MATH 400: Homework #6

Only the 'Submission Problems' listed below are due Wednesday, 10/9, before 11:30pm, via a PDF file uploaded to the Homework#6 under Assignments in the Canvas course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "HW Discussion and Solution Rules" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions through the *Canvas Discussion Forums*, during my *Office Hours*, or through *Email to me*.

PART I: Practice Problems

1. Try all the Discussion/ Review problems as you review the material for this week.
2. Make your own summary of this week's topics/ concepts/ properties: definitions, alternate forms of the definition based on theorems/ discussions, examples and non-examples, methods for showing the property holds or doesn't hold, useful ideas from examples, HW problems, etc.
3. Pay close attention to and attempt at least the following exercises from the textbook. Section 3.2: #1, #2, #3, #4 (make a note of this), #5, #7, #8, #9, #11, #13 (note this! I had asked this in class also), #14, #15.
4. Prove that the Cantor set is closed.

PART II: Submission Problems

5. **Submit written solutions to all 4 problems listed below.**
   - Section 3.2: #3, #9, #11, #13. (each worth 12.5 points)

PART III: Readings, Comments, etc.

6. Cantor Set $\mathcal{C}$ has many interesting properties. We proved/discussed some in class, try to prove the rest of them.

- $\mathcal{C}$ is non-empty.
- $\mathcal{C}$ has length zero.
- $\mathcal{C}$ is a closed set.
- $\mathcal{C}$ has no isolated points. That is, every point in $\mathcal{C}$ is a limit point of $\mathcal{C}$.
- $\mathcal{C}$ is uncountable.
- Let $x \in [0,1]$, then $x \in \mathcal{C}$ iff x can be expressed using only 0 and 2 in its ternary representation. (Ternary representation of a number means its Base 3 representation using only the numbers 0, 1, 2. For $x \in [0,1]$ this means $x = 0.a_1a_2a_3a_4 \ldots a_n \ldots = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \ldots + \frac{a_n}{3^n} + \ldots$. Here is the rule for choosing the first ternary digit a_1 is $a_1 = 0$ if $0 \leq x \leq \frac{1}{3}$, $a_1 = 1$ if $\frac{1}{3} < x \leq \frac{2}{3}$, $a_1 = 2$ if $\frac{2}{3} < x \leq 1$. And so on. Can you see the connection with $\mathcal{C}$ now?)
- Can you now give a proof for '$\mathcal{C}$ is uncountable' using the previous property?
- What if we modified the Cantor's construction for $\mathcal{C}$ by splitting each interval into four equal quarter intervals and then removing the second quarter interval, and so on. Would the set created by applying this rule to $[0,1]$ repeatedly give us something like $\mathcal{C}$? How would you describe this set using the language of k-nary representation?
- What if we modified the Cantor's construction for $\mathcal{C}$ by splitting each interval into two equal half intervals and then removing the second half interval, and so on. Would the set created by applying this rule to $[0,1]$ repeatedly give us something like $\mathcal{C}$? How would you describe this set using the language of k-nary representation?

7. The 2-dimensional version of Cantor's set is called the *Sierpinski's gasket* or triangle, or the similar Sierpinski carpet. Read more about them at https://en.wikipedia.org/wiki/Sierpi%C5%84ski_triangle and https://en.wikipedia.org/wiki/Sierpi%C5%84ski_carpet.

8. *Menger's cube* or sponge is the 3-dimensional version of Cantor's set. Read more about it at https://en.wikipedia.org/wiki/Menger_sponge.
Karl Menger was one the great mathematicians of the 20th century, and a professor at IIT for the majority of his career. Read more about him and his work at https://www.iit.edu/applied-math/about/remembering-karl-menger/about-karl-menger. Our department organizes the Karl Menger Day and Lecture every year in the Spring semester (except for a couple of years due to the pandemic).
You can make your own giant handmade Menger cube using business cards!. If you want to hold a 3D printed Menger cube in your own hands, stop by my office.

9. Cantor's set and its generalizations are examples of what are now called *Fractal sets*. There are numerous popular math and science books written about fractals. You might have even seen many posters, and picture books featuring fractal sets such as Mandelbrot set, Julia set, and more. https://en.wikipedia.org/wiki/Fractal gives a good overview.
As we informally discussed in class, Cantor's set has fractional dimension. The name 'fractal' refers to objects that have fractional dimension. The formal notion of dimension that we used is the Hausdorff dimension. Here is a list of fractals with different Hausdorff dimensions: https://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension.
You can see the visualization of many fractals on Wolfram Alpha: https://www.wolframalpha.com/examples/mathematics/applied-mathematics/fractals/. It is also easy to write simple iterative programs for generating various fractals.