Math 400: Discussion Questions/ Review #8

A statement listed with [T/F] is a True/False statement that requires a proof or a counterexample, as appropriate.

1. [T/F] If $f(x) = 3x - 2$, then $\lim_{x \to 4} f(x) = 20$.
2. [T/F] $\lim_{x \to 0} \sin(\frac{1}{x})$ exists.
3. Complete the ϵ-δ proof that $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is continuous.
4. [T/F] The function $f(x) = \sqrt{x^2 + 3}$ is continuous.
5. Give an ϵ-δ proof that $\sin x$ is a continuous function.
6. [T/F] The function $f(x) = \sin(\frac{1}{x})$ is continuous on $(0, 1)$.
7. Assuming $\sin x$ is a continuous function (as proved above), show that $[\sin^2 x + \cos^6 x]^\pi$ is continuous everywhere.
8. Assuming $\sin x$ is a continuous function (as proved above), show that $\cos x$ is continuous everywhere. What about $\tan x$?
9. When is the function $\tan x$ continuous?
10. Give an ϵ-δ proof that $f(x) = \int_0^\pi \frac{\sin(xt)}{t} dt$ is a continuous function.
11. For each $n \in \mathbb{N}$, define the function $p_n : [0, 1] \to \mathbb{R}$ as $p_n(x) = x^n$.
    (a) [T/F] p_n is continuous on $[0, 1]$.
    (b) [T/F] Define the sequence (a_n) as $a_n = p_n(1)$. Then $\lim a_n = 1$.
    (c) [T/F] Fix $c \in [0, 1)$. Define the sequence (a_n^c) as $a_n^c = p_n(c)$. Then $\lim a_n^c = 0$.
    (d) [T/F] Define the function $p : [0, 1] \to \mathbb{R}$ as $p(x) = \lim_{n \to \infty} p_n(x)$. Then p is continuous on $[0, 1]$.
12. [T/F] There exists a function $f : (0, 1) \to \mathbb{R}$ which is discontinuous at all points in $(0, 1)$.
13. [T/F] If A is open then $f(A)$ is open.
14. [T/F] If A is closed then $f(A)$ is closed.
15. Let $f(x) = x^2$. Then, what is $f(\mathbb{R})$? What is $f((-1, 1))$? What is $f([-1, 1))$? What is $f([-1, 1])$?
16. Let $f(x) = \cos x$.
    (a) [T/F] There is an interval of the form (a, b) such that $f((a, b))$ is compact.
    (b) [T/F] There is an interval of the form $[a, \infty)$ such that $f([a, \infty))$ is compact.

17. [T/F] Let $f(x) = x^2$. Then f achieves its minimum in the interval $(-2, 2)$.
18. [T/F] Let $f(x) = x^2$. Then f achieves its maximum in the interval $(-2, 2)$.
19. [T/F] $f(x) = x^3 + 3x^2 - 1$ has exactly one root in each of the intervals $[0, 1]$, $[-1, 0]$, $[-1, -2]$.
20. Given an $\epsilon > 0$, how many steps of the bisection procedure will be needed to find an approximate value of the root in $[a, b]$ with error of at most ϵ.

21. Write a poem (or find a song) about continuous functions.