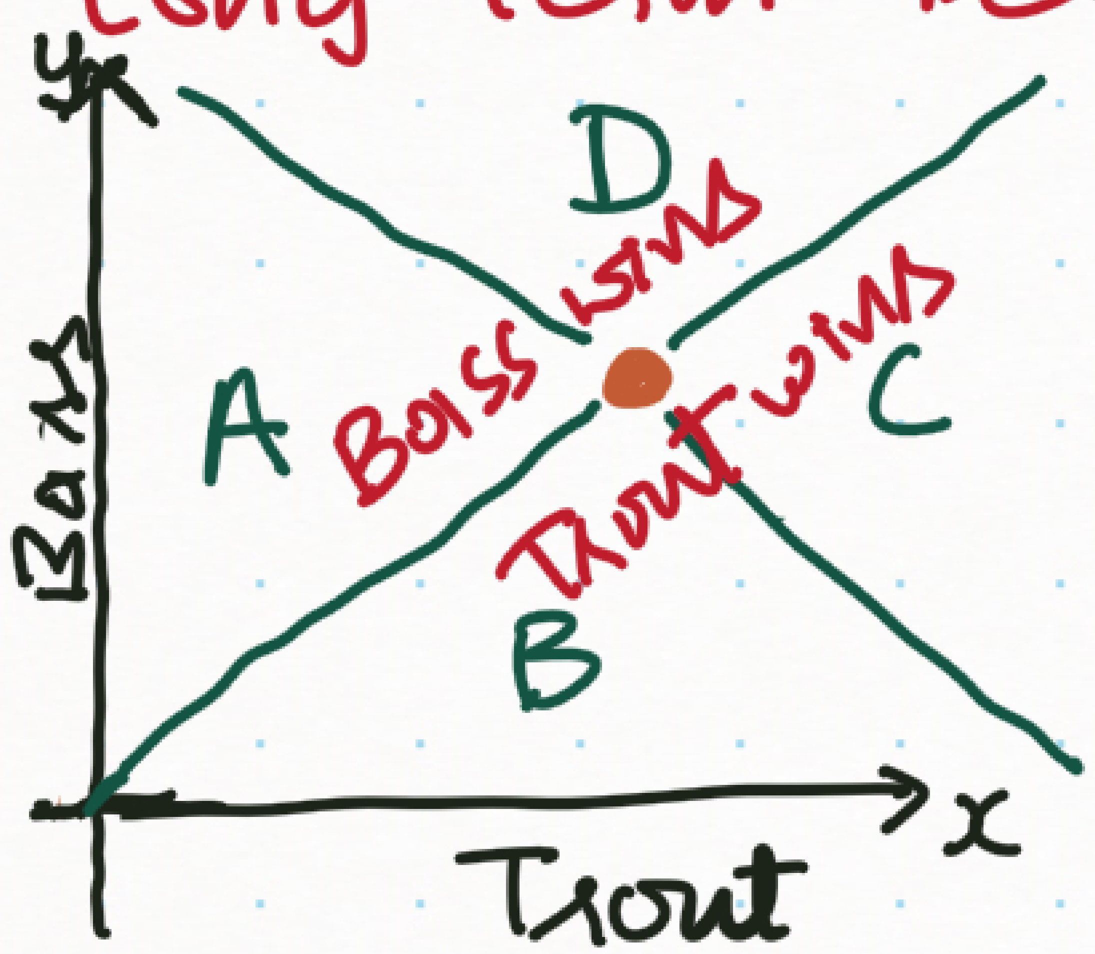

$(0,0)$ is unstable, so both bass & trout will not be extinct.

$\left(\frac{m}{n}, \frac{a}{b}\right)$ is also unstable but with peculiar behavior

If initial population (x_0, y_0) is in region A then initially both x & y increase but then y dominates & $x \to 0$

If initial population (x_0, y_0) is in region B then initially both x & y inc. but then x dominates & $y \to 0$

A Predator-Prey Model

Baleen whales around Antarctica feed on the Antarctic krill
(which feed on plankton)

When Baleen population increases, there is more consumption of the krill, until there is not enough food supply for the whales which either die or leave the region.

Krill population rebounds & whale come back, so on.

Will this cycle continue indefinitely?

What effect will killings of whales have on this "ecological balance"?

How should we manage krill (& whales) so that they are available in sufficient quantity for all animals that feed on them?

etc.

A Predator-Prey Model

Let $x(t)$ = Antarctic krill population at time t

$y(t)$ = Baleen whale population at time t

$$\frac{dx}{dt} = ax \, , \quad a > 0$$ when krill in isolation w/o whales

A Predator-Prey Model

Let $x(t)$ = Antarctic krill population at time t

$y(t)$ = Baleen whale population at time t

$$\frac{dx}{dt} = ax \, , \quad a>0 \qquad \text{when krill in isolation w/o whales}$$

$$\frac{dx}{dt} = ax - bxy \, , \quad a,b>0 \, , \qquad \text{when krill are hunted by whales.}$$

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$$\frac{dy}{dt} = -my, \quad m>0 \quad \text{when whales in isolation w/o krill}$$

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$$\frac{dy}{dt} = -my \;, \quad m>0 \quad \text{when whales in isolation w/o krill}$$

$$\frac{dy}{dt} = -my + nxy \;, \quad m,n>0 \quad \text{when whales can eat krill.}$$

A Predator-Prey Model

Let $x(t)$ = Antarctic krill population at time t

$y(t)$ = Baleen whale population at time t

Under standard assumptions, we have what?

$x(0)=x_0,\ y(0)=y_0,$

$$\frac{dx}{dt} = \underbrace{(a-by)}_{\text{intrinsic growth rate}}x, \quad \frac{dy}{dt} = \underbrace{(-m+nx)}_{\text{intrinsic growth rate}}y, \text{ where } a,b,m,n>0.$$

$k = a-by$

$k\uparrow$ when $y\downarrow$

$k\downarrow$ when $y\uparrow$

$r = -m+nx$

$r\uparrow$ when $x\uparrow$

$r\downarrow$ when $x\downarrow$

Equilibrium points?

Solution curves & behavior around the equilibria?

etc.

$$\frac{dx}{dt} = 0 \iff (a-by)x = 0 \iff x=0 \text{ or } y = a/b$$

$$\frac{dy}{dt} = 0 \iff (-m+nx)y = 0 \iff y=0 \text{ or } x = m/n$$

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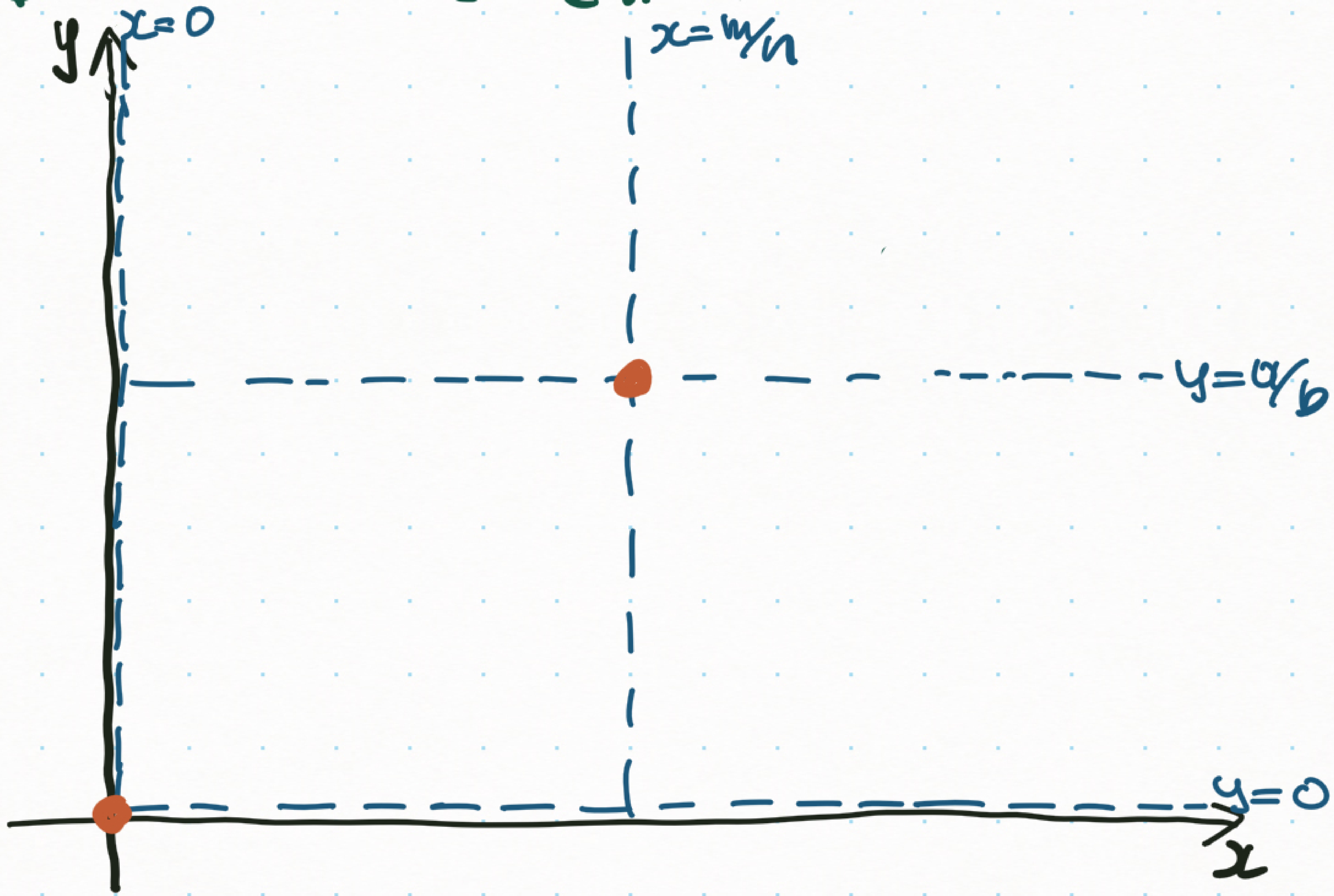

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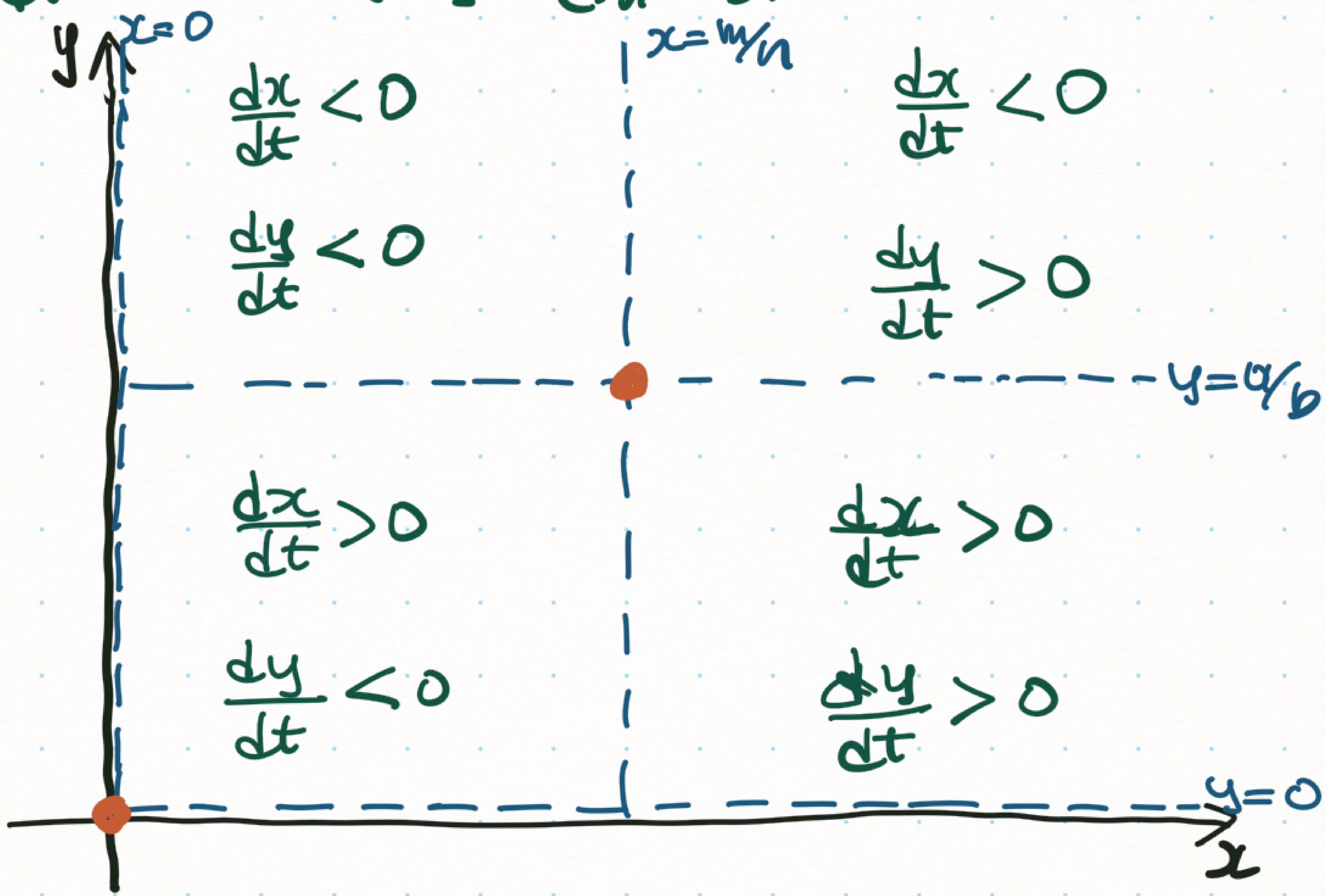

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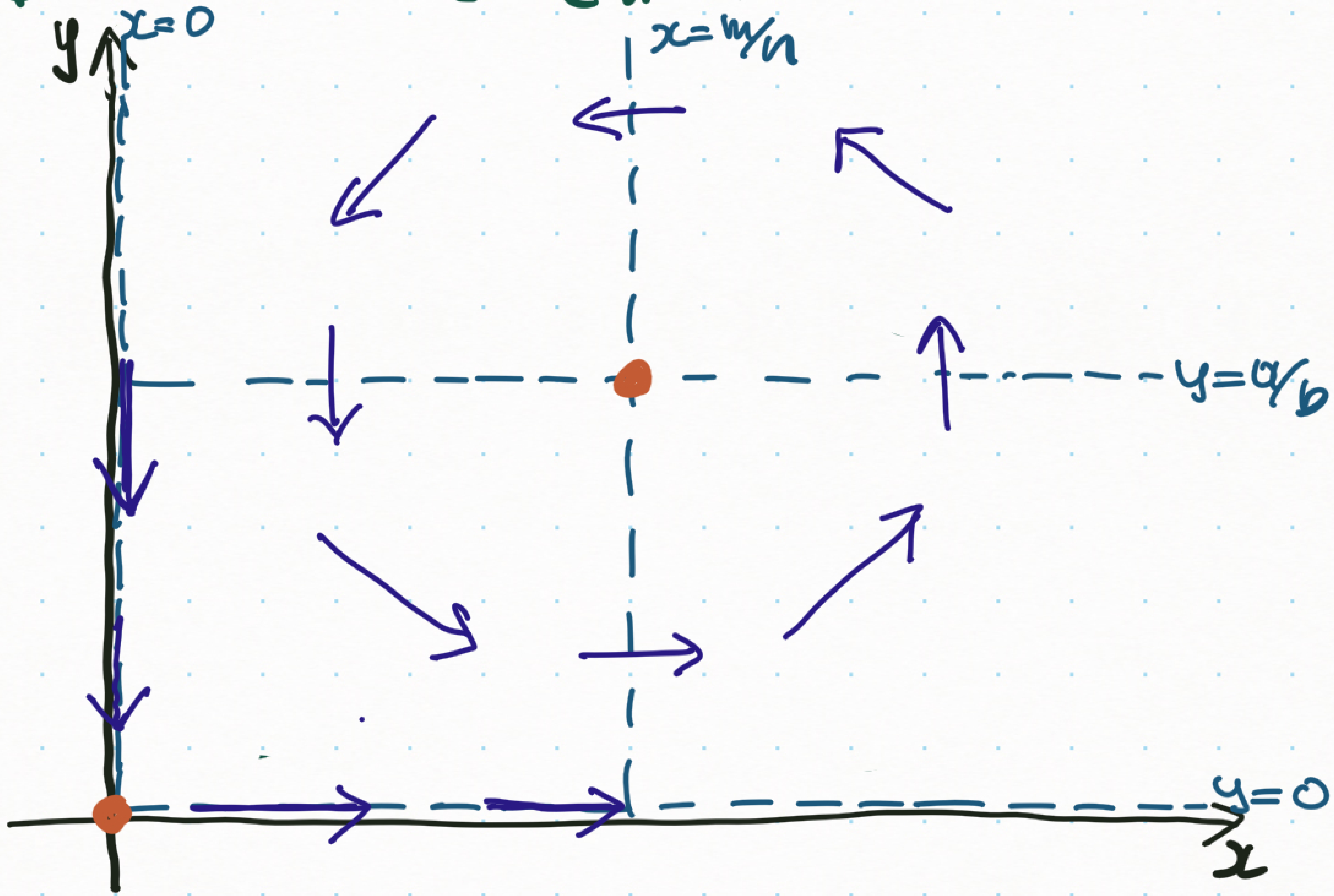

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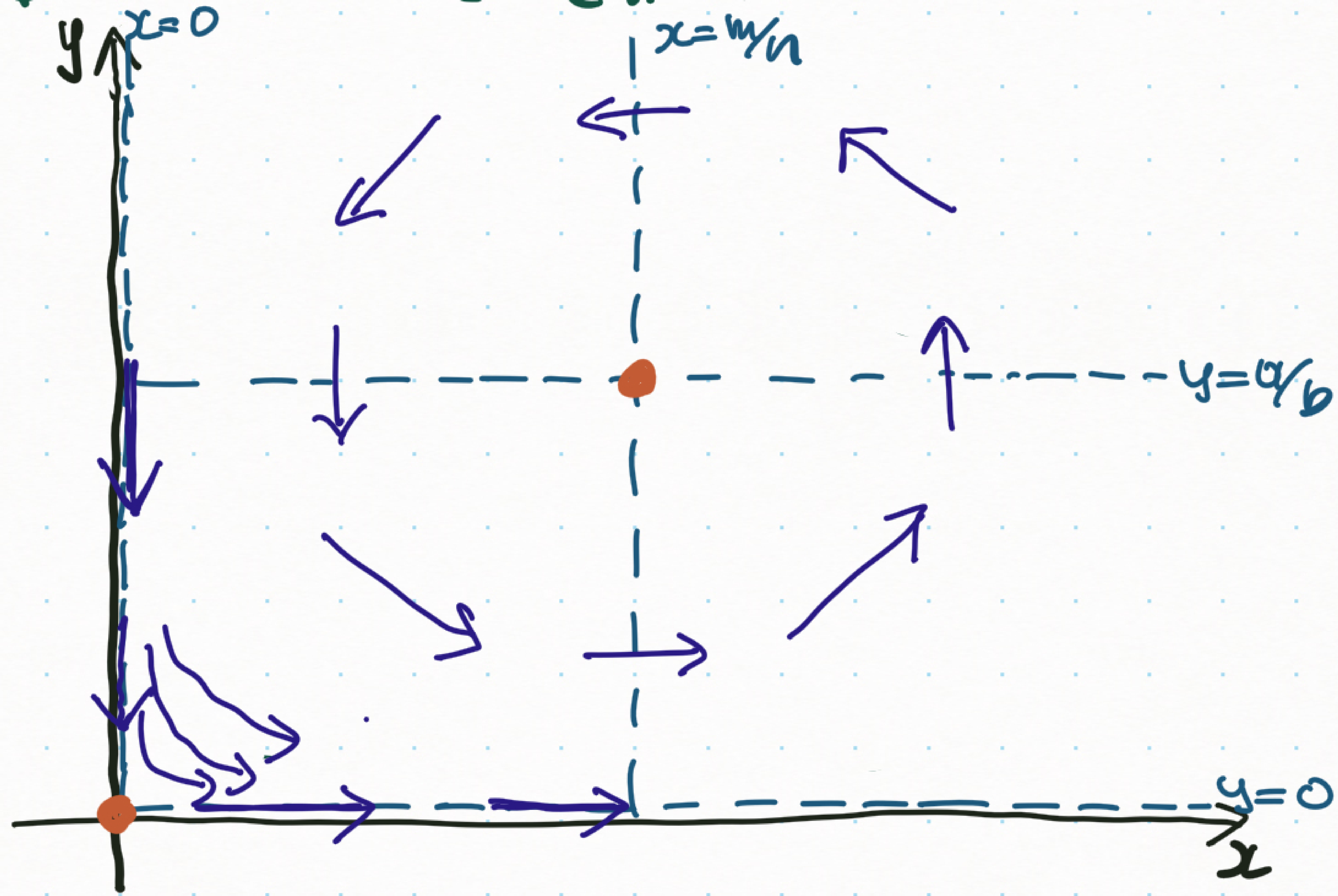

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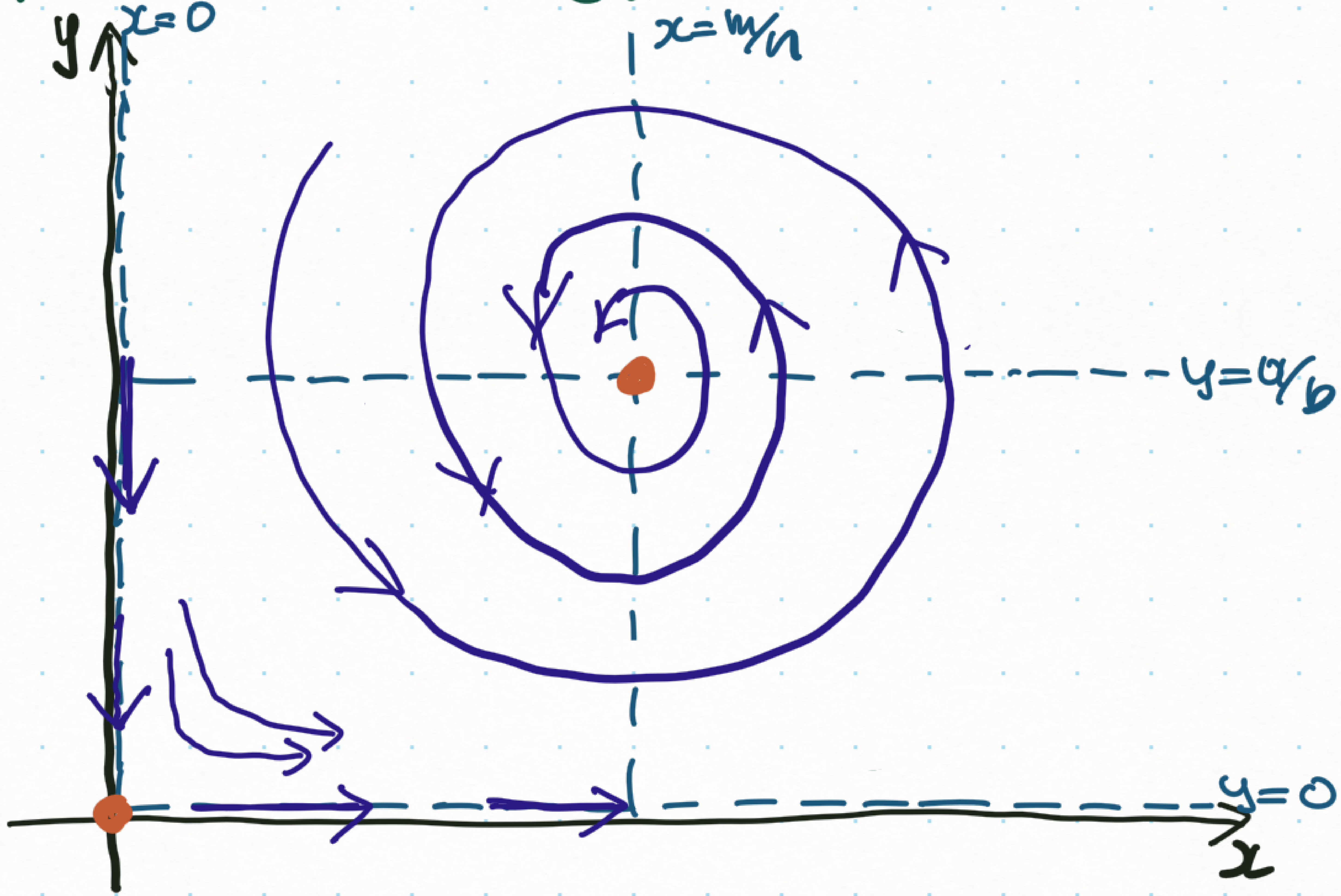
Possible scenarios



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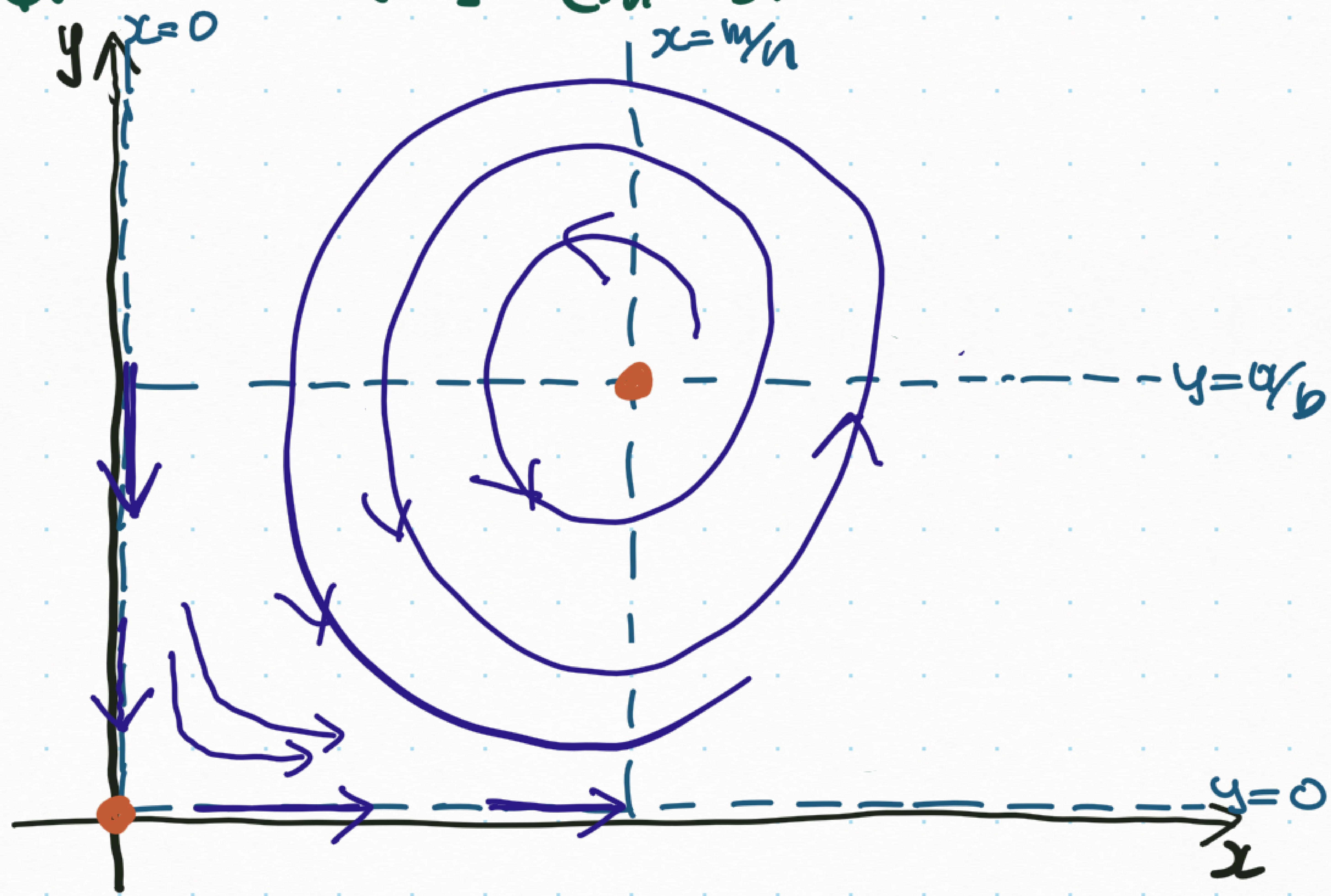
$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} = 0 \iff \begin{array}{l}(x,y) = (0,0), \text{ or} \\ (x,y) = (m/n, a/b)\end{array}$$

Possible scenarios



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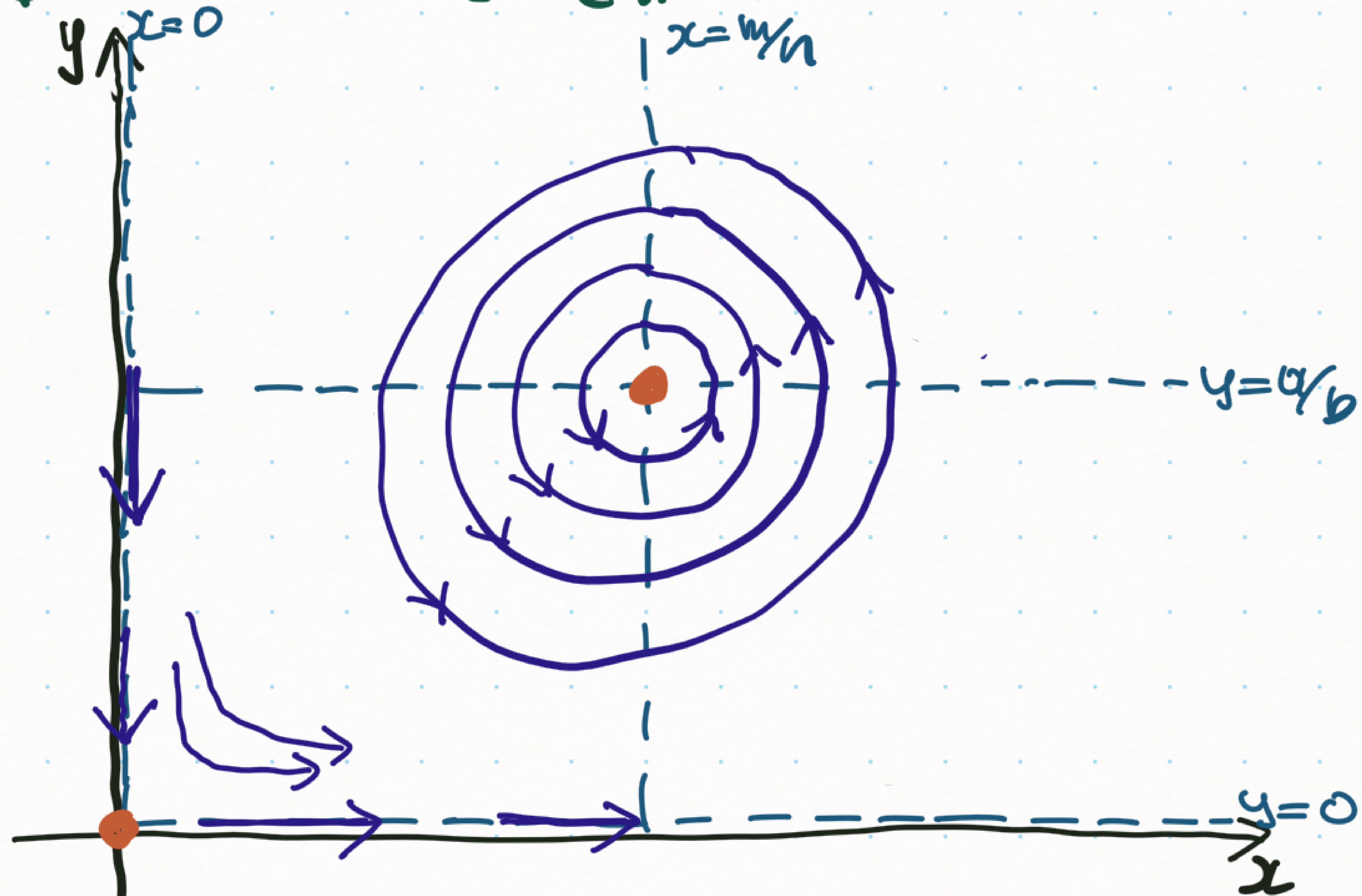

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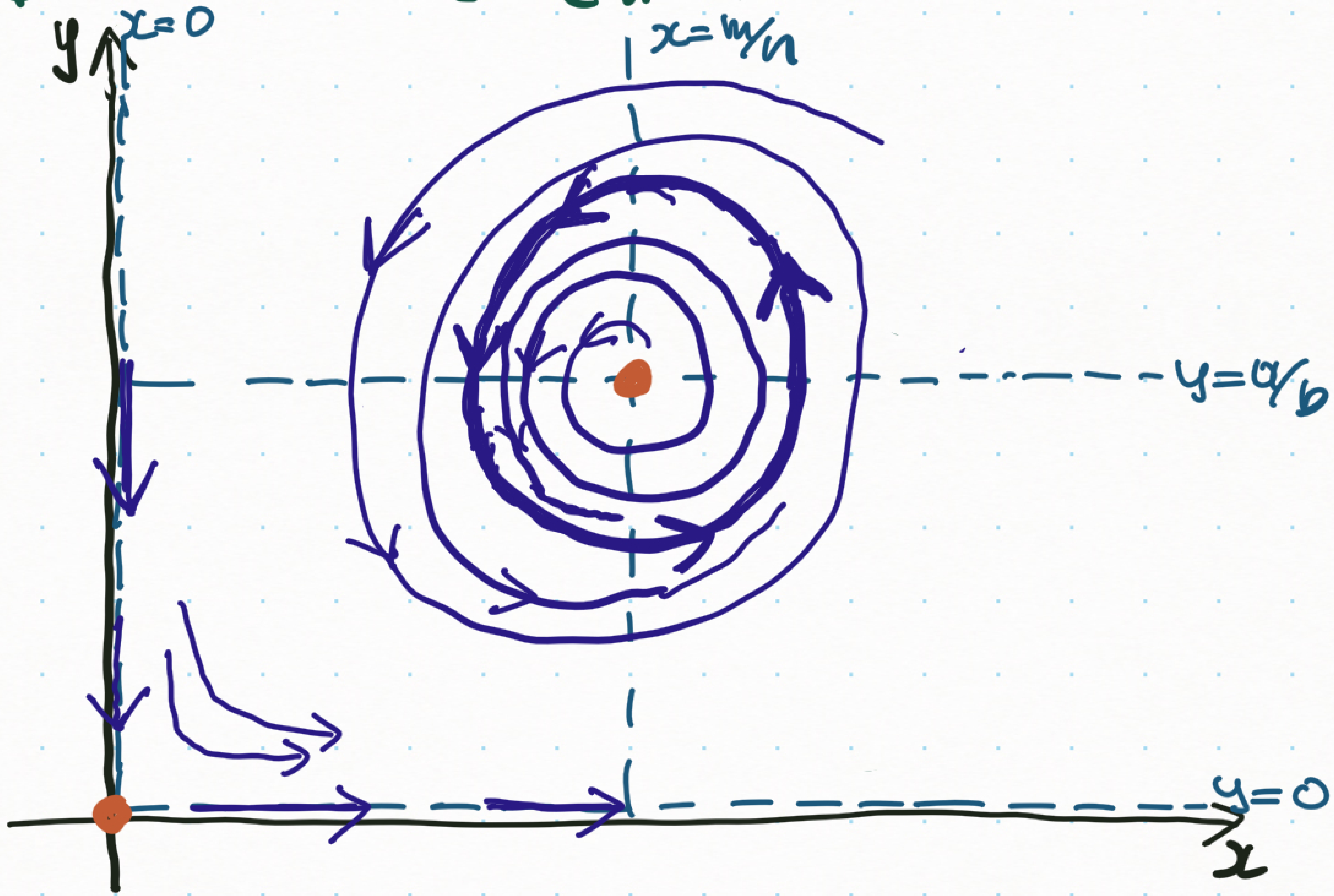

We need to know more about the solutions to say something meaningful about their behavior:

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We need to know more about the solutions to say something meaningful about their behavior:

① Use a version of <u>Euler's method</u> for system of equations to find numerical evidence for behavior of the system (we will talk about this next week).

② Solve the system analytically & then analyze the solutions as $t \uparrow$.

Not always possible, but can be done in our model.

$$\frac{dx}{dt} = (a-by)x \,, \quad \frac{dy}{dt} = (-m+nx)y$$

Chain rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(-m+nx)y}{(a-by)x}$$

By separation of variables,

$$\int \frac{a-by}{y}\,dy = \int \frac{-m+nx}{x}\,dx \,, \quad \text{i.e., } a\ln y - by = -m\ln x + nx + C$$

$$\text{i.e., } e^{a\ln y - by} = e^{-m\ln x + nx + C}$$

$$\text{i.e., } y^a e^{-by} = x^{-m} e^{nx} e^{C}$$

$$\text{i.e., } \left(\frac{y^a}{e^{by}}\right)\left(\frac{x^m}{e^{nx}}\right) = K$$

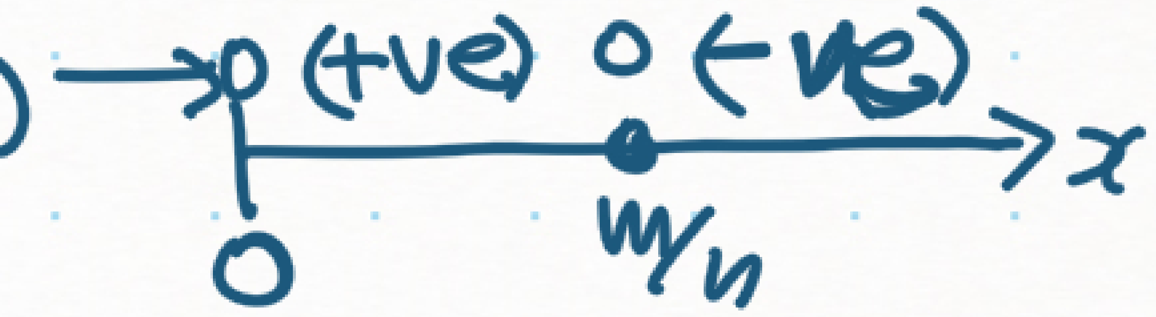
We need to understand $\underline{f(x) = x^m / e^{nx}}$ first? (Same as y^a / e^{by})

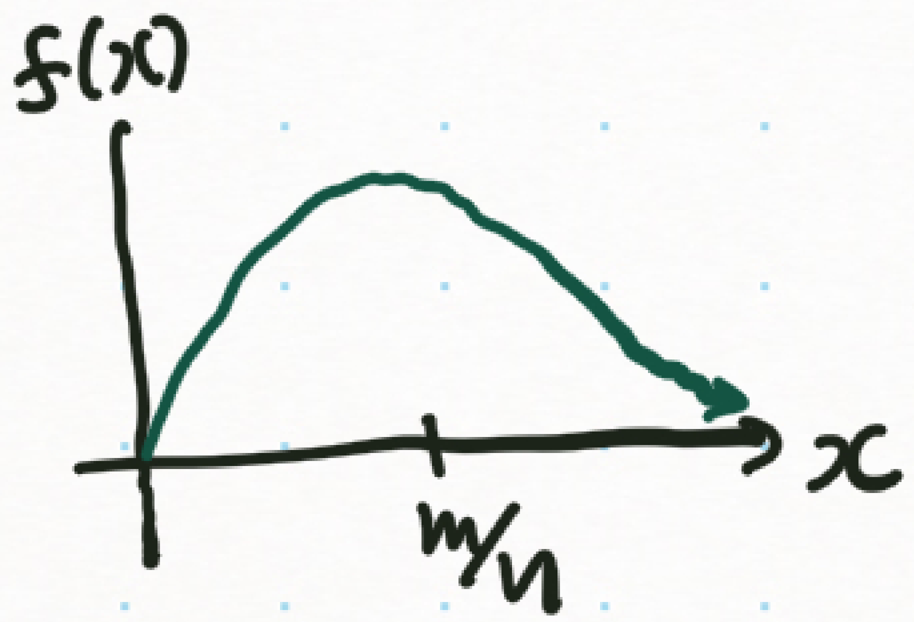
$f(x) = x^m / e^{nx}$; $\lim_{x \to \infty} f(x) = 0$

$$f'(x) = \frac{d}{dx}\left(e^{m \ln x - nx}\right) = \left(\frac{m}{x} - n\right) e^{m \ln x - nx} = \left(\frac{m}{x} - n\right) x^m / e^{nx}$$

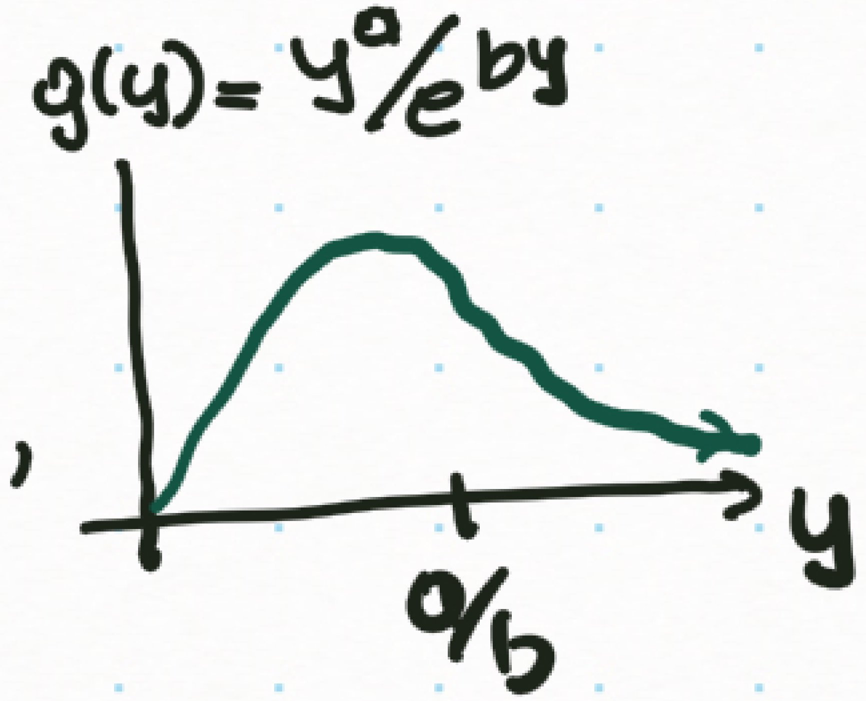
$f'(x) = 0$ when $x = 0$, $x = m/n$.

Sign of $f'(x)$ →





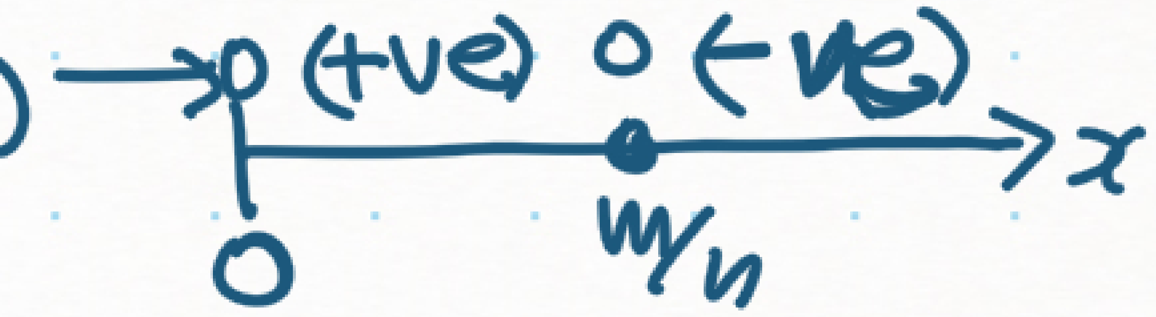
similarly,

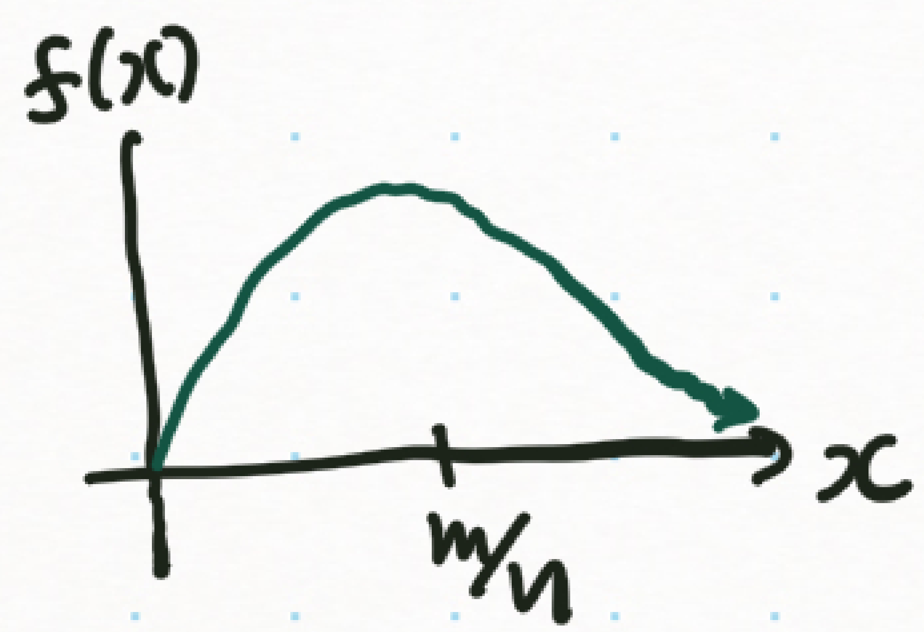


$f(x) = x^m/e^{nx}$; $\lim_{x\to\infty} f(x) = 0$

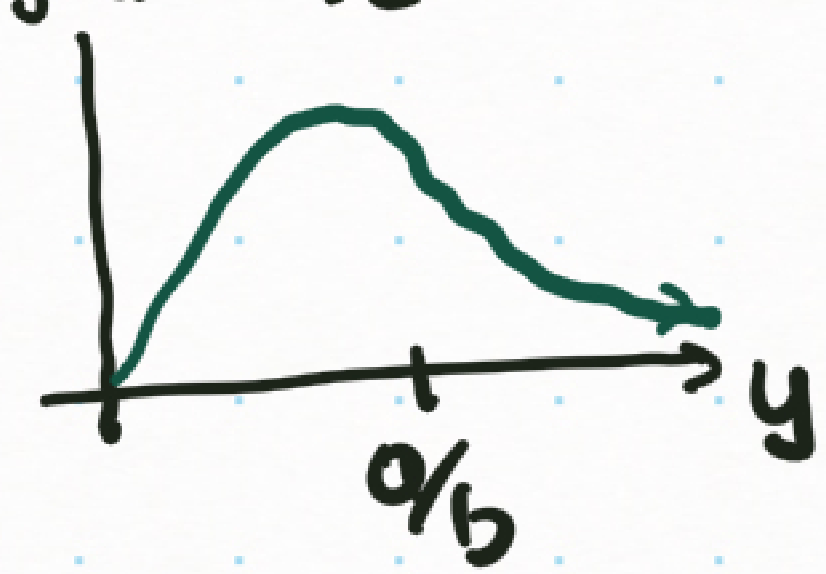
$f'(x) = \frac{d}{dx}\left(e^{m\ln x - nx}\right) = \left(\frac{m}{x} - n\right)e^{m\ln x - nx} = \left(\frac{m}{x} - n\right)x^m/e^{nx}$

$f'(x) = 0$ when $x = 0$, $x = m/n$. Sign of $f'(x) \longrightarrow$





similarly, $g(y) = y^a/e^{by}$



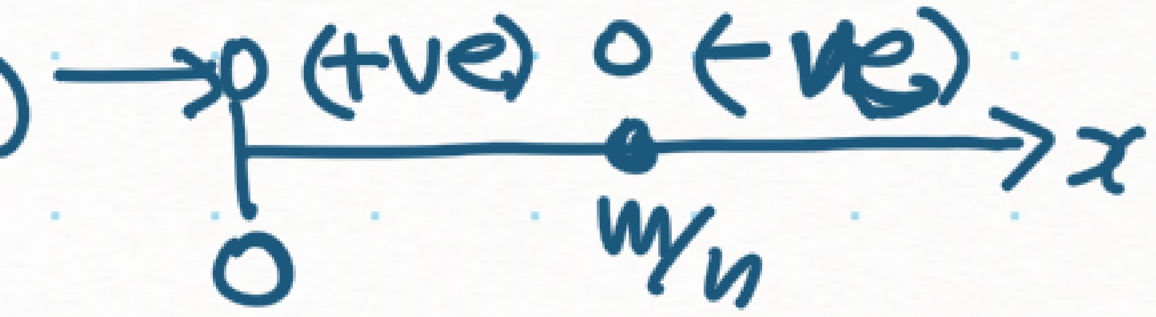
Let $M_x = \max f(x) = f(m/n) = (m/n)^m/e^m$; $M_y = \max g(y) = \cdots = (a/b)^a/e^a$

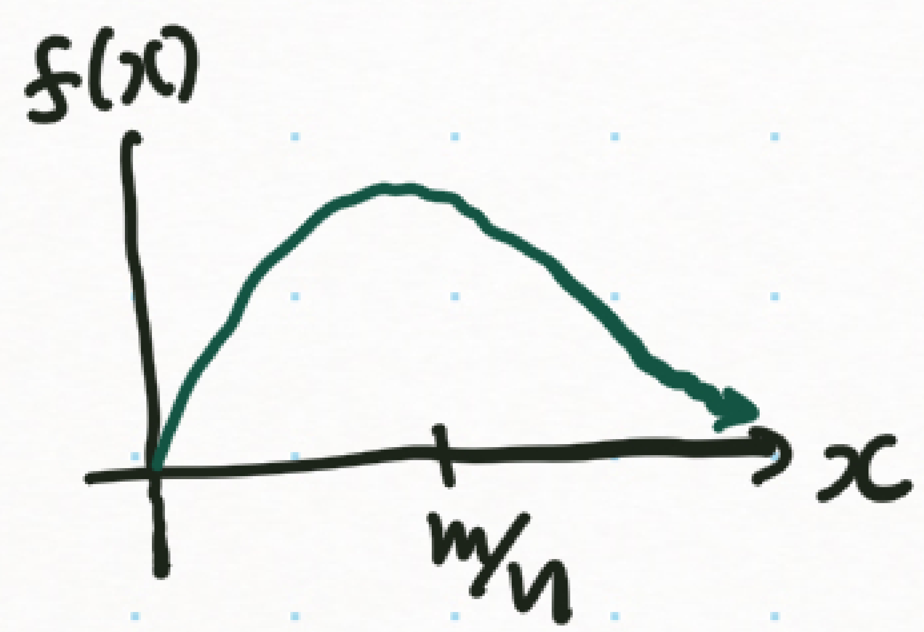
$f(x) = x^m/e^{nx}$, $\lim_{x \to \infty} f(x) = 0$

$$f'(x) = \frac{d}{dx}\left(e^{m\ln x - nx}\right) = \left(\frac{m}{x} - n\right) e^{m \ln x - nx} = \left(\frac{m}{x} - n\right) x^m/e^{nx}$$

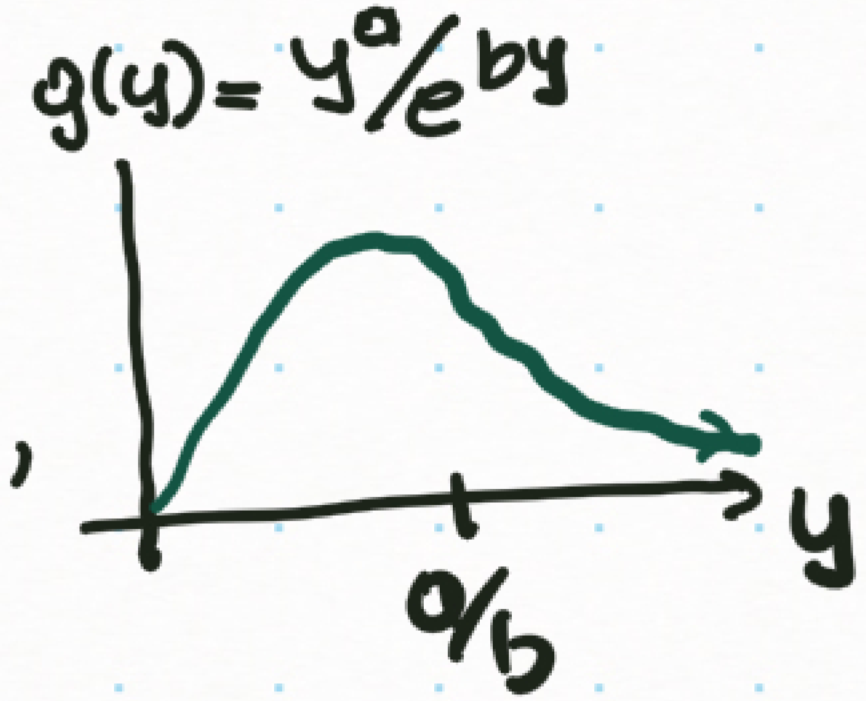
$f'(x) = 0$ when $x = 0$, $x = m/n$.

Sign of $f'(x) \to$





similarly,



And, we know

$$\boxed{f(x)\,g(y) = \left(\frac{x^m}{e^{nx}}\right)\left(\frac{y^a}{e^{nx}}\right) = K}$$

Let $M_x = \max f(x) = f(m/n) = (m/n)^m/e^m$; $M_y = \max g(y) = \cdots = (a/b)^a/e^a$

If $K > M_x M_y$ then no solution

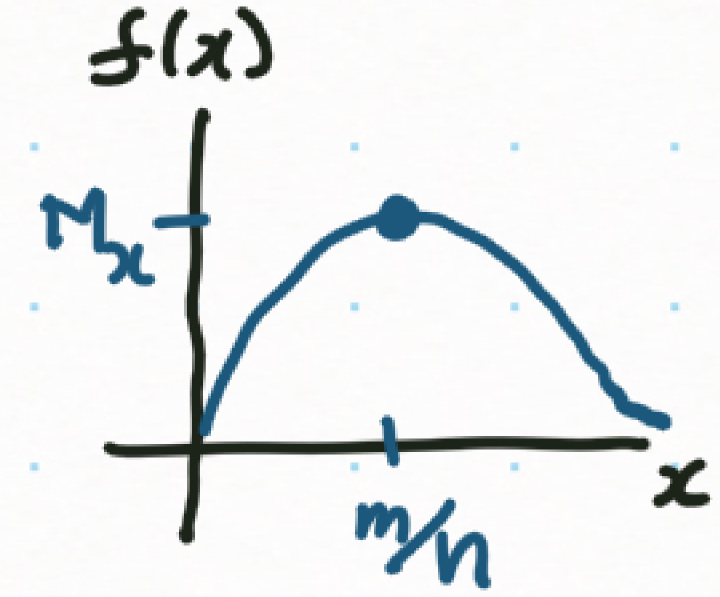
If $K = M_x M_y$ then $x = \frac{m}{n}$, $y = \frac{a}{b}$ is the soln., an equilibrium soln.

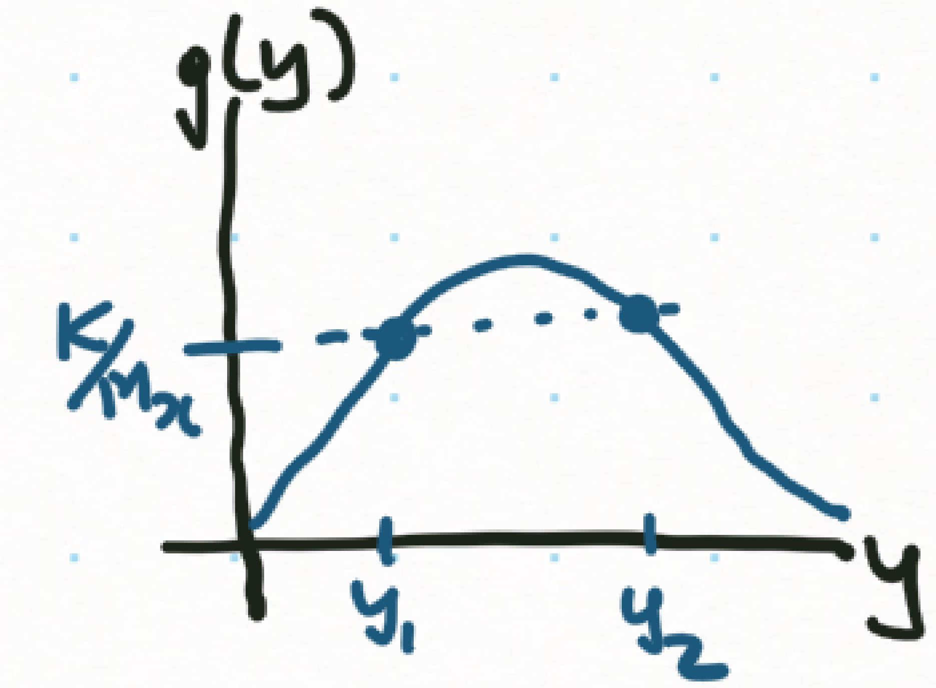
If $K < M_x M_y$ then many solutions ....

$f(x)\,g(y) = K$ & $K < M_x M_y$

max $f(x)$
max $g(y)$

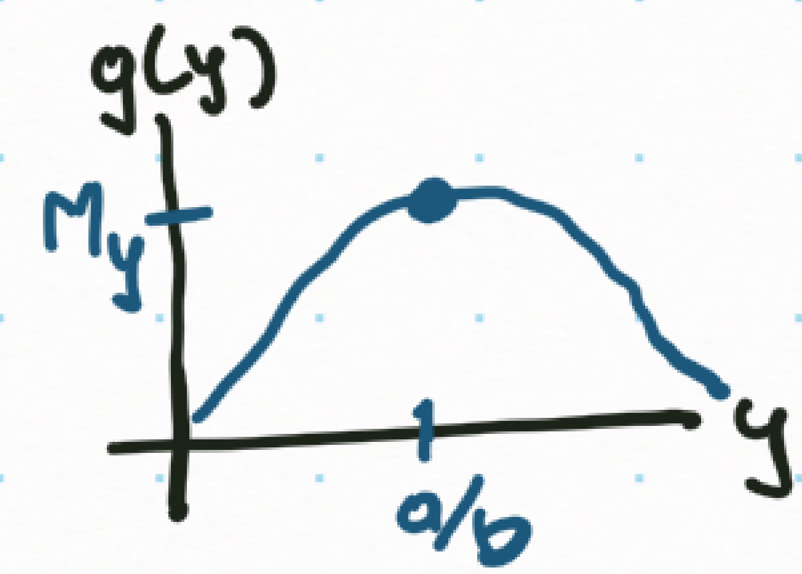
- $f(x) = M_x$, $g(y) = K/M_x$

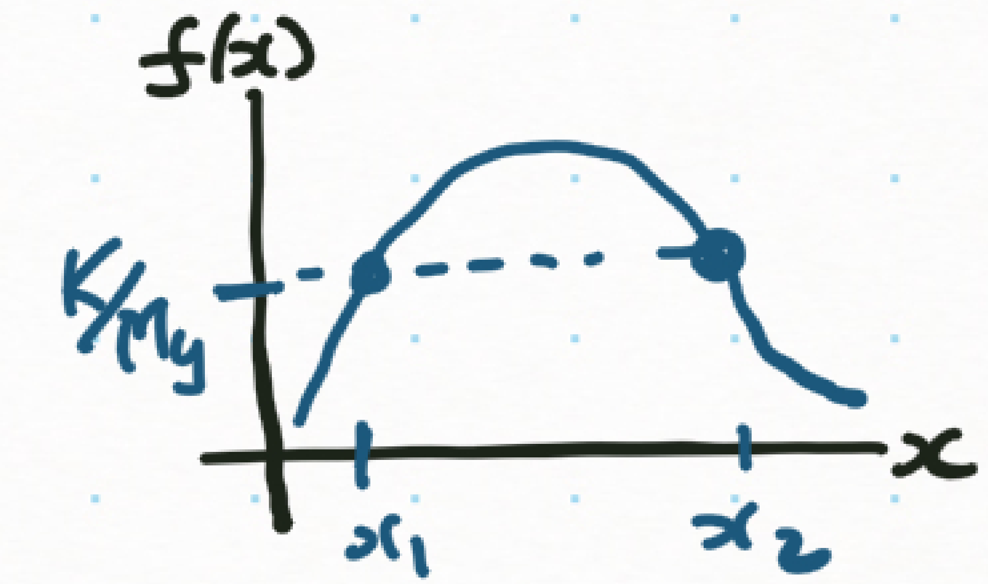




$x = m/n$, two y values

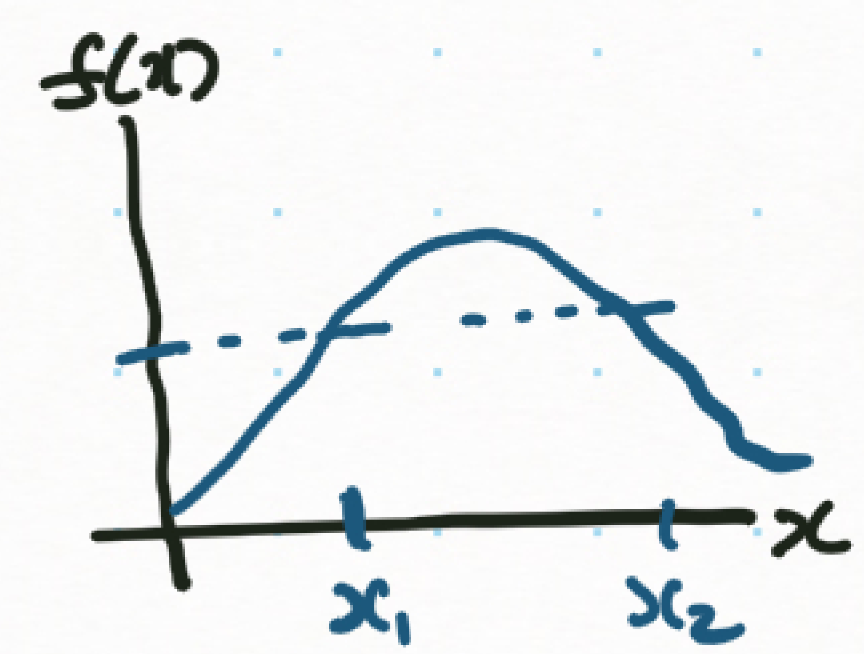
- $g(y) = M_y$, $f(x) = K/M_y$

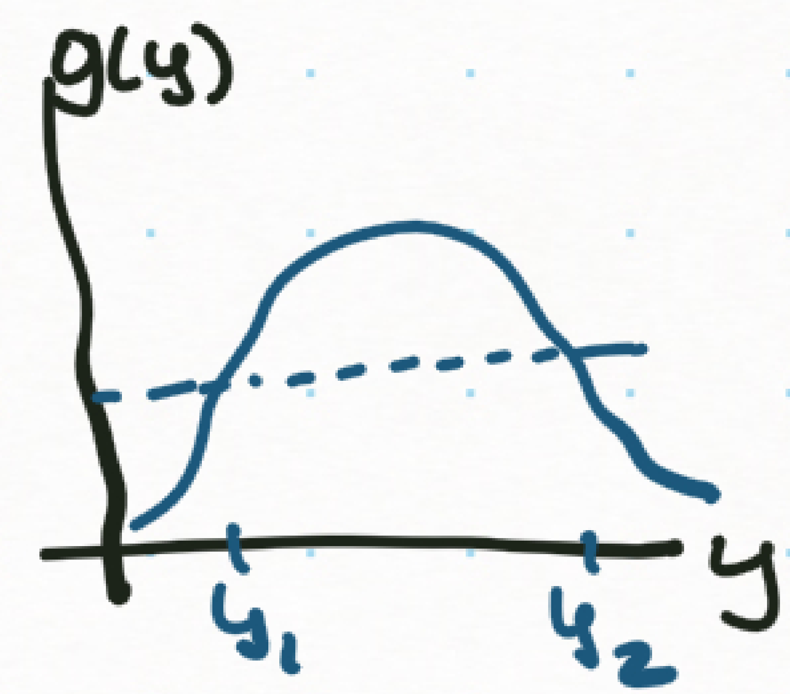




$y = a/b$, two x values

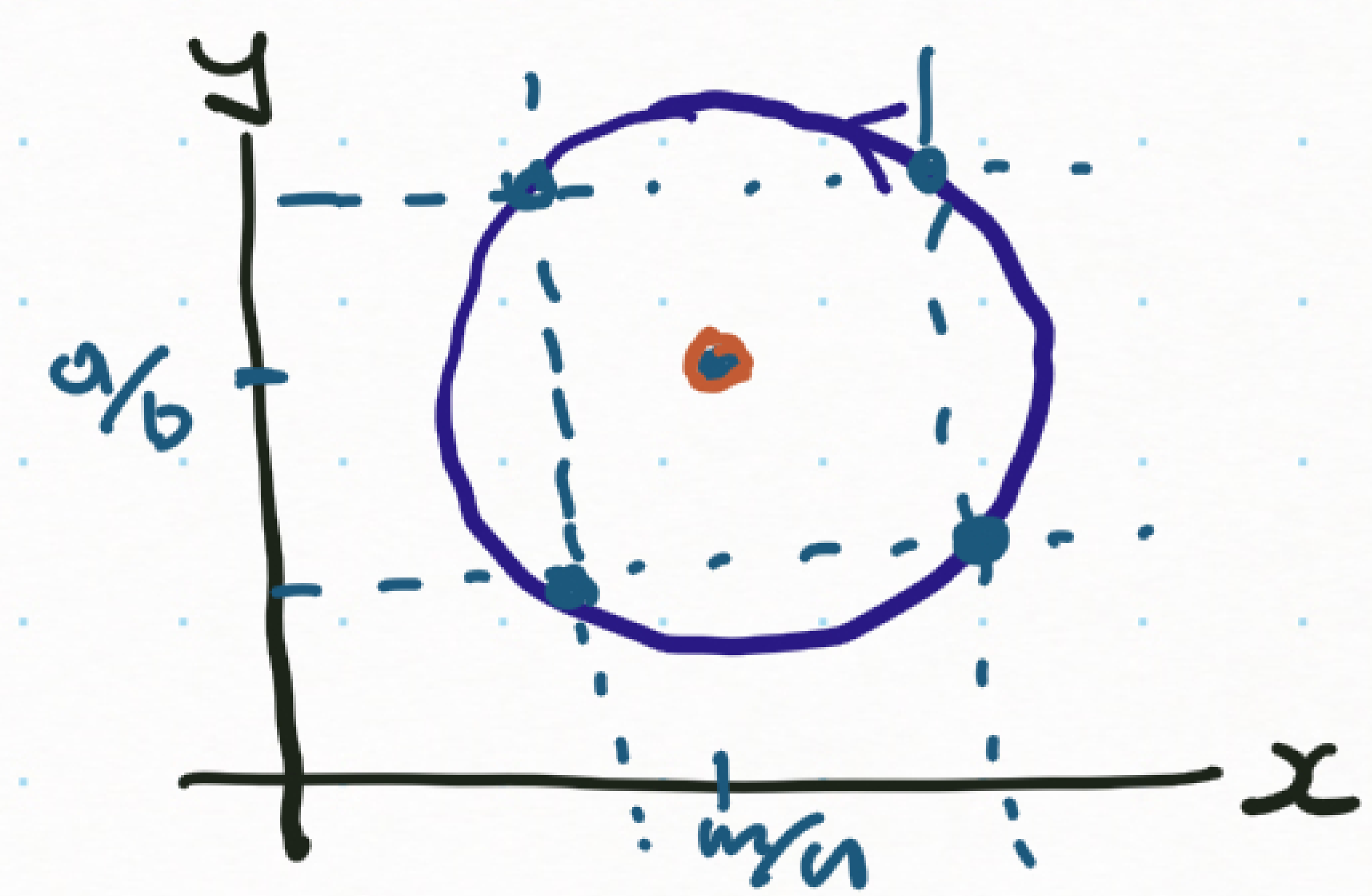
- In between, $f(x)$ & $g(y)$ balance so that their product is constant K

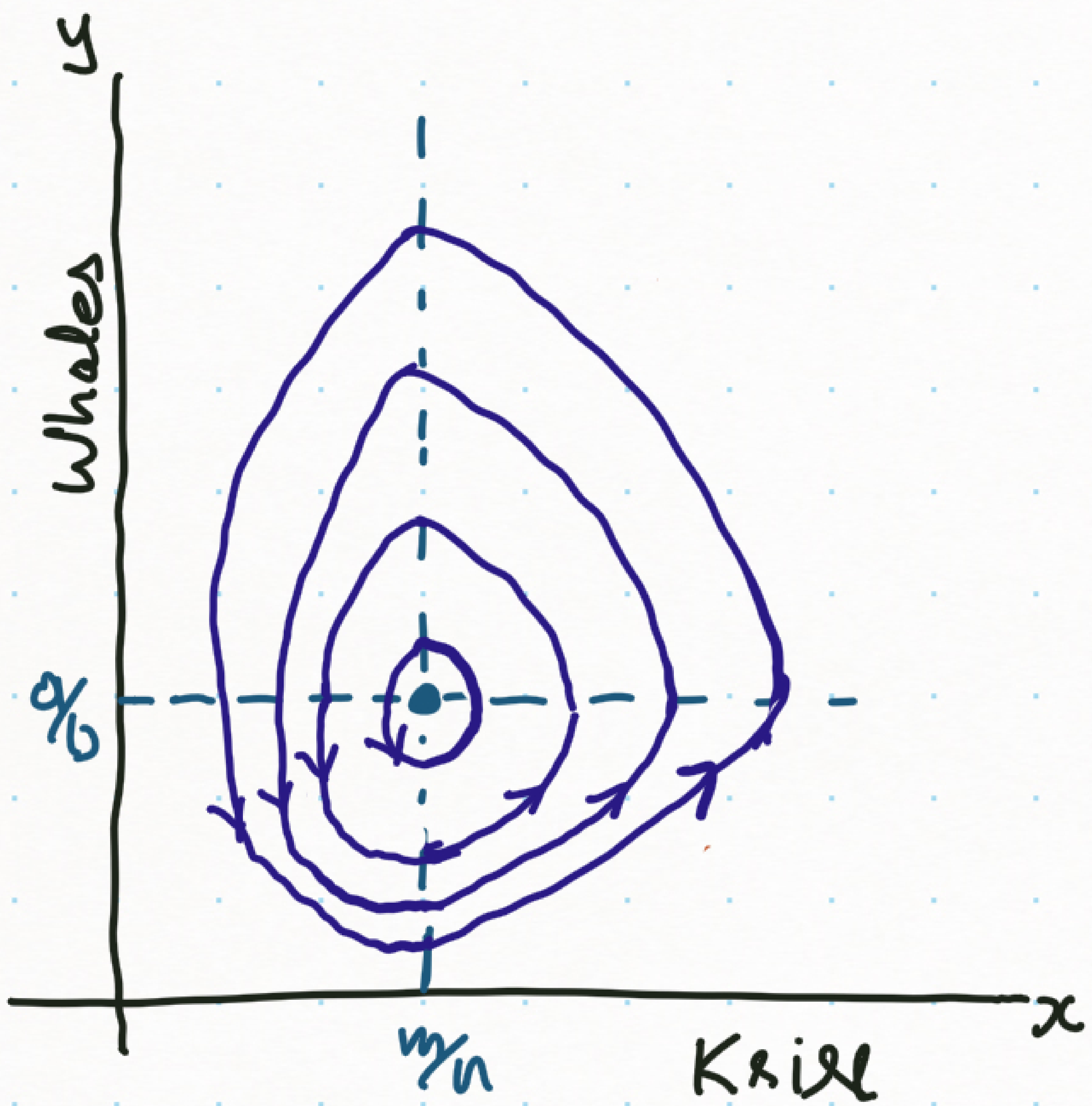


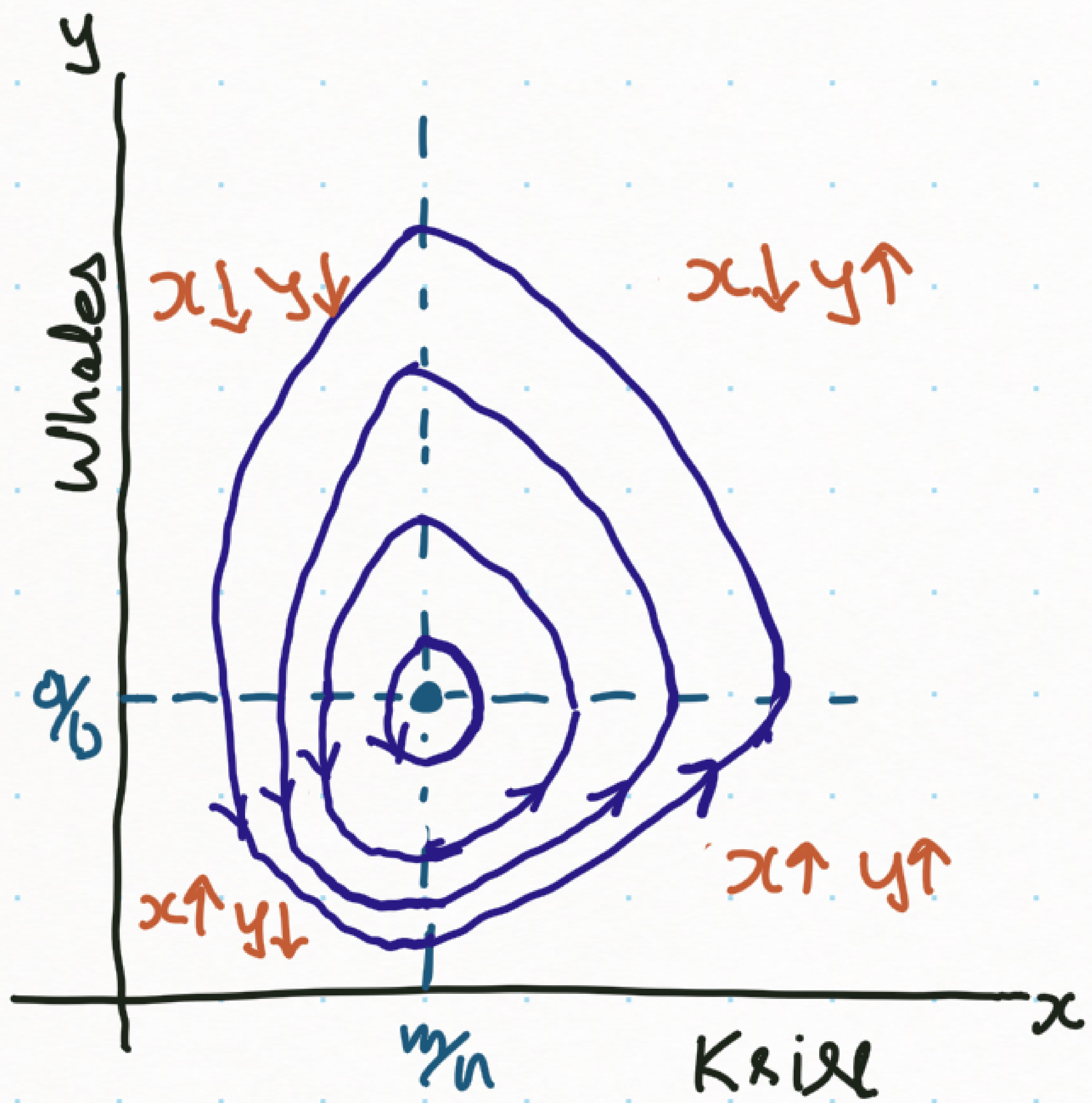


Two x values
Two y values
Four solutions:
(x_1, y_1), (x_1, y_2)
(x_2, y_1), (x_2, y_2)

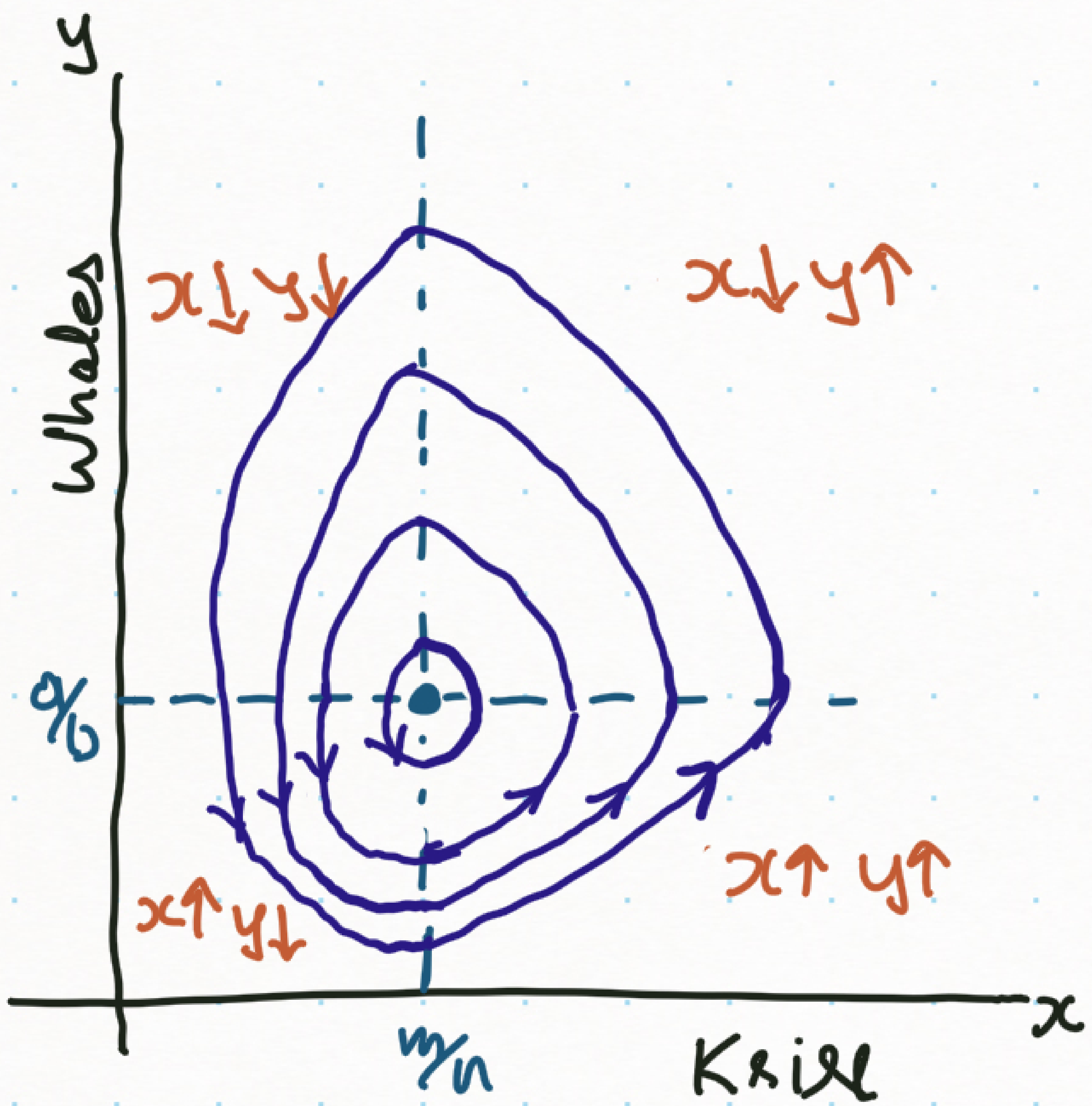
Overall solution curve in the xy-plane (for each $K < M_x M_y$) looks like

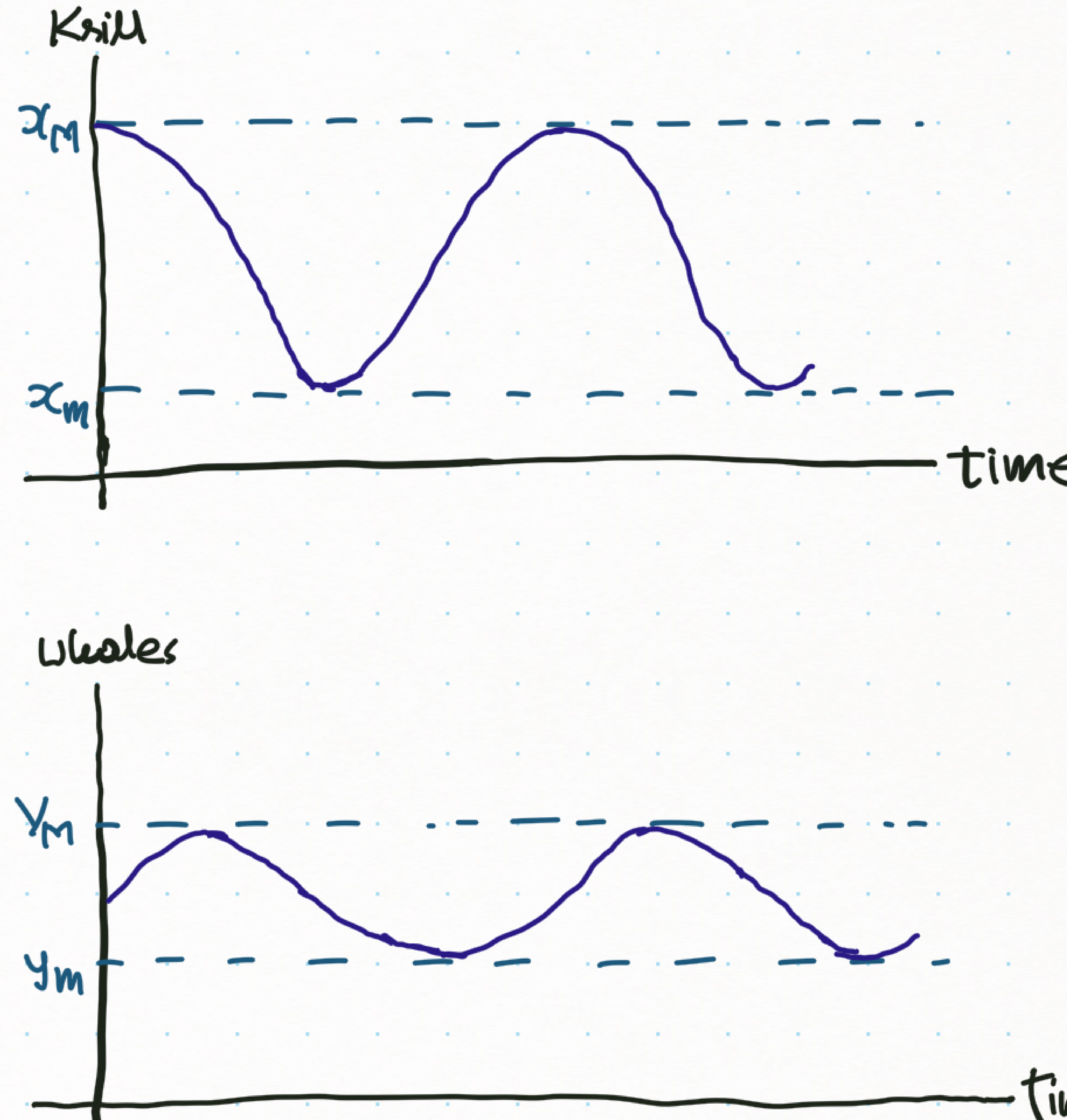


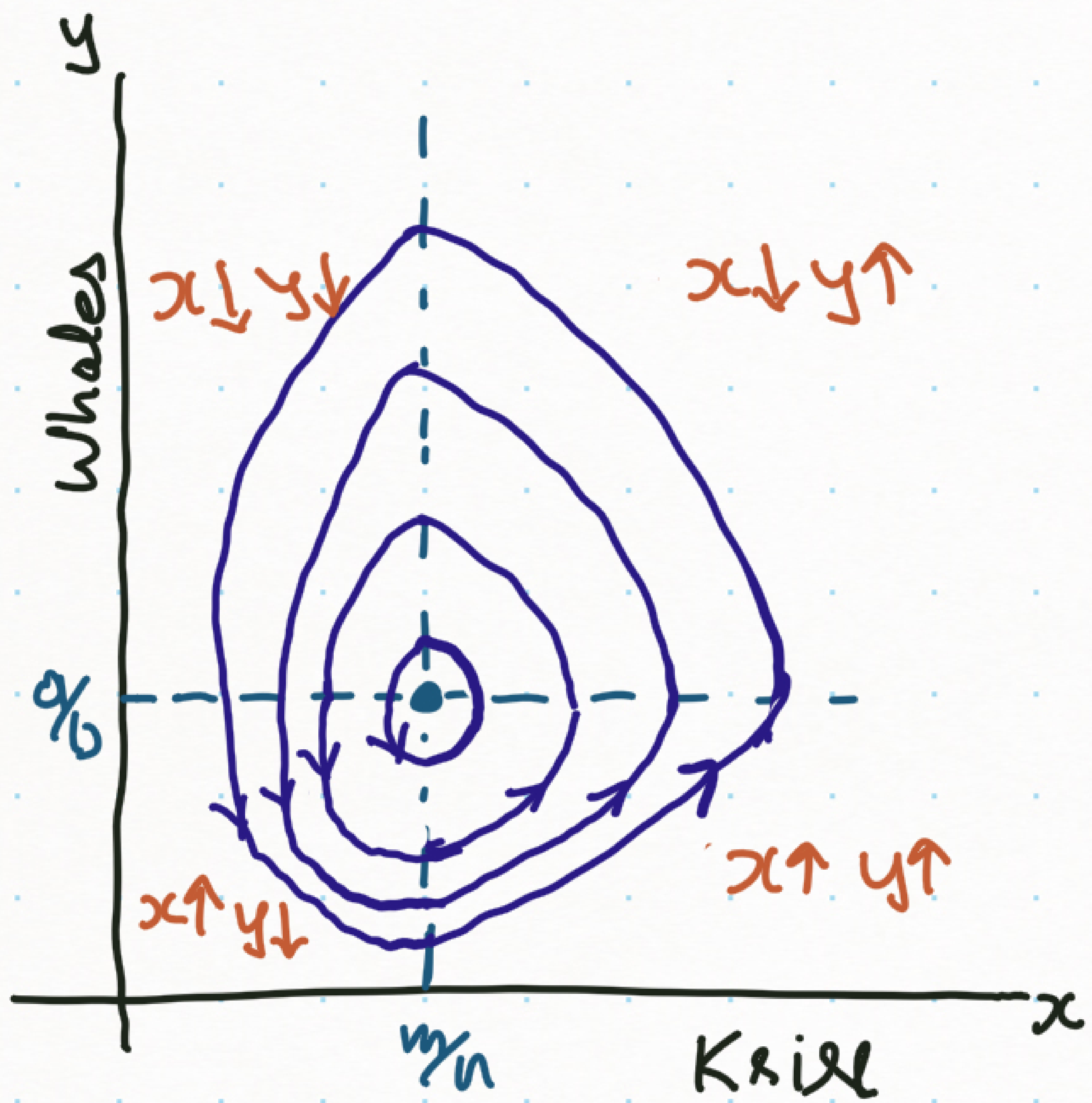

y
Whales
a/b
m/n
Krill
x

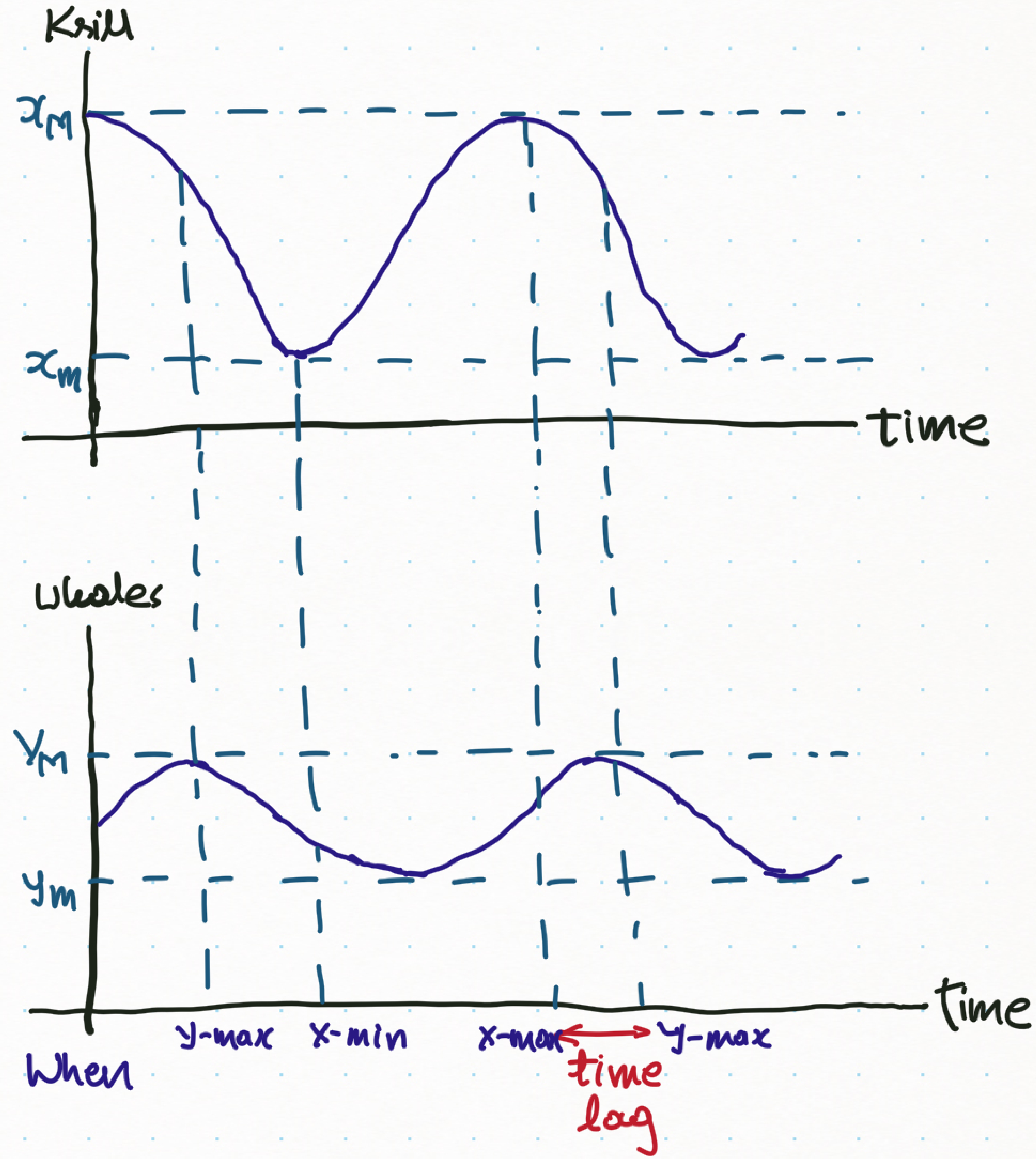


y
Whales
x↓ y↓
x↓ y↑
x↑ y↓
x↑ y↑
x
Krill


y
Whales
x↓ y↓
x↓ y↑
a/b
x↑ y↓
x↑ y↑
m/n
Krill
x


Krill
x_M
x_m
time
Whales
y_M
y_m
time





Predator lags behind prey in a cyclic fashion.

Effect of "Harvesting" Krill on the system

Average populations are $\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt$, $\bar{y} = \frac{1}{T}\int_0^T y(t)\,dt$

where T is the "period" of one time-cycle ~~of~~ population repetition.

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where T is the "period" of one time-cycle of population repetition.

Note $\frac{dx}{dt} = (a-by)x \Rightarrow \int_0^T \frac{dx}{x} = \int_0^T (a-by)\,dt$

i.e., $\ln x(T) - \ln x(0) = a(T-0) - b\int_0^T y\,dt$

(since T is the period, $x(T) = x(0)$, so)

i.e., $0 = aT - b\int_0^T y\,dt$

i.e. $\int_0^T y(t)\,dt = \frac{aT}{b}$, i.e, $\bar{y} = \frac{1}{T}\frac{aT}{b} = \frac{a}{b}$

$\therefore \bar{y} = \frac{a}{b}$ and similarly, $\bar{x} = \frac{m}{n}$

That is the average populations are the same as the equilibrium values.

Effect of "Harvesting" Krill on the system

Average populations are $\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt$, $\bar{y} = \frac{1}{T}\int_0^T y(t)\,dt$

$= \frac{m}{n}$ $\qquad = a/b$

Let's assume that harvesting krill affects both populations negatively

$\frac{dx}{dt} = (a-by)x - rx$, $\frac{dy}{dt} = (-m+nx)y - ry$; for some constant harvest rate r

i.e., $\frac{dx}{dt} = ((a-r)-by)x$, $\frac{dy}{dt} = (-(m+r)+n)y$

we get the same kind of behavior, but with
$a-r$ replacing a
$m+r$ replacing m.

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Periodic populations with $\bar{x} = \frac{m+r}{n}$, $\bar{y} = \frac{a-r}{b}$

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Periodic populations with $\bar{x} = \frac{m+r}{n}$, $\bar{y} = \frac{a-r}{b}$

Average Krill pop. is higher & whale pop. is lower when Krill harvested.

Volterra's Principle. These predator-prey models are Lotka-Volterra models.

Economics of an Arms Race (an outline)

[Read Examples 1 & 2 from Section 12.4 for non-ecological examples of ODE dynamical system models]

Two countries in an arms race.
Will spending on armaments lead to uncontrolled spending?
Will one country spend much more than the other? etc.

Let x = annual defense expenditure of country 1
y = ——————— " ——————— 2

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$$\frac{dx}{dt} = \underbrace{-ax}_{\text{economic constraint on spending}}, \text{ for } a>0 \text{ if there is peace with country 2}$$

But if that's not the case then spending increases in proportion to Country 2 defense spending: $\frac{dx}{dt} = -ax + \underbrace{b}_{\text{martial rivalry factor}}y$

Finally, $\frac{dx}{dt} = -ax+by+c$, where c indicates some spending no matter what.

Economics of an Arms Race

x = defense expenditure of country-1

y = ———————— " ———— -2

$$\frac{dx}{dt} = -ax + by + c \;, \quad \frac{dy}{dt} = mx - ny + p \;,$$ where $a, b, c, m, n, p > 0$ constants

Economics of an Arms Race

$x =$ defense expenditure of country-1
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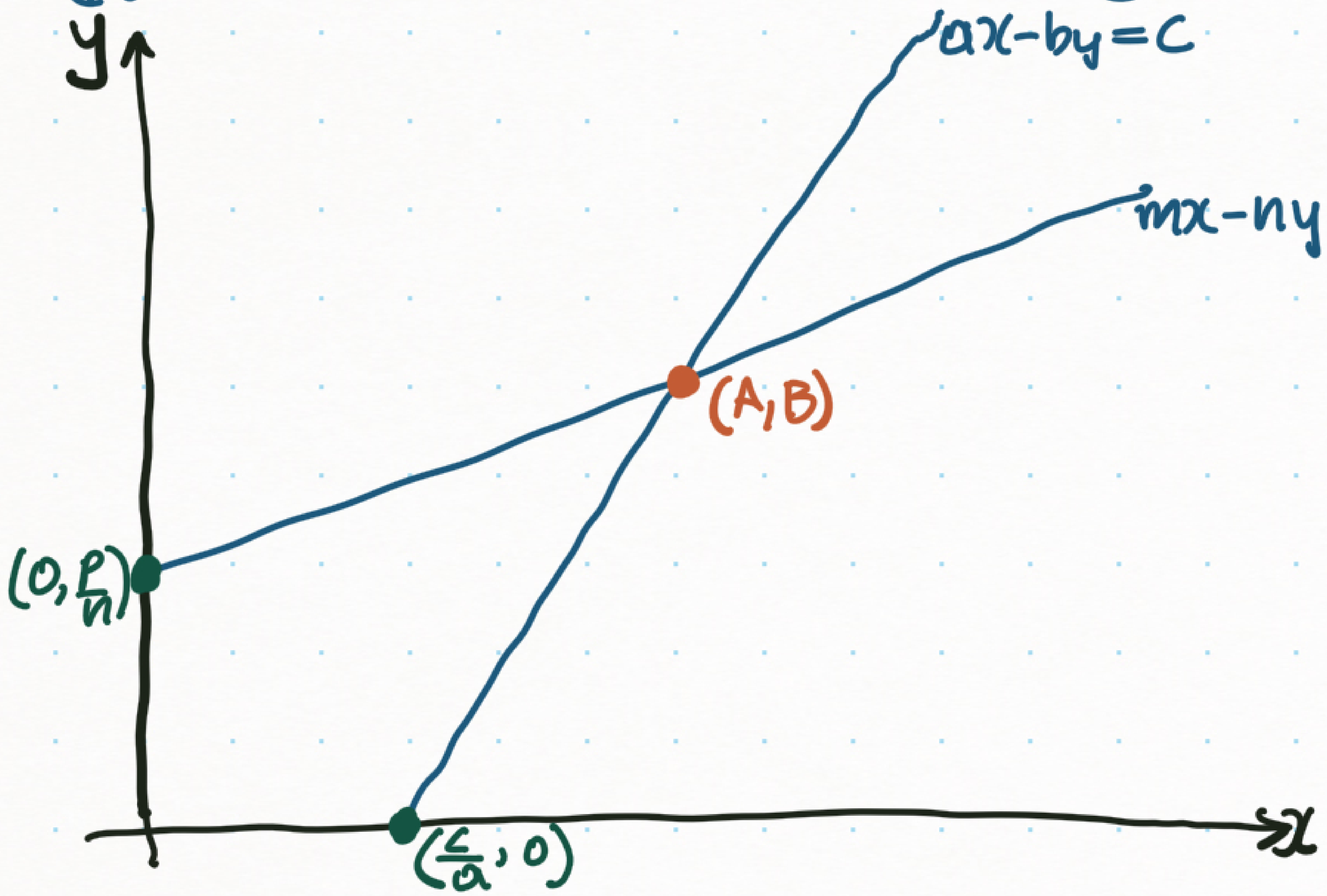
$$\frac{dx}{dt} = 0 \Leftrightarrow ax - by = c \;; \quad \frac{dy}{dt} = 0 \Leftrightarrow mx - ny = -p$$



$(A,B) = \left(\frac{bp+cn}{an-bm}, \frac{ap+cm}{an-bm}\right)$ equil. point.

Assuming $an-bm>0$, we get the phase plane on the left.

Economics of an Arms Race

x = defense expenditure of country-1
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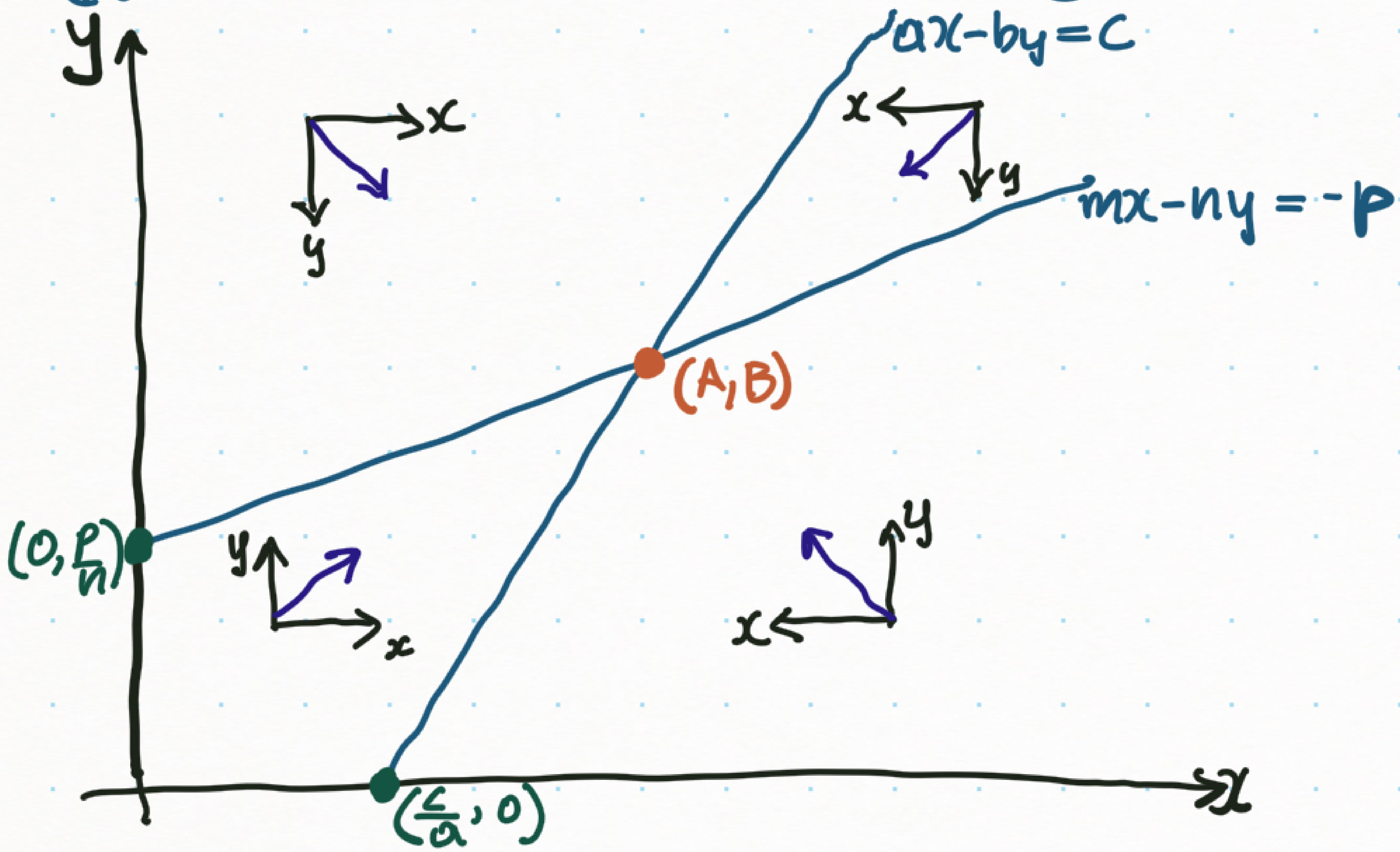

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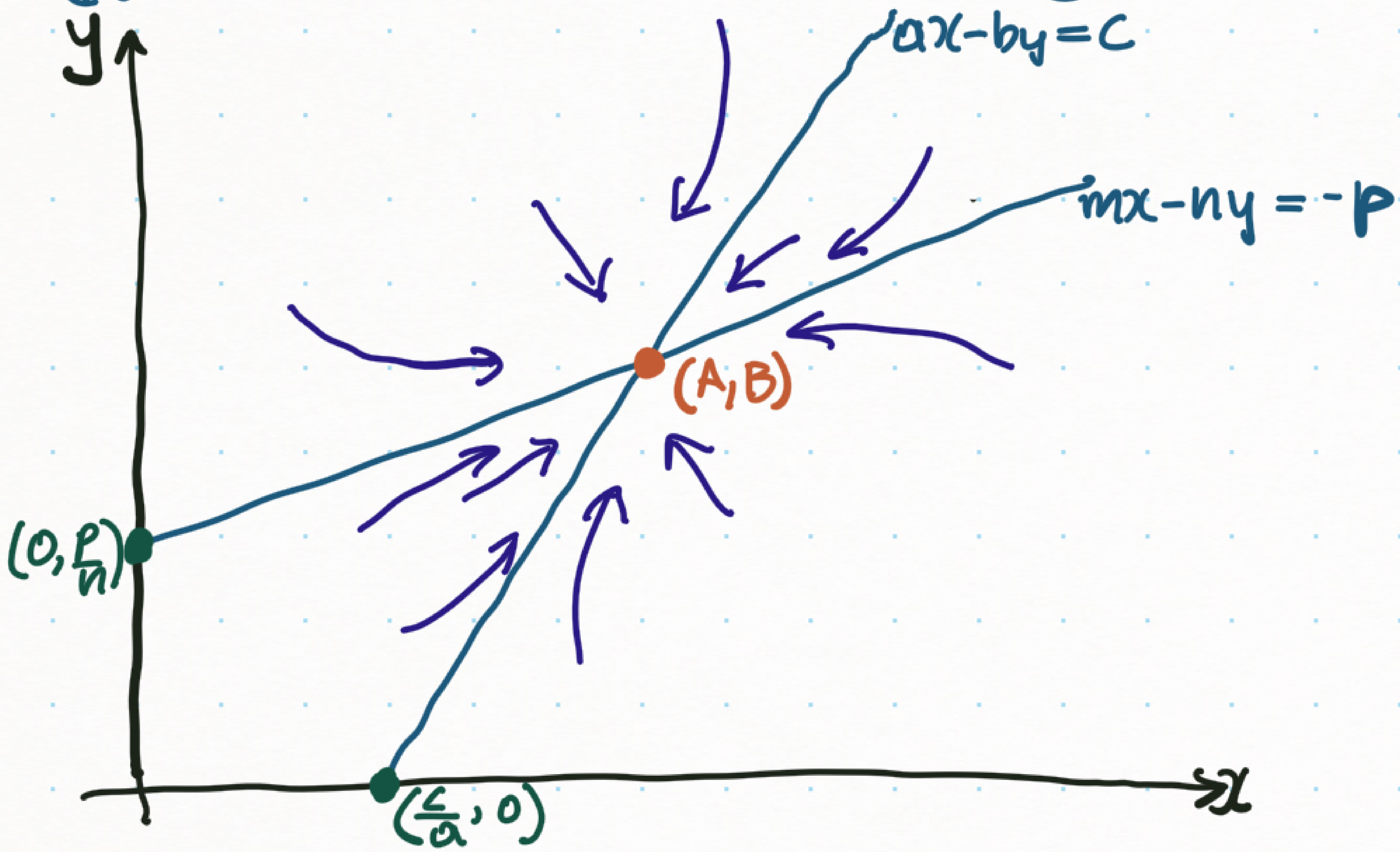

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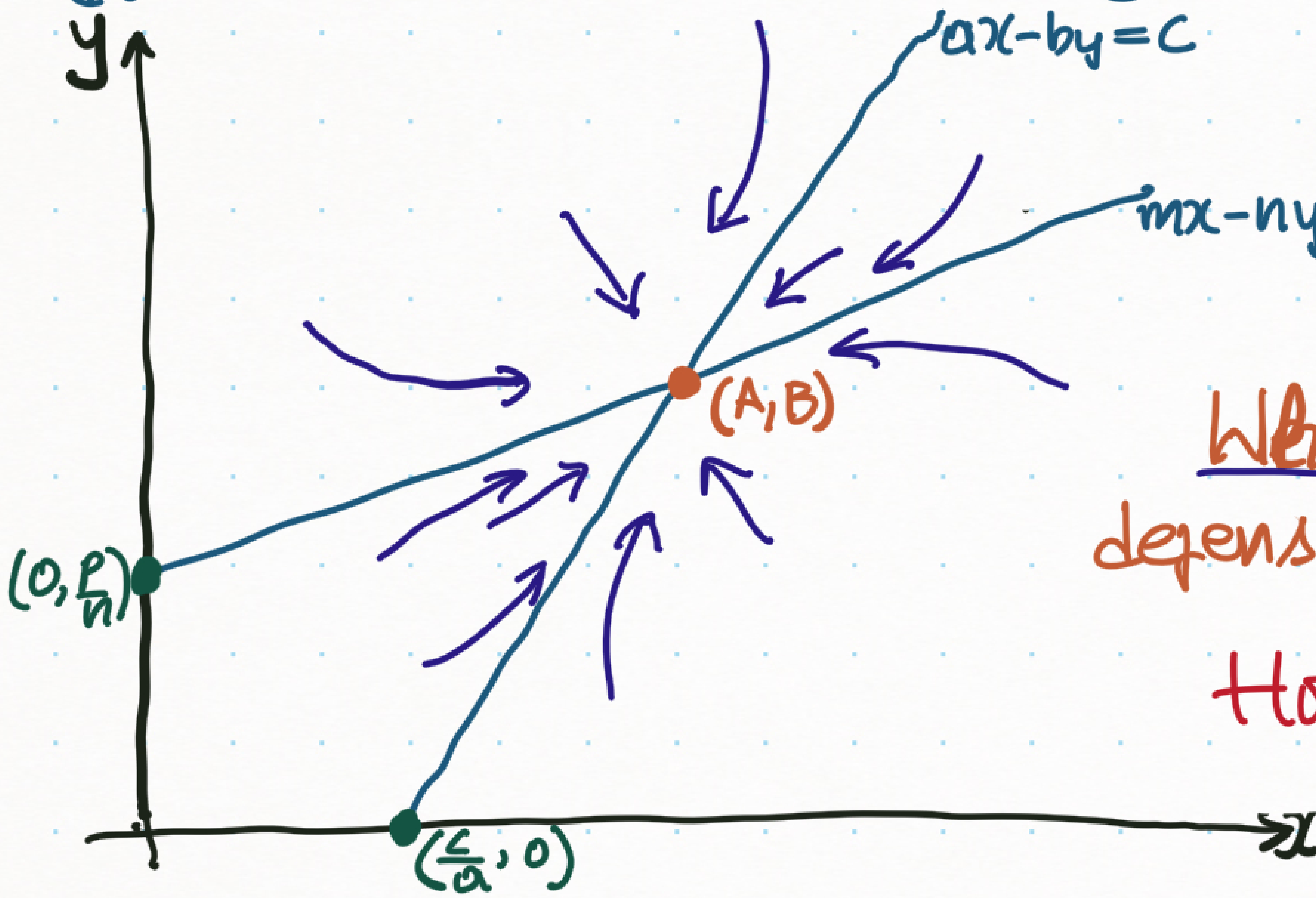
$\frac{dx}{dt} = 0 \Leftrightarrow ax - by = c$; $\frac{dy}{dt} = 0 \Leftrightarrow mx - ny = -p$



$(A, B) = \left(\frac{bp + cn}{an - bm}, \frac{ap + cm}{an - bm}\right)$ equil. point.

Assuming $an - bm > 0$, we get the phase plane on the left.

When $an > bm$, our model indicates defense spending will stabilize at $x = A, y = B$.

How would you interpret $an > bm$?

Economics of an Arms Race

$x =$ defense expenditure of country-1
$y =$ —————— " ———— -2

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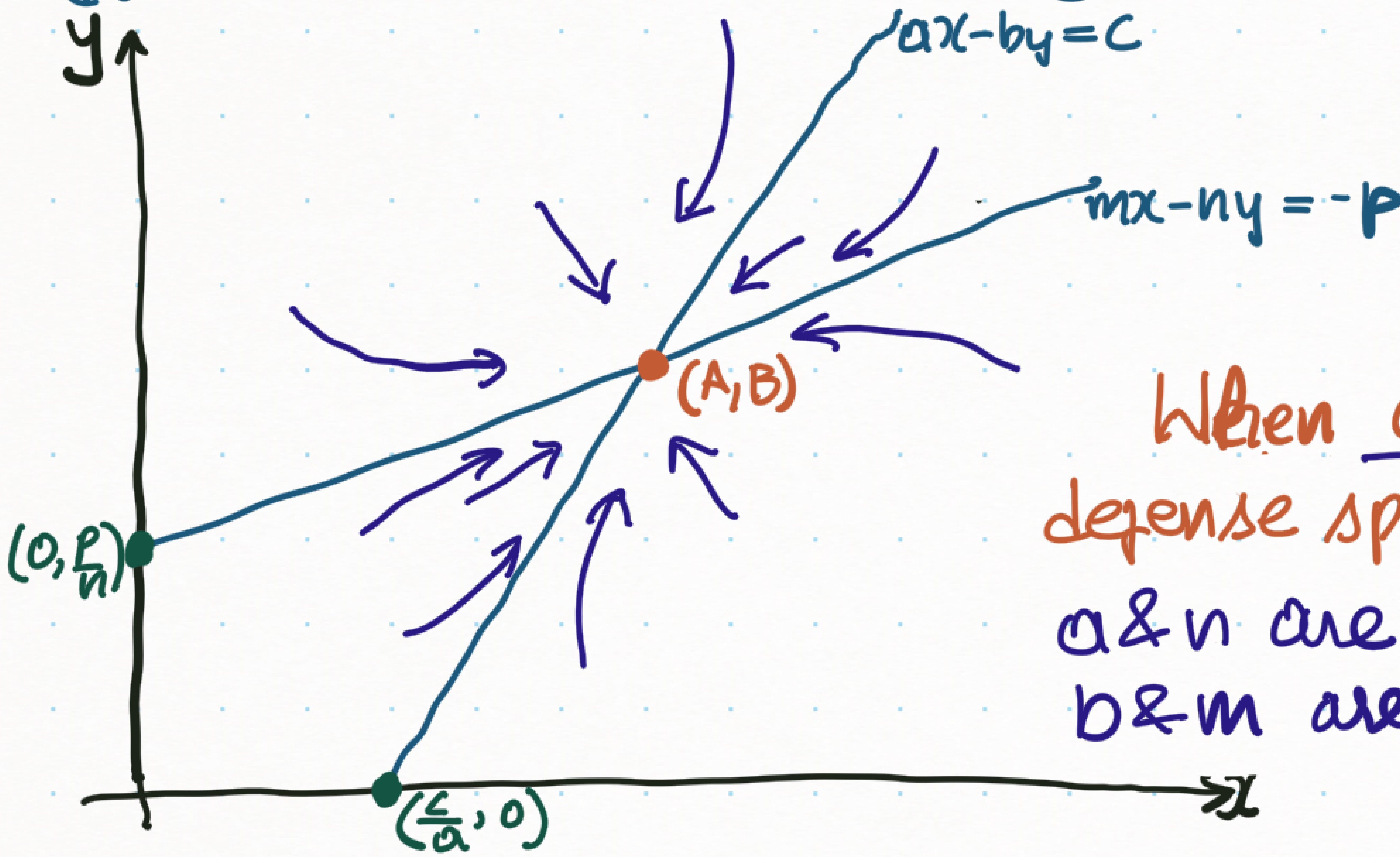

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When $an > bm$, our model indicates defense spending will stabilize at $x = A, y = B$.

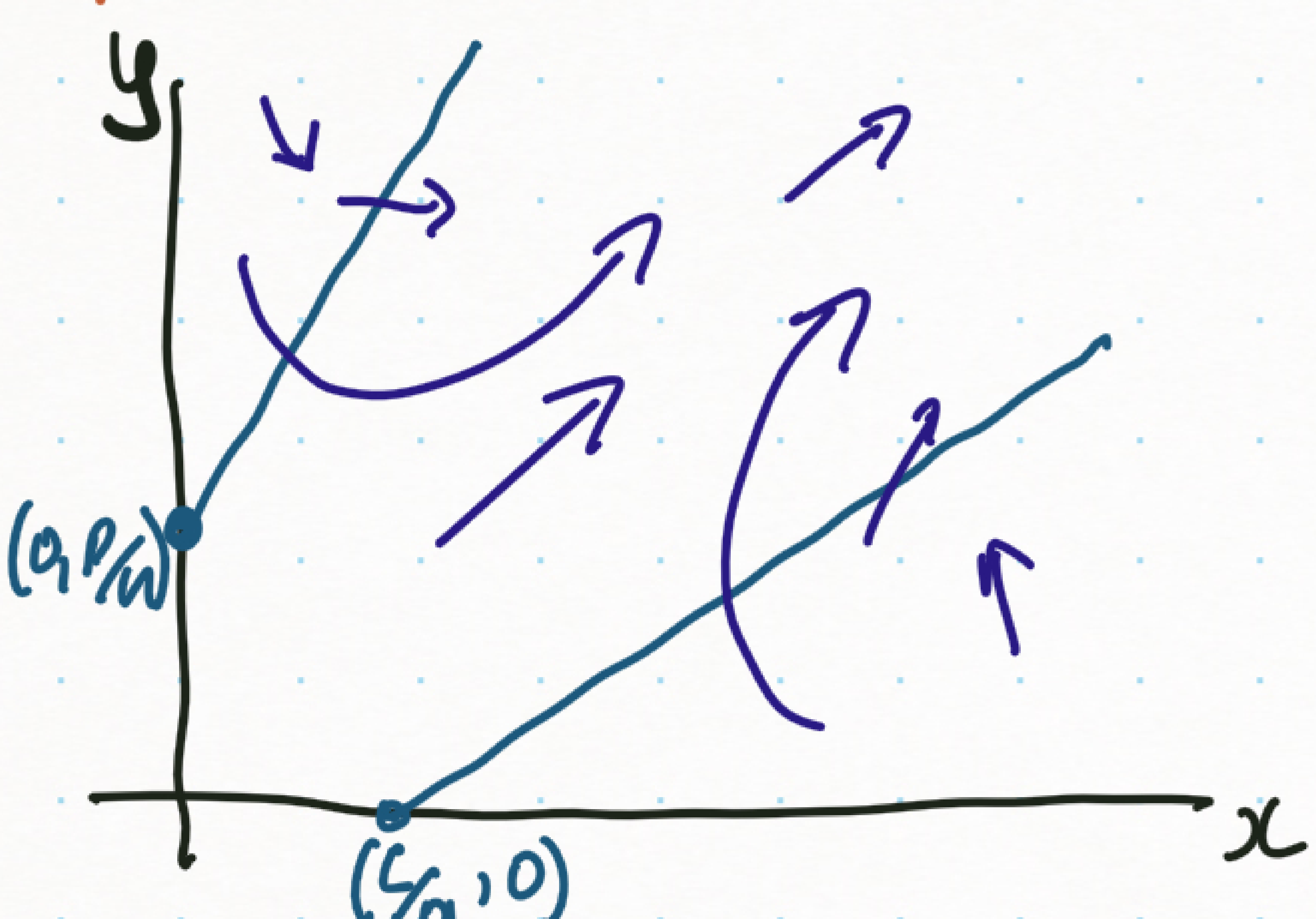
a & n are respective economic constraints.
b & m are respective martial rivalry factors.

Economics of an Arms Race

x = defense expenditure of country-1
y = ———————— " ———— -2

$$\frac{dx}{dt} = -ax + by + c \, , \quad \frac{dy}{dt} = mx - ny + p \, , \quad \text{where } a, b, c, m, n, p > 0 \text{ constants}$$

When $an \leq bm$, there is no equil. point and the phase plane looks like:



When martial rivalry outweighs economic constraints, we get ever increasing defense spending!

A competitive Species Model with carrying capacity

Blue whales and Fin whales are competing species in the same seas.

Let $x_1 = B = $ # Blue whales , $x_2 = F = $ # Fin whales.

$g_B = $ overall growth rate of blue whales (per year)

$g_F = $ overall ——— " ——— Fin —"— (per year)

$C_B = $ competition factor affecting blue w. (whales per year)

$C_F = $ ——— " ——— fin w. (whales per year)

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In the sea we are studying, upto 150000 Blue whales can be be supported by the environment, and upto 400000 fin whales. And, its been observed B have 5% p.a. intrinsic growth rate & F have 8% ——— " ———.

A competitive Species Model with carrying capacity

Blue whales and Fin whales are competing species in the same seas.

Let $x_1 = B = \#$ Blue whales , $x_2 = F = \#$ Fin whales.

$g_B =$ overall growth rate of blue whales (per year)

$g_F =$ overall ——— " ——— Fin —"— (per year)

$c_B =$ competition factor affecting blue w. (whales per year)

$c_F =$ ——————— " ——————— fin w. (whales per year)

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We model g_B & g_F as a (scaled) logistic model:

$$g_B = 0.05\, x_1 \left(1 - x_1/150000\right)$$
$$g_F = 0.08\, x_2 \left(1 - x_2/400000\right)$$

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We model the competition between the species as being proportional to the numbers of interactions between them.

$$c_B = \alpha x_1 x_2 \quad , \quad c_F = \alpha x_1 x_2 \quad ,$$ where $\alpha > 0$ is a constant.

use of same α indicates what underlying assumption?

A competitive Species Model with carrying capacity

Blue whales and Fin whales are competing species in the same seas.

Let $x_1 = B = \#$ Blue whales , $x_2 = F = \#$ Fin whales .

$$\frac{dx_1}{dt} = (0.05)\,x_1\left(1 - x_1/150000\right) - \alpha x_1 x_2$$

$$\frac{dx_2}{dt} = (0.08)\,x_2\left(1 - x_2/400000\right) - \alpha x_1 x_2 \quad , \quad \alpha > 0$$

constant (unknown)

Due to hunting & environmental effects, current populations are $x_1(0) = 5000$ B. whales & $x_2(0) = 70000$ F. whales.

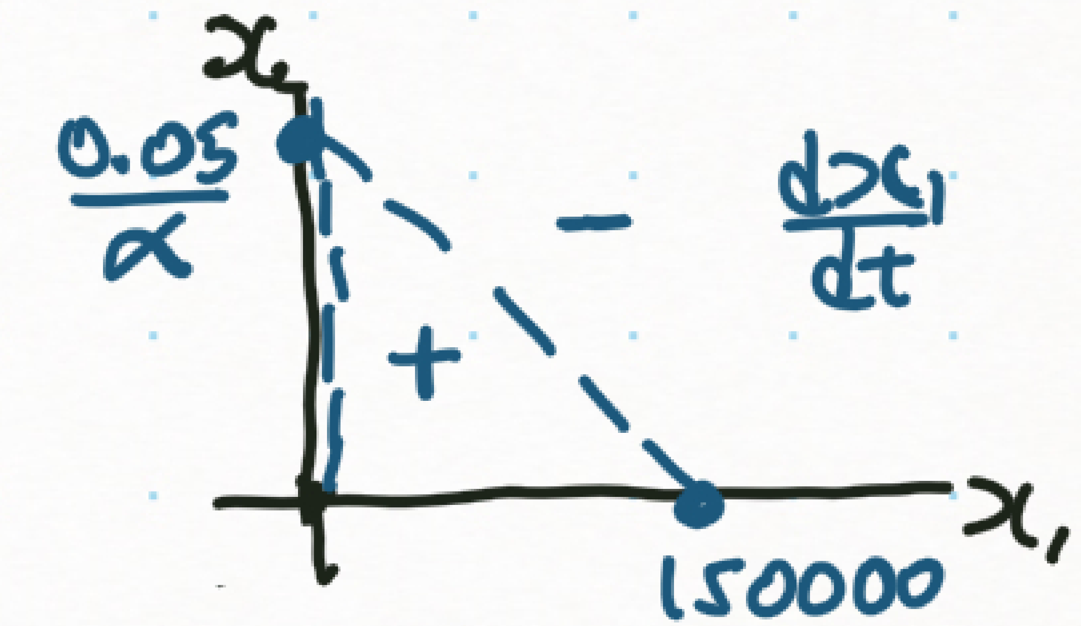
How will the two populations change over the short-term?
long-term?

Will Blue Whales become extinct? etc.

Phase Plane

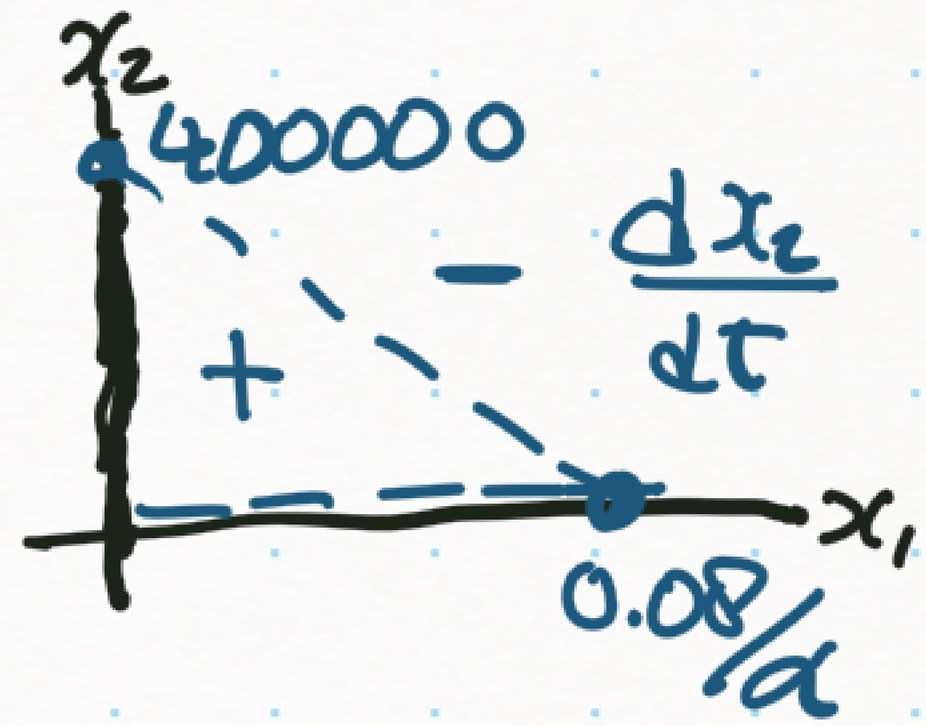
$$\frac{dx_1}{dt} = 0 \iff 0 = x_1\left(0.05 - \frac{0.05}{150000}x_1 - \alpha x_2\right)$$

i.e., $x_1 = 0$ or $\dfrac{x_1}{150000} + \dfrac{\alpha}{0.05}x_2 = 1$ ← eqn. of a line



$$\frac{dx_2}{dt} = 0 \iff 0 = x_2\left(0.08 - \frac{0.08}{400000}x_2 - \alpha x_1\right)$$

i.e., $x_2 = 0$ or $\dfrac{x_2}{400000} + \dfrac{\alpha}{0.08}x_1 = 1$



Combine these together in a single phase plane.