Math 400: Discussion Questions/ Review # 12

A statement listed with [T/F] is a True/False statement that requires a proof or a counterexample, as appropriate.

1. Let $f : [0, 2] \to \mathbb{R}$ be defined as $f(x)$ equals 0 when $x = \frac{1}{n}$ in $[0, 2]$ and 1 when $x = \frac{1}{n}$ in $[0, 2]$. Show that f is integrable on $[0, 2]$.
2. Let $f : [0, 1] \to \mathbb{R}$ be the membership function of the Cantor set $\mathcal{C}$, that is $f(x) = 1$ of $x \in \mathcal{C}$ and 0 otherwise. Show that f is integrable.
3. Let f be integrable on $[a, b]$. Prove that kf is also integrable on $[a, b]$ for any fixed real number k.
4. [T/F] Assume f and g are integrable on $[a, b]$. Then, over $[a, b]$, integral of their average is the average of their integrals.
5. [T/F] $\int_0^1 \frac{\cos x}{1+x^2} \leq \frac{\pi}{4}$.
6. Assume that "f integrable on $[a, b] \implies f^2$ integrable on $[a, b]$".
   Using this show that "f, g integrable on $[a, b] \implies fg$ integrable on $[a, b]$"
7. Prove that "f integrable on $[a, b] \implies f^2$ integrable on $[a, b]$".
8. [T/F] Assuming f, g are integrable on $[a, b]$. $\int_a^b fg = (\int_a^b f)(\int_a^b g)$.
9. When can we interchange the order of integral and limit for a sequnce of functions, that is $\lim\limits_{n\to\infty} \int_a^b f_n = \int_a^b \lim\limits_{n\to\infty} f_n$?
10. Give a proof of Integration by Parts using the Fundamental Theorem of Calculus.
11. Evaluate $\int_0^\pi x \cos x$.
12. Show that $\int_a^b h(x) - \int_{h(a)}^{h(b)} h^{-1}(u) = bh(b) - ah(a)$, where h is a 1-to-1 differentiable function on (a, b).