Dedication

To my wife Angela, and my parents.
Acknowledgements

I would like to acknowledge my academic adviser Prof. Sergey V. Lototsky who introduced me into the Theory of Stochastic Partial Differential Equations, suggested the interesting topics of research and guided me through it.

I also wish to thank the members of my committee - Prof. Remigijus Mikulevicius and Prof. Aris Protopapadakis, for their help and support.

Last but certainly not least, I want to thank my wife Angela, and my family for their support both during the thesis and before it.
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Abstract

In this work we discuss two problems related to stochastic partial differential equations (SPDEs): analytical properties of solutions and parameter estimation for SPDE’s.

We address the problem of existence, uniqueness and regularity of solutions of some parabolic SPDE’s driven by space-time white noise, either additive or multiplicative. The novelty in our study is the special form of the noise term which depends on a real parameter. We establish existence and uniqueness of weak solution in the scale of Sobolev spaces. Regularity properties of the solution are stated in terms of the real parameter involved in the noise term and spectral properties of the elliptic operator which generates the scale of Sobolev spaces.

We study the parameter estimation problem for some parabolic SPDEs with multiplicative stochastic part. Maximum Likelihood Estimators of the parameter, based on finite-dimensional approximation of the solution are found. Consistency, both in time and space variables, and asymptotic normality of these estimators are established. All theoretical results are followed by numerical simulations.
Chapter 1

Introduction

In this section we discuss some general and known results which we will need for further presentation. An overview of Sobolev spaces, Kolmogorov’s criterion, absolute continuity of measures generated by the solutions of diffusion processes, Itô’s formula in Hilbert space, existence and uniqueness of solution of parabolic SPDEs and some applications of SPDEs are presented. All results are presented without proofs, except Itô’s formula in Hilbert space for which we present a simpler proof than that from the know literature.

1.1 Sobolev spaces

Let $M$ be a $d$-dimensional compact, orientable, $C^\infty$ manifold with a smooth positive measure $dx$. Let us consider an elliptic positive definite selfadjoint differential operator $\mathcal{L}$ of order $2m$ on $M$. Then the operator $\Lambda = (\mathcal{L})^{\frac{1}{2m}}$ is well-definite, elliptic of order 1 and generates the scale $\{H^s(M)\}_{s \in \mathbb{R}}$ of Sobolev spaces on $M$ (see for instance [1], [76]). In particular, for positive integer $n$ the space $H^n(M)$ consists of all generalized functions on $M$ whose derivative up to and including the $n$th order belong to $L^2(M)$ and the norm is defined to be

$$\|u\|_{H^n(M)} = \left( \sum_{|\alpha| \leq n} \int_M |D^n u|^2 dx \right)^{\frac{1}{2}},$$
where $D^\alpha$ denotes the partial derivative of order $\alpha = (\alpha_1, \ldots, \alpha_d)$ (see [1],[9] for more details). We will write $\{H^s\}_{s \in \mathbb{R}}$ instead of $\{H^s(M)\}_{s \in \mathbb{R}}$ if there is no ambiguity in context.

In what follows we will usually use an alternative characterizations of Sobolev spaces. It is well-known (see for instance [76], Theorem 8.4) that the operator $L$ has a complete orthonormal system (CONS) of eigenvectors $\{h_k\}_{k \in \mathbb{N}}$ in the space $L_2(M, dx)$. Then for every $f \in L_2(M, dx)$ we have the representation
\[
f(x) = \sum_{k=1}^{\infty} \langle f, h_k \rangle_0 h_k(x) = \sum_{k=1}^{\infty} f_k h_k(x), \quad (x \in M)
\] (1.1)
where $\langle \cdot, \cdot \rangle_0$ denotes the inner product in $L_2(M, dx)$, and $f_k = \int_M f(x) h_k(x) dx$, $k \in \mathbb{N}$ are Fourier coefficients of $f$ with respect to $\{h_k\}_{k \in \mathbb{N}}$.

If $\mu_k > 0$ is the eigenvalue of the operator $L$ corresponding to the eigenvector $h_k$ and $\lambda_k := \mu_k^{1/2}$, then for all $s \geq 0$ we have $H^s = \{f \in L_2(M, dx) : \sum_{k=1}^{\infty} \lambda_k^{2s} |f_k|^2 < \infty\}$ and for $s < 0$, $H^s$ is the closure of $L_2(M, dx)$ in the norm $\|f\|_s = (\sum_{k=1}^{\infty} \lambda_k^{2s} |f_k|^2)^{1/2}$. Hence, every element of the space $H^s$, $s \in \mathbb{R}$, can be identified with a sequence $\{f_k\}_{k \in \mathbb{N}}$ such that $\sum_{k=1}^{\infty} \lambda_k^{2s} |f_k|^2 < \infty$. Note that the space $H^s$, equipped with the inner product
\[
\langle f, g \rangle_s = \sum_{k=1}^{\infty} \lambda_k^{2s} f_k g_k \quad (f, g \in H^s)
\]
is a Hilbert space. Moreover, for $s > 0$, $H^{-s}$ is the dual space of $H^s$ relative to inner product in $L_2(M, dx)$. It is equivalent to say that $H^s = \Lambda^{-s}(L_2(M, dx))$, $s \in \mathbb{R}$. Note that
\[
\Lambda^s h_k = \lambda_k^s h_k \quad \text{and} \quad \|h_k\|_s = \|\Lambda^s h_k\| = \lambda_k^s \quad (k \in \mathbb{N}, \ s \in \mathbb{R}).
\] (1.2)
Moreover, the functions \( h_{k,s} := \lambda_k^{-s} h_k \), \( k \in \mathbb{N} \), form a CONS in \( H^s \) for \( s \in \mathbb{R} \).

Similarly, the Sobolev Spaces \( H^s(\mathbb{R}^d) \) (\( s \in \mathbb{R} \)) are defined. Namely,

\[
H^s(\mathbb{R}^d) = \left\{ f \in L_2(\mathbb{R}^d) : \int_\mathbb{R} (1 + |y|^2)^s |\hat{f}(y)|^2 dy < \infty \right\}
\]

with norm \( ||f||_s^2 = \int_\mathbb{R} (1 + |y|^2)^s |\hat{f}(y)|^2 dy \), where \( \hat{f} \) denotes the Fourier Transform in \( \mathbb{R}^d \), i.e. \( \hat{f}(y) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-i xy)f(x)dx \). Also, we can characterize the spaces \( H^s(\mathbb{R}^d) \) using the Laplace operator \( \Delta(u) = \sum_{k=1}^d u_{x_kx_k} \). Namely, \( H^s(\mathbb{R}^d) = (I - \Delta)^{-\frac{s}{2}} L_2(\mathbb{R}^d) \). For more details on structure and some properties of operator \((I - \Delta)^s\), \( s \in \mathbb{R} \), see for instance [1], [43].

For \( p \geq 1 \) and \( \gamma \in \mathbb{R} \), we define the space \( H^p_\gamma(\mathbb{R}^d) \) as collection of all generalized functions \( f \) such that \((I - \Delta)^{-\gamma/2}f \in L_p(\mathbb{R}^d)\), and we put \( ||f||_{\gamma,p} := \|(I - \Delta)^{-\gamma/2}f\|_{L_p(\mathbb{R}^d)} \). In our notations \( H^0 = H^2 \).

The triple \((V, H, V')\) of Hilbert spaces is called normal triple if: (a) \( V \hookrightarrow H \hookrightarrow V' \) and both inclusions are dense and continuous; (b) The space \( V' \) is dual to \( V \) relative to inner product in \( H \); (c) There exists a constant \( C > 0 \) such that \( |(h,v)|_H \leq C \cdot ||v||_V \cdot ||u||_{V'} \) (\( v \in V, u \in H \)). For example the Sobolev spaces \((H^{l+\gamma}(\mathbb{R}^d), H^{l}(\mathbb{R}^d), H^{l-\gamma}(\mathbb{R}^d))\), \( \gamma > 0 \), \( l \in \mathbb{R} \), form a normal triple.

Note that \( H^p_\gamma \subset H^p_\gamma \subset H^p_\gamma \) and \( ||\cdot||_{\gamma,p} \leq ||\cdot||_{\gamma,1,p} \) if \( \gamma_2 < \gamma_1 \). Also, let us recall that by Sobolev embedding theorem \( H^p_\gamma(\mathbb{R}^d) \subset C^{r-d/p}(\mathbb{R}^d) \) if \( p \gamma > d \) (see [9], Chapter 5).

Also, we recall the interpolation theorem for spaces \( H^s \) [40], [41]. For every \( u \in H^{s+2} \) and \( s \in [\gamma, \gamma + 2] \), we have \( ||u||_s \leq C ||u||_{\gamma+2}^{\theta} ||u||_{\gamma}^{1-\theta} \), where \( \theta = \frac{s-\gamma}{2} \) and \( C \) depends only on \( M, \gamma, s \).

We will refer to inequality \( ab \leq \varepsilon \frac{|a|^p}{p} + \frac{|b|^q}{q} \), as \( \varepsilon \)-inequality, where \( a, b \in \mathbb{R}, \ 1/p + 1/q = 1, \ \varepsilon > 0 \). It follows directly from Hölder inequality.
Another central result in our investigation is Kolmogorov’s criterion. To some extent, Kolmogorov’s criterion is a version of Sobolev embedding theorem. For sake of completeness of our presentation, we will state this result here too. For the proof, see for instance [79], Corollary 1.2, or [8], Theorem 3.3.

**1.1.1 Theorem.** [Kolmogorov’s criterion] Let \((\Omega, \mathcal{F}, P)\) be a probability space, \(G \subset \mathbb{R}^d\) a bounded domain, and \(X = X(x), x \in G,\) a measurable map from \(\Omega \times G\) to \(\mathbb{R}\) so that \(X\) is continuous in \(x\) and \(\omega\). Assume that there exist positive real numbers \(K, p\) and \(q\) so that \(p > 1\) and

\[
\mathbb{E}\left|X(x) - X(y)\right|^p \leq K|x - y|^{d+q}.
\]

Then there exist a positive real number \(A\) and a random variable \(Y\) so that, with probability one,

\[
\left|X(x) - X(y)\right| \leq Y|x - y|^{q/p}\left(\ln\frac{A}{|x-y|}\right)^{2/p}, \quad (1.3)
\]

for all \(x, y \in G, q \leq p\).

### 1.2 Diffusion processes and absolute continuity of their measures

For a given probability space \((\Omega, \mathcal{F}, P)\), we consider a family of random random processes \(\xi_t(\omega)\) on this space with \(t \in [0, T]\), where \(t\) usually is understood as time variable, and \(T\) is a fixed parameter, called also terminal time. For a fixed \(\omega \in \Omega\), the time function \(\xi_t(\omega), t \in [0, T]\) is called a *trajectory or realization* corresponding to an elementary event \(\omega\). The \(\sigma\)-algebra \(\mathcal{F}_t^\xi := \{\xi_s : s \leq t\}\), being the smallest \(\sigma\)-algebra
with respect to which the random variables $\xi_s$, $s \leq t$ are measurable, are naturally associated with random process $\xi_t$. $\mathcal{F}_t^\xi$ is called the filtration generated by the random process $\xi_t$.

Let $\{\mathcal{F}_t\}_{t \in [0,T]}$ be a nondecreasing family of $\sigma$-algebras, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$, $s \leq t$. We say that a random process $\xi_t$, $t \in [0, T]$, is adapted to a family of $\sigma$-algebras $\mathcal{F}_t$, $t \in [0, T]$, if for any $t \in [0, T]$ the random variables $\xi_t$ are $\mathcal{F}_t$-measurable. Also, we will say that $\xi_t$ is $\mathcal{F}_t$-adapted or nonanticipative.

1.2.1 Definition. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. The random process $W = (W(t))$, $0 \leq t \leq T$, is called a standard Brownian Motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ if: (a) $W(0) = 0$; (b) for every partition $0 = t_0 < \cdots < t_i < t_{i+1} < \cdots t_n = T$, the random variables $W(t_{i+1}) - W(t_i)$ and $W(t_{j+1}) - W(t_j)$ are independent for every $i \neq j$ (independent increments); (c) $W(t) - W(s)$ has a Gaussian standard normal distribution with $E(W(t) - W(s)) = 0$, $\text{Var}(W(t) - W(s)) = 1$; (d) for almost all $\omega \in \Omega$ the functions $W(t) = W(t, \omega)$ are continuous on the interval $0 \leq t \leq T$.

For more details about existence of Brownian motion see for instance [13], [47]. The Brownian motion is also called Winer Process.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathcal{P})$ be a stochastic basis, $W(t)$ be a Wiener Process and $\xi(t)$ a random process, both adapted to the filtration $\mathcal{F}_t$. Denote by $\mathcal{P}_\xi$ the measure on $C(0, T)$ corresponding to the process $\xi_t$,

$$\mathcal{P}_\xi(B) = \mathcal{P}\{\omega : \xi(\omega) \in B\}.$$

In this section we will discuss the problem of the absolute continuity of measures $\mathcal{P}_\xi$ and $\mathcal{P}_\eta$, where $\xi$ and $\eta$ are some diffusion processes. As usually, by $\frac{d\mathcal{P}_\xi}{d\mathcal{P}_\eta}$ we denote the Radon-Nykodim derivative of the measure $\mathcal{P}_\xi$ w.r.t. the measure $\mathcal{P}_\eta$. 

5
In what follows all stochastic integrals are understood in sense of Itô. Let \( \xi = (\xi_1(t), \ldots, \xi_n(t)) \) and \( \eta = (\eta_1(t), \ldots, \eta_n(t)) \), \( t \in [0, T] \), be vector-processes, with Itô differentials

\[
\begin{align*}
  d\xi_t &= A_t(\xi)dt + b_t(\xi)dW_t, \\
  d\eta_t &= a_t(\eta)dt + b_t(\eta)dW_t, \\
  \xi_0 &= \eta_0,
\end{align*}
\]  

(1.4)

where \( W_t = (W_1(t), \ldots, W_k(t)) \) is a \( k \)-dimensional Wiener process with respect to \( \mathcal{F}_t \), \( A_t(x) = (A_1(t, x), \ldots, A_n(t, x)) \), \( a_t(x) = (a_1(t, x), \ldots, a_n(t, x)) \), \( b_t(x) = [b_{ij}(t, x)] \) is a matrix of order \( n \times k \), and \( \eta_0 = (\eta_1(0), \ldots, \eta_n(0)) \) is a vector of initial values such that \( \mathbb{P}\left( \sum_{j=1}^{n} |\eta_j(0)| < \infty \right) = 1 \). We suppose that equations (1.4) have a unique strong solution (for more details about conditions on the coefficients see [62], Theorem 5.5, or [47], Theorem 4.6).

We denote by \( A^* \) the transpose matrix of a given matrix \( A \). By \( b^+ \) we denote the pseudoinverse matrix with respect to the matrix \( b \). Recall that the matrix \( b^+ \) is called pseudoinverse matrix with respect to the matrix \( b \), if \( b^+bb^+ = b^+ \) and \( bb^+b = b \). Note that if \( b \) is a square invertible matrix, then \( b^+ = b^{-1} \). For more details about pseudoinverse matrices see for instance [4].

The following result about absolute continuity of measures \( \mathbb{P}_\xi \) and \( \mathbb{P}_\eta \) holds true:

1.2.2 Theorem. Assume that the system of algebraic equations

\[
b_t(x)y_t(x) = A_t(x) - a_t(x)
\]  

(1.5)
has a solution with respect to \( y_t(x) \) for every \( t \in [0, T], \ x \in \mathbb{R}, \) and

\[
\int_0^T \left( A^*_t(x)(b_t(x)b^*_t(x))^+A_t(x) + a^*_t(x)(b_t(x)b^*_t(x))^+a_t(x) \right) dt < \infty \quad \mathbb{P}_\xi - \text{a.s.}
\]

Then \( \mathbb{P}_\xi \sim \mathbb{P}_\eta \) and

\[
\frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(t, \eta) = \exp \left\{ \int_0^T \left( A_s(\eta) - a_s(\eta) \right)^* (b_t(\eta)b^*_t(\eta))^+ d\eta(s) \right. \\
- \frac{1}{2} \int_0^T \left( A_s(\eta) - a_s(\eta) \right)^* (b_t(\eta)b^*_t(\eta))^+ (A_s(\eta) + a_s(\eta)) ds \right\} .
\]

For the proof see for instance Chapter 7 in [47]. More general results are discussed in [26], [27], [28].

1.3 **Stochastic partial differential equations and their applications**

The theory of Stochastic Differential Equations (SDEs) is dealing with finite-dimensional noise, namely the perturbation consists of finite number of Wiener Processes. In the case of **Stochastic Partial Differential Equation** (SPDEs) besides of considering functions of multivariables, usually the perturbation part represents an infinite-dimensional noise, i.e. infinitely many Wiener Processes. A general theory of infinite-dimensional noise is based on the Hilbert space-valued Wiener process (see for instance [8], [70]). For sake of completeness we present here briefly the general idea and construction of infinite-dimensional noise.
Let $\mathcal{H}$ be a separable Hilbert space, and $Q$ a selfadjoint none-negative operator on $\mathcal{H}$. Suppose that $Q$ is a trace (nuclear) operator. Then there exists a complete orthonormal system (CONS) $h_k, \ k \geq 1$, in $\mathcal{H}$, and a sequence of non-negative real numbers $\{q_k\}_{k \geq 1}$ such that $Qh_k = q_k h_k, \ k \geq 1$ and $\text{Tr}(Q) = \sum q_k < \infty$. In other words, $Q$ is a selfadjoint, trace operator on $\mathcal{H}$, with eigenvalues $q_k$ and eigenvectors $h_k, \ k \geq 1$.

**1.3.1 Definition.** A $\mathcal{H}$-valued stochastic process $W(t), \ t \geq 0$ is called a $Q$-Wiener Process if (i) $W(0) = 0$; (ii) $W$ has continuous trajectories; (iii) $W$ has independent increments; (iv) $W(t) - W(s) \sim N(0, (t-s)Q), \ 0 \leq s \leq t$, where $N$ denotes a Gaussian process in Hilbert space $\mathcal{H}$ (for more details see [8], Chapter 1-4).

One can show that $\mathbb{E}(W(t)) = 0$, $\text{Cov}(W(t)) = tQ, \ t \geq 0$, and for arbitrary $t$ the following expansion holds

$$W(t) = \sum_{k=1}^{\infty} \lambda_k h_k W_k(t),$$

where $\lambda_k = \sqrt{q_k}, \ k \in \mathbb{N}$, $W_k(t)$ are real valued standard Wiener processes, mutually independent on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the last series converges in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. For complete proof see for instance [8], Proposition 4.1.

It turns out that the representation (1.7) is very convenient to use in the theory of SPDEs. Moreover, we can assume that $Q$ is a bounded, selfadjoint, non-negative operator on $\mathcal{H}$, with pure discrete spectrum (non necessary trace operator). In this case $W(t)$ will take value in a larger Hilbert space, having the same type of representation (1.7). If $\sum_{k \geq 1} \lambda_k = \infty$, then infinite-dimensional Wiener process $W(t)$ is called *cylindrical Wiener process or cylindrical Brownian motion*. In special case, if we take $\mathcal{H} = L_2(\mathbb{R}^d)$ and $\lambda_k = 1, \ k \geq 1$, the formal sum $dW(t) = \sum_{k \geq 1} h_k(x) dW_k(t)$
is called *space-time white noise*. In this work, we will mainly focus on the case \( \lambda_k = k^\alpha, \alpha \in \mathbb{R} \).

Let \((V, H, V')\) be a triple of Hilbert spaces and \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) a stochastic basis. Suppose that \(\mathcal{A}(t) : V \to V', \quad \mathcal{M}_k(t) : V \to H\) are linear bounded operators for all \(k \in \mathbb{N}, \ t \in [0, T]\), and for some finite terminal time \(T\). A general Linear Stochastic Partial Differential Equation is written as

\[
du(t) = (\mathcal{A}u(t) + f(t))dt + (\mathcal{M}_k u(t) + g_k(t))dW_k(t), \quad u(0) = u_0,
\]

where \(t \in [0, T], \ u_0 \in L^2(\Omega; H)\) and \(u_0\) is \(\mathcal{F}_0\)-measurable. We assume that \(W_k (k \in \mathbb{N})\) are independent standard Wiener processes, \(f, g_k (k \in \mathbb{N})\) are \(\mathcal{F}_t\)-adapted random processes such that \(f \in L^2(\Omega \times (0, T); V')\) and \(g_k \in L^2(\Omega \times (0, T); H), k \in \mathbb{N}\). Here and in what follows the summation over repeated indices is assumed.

Depending on the noise term the equation (1.8) is classified as follows:

- Equations with *additive noise*, if \(\mathcal{M}_k = 0\);

- Equations with *multiplicative noise*, if \(\mathcal{M}_k \neq 0\).

Similar to classical deterministic PDEs, the solution of equation (1.8) can be specified in different ways. Since the solution is a stochastic process, besides of PDE part, where we have classical solution, strong/weak generalized solution and mild solution, also we can specify strong and weak probabilistic solution. In this work we consider only solution which are strong in probabilistic sense and weak/strong/mild in PDE sense.
We say that $\mathcal{F}_t$-adapted function $u \in L^2(\Omega \times (0, T); V)$ is a \textit{weak solution} of equation (1.8), if for every $\varphi \in V$ and all $t \in [0, T]$, the equality
\[
(u(t), \varphi)_H = (u_0, \varphi)_H + \int_0^t [A u(s) + f(s), \varphi]ds + \int_0^t (\mathcal{M}_k u(s) + g_k(s), \varphi)_H dW_k(s)
\] (1.9)
holds with probability one.

We say that $\mathcal{F}_t$-adapted function $u \in L^2(\Omega \times (0, T); V)$ is a \textit{mild solution} of equation (1.8) if
\[
u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)(\mathcal{M}_k u(s) + g_k(s))dW_k(s), \quad t \in [0, T],
\]
holds true with probability one, where $S$ is a strongly continuous semigroup with infinitesimal generator $A$ (for definition of semigroup and infinitesimal operator see for instance [20]).

As we observed, stochastic evolution equations in infinite dimensions are natural generalizations of classical PDEs and Systems of Stochastic Ordinary Differential Equations. The theory related to all these equations has motivations coming from physics, chemistry, biology, medicine, finance etc. Although, the theory of SPDEs is already established and widely developed field in mathematics, and problems arising in this theory represent an interest for mathematics itself, we will mention here several classical problems related to some particular SPDEs which we are going to study latter on in Chapter 2 and 3.
Interesting and important examples represent Zakai and Kushner equations (see [8], [70], [80]) which come from diffusion filtering problem. Let $X(t)$ be the unobserved $d$-dimensional process, and $Y(t)$ be the $r$-dimensional observed process defined by the following SODEs

\[
\begin{align*}
    dX(t) &= b(X(t))dt + \sigma(X(t))dW(t) + \rho(X(t))dV(t), \\
    dY(t) &= h(X(t))dt + dV(t), \\
    X(0) &= X_0, \\
    Y(0) &= 0,
\end{align*}
\]

where $b(x) \in \mathbb{R}^d$, $\sigma(x) \in \mathbb{R}^{d \times r}$, $h(x) \in \mathbb{R}^r$, and $X_0$ has a density $u_0$. Let $f = f(x)$ be a scalar measurable function on $\mathbb{R}^d$ such that $\sup_{0 \leq t \leq T} \mathbb{E}|f(X(t))|^2 < \infty$. The filtering problem is to find the best square mean estimate $\tilde{f}$ of $f(X(t))$, $t \in [0, T]$, given the observations $Y(s)$, $s \in [0, t]$. One can show that $\tilde{f} = \mathbb{E}(f(X(t)) | \mathcal{F}_T^Y)$ and

\[
\begin{align*}
    \tilde{f}(t) &= \frac{\int_{\mathbb{R}^d} f(x)u(t, x)ds}{\int_{\mathbb{R}^d} u(t, x)dx},
\end{align*}
\]

where $u$ satisfies the following equation

\[
\begin{align*}
    du(t, x) &= Au(t, x)dt + \sum_{k=1}^{r} B_k u(t, x)dY_k(t), \\
    0 < t \leq T, \\
    x \in \mathbb{R}^d, \\
    u(0, x) &= u_0(x),
\end{align*}
\]

where $A_k, B_k$ are some partial differential operators determined by the functions $b, \sigma, \rho, h$. Equation (1.11) is called Zakai filtering equation, which is a particular case of equation (1.8). If we put $p(t, x) = u(t, x)/\int_{\mathbb{R}^d} u(t, x)dx$ then $p$ satisfies Kushner filtering equation which is a non-linear SPDE.

Helmholtz equation

\[
\begin{align*}
    du(t, x) &= (i \Delta u(t, x) + au(t, x))dt + iu(t, x)\sum_{k=1}^{\infty} q_k h_k(x)dW_k(t),
\end{align*}
\]
where $x \in \mathbb{R}^2$, $t > 0$, $a > 0$, $q_k > 0$, $k \geq 1$, $h_k$ CONS in $L_2(\mathbb{R}^2)$, is used to describe and study random media (see [70]).

The equation of the form $du(t, x) = (F(t, x, u(t, x)) u_x(t, x))_x dt + G(t, x, u(t, x)) dW(t, x)$ appears as a limit of a branching diffusion model, and has applications in biology [10], chemistry and theory of super-processes [58], [59], [63].

The simple SPDE $v_t = v_{xx} - v + dW$ with proper initial values and boundary conditions was proposed to model electrical potential at point $(x, t)$ generated by neurons (see for instance [79]).

We conclude this section with some application of SPDEs to finance. The great success of stochastic calculus in description of stock markets and valuation of options on stocks strongly influenced the research related to fixed income market, in particular modelling the term structure of interest rates. Before going into details, let us recall some notions from bond market (for more details see for instance [5], [6], [24]).

A zero coupon bond with maturity date $T$, is a contract which guarantees the holder 1 dollar to be paid on the date $T$. We denote by by $p(t, T)$ the price of a bond at time $t$. We assume that $P(t, T)$ exists for every $T > t$, $P(t, T) > 0$, $P(t, t) = 1$, and there exists $\partial P(t, T)/\partial T$.

Instantaneous forward rates at time $t$ for all time to maturity $x > 0$, $f(t, x)$, are defined by

$$f(t, x) = -\frac{\partial \log P(t, t + x)}{\partial x}, \quad (1.12)$$

which is the rate that can be contracted at time $t$ for instantaneous borrowing or lending at time $t + x$.

The spot interest rate at time $t$ is defined as

$$r(t) = f(t, 0). \quad (1.13)$$
If we want to make a model for bond market, it suffices to specify the dynamics for one of these variables: Bond Price $P(t, T)$, forward rate $f(t, x)$, spot rate $r(t)$. Knowing the dynamics of one of them, by the Itô’s formula (see for instance [62]) we can get the dynamics of rest of them (see [5], Chapter 15, page 230). There are many models for spot rate, among which we want to mention Vasicek model $dr = (b-ar)dt + \sigma dW$, $a > 0$, Cox-Ingersoll-Ross (CIR) model $dr = a(b-r)dt + \sigma \sqrt{r}dW$, Ho-Lee model $dr = \theta(t)dt + \sigma dW$. A popular model for forward rate is Heath-Jarrow-Morton (HJM) model by which the forward rate follows the dynamics

$$df(t, x) = \left( \frac{\partial f(t, x)}{\partial x} + a(t, x) \right) dt + \sigma(t, x)dW(t), \quad (1.14)$$

where $a(t, x) = \sigma(t, x)\int_0^x \sigma(t, y)dy + \varphi(t))$, and $\varphi$ is a known function (actually $\varphi$ is the market price of risk). We note that in all these models the shock is a one dimensional Brownian motion, which means that the same set of shocks affect all forward rates, which constrains the correlations between bond prices. A natural mathematical generalization of the HJM model for forward rates, since we have a function of two variables, is to consider the noise term to be an infinite-dimensional Brownian motion $W(t, x)$. In other words, this means that every instantaneous forward rate is driven by its own noise, and the equation becomes

$$df(t, x) = \left( \frac{\partial f(t, x)}{\partial x} + a(t, x) \right) dt + \sigma(t, x)dW(t, x), \quad (1.15)$$

This idea was proposed by Kennedy [34], [35], where the forward rate was modelling as a Gaussian field, and Goldstein [14], where the noise term is a specific space-time white noise. P. Santa-Clara and D. Sornette [71] generalize this approach, presenting various types of noise term, as well as, application to pricing bond derivatives.
Motivated by statistical properties of interest rates, R. Cont [7] propose a model in which the forward rate curve is decomposed in factors as follows

\[ f(t, x) = f(t, x_{\text{min}}) + s(t) \left( Y(x) + u(t, x) \right) \]  (1.16)

where \( x_{\text{min}}, x_{\text{max}} \) are shortest and longest maturity available on the market, \( s(t) = f(t, x_{\text{min}}) - f(t, x_{\text{max}}) \), \( Y \) is a deterministic shape function defining the average profile of the term structure, and \( u(t, x) \) an adapted process describing the random deviations of the term structure from its long term average shape. Empirical studies identify the level of interest rates, the steepness (slope) of term structure and its curvature as three significant parameters in the geometry of the yield (see for instance [49]). Then, under some technical assumptions on the market properties, R. Cont [7] deduces that \( u \) satisfies the following parabolic SPDE

\[ du(t, x) = \left( \frac{\partial u(t, x)}{\partial x} + b(t, x, u(t, x)) + \frac{k}{2} \frac{\partial^2 u(t, x)}{\partial x^2} \right) dt \]

\[ + \sigma(t, x, u(t, x)) dW(t, x), \]  (1.17)

where \( b \) and \( \sigma \) are some functions, and \( k \) a real parameter. In [7], author discussed the case \( b \equiv 0, \sigma \equiv \text{const.} \)

### 1.4 Itô’s formula in Hilbert space

In this section we will present a version of the Itô’s formula in Hilbert spaces. Although this result is known and can be found, for example, in [70], Chapter 2, Theorem 4.2, we are going to present here a simpler proof, using the one-dimensional Itô’s formula.
Let \( \mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P}) \) be a stochastic basis. A random variable \( \tau \) taking values in \( \mathbb{R}_+ \) is called a stopping time with respect to the filtration \( \mathcal{F}_t, \ t \in \mathbb{R}_+ \), if the random event \( \{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \), for every \( t \in \mathbb{R}_+ \). The real-valued stochastic process \( M(t) \) is called martingale relative to the filtration \( \mathcal{F}_t, \ t \in \mathbb{R}_+ \), if it \( \mathcal{F}_t \)-adaptive, \( \mathbb{P} \)-integrable, and \( \mathbb{E}[M(t)|\mathcal{F}_s] = M(s) \) for every \( s, t \in \mathbb{R}_+, \ s \leq t \). The real-valued process \( M(t) \) is called local martingale relative to the filtration \( \mathcal{F}_t \), if there exists a sequence of stoping times \( \{\tau_n\}_{n \in \mathbb{N}}, \ \tau_n \uparrow \infty \), such that for every \( n \), the process \( M(t \wedge \tau_n) \) is a martingale relative to \( \mathcal{F}_t \). We denote by \( \mathcal{M}_{loc}^c(\mathbb{R}_+, \mathbb{R}) \) the set of all real-valued continuous local martingales. One can show that for every \( M \in \mathcal{M}_{loc}^c(\mathbb{R}_+, \mathbb{R}) \), there exists a unique (up to a version) continuous increasing stochastic process \( \langle M \rangle_t \) such that \( |M(t)|^2 - \langle M \rangle_t \in \mathcal{M}_{loc}^c(\mathbb{R}_+, \mathbb{R}) \). The process \( \langle M \rangle_t \) is called the quadratic variation process of the local martingale \( M(t) \). For a predictable process \( X(t) \) one may define the integral \( \int_0^t X(t) dM(t) \) in sense of Itô. For more on these definitions, properties and proofs see for instance [48], [75].

All these definitions can be extended naturally to the stochastic processes with values in some Hilbert space (for more details see Chapter 2, Section 1.4 in [70]). For sake of completeness we briefly recall some of them. Let \( V \) be a separable Hilbert space and we denote by \( V' \) its conjugate space. An \( V \)-process \( M(t) \) is called martingale (local martingale) relative to the filtration \( \mathcal{F}_t \) if (a) \( \mathbb{E}[|M(t)|_V] < \infty \), for every \( t \in \mathbb{R}_+ \) \( \mathbb{E}[|M(t \wedge \tau_n)|_V] \), for every \( t \in \mathbb{R}_+ \) and some sequence of stoping times \( \{\tau_n\}_{n \in \mathbb{N}} \) such that \( \tau_n \uparrow \infty \); (b) for every \( h^* \in V' \), the process \( h^* M(t) \) is a martingale (local martingale) relative to the filtration \( \mathcal{F}_t \). The set of all continuous local martingales taking values in \( V \) will be denoted by \( \mathcal{M}_{loc}^c(\mathbb{R}_+, V) \). Similarly to real-valued local martingales, given \( M(t) \in \mathcal{M}_{loc}^c(\mathbb{R}_+, V) \), we denote by \( \langle M \rangle_t \) the increasing process such
that \( \|M(t)\|^2 - \langle M \rangle_t \in \mathfrak{M}^c_{loc}(\mathbb{R}_+, V) \). We say that \( \langle M \rangle_t \) is the \textit{quadratic variation} of the \( M(t) \).

Let us consider a triplet of Hilbert spaces \( (V, H, V') \) and suppose that \( M(t) \) is a continuous local martingale relative to filtration \( \{\mathcal{F}_t\} \), which takes values in the Hilbert space \( V \). Since \( V \) is a separable Hilbert space, there exists a CONS \( \{h_k\}_{k \in \mathbb{N}} \) in this space. We denote by \( \langle M_i \rangle_t \) the quadratic variation of the local continuous martingale \( M_k(t) := (M(t), h_k) \). To prove the main result of this section we will need the following result.

\textbf{1.4.1 Lemma. For all \( \omega \in \Omega_1 \subset \Omega \), where \( P(\Omega_1) = 1 \),}

\[ \langle M \rangle_t = \sum_{k=1}^{\dim V} \langle M_i \rangle_t. \]

For the proof see Lemma 8, Section 2.1.8 in [70].

For a predictable \( V \)-process \( f \) we put

\[ \int_0^t (f(s), dM(s)) := \lim_{N \to \infty} \sum_{k=1}^{N} \int_0^t (f(s), h_k) dM_k(s). \]

The last integral is well-defined [70].

Suppose that \( x(\omega, t) \) and \( y(\omega, t) \) are given functions on \( \Omega \times [0, T] \) and taking values in \( V \) and \( V' \) respectively. We assume that these functions are predictable and satisfy the inequality

\[ \int_0^T (\|x(\omega, t)\|_V^2 + \|y(\omega, t)\|_{V'}^2) \, dt < \infty \quad (P - \text{a.s.}). \] (1.18)

Let \( x(\cdot, 0) \) be an \( \mathcal{F}_0 \)-measurable function on \( \Omega \), and let \( \tau \) be a stopping time.
1.4.2 Theorem. If for every \( v \in V \) the following equality

\[
(x(t), v) = (x(0), v) + \int_0^t [v, y(s)]_{V, V'} ds + (M(t), v)
\]  

(1.19)

holds \( \mathbb{P} \times \mathcal{B} \)-a.s. on the set \( \{(\omega, t) : t < \tau(\omega)\} \), then

\[
\|x(t)\|_V^2 = \|x(0)\|_V^2 + 2 \int_0^t [x(s), y(s)]_{V, V'} ds + 2 \int_0^t (x(s), dM(s)) + \langle M \rangle_t
\]  

(1.20)

Proof. Since \( V \) is a separable Hilbert space, there exists a CONS \( \{h_k\}_{k \in \mathbb{N}} \) in this space. Then, equation (1.19) for \( v = h_k \) implies

\[
x_k(t) = x_k(0) + \int_0^t [h_k, y(s)] ds + M_k(t),
\]  

(1.21)

where \( x_k(t) = (x, h_k)_V, M_k(t) = (M, h_k)_V \), and \( k \in \mathbb{N} \).

Note that, since \( x(\omega, t) : \Omega \times [0, T] \to V \) is an adaptive process, and \( x_k(\omega, t) = (x(\omega, t), h_k)_V \), using the continuity of inner product in \( V \), we find that \( x_k \) is \( \mathcal{F}_t \)-adapted for every \( k \in \mathbb{N} \). By the same arguments, we conclude that \( M_k(t) = (M(t), h_k)_V \) is a local martingale. Also, since \([\cdot, \cdot]_V\) is a bounded bilinear form on \( V \times V' \) we get that \([v, y(t)]_{V, V'}\) is \( \mathcal{F}_t \)-adaptive, and by (1.18) we deduce that

\[
P \left( \int_0^t [v, y(s)]_{V, V'}^2 ds < \infty \quad \forall t > 0 \right) = 1.
\]
Thus, we can apply one dimensional Itô’s formula to the process $x_k$ given by (1.21), by which we get

$$
x_k^2(t) = x_k^2(0) + 2 \int_0^t x_k(s) [h_k, y(s)] ds + 2 \int_0^t x_k(s) dM_k(s) + \langle M_k(t) \rangle, \quad (k \in \mathbb{N}).
$$

(1.22)

Summing up both parts of last equalities with respect to $k = 1, \ldots, N$, we find

$$\sum_{k=1}^N x_k^2(t) = \sum_{k=1}^N x_k^2(0) + 2 \int_0^t \sum_{k=1}^N x_k(s) [h_k, y(s)] ds + 2 \int_0^t \sum_{k=1}^N x_k(s) dM_k(s) + \sum_{k=1}^N \langle M_k(t) \rangle.
$$

(1.23)

In the last equality for every $(\omega, t) \in \Omega \times [0, T]$ such that $t \leq \tau(\omega)$ we pass to the limit with $N \to \infty$. By Parseval’s equality $\sum_{k=1}^\infty x_k^2(t) = \|x(t)\|^2$. Similarly we get $\|x(0)\|^2$.

Taking into account the continuity of inner product, $\lim_{N \to \infty} \int_0^t \sum_{k=1}^N x_k(s) [h_k, y(s)] ds = \int_0^t [x(s), y(s)]_{V, V'} ds$. By the definition $\lim_{N \to \infty} \int_0^t \sum_{k=1}^N x_k(s) dM_k(s) = \int_0^t (x(s), dM(s))$. Finally, by Lemma 1.4.1 the last term from LHS of (1.23) converges to $\langle M \rangle_t$. Hence we resume (1.19). Theorem is proved.

\[ \square \]

1.5 Existence and uniqueness of solution

For every differential equation, deterministic or stochastic, ordinary or with partial derivatives, the first and fundamental question is existence and uniqueness of solution. A general theory of this problem for SPDEs can be found in [8], [43], [70], [79]. In this section we present some known results about existence and uniqueness of solution of parabolic SPDEs, which will need for future studying.
Let \((V, H, V')\) be a triple of Hilbert spaces, \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) a stochastic basis, \(\mathcal{A}(t) : V \rightarrow V', \mathcal{M}_k(t) : V \rightarrow H, k \geq 1, t \in [0, T]\) some linear bounded operators. We consider the following SPDE

\[
du(t) = (\mathcal{A}u(t) + f(t))dt + (\mathcal{M}_k u(t) + g_k(t))dW_k(t),
\]

where \(t \in [0, T], u(0) = u_0 \in L_2(\Omega; H)\) and \(u_0\) is \(\mathcal{F}_0\)-measurable. We assume that \(W_k (k \in \mathbb{N})\) are independent standard Wiener processes, \(f, g_k (k \in \mathbb{N})\) are \(\mathcal{F}_t\)-adapted random processes such that \(f \in L_2(\Omega \times (0, T); V')\) and \(g_k \in L_2(\Omega \times (0, T); H)\) \((k \in \mathbb{N})\). Let \(\mathcal{V}\) be a dense subset of the Hilbert space \(V\) relative to the topology generated by the norm \(\| \cdot \|_V\). The following theorem holds true. For complete proof see for instance [70], Chapter 3.

1.5.1 Theorem. In addition to all assumptions mentioned above, suppose that

(i) \(\sum_{k=1}^{\infty} \int_0^T \mathbb{E}\|g_k(t)\|^2_H dt < \infty;\)

(ii) For every \(\varphi \in \mathcal{V}\), the processes \(\mathcal{A}\varphi(t)\) and \(\mathcal{M}_k\varphi(t)\) are \(\mathcal{F}_t\)-adapted;

(iii) \(\sum_{k=1}^{\infty} \int_0^T \mathbb{E}\|\mathcal{M}_k\varphi(t)\|^2_H dt < \infty\) for all \(\varphi \in \mathcal{V};\)

(iv) There exist a positive constant \(\delta\) and a real number \(C_0\) such that, for all \(t \in [0, T], \omega \in \Omega\) and all \(v \in \mathcal{V}\), we have

\[
2[\mathcal{A}\varphi(t), \varphi] + \sum_{k=1}^{\infty} ||\mathcal{M}_k\varphi(t)||^2_H \leq -\delta||\varphi||^2_V + C_0||\varphi||^2_H ;
\]

(v) There exist a positive number \(C_A\) so that, for all \(t \in [0, T], \omega \in \Omega, \varphi \in \mathcal{V},\)

\[
||\mathcal{A}\varphi||_{V'} \leq C_A||\varphi||_V.
\]
Then there exists a unique weak solution $u$ of equation (1.24), the solution belongs to $L_2(\Omega; C'((0, T); H)) \cap L_2(\Omega \times [0, T]; V)$ and

$$
\mathbb{E}\left( \sup_{0 < t < T} \|u(t)\|_{H}^2 \right) + \frac{\delta}{2} \mathbb{E}\left( \int_0^T \|u(t)\|_{V}'^2 dt \right) \leq \frac{C(C_A, C_0, T) \mathbb{E}\left( \|u_0\|_{H}^2 + C_f(\delta) \int_0^T \|f(t)\|_{V}'^2 dt + C_g(\delta) \sum_{k=1}^{\infty} \int_0^T \|g_k(t)\|_{H}^2 dt \right)}{}
$$

(1.26)

We want to mention that conditions (i)-(iii), (v) mostly are related to the definition of solution and come naturally from the structure of equation (1.24). The condition (iv) represents the (strong or super-) parabolicity or coercivity assumption and the equation in this case is called parabolic SPDE. This is due to the fact that in the case when operator $\mathcal{A}$ is an elliptic differential operator and $\mathcal{M}_k$ a differential operator subordinated, in some sense, to $\mathcal{A}$, the condition (iv) is fulfilled. For example classical heat equation is the simplest deterministic parabolic equation and a stochastic counterpart can be taken as $du(t, x) = au_{xx}dt + \sigma u_x dW(t)$, which is a particular case of equation (1.24) with $\mathcal{A} = a \partial^2_x$, $f(t) = g(t) = 0$, $\mathcal{M} = \sigma \frac{\partial}{\partial x}$. This equation is solvable in the triple $(H^1(\mathbb{R}), H^0(\mathbb{R}), H^{-1}(\mathbb{R}))$ if $\delta = 2a - \sigma^2 > 0$ or in equivalent form

$$
2[\mathcal{A}\varphi(t), \varphi] + ||\mathcal{M}\varphi(t)||_{H}^2 = a \int_{\mathbb{R}} \varphi_{xx}\varphi dx + \frac{\sigma^2}{2} \int_{\mathbb{R}} \varphi_x^2 dx
$$

$$
= - (2a - \sigma^2) \int_{\mathbb{R}} \varphi_x^2 dx = -\delta \|\varphi\|_{1}^2 + c_0 \|\varphi\|_{0}^2 ,
$$

which is exactly condition (iv) from Theorem 1.5.1. Note that $\delta = 0$ corresponds to the degenerate case $2a = \sigma^2$, and although the corresponding SPDE is a parabolic type equation, special studies have to be done in this case.
Similarly to deterministic equations the estimate (1.26) of the solution is called *energy estimate* which implies uniqueness of the solution.

For all equations considered in our research, first we will establish the existence and uniqueness of the solution by applying Theorem 1.5.1. Here we will show how this theorem can be applied to stochastic heat equation. Besides the fact that this equation is one of the simplest SPDE, it will also play the role of benchmark in our investigations.

**1.5.2 Example. Stochastic heat equation with additive space-time white noise.**

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) be a stochastic basis and assume that \(\gamma < -\frac{1}{2}\). In the triple of Hilbert spaces \((H^{\gamma+1}(0, \pi), H^\gamma(0, \pi), H^{\gamma+1}(0, \pi))\) we consider the following SPDE

\[
\begin{cases}
  du(t, x) = u_{xx} dt + \sum_{k \geq 1} h_k(x) dW_k(t), \\
  u(0, x) = u(t, 0) = u(t, \pi) = 0, \quad x \in [0, \pi], t \in [0, T],
\end{cases}
\]

(1.27)

where \(h_k(x) = \sqrt{\frac{2}{\pi}} \sin(k \pi x)\) and \(W_k\) are independent standard Brownian motions. Note that \(h_k\) forms a CONS in \(L^2(0, \pi)\) and also are all eigenvalues of the Laplace operator \(\Delta = \partial_{xx}\) with zero boundary conditions on \((0, \pi)\). Using the notations from Theorem 1.5.1 we have \(V = H^{\gamma+1}, H = H^\gamma, V' = H^{\gamma+1}, A = \Delta, f = 0, M_k = 0, g_k = h_k\). Now let us check the conditions (i)-(v) from Theorem 1.5.1. Since \(\gamma < -\frac{1}{2}\) we have \(\sum_{k \geq 1} h_k(x)^2 = \sum_{k \geq 1} k^{2\gamma} < \infty\), and hence (i) is satisfied. Operators \(A\) and \(M_k\) are deterministic and hence (ii) holds true. \(M_k = 0\) and (iii) obviously is satisfied. Coercivity condition (iv): \(2[A\varphi, \varphi] = 2[\varphi_{xx}, \varphi] = -2[\varphi_x, \varphi_x] = -2\varphi_x^2 \leq -\delta \varphi_{\gamma+1}^2\), with \(\delta = 2\), so (iv) is verified. Since \(A\) is a differential operator of second order, from general theory of Sobolev spaces, we have that the operator \(A : H^\gamma \rightarrow H^{s-2}\) is a linear bounded operator for every \(s \in \mathbb{R}\). Thus assumption
(v) is verified, and we conclude that for every $\gamma \leq -\frac{1}{2}$ there exists a unique solution $u \in L_2(\Omega; C((0,T); H^\gamma)) \cap L_2(\Omega \times [0,T]; H^{\gamma+1})$ of equation (1.27), and it satisfies the following estimate (energy type estimate)

$$
\mathbb{E} \left( \sup_{t \in (0,T)} \| u(t) \|_{H^\gamma}^2 + \int_0^T \| u(t) \|_{H^{\gamma+1}}^2 dt \right) \leq C.
$$

We conclude this chapter with a version of Burkholder-Davis-Gundy (BDG) inequality [8]. Let $W(t)$ be a standard Brownian Motion, and $\mathcal{F}_t$ the filtration generated by this Brownian Motion.

**1.5.3 Theorem.** Let $\xi$ be a square integrable $\mathcal{F}_t$ adapted process. For every $p > 0$ there exists constants $c_p$ and $C_p$, such that

$$
c_p \mathbb{E} \left( \left( \int_0^T |\xi(s)|^2 ds \right)^{\frac{p}{2}} \right) \leq \mathbb{E} \left( \sup_{0 < t < T} \left( \int_0^t |\xi(s) dW(s)| \right)^p \right) \leq C_p \mathbb{E} \left( \int_0^T |\xi(s)|^2 ds \right)^{\frac{p}{2}}.
$$

(1.28)
Chapter 2

Regularity of solution

2.1 Introduction

For a given evolution equation, besides existence and uniqueness of solution, it is important to know regularity properties of solution. By regularity or smoothness we mean continuity, Hölder continuity, differentiability, etc. of the solution with respect to both time and space variables. This problem have been widely studied by many authors, and various classes of SPDEs have been considered. Here we recall some results related to parabolic evolution equations.

As an application to neurophysiology, J.B.Walsh [79] considered the stochastic heat equation on a finite interval, namely the equation of the form

\[
\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} - v + f(v, t) \dot{W} \\
0 < x < L; \\
\frac{\partial v}{\partial x}(0, t) &= \frac{\partial v}{\partial x}(L, t) = 0 \\
v(x, 0) &= v_0(x), \\
&x \in [0, L],
\end{aligned}
\]

where \( v_0 \) is \( \mathcal{F}_0 \)-measurable, \( E(v_0^2(x)) \) is bounded and \( f \) is uniformly Lipschitz continuous. In [79] was proven that the mild solution of equation (2.1) \( (t, x) \rightarrow v(t, x) \) is Hölder continuous in \( t \) with exponent \( \frac{1}{4} - \varepsilon \), and in \( x \) with exponent \( \frac{1}{2} - \varepsilon \), for all \( \varepsilon > 0 \).

A general linear parabolic evolution equation of the form

\[
du = (a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu + f)dt + (\sigma_{ik}u_{x_k} + \nu_k u + g_k) dW_k(t), \quad t > 0,
\]
with proper conditions on the coefficients, boundary values and initial conditions, has
been considered in [37]-[45], [50],[51]. In these papers the authors studied the Cauchy
problem for the whole space \(x \in \mathbb{R}^d\) and half space \(x \in \mathbb{R}^d, x_1 > 0\), the Dirichlet
problem for a bounded domain, developing \(L_p\) theory for these equations. By \(L_p\) the-
ory we mean that suitable Sobolev spaces and sufficient conditions on the coefficients
are found such that equation (2.2) has a unique solution in these spaces. From this
theory, in particular, one can conclude about regularity properties of the solution. For
example, N.V.Krylov [42] studied the Cauchy problem for equation (2.2) is case of
whole space \(\mathbb{R}^d\), and proved, as an application of \(L_p\) theory, that for \(d = 1\), the solu-
tion is Hölder continuous in \(x\) of order \(1/2 - \varepsilon\) and in \(t\) of order \(1/4 - \varepsilon\). The Dirich-
let problem on bounded domain for the same equation was discussed by N.V.Krylov,
S.Lototsky in [44],[45], where the weighted Sobolev spaces were constructed. In par-
ticular the equation (2.1) is considered, up to the boundary conditions (Dirichlet in
[45] and Neumann in [79]), and the result for \(d = 1\) agrees with that obtained in [79].

In R.Mikulevicius [54] and R.Mikulevicius, H.Pragarauskas [55] the second-order
linear parabolic SPDE with deterministic leading coefficients driven by a cylindrical
Brownian motion in some Hilbert space were considered. Following the ideas given in
B.Rozovskii [69], R.Mikulevicius [54] studied the Cauchy problem on the whole space
\((x \in \mathbb{R}^d)\) in the scale of Hölder spaces, while in [55] the Cauchy-Dirichlet problem
on the half-space in suitable weighted Hölder spaces is investigated. Since solution
belongs to Hölder space the conclusion about the smoothness of the solution follows.

Finally we want to mention the results presented in DaPrato [8], a general linear
evolution equations is investigated. More precisely, the coefficients of equation are
some linear operators which generates a strongly continuous semigroup, and the noise,
either additive (Chapter 5) or multiplicative (Chapter 6), is driven by a cylindrical
Brownian Motion in some Hilbert space. It is shown that the weak (mild) solution has a continuous path and some particular examples are considered too.

In this chapter we will study the regularity properties of equation (1.24) driven by space-time white noise, either additive or multiplicative. Similar to Chapter 1 we consider $M$ being a $d$-dimensional compact orientable $C^\infty$ manifold with a smooth positive measure $dx$. Let $\mathcal{A}$ be a differential, elliptic operator of order $2m$ on $M$, formally selfadjoint and semi-lower bounded. Without loss of generality, using the notations from Section 1.1, we consider $\mathcal{L} = -\mathcal{A}$ and apply all results mentioned therein. The operator $\Lambda = (-\mathcal{A})^{1/2m}$ is well-defined, elliptic of order 1 and generates the scale $\{H^s(M)\}_{s \in \mathbb{R}}$ of Sobolev spaces. We denote by $h_k$ and $\lambda_k$, $k \in \mathbb{N}$, the eigenfunctions and corresponding eigenvalues of the operator $\Lambda$, and by $l_k$ the eigenvalues of the operator $\mathcal{A}$. Recall that $\{h_k\}_{k \in \mathbb{N}}$ forms a CONS in $L^2(M, dx)$ (for more details, see Section 1.1).

In the triple $(H^{\gamma+m}, H^\gamma, H^{\gamma-m})$, for some $\gamma \in \mathbb{R}$, we consider the following equation

$$
\begin{cases}
  du(t) = \mathcal{A}u(t)dt + \sum_{k=1}^{\infty} \alpha_k \mathcal{B}_k(u)h_k(x)dW_k(t);
  \\
u(t, \cdot)|_{\mathcal{M}} = 0, & 0 < t < T;
  \\
u(0, x) = u_0(x), & x \in M,
\end{cases}
$$

(2.3)

where $t \in [0, T]$, $x \in M$, $u_0 \in C^\infty(0, \pi)$, $\mathcal{M}_k$ are operators acting in $H^\gamma$ and $\alpha_k = k^\alpha$ for some real parameter $\alpha$. Using results from Chapter 1, we will find sufficient conditions on parameters $\alpha$ and $\gamma$, and operators $\mathcal{A}$, $\mathcal{B}_k$, which guarantee existence and uniqueness of solution in the scale of Sobolev spaces $H^\gamma(M)$.

A major novelty is the factor $\alpha_k$ in the noise term; in all previous works $\alpha = 0$. The specific form of this factor is motivated by the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}$. It turns out that the parameter $\alpha$ controls the smoothness
of the solution $u$ of equation (2.3). As one may expect, as $\alpha$ decreases the solution becomes smoother both in time and space. We state the regularity properties (number of derivatives and Hölder continuity) of the solution in both space and time in terms of the parameter $\alpha$, the $L_\infty$ norm and some regularity properties of the eigenfunctions of the operator $A$.

In connection with this we make the following reasonable assumptions

(A1) $\|h_k\|_\infty \leq c |l_k|^\rho$, for some $\rho \in \mathbb{R}_+$ and every $k \in \mathbb{N}$;

(A2) $|h_k(x) - h_k(y)| \leq |l_k|^\delta(\beta)|x - y|^{\beta}$, where $0 < \beta \leq 1$, $x, y \in M$, $k \in \mathbb{N}$, and $\delta$ is an increasing function, with positive values.

We rely on some results from spectral theory of self-adjoint differential operators, which is a separate topic in mathematics, with many fundamental results. Even the particular question of pointwise bounds of eigenfunctions have been studied widely. There are many papers written on this subject, where various operators are considered, and this question represents a well explored field. We refer the reader to classical work [61], and as well as to some recent papers [15], [19], [17], [18], [29], [30], [31], [32], [72], [73].

Of course the estimates vary from case to case and depend on the operator $A$, as well as on the geometry of the domain $M$ and boundary conditions. One general result holds true.

2.1.1 Remark. Without making any geometric assumptions, if $A$ is the Laplacian on compact manifolds with boundary, (A1) holds true with $\rho = \frac{d-1}{4}$ (see the proof in [15]). In particular, $\|h_k\|_\infty < c |l_k|^{\frac{d-1}{4}}$ holds true for a general bounded domain in $\mathbb{R}^d$. This result is sharp for the sphere $M = S^d$. 

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We recall that regardless of the domain $M$, the eigenvalues $\mu_k$ of the operator $A$ with zero boundary conditions have the following asymptotic

$$l_k \sim k^{2m/d},$$

(2.4)

and consequently $\lambda_k \sim k^{1/d}$ (see for instance [76]).

We want to mention that generally speaking the conditions (A1) and (A2) are hard to check, and represent difficult mathematical problems, however in many applications the eigenvalues and eigenfunctions are computable explicitly and these conditions are verified by simple arithmetic evaluations. Let us consider the one dimensional Laplace operator with zero boundary conditions.

2.1.2 Example. One dimensional Laplacian.

Assume that $d = 1$, $Au = u_{xx}$, $m = 1$, $M = [0, \pi]$ and zero boundary conditions. Then $h_k(x) = \frac{2}{\pi} \sin(kx)$, and hence $\|h_k\|_\infty \leq \text{const}$. Thus (A1) is satisfied with $\rho = 0$. Also note, that $l_k = -k^2$, and thus (2.4) also holds true. Using well-known inequality $|\sin(x)| \leq |x|$, and a trivial trigonometric identity, we have $|h_k(x) - h_k(y)|^\beta \leq k^\beta |x - y|^\beta$ for all $x, y \in [0, \pi]$ and every $\beta > 0$. Obviously we have that $|h_k(x) - h_k(y)| \leq 2$, $x, y \in [0, \pi]$. From here we conclude

$$|h_k(x) - h_k(y)| = |h_k(x) - h_k(y)|^\beta |h_k(x) - h_k(y)|^{1-\beta} \leq k^\beta |x - y|^{1-\beta} \leq c |x - y|^{\beta},$$

(2.5)

where $c$ is a constant independent of $\beta$. Hence (A2) is verified with $\delta(\beta) = \frac{\beta}{2}$.

More concrete examples will be considered at the end of this chapter.
Remark. Conditions (A2) in fact represents the Hölder continuity of the eigenfunctions $h_k$. Since $h_k \in C^\infty$, the constant $\zeta$ from inequality $|h_k(x) - h_k(y)| \leq \zeta^\beta |x - y|^\beta$ corresponds to the Lipschitz constant of the function $h_k$, which can be associated with the "first derivative". On the other hand, "the first derivative" essentially is $\Lambda$ which is a differential operator of order one. Since $\Lambda(h_k) = \lambda_k h_k$ and $\lambda_k = l_k^{\frac{3}{2}}$, we get $\delta(\beta) = \frac{\beta}{2m}$. These completely heuristic arguments turn out to hold true in many concrete examples. However, we do not know if this is true in general.

The results obtained here agree with those known and mentioned above. We would like to note that equation (2.3) is a particular case of equations studied in [50], however, it turns out that to check all abstract conditions on coefficients and spaces of solution is not easier than establishing these results directly. Also, we want to note that equation (2.3) does not belong to the class of equations considered in [54] (whole space), [55] (half plane). Not only because we consider the bounded domain, which to some extent is just a technical problem as long as there are theories for the whole space and half plane, but also the noise term is different.

A similar approach to control the smoothness of the solution through some parameters of the noise term are considered in [60] where the nonlinear SPDE $du = \frac{1}{2} \Delta u dt + \sqrt{u} dW(t, x)$ is studied.

Mytnik, Perkins and Sturm in a recent paper [60] study regularity properties of the solution of the equation $du = \frac{1}{2} \Delta u dt + \sqrt{u} dW(t, x)$, where $x \in \mathbb{R}$, $t \in [0, T]$ and $W(t, x)$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$. The method of duality is used to establish the existence and uniqueness of the solution. The solution itself represents the density function for one-dimensional super-Brownian motion. The regularity properties of the solution are described in terms of asymptotic behavior of the kernel of the covariance functional on the main diagonal $y = x$. It turns out that there is
a connection between our results and results stated in [60]. We will discuss these in more details in Section 2.2.

In terms of regularity properties, the results obtained here are the same for both additive and multiplicative noise, however, the spaces in which equation (2.3) has solution differ. Also, the technical evaluations in multiplicative case are more involved. Thus, we consider separately the equations with additive noise, Section 2.2, and equations with multiplicative noise in Section 2.3. The main results of this chapter are contained in Theorem 2.2.5, 2.2.11 and 2.3.8. Every theorem will be followed by a simple example, usually stochastic heat equation. In the last section of this chapter, as applications of general results, we consider various concrete examples.

2.2 Equations with additive noise

2.2.1 Existence and uniqueness

In this section we consider equation (2.3) with additive noise, which means that we take $B_k u \equiv 1$. Thus equation (2.3) becomes

$$du(t) = Au(t)dt + \sum_{k=1}^{\infty} \alpha_k h_k(x) dW_k(t),$$

(2.6)

where $t \in [0,T], \ x \in M$, the same initial values and boundary conditions as in equation (2.3), and $\alpha_k = k^\alpha$ for some real parameter $\alpha$.

Initially we will establish some results about existence and uniqueness of the solution. For this, we make the following assumption:

(A3) Assume that $\alpha, \ \gamma \in \mathbb{R}$ such that $\gamma < -d(\alpha + \frac{1}{2})$.  


2.2.1 Theorem. If the assumption (A3) is satisfied, then equation (2.6) has a unique weak solution

\[ u \in L_2((0, T) \times \Omega; H^{\gamma+m}) \cap L_2(\Omega; C((0, T); H^{\gamma})). \]

Proof. We apply Theorem 1.5.1 with \( f = 0 \), \( M_k = 0 \), and \( g_k = \alpha_k h_k, \ k \in \mathbb{N} \). Conditions (ii), (iii), (v) of this theorem are obviously satisfied. In order to check condition (i) we write

\[
\sum_{k=1}^{\infty} \int_0^T \mathbb{E} ||g_k(t)||_{\gamma}^2 \, dt = c \sum_{k=1}^{\infty} \alpha_k^2 ||h_k||_{\gamma}^2 = c \sum_{k=1}^{\infty} k^{2\alpha} \lambda_{k}^{2\gamma}. 
\]

Since \( \lambda_k \sim k^{\frac{1}{2}} \), from the last equality we deduce

\[
\sum_{k=1}^{\infty} \int_0^T \mathbb{E} ||g_k(t)||_{\gamma}^2 \, dt = c \sum_{k=1}^{\infty} k^{2\alpha} k^{2\gamma}. 
\]

The last series converges in virtue of Assumption (A3).

Finally, we verify condition (iv). It suffices to check it for basis functions \( h_k \), which leads to

\[
2[A h_k, h_k] = 2l_k [h_k, h_k] = 2l_k ||h_k||_{\gamma}^2 = 2l_k \lambda_k^{-2m} ||h_k||_{\gamma+m}^2 = -2||h_k||_{\gamma+m}^2. 
\]

Thus (iv) from Theorem 1.5.1 is satisfied with \( \delta = 2 > 0 \). Theorem is proved.

2.2.2 Remark. Note that for \( \alpha = 0 \), which corresponds to standard space-times white noise, the solution exists if \( \gamma < -d/2 \). In particular, for \( d = 1 \), we have that the solution exists if \( \gamma < -1/2 \) (this covers well know result, see for instance [79]). Also, we want to mention that the equations driven by additive space-time white noise have
solution in some $H^\gamma$ regardless of space dimension $d$. Namely, for every $d \in \mathbb{N}$ and every $\alpha \in \mathbb{R}$ there exists $\gamma \in \mathbb{R}$ such that $u \in L_2((0, T) \times \Omega; H^\gamma)$. As will see latter on, this is not the case for the equations with multiplicative space-time white noise.

2.2.3 Example. Assume that $d = 1$, $M = [0, \pi]$ and let $A$ be the one dimensional Laplace operator with zero boundary conditions. In the triple of Hilbert spaces $(H^{\gamma+1}, H^\gamma, H^{\gamma+1})$ let us consider the following SPDE

$$du(t, x) = u_{xx}(t, x)dt + \sum_{k \geq 1} k^{-\frac{3}{2}} \sin(kx)dW_k(t), \quad (2.7)$$

where $x \in [0, \pi]$, $t \in [0, 1]$ and $W_k$ are independent standard Brownian motions. As we already mentioned (see Example 1.5.2), the functions $h_k$ are the eigenfunctions of the operator $A$ and forms a CONS in $L_2$. Note that $\alpha = -\frac{2}{3}$. By Theorem 2.2.1, for every $\gamma < \frac{1}{6}$, there exists a unique strong solution of equation (2.7) in $H^{\gamma+1}$.

Looking forward for regularity properties of the solution $u$, we present here the plot of one realizations of the field $u$ given by (2.6) for different values of the parameter $\alpha$. In the Figure 2.1, Panel (a), the solution corresponds to the equation discussed in Remark 2.2.2, i.e. $\alpha = 0$, $d = 1$. The solution of equation (2.7) from Example 2.2.3 ($\alpha = -\frac{2}{3}$) is shown in Panel (b). We used Crack-Nicolson finite difference scheme (see for instance [38], [57]) to simulate the field $u$. Note that the solution $u$ looks smoother, Panel (b), for $\alpha = -\frac{2}{3}$ than that for the parameter $\alpha = 0$, Panel (a). How smoother it is, we will answer in the next subsections.
Figure 2.1: Realization of approximated solution of stochastic heat equation with additive noise

In what follows we will use a convenient representation for the solution \( u \) of equation (2.6). Denote by \( G_t(x, y) \) the fundamental solution of the corresponding deterministic parabolic equation \( u_t = Au(t) \) with the same boundary values and initial conditions as (2.3). Then (see for instance [9], Chapter 2.7)

\[
G_t(x, y) = \sum_{k \geq 1} h_k(x)h_k(y)e^{\lambda_k t}.
\] (2.8)

Similarly to deterministic case, it can be shown that the weak solution of equation (2.6) satisfies the equality

\[
u(t, x) = \int_0^\pi G_t(x, y)u_0(y)dy + \sum_{k \geq 1} \int_0^t \int_M \alpha_k G_{t-s}(x, y)h_k(y)dW_k(s).
\] (2.9)

The proof can be found in [8], Section 5.2, page 121. Note that by (2.9), \( u \) actually is a mild solution of equation (2.6), and by specific structure of this equation it turns out that the mild solution coincides with weak solution, existence of which is provided by Theorem 2.2.1.
From (2.9) we see that without loss of generality we can suppose that \( u_0 = 0 \) and study the regularity of the second term in (2.9). Thus using (2.8), from (2.9) we get

\[
u(t, x) = \sum_{k \geq 1} \int_0^t \int_M \alpha_k \sum_{j \geq 1} h_j(x)h_j(y)e^{\ell_j(t-s)}h_k(y)dydW_k(s),
\]

and taking into account that \( \{h_k\}_{k \geq 1} \) forms an orthonormal basis in \( L_2(M) \), we have the representation

\[
u(t, x) = \int_0^t \sum_{k \geq 1} \alpha_k h_k(x)e^{\ell_k(t-s)}dW_k(s).
\]

### 2.2.2 Regularity in space

To establish the regularity properties of solution \( u \) in space variable \( x \), we will prove an auxiliary result.

#### 2.2.4 Lemma. Under assumption (A2), for any \( \beta \in (0, 1] \) and \( \alpha \in \mathbb{R} \) such that

\[
\alpha < \frac{m(1 - 2\delta(\beta))}{d} - \frac{1}{2},
\]

the solution \( u \) of equation (2.6) satisfies the following inequality

\[
E\left|u(t, x) - u(t, y)\right|^p \leq C|x - y|^{\beta p},
\]

where \( x, y \in M, t \in [0, T], \ p > 2. \)

**Proof.** From (2.10) one deduces

\[
E\left|u(t, x) - u(t, y)\right|^p = \mathbb{E}\left[\int_0^t \sum_{k \geq 1} \left( h_k(x) - h_k(y) \right) \alpha_k e^{\ell_k(t-s)}dW_k(s)\right]^p.
\]
From here by the BDG inequality (1.28), we continue
\[
\mathbb{E} \left| u(t, x) - u(t, y) \right|^p \leq \left| \sum_{k \geq 1} \int_0^t \left| h_k(x) - h_k(y) \right|^2 d\alpha_k e^{2l_k(t-s)} ds \right|^{|p/2|} \\
= c_p \left| \sum_{k \geq 1} \left| h_k(x) - h_k(y) \right|^2 \frac{\alpha_k^2}{-2l_k} \left( 1 - e^{2l_k t} \right) \right|^{|p/2|} .
\]

Note that \( l_k < 0, \ k \in \mathbb{N} \), and thus \( 1 - e^{2l_k t} \) is uniformly bounded in \( t \) and \( k \), hence
\[
\mathbb{E} \left| u(t, x) - u(t, y) \right|^p \leq c_p \left| \sum_{k \geq 1} \frac{\alpha_k^2}{|l_k|} \left| h_k(x) - h_k(y) \right|^2 \right|^{|p/2|} . \tag{2.13}
\]

From here, by assumption (A2), we get
\[
\mathbb{E} \left| u(t, x) - u(t, y) \right|^p \leq c_p |x - y|^{3p} \left\| \sum_{k \geq 1} k^{2\alpha} |l_k|^{-1} |l_k|^{2\delta(\beta)} \right\|^{|p/2|} .
\]

Finally, since \( l_k \sim k^{2m/d} \), we find that the last series converges if
\[
2\alpha + \frac{2m}{d} (2\delta(\beta) - 1) < -1 ,
\]
which essentially is assumption (2.11) on parameters \( \beta \) and \( \alpha \). Lemma is proved. \( \Box \)

Now we are ready to prove the theorem about regularity of the solution in space variable \( x \).
2.2.5 Theorem. Suppose that Assumptions (A2) and (A3) are satisfied, and \( \alpha < -\frac{1}{2} + \frac{m}{d} \). Let \( u \) be the solution of equation (2.6) and \( r \) the biggest integer such that there exists \( \beta_0 \in (0, 1] \) which satisfies the following inequality

\[
2d\alpha + 2r + 4m\delta(\beta_0) - 2m < -d.
\]

Then \( \Lambda^r(u) \) is Hölder continuous in \( x \) of order

\[
\zeta := \sup \left\{ \beta \mid \delta(\beta) < -\frac{d}{4m} - \frac{d\alpha}{2m} + \frac{1}{2} - \frac{r}{2m}, \beta \in (0, 1] \right\} - \varepsilon,
\]

for every \( \varepsilon > 0 \), as long as \( \zeta > 0 \);

Proof. By the initial assumption, the solution \( u \) of the equation (2.6) exists and satisfies Lemma 2.2.4 for some \( \beta > 0 \). Let \( r \) be the integer as stated in the theorem. Note that \( \alpha < -\frac{1}{2} + \frac{m}{d} \) implies that \( r \geq 0 \). By (2.10) we have

\[
\Lambda^r(u(t, x)) = \Lambda^r\left[ \int_0^t \sum_{k \geq 1} \alpha_k h_k(x) e^{l_k(t-s)} dW_k(s) \right]
\]

\[
= \sum_{k \geq 1} \alpha_k \Lambda^r(h_k(x)) \int_0^t e^{l_k(t-s)} dW_k(s)
\]

\[
= \sum_{k \geq 1} \alpha_k \Lambda^r h_k(x) \int_0^t e^{l_k(t-s)} dW_k(s)
\]

as long as the last series converges absolutely. Since \( \lambda_k \sim k^{\frac{3}{2}} \), we apply Lemma 2.2.4 to \( \Lambda^r u \) with \( \tilde{\alpha} = \alpha + \frac{3}{d} \) and conclude that

\[
\left| \Lambda^r(u(t, x)) - \Lambda^r(u(t, y)) \right|^p \leq c_p |x - y|^{\beta_p},
\]

(2.15)
for every $\beta > 0$ such that $\delta(\beta) < \frac{1}{2} - \frac{r}{2m} - \frac{d\alpha}{2m} - \frac{d}{4m}$. Taking into account the definition of $\zeta$ we have that the inequality (2.15) is satisfied for every $\beta \leq \zeta$. By Kolmogorov’s criterion (Theorem 1.1.1) with $d := d, p := p, q := \beta p - d$, we get that the process $\Lambda^r u(t, x)$ satisfies the following inequality

$$\left| \Lambda^r u(t, x) - \Lambda^r u(t, y) \right| \leq Y|x - y|^\beta \left( \ln \frac{A}{|x - y|} \right)^{2/p} \quad (x, y \in M, p > 2)$$

where $Y$ is a positive random variable, and $A$ is a positive real number. For every $\varepsilon > 0$ the function $|x|^\varepsilon \left( \ln \frac{A}{|x|} \right)^{2/p}$ is continuous at $x = 0$. Hence, the process $\Lambda^r u(t, x)$ is Hölder continuous in $x$ of order $\beta - \frac{1}{p} - \varepsilon$, for every $p > 2$ and $\varepsilon > 0$. Consequently, $\Lambda^r u(t, x)$ is Hölder continuous in $x$ of order less than $\beta \leq \zeta$. Theorem is proved.

2.2.6 Remark. As we already mentioned, the operator $\Lambda^r, r \in \mathbb{N}$, plays the role of the differential operator of order $r$. Generally speaking, Theorem 2.2.5 tells how many derivatives has the solution $u$ and what is the order of continuity of the $r$-th derivative. Moreover, the theorem remains true if we take any other operator $\Psi$ instead of $\Lambda^r$, as long as one can show that $\Psi h_k = c \lambda_k h_k$ (see equality (2.14)).

One special case that occurs in many application is $\delta(\beta) = \beta \frac{2}{m}$ (for more motivations about this see also Remark 2.1.3). For this particular situation Theorem 2.2.5 implies:

2.2.7 Theorem. Suppose that Assumptions (A2) and (A3) are satisfied, and assume that $\alpha < -\frac{1}{2} + \frac{m}{d}$ and $\delta(\beta) = \beta \frac{2}{m}, \beta \in (0, 1]$. Let $u$ be the solution of equation (2.6), and denote by $\eta := m - d\alpha - \frac{d}{2}$ and $r := \lfloor \eta \rfloor \lfloor \eta \rfloor$ denotes the biggest integer less than or equal to $\eta$).

(i) if $\eta \neq \mathbb{N}$ then $\Lambda^r(u)$ is Hölder continuous of order $\zeta = \eta - r - \varepsilon$, for every $\varepsilon > 0$, as long as $\zeta > 0$;
(ii) if $\eta \in \mathbb{N}$, then $\Lambda^{-1}(u)$ is Hölder continuous of order $\zeta$, for every $\zeta \in (0, 1)$.

2.2.8 Example. As an application of Theorem 2.2.7 we will consider the following SPDE: $du(t, x) = u_{xx}(t, x)dt + \sum_{k \geq 1} k^{-\frac{3}{2}} \sin(kx)dW_k(t)$ under the same setup as in Example 2.2.3, i.e. the stochastic heat equation with Dirichlet boundary conditions, zero initial values, $\alpha = -\frac{2}{3}$, $d = 1$, $x \in M = [0, \pi]$, $t \in [0, 1]$. By Example 2.2.3 we have that the solution $u \in H^{\gamma+1}$ for every $\gamma < \frac{1}{6}$. As we concluded in Example 2.1.2, Assumption (A2) is verified and $\delta(\beta) = \beta^2$. Hence we can apply Theorem 2.2.7. Recall that $h_k(x) = \frac{2}{\pi} \sin(kx)$ and $\lambda_k = k$. Since $\frac{\partial}{\partial x} h_k = c \lambda_k \tilde{h}_k$, where $\tilde{h}_k$ is either $\sin(kx)$ or $\cos(kx)$, which essentially obey the same regularity properties, by Remark 2.2.6, we can take $\Lambda = \frac{\partial}{\partial x}$. Hence, by Theorem 2.2.7 with $\eta = \frac{7}{6}$ we conclude: the solution $u$ has one derivative in $x$, and the first derivative $u_x$ is Hölder continuous of order less than $\frac{1}{6}$.

By the same arguments, we have that the solution $u(t, x)$ of equation $du(t, x) = u_{xx}(t, x)dt + \sum_{k \geq 1} \sin(kx)dW_k(t)$ (see Example 1.5.2) is Hölder continuous in $x$ of order less than $\eta = m - d\alpha - \frac{1}{2} = 1 - 0 \cdot 1 - \frac{1}{2} = \frac{1}{2}$.

By the same arguments, we have that the solution $u(t, x)$ of equation $du(t, x) = u_{xx}(t, x)dt + \sum_{k \geq 1} \sin(kx)dW_k(t)$ (see Example 1.5.2) is Hölder continuous in $x$ of order less than $\eta = m - d\alpha - \frac{1}{2} = 1 - 0 \cdot 1 - \frac{1}{2} = \frac{1}{2}$. This resume the classical result by Walsh [79]. Compare the graphs of these two stochastic fields shown on Figure 2.1.

2.2.9 Remark. We want to mention that the Theorem 2.2.5 about regularity of the solution can not be obtained by directly the Sobolev embedding theorems. For example, by Theorem 2.2.7 we have that the solution exists, is unique and $u \in H^{\gamma+m}$, for every $\gamma < -d\alpha - \frac{d}{2}$, i.e. $u \in H^{m - \frac{d}{2} - dx + \epsilon}$ for every $\epsilon > 0$. By Sobolev embedding theorem (see for instance see [9], Chapter 5) $H^s \subset C^{s - \frac{d}{2}}$, and hence $u \in C^{m - d\alpha - d - \epsilon}$. We proved that $u \in C^{m - d\alpha - \frac{d}{2} - \epsilon}$, which obviously is a stronger result than that obtained by Sobolev embedding theorems. However, if one can develop the $L_p$ theory for this equation, namely to show that the solution exists in $H^{\gamma+m}_p$ for every $p > 2$, then by Sobolev embedding theorem $u \in H^{m - \frac{d}{p} - dx + \epsilon} \subset C^{m - \frac{d}{p} - d\alpha +\frac{d}{p} - \epsilon}$. Taking $p$ large
enough we obtain our result, namely \( u \in C^{m - d\alpha - \frac{d}{2} - \varepsilon} \) for every \( \varepsilon > 0 \), as long as \( m - d\alpha - \frac{d}{2} - \varepsilon > 0 \). This is not of big surprise, since the main tool in our proof has been the Kolmogorov’s criterion 1.1.1, which to some extends is a version of Sobolev embedding theorem. In other words, one may conclude that we actually developed implicitly the \( L_p \) theory.

### 2.2.3 Regularity in time

Regularity in time is obtained in the same way as regularity in space. Initially we will prove an auxiliary result similar to Lemma 2.2.4.

**2.2.10 Lemma.** Under Assumption (A1), for every \( \alpha \in \mathbb{R} \) and \( \beta \in (0, 1] \) such that \( \beta < 1 - 2\rho - \frac{d(1 + 2\alpha)}{2m} \), the solution \( u \) of equation (2.6) satisfies the following inequality

\[
\mathbb{E} \left| u(t_1, x) - u(t_2, x) \right|^p \leq C |t_1 - t_2|^{\frac{\beta p}{2}}, \tag{2.16}
\]

where \( t_1, t_2 \in [0, T] \), \( x \in M \), \( p > 2 \).

**Proof.** Let \( 0 < t_1 < t_2 < T \). By (2.10)

\[
\mathbb{E} \left| u(t_1, x) - u(t_2, x) \right|^p = \mathbb{E} \left| \int_0^{t_1} \sum_{k \geq 1} h_k(x) \alpha_k e^{\ell_k(t_1-s)} dW_k(s) - \int_0^{t_2} \sum_{k \geq 1} h_k(x) \alpha_k e^{\ell_k(t_2-s)} dW_k(s) \right|^p \\
= \mathbb{E} \left| \int_0^{t_2} \sum_{k \geq 1} h_k(x) \alpha_k \left( e^{\ell_k(t_2-s)} - e^{\ell_k(t_1-s)} 1_{s \leq t_1} \right) dW_k(s) \right|^p.
\]
By the BDG inequality, Theorem 1.5.3, we continue

\[
E \left| u(t_1, x) - u(t_2, x) \right|^p
\leq E \left| \int_0^{t_2} \sum_{k \geq 1} h_k^2(x) \alpha_k^2 \left| e^{k(t_2-s)} - e^{k(t_1-s)} \mathbb{I}_{s \leq t_1} \right|^2 ds \right|^{p/2}
\leq E \left| \sum_{k \geq 1} h_k^2(x) \alpha_k^2 \int_0^{t_2} \left| e^{k(t_2-s)} - e^{k(t_1-s)} \mathbb{I}_{s \leq t_1} \right|^2 ds \right|^{p/2}.
\]

After direct evaluations one shows

\[
\int_0^{t_2} \left| e^{k(t_2-s)} - e^{k(t_1-s)} \mathbb{I}_{s \leq t_1} \right|^2 ds
= \frac{1}{-2l_k} \left( 2 \left( 1 - e^{k(t_2-t_1)} \right) - \left( e^{k(t_2)} - e^{k(t_1)} \right)^2 \right)
\leq \frac{1 - e^{k(t_2-t_1)}}{-l_k}.
\]

Thus,

\[
E \left| u(t_1, x) - u(t_2, x) \right|^p \leq c \left| \sum_{k \geq 1} h_k^2(x) \alpha_k^2 \frac{1 - e^{k(t_2-t_1)}}{|l_k|} \right|^{p/2}.
\]

Since \(|1 - e^{-\mu|t_2-t_1|}| \leq \mu^\beta |t_2 - t_1|^\beta\) for every \(0 < \beta \leq 1\) and \(\mu > 0\), using Assumption (A1) the last inequality can be continued

\[
\leq \left| \sum_{k \geq 1} h_k^2(x) \alpha_k^2 \beta \sum_{k \geq 1} \frac{1|t_2 - t_1|^{p\beta} }{|l_k|} \right|^{p/2} \leq |t_2 - t_1|^p \left| \sum_{k \geq 1} \frac{\alpha_k^2 \beta \cdot l_k^{2p} |l_k|^{p\beta-1} }{|l_k|} \right|^{p/2}.
\]
By asymptotic property (2.4), and the definition of $\alpha_k = k^\alpha$, from the last inequality we have the following estimation

$$
\mathbb{E}\left| u(t_1, x) - u(t_2, x) \right|^p \leq c|t_2 - t_1|^{\frac{\beta p}{2}} \sum_{k \geq 1} k^{2\alpha + \frac{2m}{d} (\beta - 1 + 2\rho)}.
$$

(2.17)

The initial assumption $\beta < 1 - 2\rho - \frac{d}{2m}(1 + 2\alpha)$ implies that the last series converges, hence the lemma is proved.

Now we are ready to prove the Theorem about regularity in $t$ of the solution $u(t, x)$

2.2.11 Theorem. Suppose that $\alpha < \frac{m}{d} - \frac{2mp}{d} - \frac{1}{2}$, and assume that the assumptions (A1) and (A3) are satisfied. Then the solution $u$ of equation (2.6) is Hölder continuous in $t$ of order less than $\eta := \min\left\{ \frac{1}{2}, \frac{1}{2} - \rho - \frac{d(1 + 2\alpha)}{4m} \right\}$.

Proof. Assumptions (A1) and (A3) imply that the equation (2.6) has a solution. Let $\beta = 2\tau$. By initial assumption $\alpha < \frac{m}{d} - \frac{2mp}{d} - \frac{1}{2}$ we have that $\eta > 0$, and hence $\beta \in (0, 1]$. Consequently, by Lemma 2.2.10 the solution $u$ of equation (2.6) satisfies the inequality $\mathbb{E}\left| u(t_1, x) - u(t_2, x) \right|^p \leq C|t_1 - t_2|^{\frac{\beta p}{2}}$, where $t_1, t_2 \in [0, T]$, $x \in M$, $p > 2$. Applying the Kolmogorov’s criterion (Theorem 1.1.1) with $d := d, p := p, q := \frac{\beta p}{2} - d$, we get

$$
\left| u(t_1, x) - u(t_2, x) \right| \leq Y|t_1 - t_2|^{\frac{\beta p}{2} - \frac{d}{\tau}} \left( \ln \frac{A}{|t_1 - t_2|} \right)^{2/p}.
$$

(2.18)

By the same arguments as in Theorem 2.2.7 we conclude that the process $u$ is Hölder continuous in $t$ of any order less than $\frac{\beta p}{2} = \eta$. Theorem is proved.

Taking $\mathcal{A} = \Delta$, and without making any geometric assumption on $M$, a direct consequence of Theorem 2.2.11 gives:
2.2.12 Corollary. Suppose that $A$ is the Laplacian on a compact manifold with boundary, and assume that $γ < −dα − \frac{d}{2}$ and $α < \frac{3}{2d} − 1$. Then the solution $u$ of equation (2.6) is Hölder continuous in $t$ of order less than $η = \min\{\frac{1}{2}, \frac{3}{4} − \frac{dα}{2} − \frac{d}{2}\}$.

**Proof.** By Remark 2.1.1 $ρ ≤ \frac{d−1}{4}$. We apply the previous theorem, with $m = 1$ and above mentioned $ρ$ and conclude that for every $α < \frac{m}{d} − \frac{2mρ}{d} − \frac{1}{2} = \frac{3}{2d} − 1$ the solution $u$ is Hölder continuous in $t$ of any order less then $η = \min\{\frac{1}{2}, \frac{1}{2} − ρ − \frac{d(1+2α)}{4m}\} = \min\{\frac{1}{2}, \frac{3}{4} − \frac{d}{2} − \frac{da}{2}\}$. □

2.2.13 Example. Let us consider the same equation as in Example 2.2.8: $du(t, x) = u_{xx}(t, x)dt + \sum_{k≥1} k^{−\frac{2}{3}} \sin(kx)dW_k(t)$, i.e. the stochastic heat equation with Dirichlet boundary conditions, zero initial values, $α = −\frac{2}{3}$, $d = 1$, $x ∈ M = [0, π]$, $t ∈ [0, 1]$. Recall that the solution $u ∈ H^{γ+1}$ for every $γ ≤ \frac{1}{6}$. Since $u_{xx}$ is the Laplace operator, by Corollary 2.2.12 we have that the solution $u$ is Hölder continuous in time variable $t$, and the order of continuity is $η = \min\{\frac{1}{2}, \frac{7}{12}\} − ε = \frac{1}{2} − ε$ for every $ε ∈ (0, 1/2)$.

2.2.14 Remark. Theorem 2.2.7 implies that order of continuity of the solution $u$ can not exceed $\frac{1}{2}$. This property comes natural since the Brownian motion $W$ is a continuous, nowhere differentiable process, Hölder continuous of order $\frac{1}{2} − ε$ for every $0 < ε < \frac{1}{2}$.

2.3 Equations with multiplicative noise

2.3.1 Existence and uniqueness

In this section we study the equation (2.3) in the case of multiplicative noise. For simplicity of writing we take $B_ku = u$, but the results remain valid for $B_ku = g(u)$ where $g$ is global Lipschitz continuous. Generally speaking, the methods used here
can be applied to general operators $B_k : \mathbb{H}^{\gamma+m} \to \mathbb{H}^\gamma$, however we postpone these
discussion for our future research and the results will be published somewhere else.

Thus, keeping the same notations and boundary conditions as in the previous sec-
tion, we consider the following equation

$$du(t) = Au(t)dt + \sum_{k=1}^{\infty} \alpha_k u(t) h_k(x) dW_k(t) ,$$

(2.19)

where $t \in [0, T]$, $x \in M$, $a \in \mathbb{R}$, $\alpha_k = k^\alpha$ and $\alpha \in \mathbb{R}_-$. 

First, we will establish the existence and uniqueness of the solution. For this, we
make the following assumptions

(A4) Assume that $\alpha, \gamma \in \mathbb{R}$ are such that $-\frac{m}{2} < \gamma < -\frac{d}{2} - d\alpha$.

We will prove an auxiliary results.

2.3.1 Lemma. For every $\varepsilon > 0$ and $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_0 \in [\gamma_2 - 2, \gamma_2]$, $\gamma_1 \leq \gamma_2 - 2$, there exists $M_\varepsilon \in \mathbb{R}$ such that

$$\|v\|_{\gamma_0}^2 \leq \varepsilon \|v\|_{\gamma_2}^2 + N_\varepsilon \|v\|_{\gamma_1}^2 ,$$

(2.20)

for every $v \in \mathbb{H}^{\gamma-2}$.

Proof. By interpolation theorem (see Section 1.1 for more details) $\|v\|_{s}^2 \leq C_1 \|v\|_{s_1+2}^{2\theta_1} \|v\|_{s_1}^{2(1-\theta_1)}$, where $s \in [s_1, s_1 + 2]$, $\theta_1 = \frac{s-s_1}{2}$ and $C_1$ is a constant which depends only on $M, s, s_1$. Consequently, by $\varepsilon$-inequality (see Section 1.1) with $\varepsilon := \varepsilon_1$, $p := \frac{1}{\theta_1}$, $q = \frac{1}{1-\theta_1}$ we have

$$\|v\|_{s_1}^2 \leq C_1 \varepsilon_1 \|v\|_{s_1+2}^{\frac{1}{\theta_1}} + C_1 \varepsilon_1^{\frac{1-\theta_1}{\theta_1}} \|v\|_{s_1}^2 ,$$

(2.21)
Applying inequality (2.21) to $\|v\|_{s_1}^2$ with $s_1 \in [s_2, s_2 + 2]$ we get

$$
\|v\|_{s_1}^2 \leq C_1 \varepsilon_1 \|v\|_{s_1 + 2}^2 + C_1 C_2 \varepsilon_1 \varepsilon_2 \|v\|_{s_2 + 2}^2 + C_1 C_2 \varepsilon_1 \varepsilon_2 \|v\|_{s_2}^2.
$$

Note that $s_1 \geq s_2$, hence $\|v\|_{s_2 + 2} \leq \|v\|_{s_1 + 2}$, and consequently by the above inequality we deduce

$$
\|v\|_{s_1}^2 \leq \left( C_1 \varepsilon_1 + C_1 C_2 \varepsilon_1 \varepsilon_2 \right) \|v\|_{s_1 + 2}^2 + C_1 C_2 \varepsilon_1 \varepsilon_2 \|v\|_{s_2}^2.
$$

Similarly, by indication with $s_1 > s_2 > \ldots > s_n$, one has

$$
\|v\|_{s_1}^2 \leq \left( C_1 \varepsilon_1 + \ldots + C_1 \cdots C_n \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n \right) \|v\|_{s_1 + 2}^2 + C_1 \cdots C_n \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n \|v\|_{s_n}^2.
$$

(2.22)

Note that $\varepsilon_1, \ldots, \varepsilon_n$ are mutually independent, so we can take $\varepsilon_s, s = 1, \ldots, n$ such that $C_1 \varepsilon_1 + \ldots + C_1 \cdots C_n \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{n-1} \varepsilon_n \leq \varepsilon$. Also, we put $s_1 = \gamma_2 - 2$, and continue with $s_2, \ldots, s_n$ until $s_n \leq \gamma_1$. Finally denote by $N_\varepsilon := C_1 \cdots C_n \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n$. Lemma is proved.

2.3.2 Theorem. If Assumption (A4) is satisfied and $u_0 \in L_2(\Omega, H^\gamma)$, then equation (2.19) has a unique solution $u \in L_2((0, T) \times \Omega; H^{\gamma + m} \cap L_2(\Omega; C((0, T); H^\gamma))$.

Proof. Similar to Section 2.2, the existence and uniqueness is established by Theorem 1.5.1. See also [79], Theorem 3.2, pag. 313. For the sake of completeness, let us verify conditions (iii) and (iv) of Theorem 1.5.1. In our case, (iii) becomes

$$
I_1 = \sum_{k \geq 1} k^{2\alpha} \|h_k v\|_{\gamma}^2 < \infty \quad \text{for all} \quad v \in V = H^{\gamma + m}.
$$

(2.23)
For every \( v_1 \in H^s \) and every \( v_2 \in H^{s+\varepsilon} \) we have \( \|v_1v_2\|_s \leq \|v_1\|_s \cdot \|v_2\|_{s+\varepsilon} \) for every \( \varepsilon > 0 \) (see for instance [78] Corollary 2.8.2). We continue (2.23)

\[
I_1 \leq \sum_{k=1}^{\infty} k^{2\alpha} \|h_k\|_{\gamma}^2 \cdot \|v\|_{\gamma+\varepsilon}^2.
\]

Since \( v \in H^{\gamma+m} \), \( \|v\|_{\gamma+\varepsilon} \) is finite if \( \gamma + m > |\gamma| \) which holds true by initial assumption. Hence,

\[
I_1 \leq c \sum_{k=1}^{\infty} k^{2\alpha} \|h_k\|_{\gamma}^2.
\]

Recall that \( \|h_k\|_{\gamma} = \lambda_k^\gamma \), and \( \lambda_k \sim k^{\frac{1}{d}} \) (see notes on page 2), and we continue

\[
I_1 \leq c \sum_{k=1}^{\infty} k^{2\alpha} k^{\frac{2\gamma}{d}} = C.
\]

By assumption (A4) this series converges and \( C < \infty \).

Finally let us check the condition (iv). Assume that \( v \in H^{\gamma+m} \). Then,

\[
I_2 = 2[Av, v] + \sum_{k \geq 1} \alpha_k^2 \|h_kv\|_{\gamma}^2 = -2[\Lambda^m v, v] + \sum_{k \geq 1} k^{2\alpha} \|h_kv\|_{\gamma}^2
\]

\[
= -2\|v\|_{\gamma+m}^2 + \sum_{k \geq 1} k^{2\alpha} \|h_kv\|_{\gamma}^2.
\]

Using the estimates for \( I_1 \) established above, we continue

\[
I_2 \leq -2\|v\|_{\gamma+m}^2 + C'\|v\|_{\gamma+\varepsilon_1}^2.
\]
Since $|\gamma| \leq \gamma + m$, we can take $\varepsilon_1 > 0$ such that $\gamma + m - 2 < |\gamma| + \varepsilon < \gamma + m$ and apply Lemma 2.3.1 to $\|v\|^2_{|\gamma|+\varepsilon_1}$ with $\gamma_0 = |\gamma| + \varepsilon_1$, $\gamma_1 = \min\{\gamma, \gamma + m - 2\}$, $\gamma_2 = \gamma + m$.

From here we have

$$I_2 \leq -2\|v\|^2_{\gamma + m} + C\varepsilon\|v\|^2_{|\gamma|+m} + N\varepsilon\|v\|^2_{\gamma_1},$$

and since $\gamma_1 \leq \gamma$, we continue

$$I_2 \leq (-2 + C\varepsilon)\|v\|^2_{|\gamma|+m} + N\varepsilon\|v\|^2_{\gamma_1}.$$ 

Put $\delta = 2 - C\varepsilon$, and take $\varepsilon > 0$ small enough, such that $\delta > 0$. Thus condition (iv) from Theorem 1.5.1 is fulfilled. Theorem is proved. 

Note that $\gamma < -\frac{d}{2} - d\alpha$ is Assumption (A3), which guaranties the existence and uniqueness of the solution for equation with additive noise. It should be mentioned that for $d = 1$, $m = 1$, the conditions on $\alpha$ and $\gamma$ from Theorem 2.3.2 imply $-\frac{1}{2} < \gamma < -\alpha - \frac{1}{2}$. Consequently, $\alpha < 0$. In other words, Theorem 2.3.2 ”almost” cover the results from Walsh [79] where $\alpha = 0$ is considered. However, we will present here the corresponding result for $\alpha = 0$ and $\mathcal{A}$ being the Laplacian. Although the result is known [79], our proof is different and simpler in our opinion.

**2.3.3 Theorem.** Suppose that $-1 \leq \gamma < -\frac{d}{2}$ and $u_0 \in L_2(\Omega; H^\gamma)$. Then the equation

$$du(t, x) = \Delta u(t, x) dt + \sum_{k \geq 1} h_k(x) u(x, t) dW_k(t)$$

has a unique solution $u \in L_2((0, T) \times \Omega; H^{\gamma+1}) \cap L_2(\Omega; C((0, T); H^\gamma))$. 

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Proof. Similar to previous proof, we will apply Theorem 1.5.1. Condition (iii) gives

\[
I_1 = \sum_{k \geq 1} \|h_k v\|_\gamma^2 = \sum_{j \geq 1} \sum_{k \geq 1} \lambda_j^2 \left| \langle h_k v, h_j \rangle_0 \right|^2 = \sum_{j \geq 1} \lambda_j^2 \|vh_j\|_0^2 \leq \sum_{j \geq 1} \lambda_j^2 \|h_j\|_\infty \|v\|_0^2.
\]

Since \( \gamma > -1 \) and \( v \in H^{\gamma+1} \), we have \( \|v\|_0 < \infty \). By Remark 2.1.1 \( \|h_k\|_\infty \leq c \lambda_k^\frac{d-1}{2} \).

Hence, we continue

\[
I_1 \leq c \|v\|_0^2 \sum_{j \geq 1} \lambda_j^{2\gamma+d-1}.
\]

Since \( \gamma < -\frac{d}{2} \), the last series converges. Condition (iii) from Theorem 1.5.1 is verified.

Now let us check coercivity condition (iv) from Theorem 1.5.1. Suppose that \( v \in H^{\gamma+1} \). Then, by the same arguments as above and using that \( \Lambda = \sqrt{-\Delta} \), we get

\[
I_2 = 2[\Delta v, v] + \sum_{k \geq 1} \|h_k v\|_\gamma^2 \leq -2\|v\|_{\gamma+1}^2 + c\|v\|_0^2 \sum_{k \geq 1} \lambda_k^{2\gamma+d-1}.
\]

The last series converges, since \( \gamma < -\frac{d}{2} \). Hence

\[
I_2 \leq -2\|v\|_{\gamma+1}^2 + C\|v\|_0^2.
\]

Note that \( \gamma - 1 < -1 < \gamma < 0\gamma + 1 \). By Lemma 2.3.1 we have

\[
I_2 \leq -2\|v\|_{\gamma+1}^2 + C\varepsilon\|v\|_{\gamma+1}^2 + N\varepsilon\|v\|_{\gamma-1}^2 \leq \left( -2 + C\varepsilon \right)\|v\|_{\gamma+1}^2 + N\varepsilon\|v\|_{\gamma}^2.
\]

Take \( \varepsilon > 0 \) such that \( \delta := 2 - C\varepsilon > 0 \), and conditions (iv) is verified. Theorem is proved. \( \square \)
2.3.4 Remark. Recall that for the equations with additive noise, the solution exists regardless of the space dimension and the order of the differential operator $\mathcal{A}$ (see Remark 2.2.2). The situation is different for the equations with multiplicative noise. It is known that the equation $du = \Delta u dt + udW(t, x)$ has solution only for dimension $d = 1$. This also follows from Theorem 2.3.3. Indeed, the condition $-1 \leq \gamma \leq -\frac{d}{2}$ implies $d < 2$. For initial equation (2.19) the restrictions are given by Assumption (A4), which also imply that the existence of the solution depends on both space dimension and order of the differential operator $\mathcal{A}$. However, for any $m, d \in \mathbb{N}$, we can choose $\alpha \ll -1$ such that Assumption (A4) is satisfied, and hence the solution of equation (2.19) exists.

We conclude this subsection by presenting the multiplicative counterpart of equations from Example 2.2.3.

2.3.5 Example. Assume that $d = 1$, $M = [0, \pi]$ and let $\mathcal{A}$ be the one dimensional Laplace operator with zero boundary conditions. In the triple of Hilbert spaces $(H^{\gamma+1}, H^{\gamma}, H^{\gamma+1})$ let us consider the following SPDE

$$du(t, x) = u_{xx}(t, x)dt + \sum_{k \geq 1} k^{-\frac{2}{3}} \sin(kx)u(t, x)dW_k(t),$$

(2.24)

where $x \in [0, \pi]$, $t \in [0, 1]$ and $W_k$ are independent standard Brownian motions. Note that $\alpha = -\frac{2}{3}$. By Theorem 2.3.2, for every $-\frac{1}{2} < \gamma < \frac{1}{6}$, there exists a unique strong solution of equation (2.7) in $H^{\gamma+1}(0, \pi)$.

In Figure 2.2 we show realizations of the solution $u$ of some SPDE’s with multiplicative noise (compare to Figure 2.1, where additive noise is discussed). Panel (a) corresponds to the equation $du = u_{xx}dt + h_k udW_k$, investigated in Theorem 2.3.3. One realization of the solution of equation (2.24) is shown Panel (b). For both equations we
considered zero boundary conditions and the initial values $u(0, x) = \sin(3x) + 1$, $x \in [0, \pi]$. To simulate the field $u$ we used Crank-Nicolson finite difference scheme (see for instance [38], [57]). Similar to additive noise (see Figure 2.1 and Example 2.2.3), the solution $u$ looks smoother for $\alpha = -\frac{2}{3}$ than the solution of equation with corresponding noise parameter $\alpha = 0$. Actually for a fixed parameter $\alpha$, the solutions of equations with additive and multiplicative noise, if exist, have the same order of continuity. We will show this in the next subsection.

![Figure 2.2: Realization of approximated solution of stochastic heat equation with multiplicative noise](image)

2.3.2 Regularity in space and time

Similar to additive case, to establish the regularity properties of the solution $u$, we will prove an auxiliary result. In what follows we will assume that $u_0 \in L_2(\Omega, H^0(M))$.

2.3.6 Lemma. Suppose that Assumptions (A2) and (A4) are satisfied and assume that $\gamma > -m - d\alpha$. Then for any $\beta \in (0, 1]$ and $\alpha \in \mathbb{R}$ such that

$$\alpha < \frac{m(1 - 2\delta(\beta))}{d} - \frac{1}{2},$$

(2.25)
the solution $u$ of equation (2.6) satisfies the following inequality

$$
\mathbb{E} \left| u(t, x) - u(t, y) \right|^p \leq C |x - y|^{3p},
$$

(2.26)

where $x, y \in M$, $t \in [0, T]$, $p > 2$.

Proof. Following the same arguments as in the previous section, the solution $u$ of equation (2.19) can be represented as

$$
u(t, x) = \int_M G_t(x, y)u_0(y)dy + \sum_{k \geq 1} \int_0^t \int_M G_{t-s}(x, y)h_k(y)\alpha_k u(s, y)dydW_k(s) \quad (2.27)
$$

where $x \in M$, $t \in [0, T]$.

Since $h_k \in C^\infty(M)$ and $u_0 \in L_2(\Omega \times M)$ the first term is a smooth function and (2.36) is satisfied. Hence we will study the regularity properties of the second term. Using (2.27), we get

$$
\mathbb{E} \left| u(t, x) - u(t, z) \right|^p \leq \mathbb{E} \left| \sum_{k \geq 1} \int_0^t \int_M \alpha_k \tilde{G}(t-s, x, y, z)h_k(y)u(s, y)dydW_k(s) \right|^p,
$$

(2.28)

where $\tilde{G}(t-s, x, y, z) = G_{t-s}(x, y) - G_{t-s}(z, y)$.

By BDG (1.5.3), we continue

$$
\mathbb{E} \left| u(t, x) - u(t, z) \right|^p \leq c_p \mathbb{E} \left| \sum_{k \geq 1} \int_0^t \int_M \tilde{G}(t-s, x, y, z) \cdot \alpha_k h_k(y)u(s, y)dyds \right|^{\frac{p}{2}}.
$$

(2.28)
Define
\[
I_1 = \sum_{k \geq 1} \left| \int_M \left( G_{t-s}(x,y) - G_{t-s}(z,y) \right) \alpha_k h_k(y) u(s,y) \, dy \right|^2 \\
= \sum_{k \geq 1} \left| \langle u \widetilde{G}(t-s,x,y,z), \alpha_k h_k \rangle_{L^2(M)} \right|^2 .
\]

Since \( \lambda_k \sim k^{\frac{1}{2}} \), \( \Lambda^s(h_k) = \lambda_k^s h_k \) and \( \alpha_k = k^\alpha \), we continue
\[
I_1 \leq c \sum_{k \geq 1} \left| \langle u \widetilde{G}(t-s,x,y,z), \Lambda^s_{\alpha}(h_k) \rangle_{L^2(M)} \right|^2 .
\]

Using the fact that \( \Lambda \) is a selfadjoint operator we get
\[
I_1 \leq c \sum_{k \geq 1} \left| \langle \Lambda^s_{\alpha}(u \widetilde{G}), h_k \rangle \right|^2 .
\]

Finally, by Parseval’s equality and property (1.2) of the operator \( \Lambda \) we have
\[
I_1 \leq c \left\| \Lambda^s_{\alpha}(u \widetilde{G}) \right\|_0^2 = c \left\| u \widetilde{G} \right\|_{d\alpha}^2 ,
\]
and thus, from here and (2.28) one deduces
\[
\mathbb{E} |u(t,x) - u(t,z)|^p \leq c \mathbb{E} \left| \int_0^t \left\| \widetilde{G}(t-s,x,\cdot, z) u(s,\cdot) \right\|_{d\alpha}^2 \, ds \right|^\frac{p}{2} . \tag{2.29}
\]

By the same arguments as in Theorem 2.3.2 from (2.28) we have
\[
\mathbb{E} |u(t,x) - u(t,z)|^p \leq c \mathbb{E} \left| \int_0^t \left\| \widetilde{G}(t-s,x,\cdot, z) \right\|_{d\alpha}^2 \left\| u(s,\cdot) \right\|_{d\alpha+e}^2 \, ds \right|^\frac{p}{2} . \tag{2.30}
\]
By Hölder inequality with $p_1 := \frac{p}{p-2}$, $q_1 := \frac{q}{2}$, and (2.30) we get

$$E |u(t, x) - u(t, z)|^p \leq c E \left\{ \left| \int_0^t |u(s, \cdot)|^{2q_1} ds \right|^{\frac{p}{2q_1}} \cdot \left| \int_0^t \|G(t - s, x, \cdot, z)\|_{da_0}^{2p_1} ds \right|^{\frac{p}{2p_1}} \right\}.$$  

By initial assumption $\gamma + m > |da_0|$ which implies that the first factor in the last inequality is finite.

Now, we are going to estimate the last term on the right-hand side of (2.31), and show that it is finite too. Using the form (2.8) of the function $G$ we get

$$I_2 := \int_0^t \|G(t - s, x, \cdot, z)\|_{da_0}^{2p_1} ds = \int_0^t \left( \sum_{j \geq 1} \left| \langle G(t - s, x, \cdot, z), h_j \rangle_0 \right|^2 \lambda_j^{2da_0} \right)^{p_1} ds$$

$$= \int_0^t \left( \sum_{j \geq 1} \sum_{k \geq 1} (h_k(x) - h_k(z)) e^{l_k(t-s)} \left| \langle h_k, h_j \rangle_0 \right|^2 \lambda_j^{2da_0} \right)^{p_1} ds$$

$$= \int_0^t \left( \sum_{k \geq 1} (h_k(x) - h_k(z))^2 e^{2l_k(t-s)} \lambda_k^{2da_0} \right)^{p_1} ds .$$

By Hölder inequality with $p_2 := p_1 = \frac{p}{p-2}$ and $q_2 = \frac{q}{2}$ we continue

$$I_2 \leq \int_0^t \sum_{k \geq 1} (h_k(x) - h_k(z))^{2p_2} \lambda_k^{2da_2} e^{2l_k(t-s)p_2} \left( \sum_{j \geq 1} e^{l_j(t-s)q_2} \right)^{\frac{p_1}{2q_2}} ds ,$$

and consequently

$$I_2 \leq \sum_{k \geq 1} (h_k(x) - h_k(z))^{2p_2} \lambda_k^{2da_2} \int_0^t e^{l_k(t-s)p_2} \left( \sum_{j \geq 1} e^{l_j(t-s)q_2} \right)^{\frac{p_1}{2q_2}} ds . \quad (2.32)$$
Let us estimate the inner integral of the right-hand side of (2.32). Again by Hölder inequality with $p_3, q_3$ such that $\frac{1}{p_3} + \frac{1}{q_3} = 1$ we get

$$I_3 := \int_0^t e^{\ell_k(t-s)p_2} \left( \sum_{j \geq 1} e^{l_j(t-s)q_2} \right)^{\frac{p_1}{q_2}} ds$$

$$\leq \left( \int_0^t e^{\ell_k(t-s)p_2} ds \right)^{\frac{1}{p_3}} \left( \int_0^t \left[ \sum_{j \geq 1} e^{l_j(t-s)q_2} \right]^{\frac{p_1}{q_2}} ds \right)^{\frac{1}{q_3}}.$$  

If $q_3 := \frac{q_2}{p_1} = \frac{p-2}{p}$ then the last factor is finite for every $p > 2$. Indeed,

$$I_4 = \int_0^t \left[ \sum_{j \geq 1} e^{l_j(t-s)q_2} \right]^{\frac{p_1}{q_2}} ds = \int_0^t \sum_{j \geq 1} e^{l_j(t-s)q_2} ds = \sum_{j \geq 1} \frac{e^{l_jtq_2} - 1}{l_jq_2}.$$  

Recall that $- l_j \sim j^{\frac{2m}{d}}$ and since $l_j < 0$ we conclude that $1 - e^{l_j t q_2} < 1$. Hence,

$$I_4 \leq c \sum_{j \geq 1} \frac{1}{j^{\frac{2m}{d}}}.$$  

and since $m > \frac{d}{2}$, we have that $I_4$ is finite. Consequently, from (2.33) we get

$$I_3 \leq I_4 \left( \int_0^t e^{\ell_k(t-s)p_2p_3} ds \right)^{\frac{1}{p_3}} = I_4 \left[ \frac{1 - e^{\ell_k t p_2 p_3}}{|l_k| p_2 p_3} \right]^{\frac{1}{p_3}} \leq c_p |l_k|^{-\frac{1}{p_3}},$$

where $p_3 = \frac{p-2}{p-4}$.
From (2.31) using (2.32), (2.33) and (2.34) and Assumption (A2), for \( \beta > 0 \) one deduces

\[
E \left| u(t, x) - u(t, z) \right|^p \leq c \left| x - y \right|^\beta \sum_{k \geq 1} \lambda_k^{2d \alpha p_2} \left| x - y \right|^{2p_2 \beta} l_k^{2\delta(\beta) p_2} l_k^{\frac{1}{3}} \left| x - y \right|^{\frac{p}{2p_1}} \leq c \left| x - y \right|^\beta \left| x - y \right|^{\frac{p-2}{2}}.
\]

and since, by initial assumption \( \alpha < -\frac{1}{2} + \frac{m(1 - 2\delta(\beta))}{d} \), we conclude that for sufficiently large \( p \) the last series converges. Lemma is proved.

Now, we will prove a similar result for the time variable \( t \).

**2.3.7 Lemma.** Suppose that \( \gamma > -m - d\alpha \) and let Assumptions (A1) and (A4) be satisfied. Then for every \( \beta \in (0, 1] \) such that

\[
\beta < 1 - 2\rho - \frac{d(1 + 2\alpha)}{2m}
\]

the solution \( u \) of the equation (2.19) satisfies the following inequality

\[
E \left| u(t_1, x) - u(t_2, x) \right|^p \leq C |t_1 - t_2|^\frac{\beta p}{2},
\]

where \( t_1, t_2 \in [0, T], \ x \in M, \ p > 2. \)
Proof. Suppose that $0 \leq t_1 \leq t_2 \leq T$. The existence of the solution $u$ is provided by the above assumptions and Theorem 2.3.2, and by representation (2.27) we have

$$J = \mathbb{E} \left| u(t_1, x) - u(t_2, x) \right|^p = \mathbb{E} \left[ \int_0^{t_2} \sum_{k \geq 1} \int M^\hat{G}(t_1, t_2, s, x, y) \alpha_k h_k(y) u(s, y) dy dW_k(s) \right]^p$$

where $\hat{G}(t_1, t_2, s, x, y) := G_{t_1-s}(x, y)\mathbb{I}_{(s \leq t_1)} - G_{t_2-s}(x, y)$. By similar evaluations as in Lemma 2.3.6 we deduce

$$J \leq C \left[ \int_0^{t_2} \left\| \hat{G}(t_1, t_2, s, x, \cdot) \right\|^2_{\text{d}a} ds \right]^{\frac{p}{2} \frac{1}{p_1}}, \quad (2.38)$$

where $p_1 = \frac{p}{p-2}$ and $C$ depends only on $p$ and $\mathbb{E}(\|u\|_\gamma)$ (compare to (2.31)). Taking into account the definition (2.8) of the function $G$ and properties of the norm $\| \cdot \|_\gamma$, we get

$$J_1 := \int_0^{t_2} \left\| \hat{G}(t_1, t_2, s, x, \cdot) \right\|^2_{\text{d}a} ds = \int_0^{t_2} \left( \sum_{j \geq 1} \left( \hat{G}(t_1, t_2, s, x, \cdot), h_j(\cdot) \right)_0 \right)^2 \lambda_j^{2\alpha} ds$$

$$= \int_0^{t_2} \left( \left\| \sum_{k \geq 1} \left( e^{\lambda_j(t_1-s)}I_{(s \leq t_1)} - e^{\lambda_j(t_2-s)} \right) h_k(x) h_k(\cdot), h_j(\cdot) \right\|_0^2 \lambda_j^{2\alpha} \right) ds$$

and since $h_j$ forms a CONS in $L_2(M)$, we continue

$$J_1 = \int_0^{t_2} \left| \sum_{k \geq 1} \left( e^{\lambda_j(t_1-s)}I_{(s \leq t_1)} - e^{\lambda_j(t_2-s)} \right)^2 |h_j(x)|^2 \lambda_j^{2\alpha} \right| ds.$$
By Assumption (A1) and Hölder inequality with $p_2 := p_1 = \frac{p}{p-2}$ and $q_2 = \frac{p}{2}$ we get

$$J_1 \leq c \int_0^{t_2} \sum_{j \geq 1} |l_j|^{2p_1} \lambda_j^{2d\rho p_1 + \epsilon} \left( e_j^{(t_1 - s)} \mathbb{I}_{(s \leq t_1)} - e_j^{(t_2 - s)} \right)^{2p_1} \left( \sum_{k \geq 1} \lambda_k^{-\epsilon q_2} \right)^{\frac{1}{q_2}} ds .$$

For sufficiently large $p$ and respectively large $q_2$ the series $\sum_{k \geq 1} \lambda_k^{-\epsilon q_2}$ converges, and hence

$$J_1 \leq c(p, \varepsilon) \sum_{j \geq 1} |l_j|^{2p_1} \lambda_j^{2d\rho p_1 + \epsilon} \int_0^{t_2} \left( e_j^{(t_1 - s)} \mathbb{I}_{(s \leq t_1)} - e_j^{(t_2 - s)} \right)^{2p_1} ds \quad (2.39)$$

Direct arithmetic evaluations yield to

$$\int_0^{t_2} \left( e_j^{(t_1 - s)} \mathbb{I}_{(s \leq t_1)} - e_j^{(t_2 - s)} \right)^{2p_1} ds \leq c_p |t_2 - t_1|^{p_1 |l_k|^{p_1 (\beta - 1)}} \left( \beta \in (0, 1) \right).$$

From here and (2.39) we conclude

$$J_1 \leq c_p |t_2 - t_1|^{p_1 |l_k|^{p_1 (2p + \beta - 1)}} \lambda_j^{2d\rho p_1 - \epsilon} . \quad (2.40)$$

Since $|l_k| = \lambda_k^{2m}$ and $\lambda_k \sim k^{\frac{1}{d}}$, the last series converges if $\frac{2m(2p + \beta - 1)}{d} + \frac{2d\rho p_1 - \epsilon}{d} < -1$. Recall that $p_1 = \frac{p}{p-2}$, so $\lim_{p \to \infty} p_1 = 1$. Hence, the series converges if $\beta < 1 - 2\rho - \frac{d\alpha}{m} - \frac{d}{2m}$, which holds true by our initial assumption. Finally, by (2.38) and (2.40) we conclude

$$J \leq c_p |J_1|_{p_1}^{\frac{p}{p_1}} \leq c_p |t_1 - t_2|^{\frac{2p}{p_1}} .$$

Lemma is proved. \qed

Now, we are ready to formulate the main result of this section.
2.3.8 **Theorem.** Assume that Assumption (A4) is satisfied and let $u$ be the solution of the equation (2.19)

(i) if Assumption (A2) is fulfilled and $r$ is the biggest integer such that there exists $\beta_0 \in (0, 1]$ which satisfies the inequality $2d\alpha + 2r + 4m\delta(\beta_0) - 2m < -d$, then $N'(u)$ is Hölder continuous of order

$$
\eta := \sup \left\{ \beta \mid \delta(\beta) < -\frac{d}{4m} - \frac{d\alpha}{2m} + \frac{1}{2} - \frac{r}{2m} \right\} - \varepsilon,
$$

for every $\varepsilon > 0$, as long as $\eta > 0$;

(ii) If Assumption (A1) is fulfilled and $\alpha < \frac{m}{d} - \frac{2m\rho}{d} - \frac{1}{2}$, then the solution $u$ is Hölder continuous in $t$ of order less than $\zeta = \min\left\{\frac{1}{2}, \frac{1}{2} - \rho - \frac{d(1+2\alpha)}{4m}\right\}$.

As long as the solutions of equations (2.2) and (2.19) exist, Lemma 2.3.6 and 2.3.7 coincide with Lemma 2.2.4 and 2.2.10 from additive noise. Under assumption that the solution of the equation 2.19 exists, which is guaranteed by Assumption (A4), the proof of Theorem 2.3.8 is the similar to the proof of Theorem 2.2.5 and 2.2.11.

2.3.9 **Example.** Let us consider the following SPDE

$$
\begin{align*}
    du(t,x) &= u_{xx}(t,x)dt + \sum_{k \geq 1} k^{-\frac{2}{3}} \sin(kx)u(t,x)dW_k(t),
\end{align*}
$$

where $x \in [0, \pi]$, $t \in [0,1]$, $u(t,0) = u(t,\pi) = 0$ and $W_k$ are independent standard Brownian motions. We showed in Example 2.3.5 that for every $-\frac{1}{2} < \gamma < \frac{1}{6}$, there exists a unique strong solution $u$ in the triple of Hilbert spaces $(H^{\gamma+1}, H^\gamma, H^{\gamma+1})$. Recall that in this case $\rho = 0, \alpha = -2/3$, $\delta(\beta) = \frac{\beta}{2}$ (See Example 2.1.2). By Theorem 2.3.8 we have that the solution $u$ has one derivative in $x$ and the first derivative is Hölder continuous of order less than $\eta := \sup \left\{ \beta \mid \delta(\beta) < -\frac{d}{4m} - \frac{d\alpha}{2m} + \frac{1}{2} - \frac{r}{2m} \right\}$.
\[ (0, 1] = \sup \{ \beta \mid \frac{\beta}{2} < -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \} = \frac{1}{6}. \] Respectively \( u \) is Hölder continuous in time variable \( t \) of order less than \( \min \{ \frac{1}{2}, \frac{7}{12} \} = \frac{1}{2} \).

### 2.3.10 Remark

In both cases, additive and multiplicative noise, the solutions, if exists, has the same order of regularity. However, the existence and uniqueness of solution, in terms of the scale of Sobolev spaces, is more restrictive in multiplicative case.

### 2.4 Covariance functional and relation with some known results

As we already mentioned the novelty in our approach is the factor \( \alpha_k = k^\alpha \) in the noise term. In this section we will show how our results are related to some known results, and also we will find the analogous of parameter \( \alpha \) in the known literature in terms of the asymptotic of kernel of covariance functional.

In recent paper Mytnik, Perkins, Sturm \[60\] investigated the equation of the form

\[ du(t, x) = \frac{1}{2}\Delta u(t, x)dt + \sqrt{u(t, x)}dW(t, x), \tag{2.41} \]

where \( \Delta \) is Laplacian, \( x \in \mathbb{R}^d \), \( t \in [0, T] \) for some finite \( T \). \( W \) is space-time white noise on \( \mathbb{R}_+ \times \mathbb{R}^d \), namely \( W \) is defined on a filtrated probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) and is a Gaussian martingale measure on \( \mathbb{R}_+ \times \mathbb{R}^d \) in sense of Walsh \[79\] or that introduced in Section 1.3. In \[60\] study the existence of mild solution of equation (2.41) and it regularity properties. The special feature of this equation is the nonlinear factor \( \sqrt{u} \). The solution \( u \) is the density for one-dimensional super-Brownian motion \[63\]. Of course this equation is different from those considered in our investigations.
Equation (2.41) is nonlinear on the whole space, in our notations $M = \mathbb{R}^d$. Our equations are linear SPDE’s, but $M$ is a bounded domain, and we allow operators of arbitrary order (not necessarily Laplacian). Here we will discuss the similarities in terms of noise $W(t, x)$. Since we are working on different domains, some evaluations will be pure heuristic.

Denote by $C^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d)$ the space of compactly supported, infinitely differentiable functions on $\mathbb{R}_+ \times \mathbb{R}^d$, and put

$$W_t(\varphi) = \int_0^t \int \varphi(s, x)W(dxds).$$

If $W(\varphi) = W_\infty(\varphi)$, $W$ can be characterize by its covariance functional

$$J_\Psi(\varphi, \psi) := \mathbb{E}\left[W(\varphi)W(\psi)\right] = \int_0^\infty \int \int \varphi(s, x)\Psi(x, y)\psi(s, y)dxdyds,$$  

(2.42)

for $\varphi, \psi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d)$. The function $\Psi : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ is called the correlation kernel of $W$. A general classes of martingale measures, spatially homogeneous noises, can be describe by (2.42) with $\Psi(x, y) = \tilde{\Psi}(x - y)$. We note that the standard space-time white noise $\sum_{k \geq 1} h_k(x)W_k(t)$ will correspond to the case $\tilde{\Psi} = \delta_0$, where $\delta$ is the Dirac function. Suppose that the correlation is bounded by a Riesz kernel

$$|\Psi(x, y)| \leq c \left[|x - y|^{-\vartheta} + 1\right] \quad (x, y \in \mathbb{R}^d)$$

(2.43)

where $\vartheta > 0$.

In [60] is proved that for every $\xi \in (0, 1 - \frac{\vartheta}{2})$ the solution of equation (2.41) is uniformly H"{o}lder continuous on compacts in $[0, \infty) \times \mathbb{R}^d$, with H"{o}lder coefficients $\frac{\xi}{2}$ in time and $\xi$ in space.

Now, let us find the correlation functional and kernel that corresponds to the equations in our research. For simplicity of writing we will consider the one-dimensional
stochastic heat equation $du(t, x) = u_{xx}(t, x)dt + \sum_{k \geq 1} k^\alpha u(t, x) \sin(kx)dW_k(t)$, where $x \in [0, \pi]$, $t \in [0, 1]$. In this case $W_t(\varphi) = \sum_{k \geq 1} ^t \int_0^1 \varphi(s, x)h_k(x)k^\alpha dx dW_k(ts)$. Consequently,

$$J_{\psi}(\varphi, \psi) = \mathbb{E} \left[ \sum_{k \geq 1} ^\infty \int_0^\pi \int_0^\pi k^\alpha \varphi(s, x)h_k(x)dx dW_k(s) \right] \cdot \sum_{n \geq 1} ^\infty \int_0^\pi \int_0^\pi n^\alpha \psi(s, y)h_n(y)dy dW_n(s) .$$

Taking into account that $W_k$ are independent, we continue

$$J_{\psi_0}(\varphi, \psi) = \mathbb{E} \left[ \sum_{k \geq 1} k^{2\alpha} \int_0^\infty \langle \varphi(s, \cdot), h_k \rangle_0 dW_k(s) \int_0^\infty \langle \psi(s, \cdot), h_k \rangle_0 dW_k(s) \right] .$$

Suppose for simplicity that the functions $\varphi, \psi$ do not depend on time variable $t$. Then

$$J_{\psi_0}(\varphi, \psi) = c \sum_k k^{2\alpha} \langle \varphi, h_k \rangle_0 \langle \psi, h_k \rangle_0 = c \langle \varphi, \psi \rangle_\alpha ,$$

where $c$ is a constant depending on the support of functions $\varphi$ and $\psi$. Note that $\langle \varphi, \psi \rangle_\alpha = c \sum_{k \geq 1} k^{2\alpha} \int_0^\pi \int_0^\pi h_k(x)h_k(y)\varphi(x)\psi(y)dxdy$. Hence,

$$\Psi_0(x, y) = \sum_{k \geq 1} k^{2\alpha} h_k(x)h_k(y) = \sum_{k \geq 1} k^{2\alpha} \sin(kx) \sin(ky) .$$

If $\alpha < -\frac{1}{2}$, then the last series converges uniformly, and the covariance kernel is bounded. Hence, in this case there is no connection with the estimates (2.43). Suppose that $-\frac{1}{2} < \alpha < 0$. By Abel’s criterion about convergence of series, one can show that $\sum_{k \geq 1} k^{2\alpha} \sin(kx) \sin(ky)$ converges if $x \neq y$ and diverges if $x = y$. So, the kernel $\Psi$ seems to have a similar asymptotic behavior to that from (2.43). To find $\vartheta$ we note
that \( J_{\psi_0}(\varphi, \psi) = \langle \Lambda^{2\alpha} \varphi, \psi \rangle_0 \), where \( \Lambda = \sqrt{-\Delta} \) (see Chapter 1 for more details).

The counter part for the whole line will be the operator \( \Lambda = (I - \Delta)^{1/2} \) which can be represented as an integral operator with kernel the Bessel potential. Namely,

\[
\langle \Lambda^{-2\alpha} \varphi, \psi \rangle_0 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \int_{0}^{\infty} \left( t^{-\alpha - \frac{1}{2}} e^{-\frac{|x-y|^2}{4t} - t} \right) \varphi(x) \psi(y) dy dx \right) .
\]

Hence, the correlation kernel \( \tilde{\Psi}_0(x) = \int_{0}^{\infty} t^{-\alpha - \frac{3}{2}} e^{-\frac{|x|^2}{4t} - t} dt \). Note that Fourier transform \( F[\tilde{\Psi}_0](x) = (1 + |x|^2)^{\alpha} \), and \( F[|x|^{-\vartheta}] = |x|^{-1+\vartheta} \) (see for instance [77], Chapter 5).

From here we conclude that \(-2\alpha = 1 - \vartheta\). Hence we can say that the parameter \( \vartheta \) from (2.43) corresponds to \( 1 - 2\alpha \) where \( \alpha \) is parameter involved in our research. Note that \( \alpha \in (-\frac{1}{2}, 0) \) is equivalent to \( \vartheta \in (0, 1) \). Finally we will show that regularity properties coincide. By Theorem 2.3.8 the solution of corresponding equation is Hölder continuous in space variable \( x \) with the order of continuity less than \( \xi = \frac{1}{2} - \alpha = 1 - \frac{\vartheta}{2} \), and in time variable with order of continuity less than \( \eta = \frac{1}{4} - \frac{\alpha}{2} = \frac{1}{2} - \frac{\vartheta}{4} \), which is exactly the statement of Theorem 1.8 from [60].

### 2.5 Examples

#### 2.5.1 Example. Stochastic heat equation on \([0, \pi]\) with zero boundary conditions.

First we consider the classical heat equation \( du = u_{xx} dt + f(u) dW(t, x) \), as a benchmark for testing our results. We recall that this problem was investigated by Walsh [79]. In this case, the solution exists, is unique and Hölder continuous in \( x \) of order \( \frac{1}{2} - \varepsilon \), and in \( t \) of order \( \frac{1}{4} - \varepsilon \) for every \( \varepsilon > 0 \) (see Section 2.1 for more details).

Let \( \mathcal{A} \) be the one dimensional Laplace operator on \( C^\infty(0, \pi) \). In our notations \( d = 1, \mathcal{A}u = u_{xx}, m = 1, M = [0, \pi] \). It is well-known that the functions
$h_k(x) = \frac{2}{\pi} \sin(kx), \ k \in \mathbb{N},$ are the eigenfunctions of the operator $A$, with corresponding eigenvalues $l_k = -k^2$. Moreover, Assumptions (A1) and (A2) are satisfied, and $\rho = 0, \ \delta(\beta) = \frac{\beta}{2}$ (see Section 2.1, Example 2.7, page 27 for the proof).

2.5.1.a. Additive noise

Let us consider the heat equation with additive noise term

$$
\begin{cases}
du(t, x) = u_{xx}(t, x)dt + \sum_{k=1}^{\infty} k^\alpha h_k(x)dW_k(t) & t \in (0, T), \ 0 < x < \pi; \\
u(t, 0) = u(t, \pi) = 0 & t \in (0, T); \\
u(x, 0) = 0, & x \in [0, \pi],
\end{cases}
$$

(2.44)

where $\alpha$ is a real parameter.

Existence and Uniqueness. By Theorem 2.2.1, if $\gamma < -\alpha - \frac{1}{2}$ then the equation (2.44) has a unique solution $u \in L_2((0, T) \times \Omega; \mathbb{H}^{\gamma+1}(0, \pi)) \cap L_2(\Omega; C((0, T); \mathbb{H}^{\gamma}(0, \pi)))$. Note that regardless of $\alpha$ the solution exists in some $\mathbb{H}^\gamma$.

Regularity in $x$. Since $\delta(\beta) = \frac{\beta}{2m} = \frac{\alpha}{2}$, we will apply Theorem 2.2.7. Suppose that $\gamma < -\alpha - \frac{1}{2}$ and denote $\alpha < \frac{1}{2}$. By Remark 2.2.6 we can take the operator $\Lambda$ from Theorem 2.2.7 being the partial derivative operator $\frac{\partial}{\partial x}$ (see also Example 2.2.8). Let $\eta = \frac{1}{2} - \alpha$ and denote by $[a]$ the integer part of real number $a$ (largest integer smaller than $a$). If $\eta \notin \mathbb{N}$, then the solution $u$ has $r := [\eta]$ derivatives in $x$ and the $r$-th derivative is Hölder continuous of order $\zeta := \eta - r - \varepsilon$, for every $\varepsilon > 0$ as long as $\zeta > 0$. If $\eta \in \mathbb{N}$, then $u$ has $r := [\eta] - 1$ derivatives in $x$ and the $r$-th derivative is Hölder continuous of order $\zeta$ for every $\zeta \in (0, 1)$.

Regularity in $t$. Suppose that $\alpha < \frac{1}{2}$. Then, by Theorem 2.2.11, the solution $u$ of equation (2.44) is Hölder continuous in time variable $t$ with order of continuity $\eta = \min\{\frac{1}{2}, \frac{1}{4} - \frac{\alpha}{2}\} - \varepsilon$, for every $\varepsilon > 0$ as long as $\eta > 0$. 

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If \( \alpha = 0 \), we resume the results by Walsh [79]: the solution exists if \( \gamma \leq -\frac{1}{2} \), and \( u(t, x) \in C^{\frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon}_{t, x} \).

If \( \alpha \) decreases, the solution becomes smoother, both in time and space variable. However, the maximum smoothness in time is achieved for \( \alpha = -\frac{1}{2} \). In this case, the solution \( u \) has an order of continuity less than \( \frac{1}{2} \) and remains the same for every \( \alpha \leq -\frac{1}{2} \). For \( \alpha = -\frac{1}{2} \), the solution is Lipschitz continuous in \( x \). Generally speaking, there are \( r = \lfloor \frac{1}{2} - \alpha \rfloor \) derivatives in \( x \), and \( u^{(r)}_x \in C^{\frac{1}{2} - \alpha - r} \).

2.5.1.b. Multiplicative noise.

Now, let us consider the equation

\[
\frac{du}{dt}(t, x) = u_{xx}(t, x) dt + \sum_{k=1}^{\infty} k^{\alpha} u(t, x) h_k(x) dW_k(t) \quad t \in (0, T), \ 0 < x < \pi ,
\]

with same initial values and boundary conditions as in (2.44). By Theorem 2.3.2 this equation has a solution \( u \in H^{\gamma + 1} \) if \( -\frac{1}{2} < \gamma < -\frac{1}{2} - \alpha \). If the solution exists, then the regularity in time and space are the same as for additive noise, discussed above.

2.5.2 Example. Stochastic heat equation on torus \( \mathbb{T}^2_\pi \).

We consider the operator \( A \) being the Laplace operator \( \Delta \) on two dimensional torus \( \mathbb{T}^2_\pi \). Following our previous notations \( d = 2, \ m = 1, \ M = \mathbb{T}^2_\pi \). The eigenfunctions of this operator are \( h_{j,n}(x_1, x_2) = \frac{1}{\pi} e^{ijx_1} e^{inx_2} \) where \( x = (x_1, x_2) \in M, \ j, n \in \mathbb{Z}, \ i = \sqrt{-1} \), with corresponding eigenvalues \( l = j^2 + m^2 \). Obviously Assumption (A1) is satisfied, \( ||h_k||_{\infty} \leq \frac{1}{\pi} \), so \( \rho = 0 \). Let us check Assumption (A2) and find the function \( \delta \). Denote \( x = (x_1, x_2), \ y = (y_1, y_2) \). Then \( R := |h_{j,k}(x) - h_{j,k}(y)| = |e^{i(jx_1 + nx_2)} - e^{i(jy_1 + ny_2)}| = \left| 2 - 2(\cos(z_1) \cos(z_2) + \sin(z_1) \sin(z_2)) \right|^{\frac{1}{2}}, \) where \( z_1 = jx_1 + nx_2, \ z_2 = jy_1 + ny_2 \). Consequently, \( R = \sqrt{2} \left| 1 - \cos(\frac{z_1 - z_2}{2}) \right|^{\frac{1}{2}} = 2 \left| \sin(\frac{z_1 - z_2}{4}) \right| \leq c |z_1 - z_2|^\beta \), for every
\( \beta \in (0, 1] \). Hence, \( R \leq c |j(x_1 - y_1) + n(x_2 - y_2)|^\beta \leq c (j^2 + n^2)^{\frac{\beta}{2}} ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{\frac{\beta}{2}} = c l^2 |x - y|^\beta \). Thus, \( \delta(\beta) = \frac{\beta}{2} \), \( \beta \in (0, 1] \).

Let us consider the corresponding equation with additive noise, i.e. \( du = \Delta u dt + \sum_{k \geq 1} h_k dW_k(t) \). Note that \( \delta(\beta) = \frac{1}{2m} = \frac{\beta}{2} \), and hence Remark 2.2.6 holds true. By Theorem 2.2.1 there exists a unique solution \( u \in H^{\gamma + 1} \) if \( \gamma < -2\alpha - 1 \). Theorem 2.2.5 implies: if \(-2\alpha \notin \mathbb{N}\), then the solution has \( r := \lfloor -2\alpha \rfloor \) derivatives in \( x \) and the \( r \)-th derivative \( D^r(u) \) is Hölder continuous of order \( \eta = -2\alpha - |2\alpha| \); if \(-2\alpha \in \mathbb{N}\), then there exists \( D^{r-1}(u) \) with order of continuity \( \tau \), for every \( \eta \in (0, 1) \).

Regularity in time is given by Theorem 2.2.11. The solution \( u(\cdot, x) \) is Hölder continuous of order \( \eta = \min\{\frac{1}{2}, -\alpha\} \), for every \( \alpha \leq 0 \).

2.5.3 Example. SPDE of order \( 2m \).

Let consider the equation of the form

\[
 du(t, x) = \frac{\partial^{2m} u(t, x)}{\partial x^{2m}} dt + \sum_{k \geq 1} k^\alpha h_k(x) dW_k(t) \quad x \in [0, \pi], \ t \in [0, T],
\]

where \( h_k(x) = \frac{2}{\pi} \sin(kx) \) and \( m \in \mathbb{N} \). We assume zero boundary conditions, and zero (or any smooth) initial values. In fact this equation is similar to the equation (2.44) from Example 1, but \( \mathcal{A} = \Delta^m \). Since the Laplace operator \( \Delta \) is a selfadjoint operator, we conclude that the functions \( h_k \) are the eigenfunctions of the operator \( \mathcal{A} \), with corresponding eigenvalues \( l_k = -k^{2m} \). Thus, Assumptions (A1) and (A2) are satisfied, and \( \rho = 0 \), \( \delta(\beta) = \frac{\beta}{2m} \).

Existence. Applying Theorem 2.2.1, we have: if \( \gamma < -\alpha - \frac{1}{2} \) then there exists a unique solution \( u \in L_2((0, T) \times \Omega; H^{\gamma + 1}(0, \pi)) \cap L_2(\Omega; C((0, T); H^\gamma(0, \pi))) \)

Regularity in \( x \). By Theorem 2.2.5, if \( \alpha < -\frac{1}{2} + \frac{m}{2} \) and \( m - \alpha - \frac{1}{2} \notin \mathbb{N} \), then the solution \( u \) has \( r = \lfloor m - \alpha - \frac{1}{2} \rfloor \) derivatives, and the last derivative is Hölder continuous.
of every order less than $m - \alpha - \frac{1}{2} - r$. Similar for $m - \alpha - \frac{1}{2} \in \mathbb{N}$.

**Regularity in $t$.** If $\alpha < m - 1$, then the solution $u$ is Hölder continuous in time variable, with order of continuity less than $\min\{\frac{1}{2}, \frac{1}{2} - \frac{1-2\alpha}{4m}\}$.

To show the impact of the parameter $\alpha$, we summarize these results from the examples considered in this Chapter in the Table 2.1:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Existence</th>
<th>Reg. in $x$</th>
<th>Reg. in $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Walsh [79] multiplicative</td>
<td>$d = 1, m = 1, \alpha = 0$</td>
<td>-</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Ex. 2.1.2, 2.2.3, 2.2.8, additive</td>
<td>$d = 1, m = 1$</td>
<td>$\gamma &lt; \frac{1}{6}$</td>
<td>$\frac{7}{6}$</td>
</tr>
<tr>
<td>$\alpha = -\frac{2}{3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ex. 2.3.5, 2.3.9 multiplicative</td>
<td>$d = 1, m = 1$</td>
<td>$-\frac{1}{2} &lt; \gamma &lt; \frac{1}{6}$</td>
<td>$\frac{7}{6}$</td>
</tr>
<tr>
<td>$\alpha = -\frac{2}{3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ex. 2.5.1.a additive</td>
<td>$d = 1, m = 1$</td>
<td>$\gamma &lt; -\alpha - \frac{1}{2}$</td>
<td>$\frac{1}{2} - \alpha$</td>
</tr>
<tr>
<td>Ex. 2.5.1.b multiplicative</td>
<td>$d = 1, m = 1$</td>
<td>$-\frac{1}{2} &lt; \gamma &lt; -\alpha - \frac{1}{2}$</td>
<td>$\frac{1}{2} - \alpha$</td>
</tr>
<tr>
<td>Ex. 2.5.2 additive</td>
<td>$d = 2, m = 1$</td>
<td>$\gamma &lt; -2\alpha - 1$</td>
<td>$-2\alpha$</td>
</tr>
<tr>
<td>Ex. 2.5.3 additive</td>
<td>$d = 1, m &gt; 1$</td>
<td>$\gamma &lt; -2\alpha - 1$</td>
<td>$m - \alpha - \frac{1}{2}$</td>
</tr>
</tbody>
</table>

Table 2.1: Existence and Regularity of Solution. Summary
Chapter 3

Parameter estimation problems for some classes of SPDE’s

3.1 Introduction

Stochastic Partial Differential Equations (SPDE’s) are of big interest in last decades, primarily as a modelling of various phenomena from fluid mechanics [74], oceanography [64], temperature anomalies [11], [65], finance [7], [12], [14], and other domains. Every model is describe by a class of SPDE’s, with some specific properties, which are related to the corresponding phenomena, and usually this is a broad class of equations with some unspecified/unknown coefficients also called parameters. In practice we observe the process, which we believe is described by some equation, and we want to find all unknown parameters of this model by using the past data such that the equation fits and predicts, in some sense, as much as possible the real data. In mathematical terms, we want to solve a parameter estimation problem.

The parameter estimation problem, generally speaking, is a particular case of inverse problem when the solution is known (observed) and an inference about the coefficients of the equation is made. Because of general setup of this problem, there are numerous methods in modern literature dedicated to this topic, covering different classes of equations. We want to emphasize that the solution of a Stochastic PDE is a random variable, and the parameter estimation problem needs to be solved by
some statistical methods. In contrast, the similar problem for the models described by a deterministic PDE is usually reduced to some inverse spectral problems (see for instance [68] and the references therein).

In this chapter we will investigate a parameter estimation problem for evolution equations of the following form

\[
\begin{aligned}
& du(t) = (A_0 + \theta A_1)u(t)dt + (M u(t) + g(t))dW(t), \\
& u(0) = u_0,
\end{aligned}
\]  

(3.1)

where \(t \in [0, T]\), \(A_0, A_1, M\) are linear operators, \(g\) a real-valued function, \(W\) is a cylindrical Brownian motion, and \(\theta\) is the unknown parameter belonging to an open subset of the real line.

Parameter estimation for the ordinary stochastic differential equations (SDE’s), finite dimensional counterpart of equation (3.1), has been studied by many authors under general assumptions on coefficients, with very subtle results (see for instance [25], [36], [46], [47] and references therein). Some infinite dimensional models with long time and small noise asymptotics, using system theory, have been studied by Aihara [2], Bagchi and Borkar [3], Ibragimov and Khasminskii [26]-[28] etc. Using projection-based methods, the parameter problem for SPDE (3.1) driven by additive noise was studied for the first time by Khasminskii, Rozovskii and Huebner [22] [23]. The key point is that in reality, one may not assume to observe the whole field \(u(t, x)\), so a finite dimensional approximation of the solution is used to obtain consistent estimates of unknown parameter \( \theta \). The idea is to choose a basis in the suitable Hilbert space, such that the finite dimensional projection of the solution \(u\), i.e. Fourier coefficients w.r.t. this basis, will lead to a system of independent processes. Then, using first
Fourier coefficients, an explicit and simple form of the Maximum Likelihood Estimator (MLE) $\hat{\theta}_N$ for $\theta$ is obtained. As one may expect, $\hat{\theta}_N$ converges (in some sense) to the true parameter as $N$ increases. The equation (3.1) driven by additive noise, and with $A_0, A_1$ being elliptic differential selfadjoint operators on a bounded domain in $\mathbb{R}^d$, with a common system of eigenfunctions, was studied in [21], [22], [23], [66]. It was shown that if $\text{ord}(A_1) \geq (\text{ord}(A_0) - d)/2$, then MLE $\hat{\theta}_N$ is strongly consistent as $N \to \infty$. We note here, that operators $A_0, A_1$ commute, since they have the same system of eigenfunctions, and hence the random field is diagonalizable. Consequently, direct application of Galerkin type of approximations will lead to a system of ODE’s mentioned above.

The equation with none-commuting operators was considered by Lototsky and Rozovskii [52],[53]. Even though, applying the same technics, formally computed estimator is not an MLE, it was shown that under some technical assumptions the estimator is consistent and asymptotically normal.

We would like to mention that for ordinary SDE consistent estimates are obtained if either observation time $T$ goes to infinity or the noise amplitude tends to zero (for some examples see [36]). Although, for SPDE’s within the above framework consistency is archived on any time interval $[0, T]$ as the dimension of the projection increases. This is due to the fact that generally speaking the probability measures generated by the solution $u^\theta$ of an ordinary SDE are absolute continues for different $\theta$, while for an SPDE with additive space-time white noise, under some conditions on the order of differential operators, the measures are singular (see for instance [56], [39]). However, the measures generated by the projection are absolute continuous for different values of the parameter, which implies the existence of MLE computed from likelihood ratio (Radon-Nikodym derivative).
While the SPDE’s driven by additive noise have been studied by many authors, the corresponding equations with multiplicative noise have not been covered at all (as far as we know), in particular the equation with multiplicative space-time white noise. In this work, we will address this question for several classes of equations, including space-time white noise. Despite similarity of these equations, it turns out that the measures generated by the projection of the solution with multiplicative noise, generally speaking, are not anymore absolute continuous and the likelihood ratio does not exist. For example, this is the case when $W$ is space-time white noise, and we will discuss this in Section 3.2. Regarding this, we consider equations with some noises similar to cylindrical Brownian motion, but for which we are able to find consistent estimates of the unknown parameter, based on finite projection of the solution on some basis. For simplicity, we suppose that operators $A_0, A_1, M$ commute, however the results can be extended to non-commutative ones. In Section 3.3 we set up the problem, and study existence, uniqueness and finite dimensional approximation of the solution. The estimate and its properties are discussed in Section 3.4. We find sufficient conditions on operators $A_0, A_1, M$ that implies consistency, asymptotic efficiency, and normality of estimates. We conclude this chapter with some application of abstract results to general stochastic parabolic equations, presented in Section 3.5. Some numerical results are presented at the end of Section 3.5, where we show some practical applications of theoretical results, and provide some counter-examples related to sufficiency of the conditions imposed in on the original equation.
3.2 Special case: space-time white noise

In this section we will analyze the simplest SPDE driven by space-time white noise, and give some insights about the problems arising in parameter estimation for equations with multiplicative noise. We will follow the notations and definitions from Chapter 1.

Let us consider the stochastic one dimensional heat equation

\[ du(t, x) = \theta u_{xx}(t, x)dt + u(t, x)dW(t, x) , \quad (3.2) \]

where \( t \in [0, T] \), \( x \in [0, 1] \), \( u(0) = u_0 \in L_2(0, 1) \) and periodic boundary conditions.

The elliptic operator \( Au = -u_{xx} \), with eigenfunctions \( h_k(x) = e^{ik\pi x} \) \( (k \in \mathbb{Z}) \), generates the scale of Sobolev spaces \( H^s = H^s(0, 1) \) \( (s \in \mathbb{R}) \). The system \( h_k, k \in \mathbb{Z} \), forms a CONS in \( L_2(0, 1) \) and \( Ah_k = -(\lambda_k)^2 h_k \), with \( \lambda_k = k\pi \). By Theorem 1.5.1, if \( s < -1/2 \) then equation (3.2) has a unique solution \( u \in L_2((0, T) \times \Omega; H^{s+1}) \cap L_2(\Omega; C((0, T); H^s)) \) (see also [79]).

It is assumed that the observed field \( u \) satisfies (3.2) for some unknown but fixed value \( \theta_0 \) of parameter \( \theta \). We suppose that the first \( N \) Fourier coefficients of \( u \), relative to \( \{h_k\}_{k \in \mathbb{Z}} \), are known, and we would like to find a consistent estimate of \( \theta_0 \) as \( N \to \infty \). Denote by \( \{u_k\}_{k \in \mathbb{Z}} \) the Fourier coefficients of the solution \( u \) w.r.t. \( h_k, k \in \mathbb{Z} \), and by \( \Pi^N \) the projection of \( H^s \), \( s \in \mathbb{R} \), on \( \text{Span}\{h_k\}_{k=-N}^N \). Hence, \( u^N(t, x) := \Pi^N u = \sum_{k=-N}^{N} h_k(x)u_k(t) \). We consider the following finite-dimensional approximations of (3.2)

\[ dv^N = \theta v_{xx}^N dt + \Pi^N(v^N dW^N) , \quad (3.3) \]

which to some extent corresponds to Galerkin-type approximation.
It should be mentioned that since operator \( \Pi^N \) and multiplication operator \( v^N \rightarrow v^N dW^N \) do not commute, \( v^N \) from (3.3) does not coincide with projection \( u^N \) of original solution \( u \). In other words, generally speaking \( \Pi^N v^N dW^N \neq u^N dW^N \). However, we will show that approximation sequence \( v^N = \sum_{k=-N}^{N} v_k^N \) converges to the solution \( u \), as \( N \to \infty \), which means that from practical point of view, we can substitute in our estimates the values of \( v_n \) by Fourier coefficients \( u_n, n = -N, \ldots, N \).

Taking into account that operators \( \Pi^N \) and \( \Delta v = v_{xx} \) have a common system of eigenvalues, hence commute, we rewrite (3.3) as follows

\[
\sum_{k=-N}^{N} h_k(x) dv_k(t) = -\theta \sum_{k=-N}^{N} \lambda_k^2 h_k(x) v_k(t) dt + \Pi^N \left( \sum_{k=-N}^{N} h_k(x) v_k \sum_{j=-N}^{N} h_j(x) dW_j(t) \right). \tag{3.4}
\]

Observe that \( h_k h_m = h_{k+m}, k, m \in \mathbb{Z} \), which implies that (3.4) is equivalent to a system of \( 2N + 1 \) stochastic ODEs,

\[
dv_{-N} = -\theta \lambda_{-N}^2 v_{-N} dt + \left( v_{-N} dW_0 + v_{-N+1} dW_{-1} + \cdots + v_0 dW_{-N} \right)
\]
\[
dv_{-N+1} = -\theta \lambda_{-N+1}^2 v_{-N+1} dt + \left( v_{-N} dW_1 + v_{-N+1} dW_0 + \cdots v_1 dW_{-N} \right)
\]

\[
\vdots
\]
\[
dv_0 = -\theta \lambda_0^2 v_0 dt + \left( v_{-N} dW_N + v_{-N+1} dW_{N-1} + \cdots + v_{N-1} dW_{-N} \right)
\]
\[
\vdots
\]
\[
dv_N = -\theta \lambda_N^2 v_N dt + \left( v_0 dW_N + v_1 dW_{N-1} + \cdots + v_N dW_0 \right)
\]

or, in matrix form

\[
dv^{\theta,N} = \theta A(v^{\theta,N}) dt + b(v^{\theta,N}) dW^N_t, \tag{3.6}
\]
where \( A(v^N) = -[\lambda^{-N} v_{-N}, \ldots, \lambda^N v_N]' \), and

\[
b(v^N) = \begin{pmatrix}
v_0 & v_{-1} & \ldots & v_{-N+1} & v_{-N} & 0 & \ldots & 0 \\
v_1 & v_0 & \ldots & \ldots & v_{-N+1} & v_{-N} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
v_N & v_{N-1} & \ldots & v_1 & v_0 & v_{-1} & \ldots & v_{-N} \\
0 & 0 & \ldots & 0 & v_N & v_{N-1} & \ldots & v_0
\end{pmatrix},
\]

with \( \dim(b) = (2N + 1) \times (2N + 1) \) and the element \( b_{00} \) is boldfaced. For conveniens of writing, we index the coordinates in \( \mathbb{R}^{2N+1} \) from \(-N\) to \(N\).

As we mentioned above, it is natural to have some convergence of the approximation \( v^N \) to the real solution \( u \) as dimension of approximation increases. To show this we prove an auxiliary result.

**3.2.1 Lemma.** If \( M_N = (m_{kj})_{k,j=-N}^{N} \) is a \( (2N + 1) \times (2N + 1) \) real matrix such that \( m_{jj} = -|j|^2 \), \( j = -N, \ldots, N \), \( m_{kj} = 1 \) if \( |k - j| \leq N \), and \( \mu_N \) is the maximum eigenvalues of the matrix \( M_N \), then there exist a constant \( C \) such that \( \mu_N < C \) for all \( N \in \mathbb{N} \).

**Proof.** It suffices to show that \( \langle M_N x, x \rangle \leq C \) for some real constant \( C \), and for every \( x \in \mathbb{R}^{2N+1} \) with \( \|x\| = 1 \), where \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) are usual (Euclidian) inner product and norm in \( \mathbb{R}^{2N+1} \).

Let us denote \( \alpha_n = m_{nn}, n = 0, 1, \ldots, N \) and \( f(x) = \langle M_N x, x \rangle \). We rewrite \( f(x) \) as follows:
\[ f(x) = \alpha_N x_N^2 + 2x_N(x_{N+1} + \cdots + x_0) + \alpha_{N-1} x_{N+1}^2 + 2x_{N+1}(x_{N+2} + \cdots + x_0 + x_1) + \cdots + \alpha_1 x_{-1}^2 + 2x_{-1}(x_0 + \cdots + x_{N-1}) + \alpha_0 x_0^2 + 2x_0(x_1 + \cdots + x_N) + \alpha_1 x_1^2 + 2x_1(x_2 + \cdots + x_N) + \cdots + \alpha_{N-1} x_{N-1}^2 + 2x_{N-1}x_N + \alpha_N x_N^2. \] (3.7)

So, we want to maximize \( f \) under constrain \( \| x \| = 1 \). Using the method of Lagrange multipliers we find that \( x \) should be a non-trivial solution of the system of linear equations \( L_\lambda x = 0 \), where

\[
L_\lambda = \begin{pmatrix}
\alpha_N + \lambda & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & \alpha_{N-1} + \lambda & \ldots & 1 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \alpha_0 + \lambda & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \alpha_N + \lambda
\end{pmatrix},
\]

and \( \lambda \) is the multiplier parameter. Obviously, \( L_\lambda x = 0 \) has a non-zero solution, iff \( \alpha_j + \lambda = 0 \) for some \( j \). Suppose \( \alpha_N + \lambda = 0 \), since otherwise, we get the same system, but of smaller dimension. Let \( a = \max\{|x_{-N}|, |x_N|\} \). Note that since \( \| x \| = 1 \) we
have \( a \leq 1 \). By direct arithmetic evaluations, one can deduce that \( |x_{\pm j}| \leq \frac{a}{n+j}, \; j = -N + 1, \ldots, N + 1 \). Hence, by (3.7), we get \( f(x) \leq C \).

Now we can prove the following result

3.2.2 Theorem. For every \( \gamma < -1/2 \), the sequence \( \{v^N\}_{n \geq 1} \) converges weakly in \( H^{\gamma+1} \) to the weak solution of the equation (3.9).

Proof. Similar to deterministic case, we will use weak sequential compactness of Hilbert spaces (see for instance [9]). The main part of the proof is to establish that the sequence \( \|v^\theta,N\|^2_{L_2(\Omega \times [0, T]; H^\gamma)} \) is uniformly bounded. Since \( \gamma < -1/2 \), it suffices to show that \( \mathbb{E} \int_0^T \|v^\theta,N\|_0^2 dt \) is uniformly bounded in \( N \). By Itô’s formula for \( f(x) = \sum_{k=-N}^N x_k^2 \) we deduce

\[
dY = AYdt,
\]

where \( Y = \left[ \mathbb{E}(v_{-N}^2), \ldots, \mathbb{E}(v_N^2) \right]' \) and

\[
A = \begin{pmatrix}
1 - 2\theta \lambda_N^2 & \ldots & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \ldots & 1 - 2\theta \lambda_0^2 & \ldots & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & \ldots & 1 - 2\theta \lambda_N^2
\end{pmatrix}.
\]

By Lemma 3.2.1 the maximal eigenvalue of the matrix \( A \) is uniformly (in \( N \)) bounded from above. Consequently, by Theorem 4.2 from [16] the norm \( \|v^\theta,N\|_0 \) is also uniformly bounded in \( N \). Hence, there exists a subsequence of \( v^\theta,N \) that converges weakly in \( L_2(\Omega \times [0, T]; H^{\gamma+1}) \). Let us denote this limit by \( \tilde{v} \). By (3.3), for every \( \varphi \in H^{\gamma+1} \) we have \( (v^\theta,N, \varphi) = (u_0^N, \varphi) + \theta \int_0^T [v^\theta,N, \varphi] + \int_0^T (\Pi^N (v^\theta,N dW^N(s), \varphi)) \). Taking \( N \to \infty \), we conclude...
we deduce \((\tilde{v}, \varphi) = (u_0, \varphi) + \theta \int_0^t [\tilde{v}_{xx}, \varphi] + \int_0^t (\tilde{v}dW(s), \varphi)]\). Since the last equality holds for every \(\varphi \in H^{\gamma+1}\), we get \(\tilde{v} = u\). Theorem is proved. 

Let us denote by \(P^{\theta,N}\) the measure in \(\mathcal{X} = C([0, T]; \Pi^N(H^0))\), generated by the solution \(v^{\theta,N}(s), 0 \leq s \leq T\) of equation (3.6), i.e. \(P^{\theta,N}(a) = P(v^{\theta,N} \in a)\) for all \(a \in \mathcal{B}_T\), where \(\mathcal{B}_T\) is the Borel \(\sigma\)-algebra on \(\mathcal{X}\).

By Theorem 1.2.2 (see also Theorem 7.6.4 in [47]) the measures \(P^{\theta,N}\) and \(P^{\theta_0,N}\) are mutually absolutely continuous if the algebraic system of equations

\[
b(x)y = (\theta - \theta_0)A(x) \tag{3.8}
\]

has a bounded solution (in \(y \in \mathbb{R}^{2N+1}\)) for each \(t \in [0, T], x \in \mathbb{R}^{2N+1}\). It turns out that this algebraic equation with apparently nice structure does not have an obvious solution. Direct evaluations show that for \(N = 1, 2, 3\), the system (3.8) does not have solution for some values of \(x\), which means that we can not apply the general results about absolute continuity of measures. On the other hand, we can not conclude that these measures are singular for different parameters \(\theta\). Some examples and generalization of Theorem 7.6.4 from [47] are discussed in [36]. In particular, it is shown that the corresponding measures can be singular as well as absolute continuous, if the system (3.8) does not have a finite solution. Unfortunately, these results can not be applied to our system of SDE’s, nor the proofs can be adapted (by best of our knowledge).

We want to mention, that simple structure of matrix \(b\) is due to the unique property of eigenfunctions, \(h_k h_m = h_{k+m}\). In general, \(b\) will be a full matrix, without any specific properties, which makes impossible to find a solution of equation (3.8).

We estimated naively the MLE \(\hat{\theta}_N\) of \(\theta\), by maximizing the likelihood ratio

\[
\frac{dP^{\theta,N}}{dP^{\theta_0,N}}(v^{N,\theta_0}),
\]

and used it for some numerical simulations. The results suggest us that
the measures look like to be singular, which consequently implies that some different approach needs to be considered for the equations with multiplicative space-time white noise.

### 3.3 General case. Approximation of the solution

In this section we will set up the parameter estimation problem, discuss the existence and uniqueness of the solution of the corresponding equations, and consider the finite dimensional approximation of the solution.

Let $H$ be a separable Hilbert space with the inner product $(\cdot, \cdot)_0$, and the corresponding norm $\| \cdot \|_0$. Suppose $\Lambda$ is a linear operator on $H$ such that $\| \Lambda u \|_0 \geq c \| u \|_0$ for every $u$ from the domain of $\Lambda$. Then the operators $\Lambda^\gamma$, $\gamma \in \mathbb{R}$, are well defined, and generate the spaces $H^\gamma$ as follows: the domain of $\Lambda^\gamma$ coincides with $H^\gamma$ for every $\gamma \geq 0$; $H^\gamma$ is the completion of $H$ with respect to the norm $\| \cdot \|_{\gamma} := \| \Lambda^\gamma \cdot \|_0$, for all $\gamma < 0$ (see Chapter 1 here, or [41]). The set $\{H^\gamma\}_{\gamma \in \mathbb{R}}$ is called a Hilbert scale. The spaces $(H^{\gamma+m}, H^\gamma, H^{\gamma-m})$ form a normal triple with canonical bilinear form $\langle u_1, u_2 \rangle = (\Lambda^{\gamma-m} u_1, \Lambda^{\gamma+m} u_2)_0$, where $u_1 \in H^{\gamma-m}, u_2 \in H^{\gamma+m}$; $\Lambda^\gamma(H^r) = H^{r-\gamma}$ for every $\gamma, r \in \mathbb{R}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a stochastic basis. In the triple $(H^{\gamma+m}, H^\gamma, H^{\gamma-m})$, for some $\gamma \in \mathbb{R}$, $m \in \mathbb{N}$, we consider the following SPDE

$$
\begin{cases}
  du(t) = ((A_0 + \theta A_1)u(t) + f(t))dt + \sum_{j=1}^{n}(M_j u(t))dW_j(t) \\
u(0) = u_0,
\end{cases}
$$

(3.9)
where $A_0, A_1, M_s$ are linear operators, $f$ is a given vector-valued function, $W_j$ are independent standard Brownian motions, $n \in \mathbb{N}$ or $n = \infty$, $t \in [0, T]$, and $\theta$ is a scalar parameter subject to estimation belonging to an open set $\Theta \subset \mathbb{R}$.

The first question we will address is the existence and uniqueness of the solution of equation (3.9). In connection with this, throughout in what follows, we will assume that:

(H1) there exists $m > 0$ such that the linear operator $A_\theta := A_0 + \theta A_1$, is bounded from $H^{\gamma+m}$ to $H^{\gamma-m}$ for every $\gamma \in \mathbb{R}$;

(H2) $M_j (j = 1, \ldots, n)$ are linear bounded operators acting from $H^{\gamma+m}$ into $H^{\gamma}$;

(H3) $u_0 \in H^{\gamma}$;

(H4) $f \in L^2(\Omega \times (0, T); H^{\gamma-m})$;

(H5) The operators $\Lambda, A_1, A_0, M_j$ have the same system of eigenfunctions $\{h_k\}_{k=1}^\infty$, the system $\{h_k\}_{k=1}^\infty$ is a complete system in $H$, and $h_k \in \cap_{\gamma \geq 0} H^{\gamma}$.

Condition (H5) is equivalent to say that operators $A_1, A_0, M$ commute, which consequently implies that eigenfunctions and eigenvalues of these operators does not depend on the parameter $\theta$. We want to note that this is the case in many applications. Usually the operator $-A_\theta$ is a differential, selfadjoint, positive defined operator of order $2m$ defined on the scale of Sobolev spaces $\{H^{\gamma}(M)\}_{\gamma \in \mathbb{R}}$, where $M$ is a $d$-dimensional compact orientable $C^\infty$ manifold. In this case $\Lambda = (-A_\theta)^{\frac{1}{2m}}$ and with $M_j = \Lambda^\gamma$ the assumption (H5) holds true. We will discuss this example in details later on (see Section 3.5).

Most of the results remain true in more general situations, in particular when operators $A_1, A_0, M$ do not commute, and will be published elsewhere.
Denote by $l^r_k$, $l^0_k$, $\lambda_k$ and $\mu^j_k$ the eigenvalues of the operators $A_1, A_0, \Lambda$ and $M_j$ respectively, which correspond to the eigenfunction $h_k$ for every $k \in \mathbb{N}$.

Assume that:

(H6) $l^r_k < 0$, $\mu^j_k \neq 0$ for $r = 0, 1$, $j = 1, \ldots, n$, $k \in \mathbb{N}$, and there exist $\epsilon > 0$, $M \in \mathbb{R}$ such that $\lambda_k^{-2m} \sum_{s=1}^{n} (\mu^s_k)^2 \leq \epsilon - 2\lambda_k^{-2m}(\theta l^1_k + l^0_k) \leq M$ for every $\theta \in \Theta$, $k \in \mathbb{N}$.

Suppose $l^r_k$, are enumerated in order of magnitude

$$l^r_1 \geq l^r_2 \cdots \geq l^r_k \geq \ldots ,$$

where $r = 0, 1$.

3.3.1 Theorem. Under Assumptions (H1)-(H6), there exists a unique weak solution $u$ of equation (3.9), the solution belongs to $L^2(\Omega, C((0, T); H^\gamma)) \cap L^2(\Omega \times [0, T]; H^{\gamma+m})$ and

$$E \left( \sup_{0 \leq t < T} \|u(t)\|_\gamma^2 \right) + E \left( \int_0^T \|u(t)\|_{\gamma+m}^2 dt \right) \leq C E \left( \|u_0\|_\gamma^2 + \int_0^T \|f(t)\|_{\gamma+m}^2 dt \right), \quad (3.10)$$

for some constant $C$.

Proof. This theorem follows from Theorem 1.5.1, a general result from the theory of SPDEs (see also [70], Theorem 3.1.4). Conditions (i)-(iii) and (v) of Theorem 1.5.1 come naturally from the structure of equation (3.9) and follow immediately from Assumptions (H1)-(H4).
Condition (iv) represents strong parabolicity or coercivity assumption. It suffices to check this condition for the vectors \( h_k \). Put \( l_k(\theta) = l_k^0 + \theta l_k^1 \), then using above assumptions we have

\[
2[A\theta h_k, h_k] + \sum_{j=1}^{n} ||M_j h_k||^2_{\gamma} = 2l_k(\theta)[h_k, h_k] + \sum_{j=1}^{n} (\mu_k^j)^2 ||h_k||^2_{\gamma} \\
\leq 2l_k(\theta)||h_k||^2_{\gamma} + \sum_{j=1}^{n} (\mu_k^j)^2 ||h_k||^2_{\gamma} = \left( 2l_k(\theta) + \sum_{j=1}^{n} (\mu_k^j)^2 \right) \lambda_k^{-2m}||h_k||^2_{\gamma+m}.
\]

(3.11)

Denote by \( \delta = -\max_k \{\lambda_k^{-2m} (2l_k(\theta) + \sum_{j=1}^{n} (\mu_k^j)^2)\} \). By Assumption (H6) we have that \( \delta > 0 \), thus the condition (v) is verified, and the Theorem is proved.

To make an inference about parameter \( \theta \) we will project the solution \( u \) on some finite-dimensional subspace of \( H^\gamma + m \), and essentially reduce the SPDE to a system of SODE’s. As we mentioned, a natural approach is to project on the subspace generated by the eigenfunctions \( h_k \), and use the Fourier coefficients of the solution \( u \) with respect to \( \{h_k\}_{k=1}^{\infty} \).

Suppose that the observed field \( u \) satisfies (3.9) for some unknown but fixed value \( \theta_0 \) of the parameter \( \theta \). Denote by \( \{u_k\}_{k \in \mathbb{N}} \) the Fourier coefficients of the solution \( u \) w.r.t. \( \{h_k\}_{k \in \mathbb{N}} \), and by \( \Pi^N \) the projection of \( H^\gamma, \gamma \in \mathbb{R} \), on \( \text{Span}\{h_k\}_{k=1}^{N} \). Hence, \( u^N(t, x) := \Pi^N u = \sum_{k=1}^{N} u_k(t)h_k(x) \).

We consider the following finite-dimensional approximation of (3.9)

\[
du^N(t, x) = \Pi^N((A_0 + \theta A_1)u(t, x) + f(t)) dt \\
+ \sum_{j=1}^{n} \Pi^N(M_j u(t, x))dW_j(t).
\]

(3.12)
3.3.2 Remark. By Theorem 3.3.1, \( u \in L_2(\Omega \times [0, T]; H^{\gamma+m}) \), which implies that

\[
\lim_{N \to \infty} \mathbb{E} \int_0^T ||u^N - u||_{\gamma+m}^2 dt = 0.
\]

Denote by \( f_k \) the Fourier coefficients of the function \( f \). From (3.12) and by assumption (H6), we deduce that Fourier coefficient \( u_k \) satisfies the following equation

\[
du_k(t) = \left( (l_0^k + \theta l_1^k)u_k(t) + f_k(t) \right) dt + \sum_{j=1}^n \mu_k^j u_k(t) dW_j(t),
\]

(3.13)

with \( u_k(0) = (u_0, h_k)_{\gamma} \), and \( k \in \mathbb{N} \).

3.4 The estimate and its properties

In this section we will find a Maximum Likelihood Estimate (MLE) of the parameter \( \theta \) by using the approximation (3.12) of the solution to the original equation.

Let us fix a number \( \theta_0 \in \Theta \), that represents the true value of the parameter \( \theta \) subject to estimation. In what follows we will suppose that the first \( k \) Fourier coefficients of the field \( u \) are observed, and we will estimate \( \theta_0 \) from them. For simplicity of writing we will drop the index \( \theta_0 \) where there is no room for ambiguity. Namely, we will write \( u_k \) instead of \( u_{\theta_0,k} \). Let \( B_T \) be the Borel \( \sigma \)-algebra on the space \( C([0, T], H^{\gamma+m}) \). Denote by \( P_\theta^k \) the measure on \( B_T \) generated by the solution \( u_{\theta,k} \) of equation (3.13). Note that the equation (1.5) from Theorem 1.2.2 in this case becomes \( \sum_{j=1}^n \mu_k^j y = (\theta - \theta_0) l_1^k \).
which has a finite solution for every $k \in \mathbb{N}$, and hence the measures $P^\theta_k$ and $P^{\theta_0}_k$ are mutually absolute continuous, with Radon-Nikodym derivative

$$
\frac{dP^\theta_k}{dP^{\theta_0}_k}(u_k) = \exp \left\{ \int_0^T \frac{l_1^k(\theta - \theta_0)}{u_k} \frac{\sum_{j=1}^n (\mu^j_k)^2}{l_1^k} \frac{du_k}{\sum_{j=1}^n (\mu^j_k)^2} \right\} .
$$

The maximum likelihood estimate $\hat{\theta}^N$ of $\theta_0$ is obtained by maximizing the Radon-Nikodym derivative (likelihood ratio) (3.14) with respect to the parameter of interest $\theta \in \Theta$. Direct computations yield

$$
\hat{\theta}_k = \frac{1}{l_1^k T} \int_0^T \frac{du_k}{u_k} - \frac{1}{2} \sum_{j=1}^n (\mu^j_k)^2 dt ,
$$

(3.15)

By Itô’s Lemma, we find that $d \ln(u_k) = \frac{du_k}{u_k} - \frac{1}{2} \sum_{j=1}^n (\mu^j_k)^2 dt$, and hence from (3.15) we get

$$
\hat{\theta}_k = \frac{1}{l_1^k T} \ln \frac{u_k(T)}{u_k(0)} + \frac{1}{2l_1^k T} \sum_{j=1}^n (\mu^j_k)^2 - \frac{1}{l_1^k T} \int_0^T \frac{f_k(s)}{u_k(s)} ds .
$$

(3.16)

Also, from (3.15) and taking into account (3.13) we have the following representation for the estimate

$$
\hat{\theta}_k = \theta_0 + \frac{1}{l_1^k T} \sum_{j=1}^n \mu^j_k W_j(T) ,
$$

(3.17)

which will play the key role in our future investigations. Note that by (3.15) $\hat{\theta}_k$ is an unbiased estimate of $\theta_0$. 

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In addition to Assumptions (H1)-(H6), we will assume the following:

(H7) There exists the limit \( \lim_{k \to \infty} \frac{\sum_{j=1}^{n} (\mu_j^k)^2}{(l_k^j)^2} = \mu < \infty. \)

We want to mention that assumption (H7) does not follow from (H6) neither converse is true. For example, take \( l_0^k = -\lambda_k^{2m}, \ l_1^k = -\lambda_k^m, \mu_k = \lambda_k^m, \ n = 1. \) Then (H6) is satisfied, while (H7) does not hold. If we put \( l_0^k = 0, \ l_1^k = -\lambda_k^{m-1}, \mu_k = \lambda_k^{2m-1}, \ n = 1, m > 2, \) then (H7) holds, and \( \mu = 0, \) but (H6) is not satisfied. Both assumptions are related to the order of operators \( A_0, A_1, M_j, \) when all of them are some pseudo-differential operators. The relationship between orders of operators will be discussed in details in the next section.

Denote by \( L \) the set of all real-valued, nonnegative functions \( w \) defined on \( \mathbb{R}, \) which are symmetric, \( w(0) = 0 \) and monotone for \( x > 0. \) These functions are called loss functions.

3.4.1 Theorem. Under Assumptions (H1)-(H6) the following holds true:

(i) \( \lim_{T \to \infty} \hat{\theta}_k = \theta_0, \ P\text{-a.s., for every } k \in \mathbb{N} \) (consistency in time);

Suppose in addition that (H7) is satisfied. Then

(ii) if \( \mu = 0, \) then \( \lim_{k \to \infty} \hat{\theta}_k = \theta_0, \ P\text{-a.s., for every } T > 0 \) (consistency in \( k \)). Moreover, in this case, MLE \( \hat{\theta}_k \) is asymptotic efficient, that is for any loss function \( w \in L \)

\[
\lim_{k \to \infty} \mathbb{E} w \left( \frac{|l_k^1| \sqrt{T}}{\sqrt{\sum_{j=1}^{n} (\mu_j^k)^2}} (\hat{\theta}_k - \theta_0) \right) = \mathbb{E} w(\xi),
\]

were \( \xi \) a Gaussian random variable with zero mean and unit variance;
(iii) if $\mu \neq 0$, then $\lim_{k \to \infty} \hat{\theta}_k = \theta_0 + \xi(\mu)$, $\mathcal{P}$-a.s., for every $T > 0$, where $\xi(\mu) \sim \mathcal{N}(0, \frac{\mu}{T})$.

**Proof.** Property (i) follows from (3.17) and the fact that $\lim_{t \to \infty} \frac{W(t)}{t} = 0$.

If $\mu = 0$ then $\lim_{k \to \infty} \frac{\mu_j}{T_k} = 0$ for every $j = 1, \ldots, n$. Thus, from equality (3.17) we conclude that $\lim_{k \to \infty} \hat{\theta} = \theta_0$. Also, by (3.17) we note that

$$
\xi_k := \frac{|l_k| \sqrt{T}}{\sqrt{\sum_{j=1}^{n} (\mu_j^2)}} (\hat{\theta}_k - \theta_0) \sim \xi,
$$

where $\xi \sim \mathcal{N}(0, 1)$. Hence $\mathbb{E}(w(\xi_k)) = \mathbb{E}(w(\xi))$, that consequently implies $\lim_{k \to \infty} \mathbb{E}(w(\xi_k)) = \mathbb{E}(w(\xi))$, and (ii) is proven.

Property (iii) follows from (3.18) and Assumption (H7).

3.4.2 Remark. By (3.18) one gets at once that $\hat{\theta}_k$ is also asymptotic efficient in $T$.

We live this property apart since the main goal is to get a consistent estimate on every small time interval $[0, T]$.

By the above theorem, we can use every individual $\hat{\theta}_k$ as an estimate for $\theta_0$, and by increasing $T$ to get a good estimate of the true parameter, regardless of Assumption (H7). However in practice we would like to find $\theta_0$ with any precision in a small interval of time. As we can see from (ii), Theorem 3.4.1, this is possible by increasing the number of Fourier coefficients. This is due to the fact that the measures $\mathbb{P}_{\theta}^u$, that correspond to the solution of the original equation (3.9), are singular for different values of the parameter $\theta$. Comparing (ii) and (iii), we also note that for $\mu = 0$ we get consistent estimates, while for $\mu \neq 0$, the limit estimate is biased. The speed of convergence is of order $\frac{\sum_{j=1}^{n} (\mu_j^2)}{|l_k| \sqrt{T}}$. Although for $\mu = 0$, $\hat{\theta}_k$ is a consistent estimate of the true values as $k \to \infty$, we want to mention that in practice the estimates
will be evaluated by formula (3.16), that involves the values of \( u_k \) and \( f_k \). Note that \( u_k \) and \( f_k \) being Fourier coefficients of some functions in \( \mathcal{H}^\gamma \) will have at least polynomial decay to zero. This implies that for large \( k \) we will not be able to observe and compute \( u_k \), \( f_k \) reliably. However, in many applications \( f \equiv 0 \), and in this case to apply (3.16) we need to know only the values of \( \log \frac{u_k(T)}{u_k(0)} \). In this situation, equation (3.13) becomes the well-known geometrical Brownian motion, with solution

\[
\begin{align*}
    u_k(t) &= u_k(0) \exp\left\{ (\theta_0 l_k(\theta) - \sum_{j=1}^{n} (\mu_j^{k})^2/2) t + \sum_{j=1}^{n} \mu_j^{k} W_j(t) \right\}.
\end{align*}
\]

Obviously,

\[
\log \frac{u_k(T)}{u_k(0)} = \left( \theta_0 l_k(\theta) - \sum_{j=1}^{n} (\mu_j^{k})^2/2 \right) T + \sum_{j=1}^{n} \mu_j^{k} W_j(T) \tag{3.19}
\]

which implies that \( \log \frac{u_k(T)}{u_k(0)} \) has the same magnitude as \( l_k \) for a reasonable time interval \([0, T]\), and thus the estimator given by (3.16) is computable for sufficiently large \( k \). As will see later on, it suffices to take \( k \) being around ten, to get an approximation of order \( 10^{-4} \).

On the other hand, a reasonable and natural approach is to consider a weighted average of the estimates \( \hat{\theta}_k \). For example, if \( n = 1 \), and the sign of \( \mu_k \) varies with \( k \), using (3.17) we observe that the estimates \( \hat{\theta}_k \) will approach the true value by oscillating around it (depending on the sign of \( W \) and \( \mu_k \)). Hence, some proper chosen weighted average of \( \theta_k \) will converge faster to the true value. For the rest of this section we will discuss how to choose the weights such that to have at least the same rate of convergence of new estimates comparative to \( \hat{\theta}_k \).

Suppose \( \{\beta_k\}_{k=1}^{\infty} \subset \mathbb{R}_+ \), and put

\[
\hat{\theta}_{(N)} = \frac{\sum_{k=1}^{N} \beta_k \hat{\theta}_k}{\sum_{k=1}^{N} \beta_k}. \tag{3.20}
\]
Since $\mathbb{E}(\hat{\theta}_k) = \theta_0$, by (3.20) obviously we have that $\mathbb{E}(\hat{\theta}(N)) = \theta_0$, i.e. $\hat{\theta}(N)$ is an unbiased estimator of $\theta_0$.

From (3.16), we get

$$
\hat{\theta}(N) = \frac{\sum_{k=1}^{N} \frac{\beta_k}{T} \left( \ln \frac{u_k(T)}{u_k(0)} - \int_0^T \frac{f_k(s)}{u_k(s)} ds \right)}{T \sum_{k=1}^{N} \beta_k} + \frac{\sum_{k=1}^{N} \frac{\beta_k}{T} \sum_{j=1}^{n} (\mu_j^k)^2}{2 \sum_{k=1}^{N} \beta_k} - \frac{\sum_{k=1}^{N} \frac{\beta_k \mu_k^0}{\beta_k}}{\sum_{k=1}^{N} \beta_k},
$$

(3.21)

which will be the main formula used in computational tasks.

Similarly, by (3.17) we deduce

$$
\hat{\theta}(N) = \theta_0 + \frac{\sum_{k=1}^{N} \beta_k \sum_{j=1}^{n} \eta_{jk}^k W_j(T)}{T \sum_{k=1}^{N} \beta_k},
$$

(3.22)

where $\eta_{jk}^k = \mu_j^k / l_k^1$ for $j = 1, \ldots, n$ and $k = 1, \ldots, N$.

It turns out, to preserve consistency and normality, it suffices to make the following assumption:

(H8) $\beta_k \geq 0$, $k \in \mathbb{N}$ and $\lim_{N \to \infty} \sum_{k=1}^{N} \beta_k = \infty$.

3.4.3 Theorem. If Assumptions (H1)-(H8) are fulfilled then:

(i) $\lim_{T \to \infty} \hat{\theta}(N) = \theta_0$, $\mathbb{P}$-a.e., for every $N \in \mathbb{N}$ (consistency in time) and MLE is asymptotic efficient as $T \to \infty$;

(ii) if $\mu = 0$, then $\lim_{N \to \infty} \hat{\theta}(N) = \theta_0$, $\mathbb{P}$-a.e., for every $T > 0$ (consistency in $N$),

and MLE $\hat{\theta}(N)$ is asymptotic efficient;

(iii) if $\mu \neq 0$, then $\lim_{N \to \infty} \hat{\theta}(N) = \theta_0 + \xi(\mu)$, $\mathbb{P}$-a.s., for every $T > 0$, where $\xi(\mu) \sim \mathcal{N}(0, \frac{\mu}{T})$. 

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Proof. Similar to Theorem 3.3.1, (i) follows from (3.22) and the fact that \( \lim_{t \to \infty} \frac{W(t)}{t} = 0 \).

By Stolz-Cesaro Theorem (discrete version of L’Hospital’s rule, see for instance [33], p.35 or [67], p.17), under Assumption (H8) we have

\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} \beta_k \eta_k^j}{\sum_{k=1}^{N} \beta_k} = \lim_{k \to \infty} \eta_k^j,
\]

for every \( j = 1, \ldots, n \).

If \( \mu = 0 \), then \( \lim_{k \to \infty} \eta_k^j = 0 \) for every \( j = 1, \ldots, n \), hence by (3.23) and (3.24) it follows that \( \lim_{N \to \infty} \hat{\theta}(N) = \theta_0 \).

Using (3.22) one can show that

\[
\frac{\sum_{k=1}^{N} \beta_k \sqrt{T}}{\sum_{k=1}^{N} \beta_k \sqrt{\sum_{j=1}^{n} (\eta_k^j)^2}} (\hat{\theta}_N - \theta_0) \sim \mathcal{N}(0, 1)
\]

and similar to Theorem 3.3.1 the rest of the proof follows. \( \square \)

In practice, ideally we want to choose the weights \( \beta_k \) such that the rate of convergence of \( \hat{\theta}_N \to \theta_0 \) to be at least the same (or faster) than the rate of convergence of \( \hat{\theta}_k \to \theta_0 \) and also, as we mentioned before, \( \beta_k \) should offset somehow the fast decay of the Fourier coefficients \( u_k \). It should be mentioned that Assumption (H8) does not guarantee the same rate of convergence. For example, if \( \eta_k = k^{-2} \) and \( \beta_k = k \), then

\[
\lim_{k \to \infty} \frac{\sum_{k=1}^{N} \beta_k \eta_k}{\eta_N \sum_{k=1}^{N} \beta_k} = \infty. \]

However, if \( \beta_k = k^\delta \) with \( \delta > 1 \), or \( \beta_k = \exp(k^\epsilon) \) with \( \epsilon > 0 \), then the rate of convergence is preserved. These heuristic discourses lead to the conclusion that \( \beta_k \) should grow fast enough. On the other hand, we have to keep in mind that
Finally, we will prove the following technical lemma about rate of convergence.

**3.4.4 Lemma.** Suppose \( \{a_n\}_{n=1}^{\infty} \) is an increasing sequence of positive numbers such that
\[
\lim_{n \to \infty} a_n = \infty, \quad \left| \frac{\Delta^{(2)}(a_n)}{\Delta^{(3)}(a_n)} \right| < M,
\]
where \( \Delta^{(k)} \) is the \( k \)-th finite difference. Then,
\[
\sum_{n=1}^{N} \frac{a_n^{-1} \exp(a_n)}{\sum_{n=1}^{N} \exp(a_n)} \sim a_N^{-1}.
\]

**Proof.** We will apply several times the Stolz-Cesaro Theorem to the sequence
\[
a_N \sum_{n=1}^{N} a_n^{-1} b_n / \sum_{n=1}^{N} b_n,
\]
where \( b_n := \exp(a_n) \), and show that this sequence has a finite limit.

\[
A = \lim_{N \to \infty} \frac{a_N \sum_{n=1}^{N} a_n^{-1} b_n}{\sum_{n=1}^{N} b_n} = \lim_{N \to \infty} \frac{a_N \sum_{n=1}^{N} \frac{b_n}{a_n} - a_N^{-1} \sum_{n=1}^{N-1} \frac{b_n}{a_n}}{b_N}
\]
\[
= 1 + \lim_{N \to \infty} \frac{(a_N - a_{N-1}) \sum_{n=1}^{N-1} a_n^{-1} b_n}{b_N}
\]
\[
= 1 + \lim_{N \to \infty} \frac{(a_N - a_{N-1}) \sum_{n=1}^{N-1} \frac{b_n}{a_n} - (a_{N-1} - a_{N-2}) \sum_{n=1}^{N-2} \frac{b_n}{a_n}}{b_N - b_{N-1}}
\]
\[
= 1 + \lim_{N \to \infty} \frac{(a_N - a_{N-1}) b_{N-1}}{a_{N-1}(b_N - b_{N-1})} + \lim_{N \to \infty} \frac{(a_N - 2a_{N-1} + a_{N-2}) \sum_{n=1}^{N-2} \frac{b_n}{a_n}}{(b_N - b_{N-1})}.
\]
We claim that both limits in the last expression are zero. Recall that \( b_n = \exp(a_n) \).

Thus,

\[
A_1 := \lim_{N \to \infty} \frac{a_N - a_{N-1}}{a_{N-1}} \cdot \frac{b_{N-1}}{b_N - b_{N-1}} = \lim_{N \to \infty} \frac{a_N - a_{N-1}}{a_{N-1}} \cdot \frac{1}{\exp(a_N - a_{N-1}) - 1},
\]

and since the sequence \( a_n \) is increasing, by Taylor series expansion for exponential functions, from the last equality we get

\[
A_1 \leq \lim_{N \to \infty} \frac{1}{a_{N-1}} = 0,
\]

hence \( A_1 = 0 \). Similarly,

\[
A_2 = \lim_{N \to \infty} \frac{(a_N - 2a_{N-1} + a_{N-2}) \sum_{n=1}^{N-2} \frac{b_n}{a_n}}{b_N - b_{N-1}} = \lim_{N \to \infty} \frac{\Delta^{(2)}(a_N) \sum_{n=1}^{N-2} \frac{b_n}{a_n}}{\Delta^{(1)}(a_N) \exp(a_N)}.
\]

By the initial assumption, \( \left| \frac{\Delta^{(2)}(a_N)}{\Delta^{(1)}(a_N)} \right| < M \), that implies

\[
A_2 \leq M \lim_{N \to \infty} \frac{\sum_{n=1}^{N-2} \frac{b_n}{a_n}}{\exp(a_N)}
\]

and again by the Stolz-Cesaro Theorem, we continue

\[
A_2 \leq M \lim_{N \to \infty} \frac{\exp(a_{N-2})}{a_{N-2}(\exp(a_N) - \exp(a_{N-1}))} \leq M \lim_{N \to \infty} \frac{1}{a_{N-2}} = 0.
\]

Finally by (3.25) we have \( A = 1 \), and lemma is proved.
Similarly to Lemma 3.4.4, one can prove that
\[ \frac{\sum_{n=1}^{N} a_n^{-1} b_n}{\sum_{n=1}^{N} b_n} \sim a_n^{-1}, \] if \( a_n^{-1} \) has polynomial growth, and \( b_n \) grows faster than \( a_n \).

**3.4.5 Remark.** Observe that \( \hat{\theta}_k \) involves only the \( k \)-th Fourier coefficient, while evaluating \( \hat{\theta}_{(N)} \) we use all \( u_k \) with \( k = 1, \ldots, N \). One should expect that using \( \hat{\theta}_{(N)} \) we will get more precise estimates since more information is used. However, we get actually the same precision in our estimates by applying either simple estimates \( \hat{\theta}_k \) or weighted average estimates \( \hat{\theta}_{(N)} \), since the rates of convergence are the same. Generally speaking, the filtration \( \mathcal{F}_t^{u_k} \) (see Section 1.2) generated by the solution \( u_k \) is smaller then the filtration \( \mathcal{F}_t \) generated by the Brownian Motions \( W_j(t) \), and could be different for different \( k \)'s. In our case, each \( u_k \) is a geometrical Brownian motion driven by the same processes \( W_j(t) \) \( (j = 1, \ldots, n) \), hence are given by the same function with different parameters but with same noise. This implies that the filtrations \( \mathcal{F}_t^{u_k} \) are the same for all \( k \in \mathbb{N} \). In other words, each Fourier coefficient \( u_k \) contains the same amount of information about \( \theta_0 \), that explains why the precisions of \( \hat{\theta}_k \) and \( \hat{\theta}_{(N)} \) are the same.

### 3.5 Applications to stochastic parabolic differential equations

In this section, we will present some applications of Theorems 3.4.1 and 3.4.3 to equation (3.9) with \( A_{\theta}, M_j \) being some pseudo-differential operators. We will begin with an abstract case, and continue with some particular examples, including some numerical results for the initial parameter estimation problem. If otherwise it is not mentioned all notations from Section 3.3 will be preserved.
Let $M$ be a $d$-dimensional compact, orientable, $\mathbb{C}^\infty$ manifold with a smooth positive measure $dx$, $\{H^s(M)\}_{s \in \mathbb{R}}$ be the scale of Sobolev spaces on $M$ (see for instance [1], [76]), and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a stochastic basis.

Suppose that $A_\theta = A_0 + \theta A_1$, is a differential, elliptic operator of order $2m$, formally self-adjoint, and the operator $A_\theta$ is lower semi-bounded for every $\theta \in \Theta$ (uniformly in $\theta$). The latter means that there exists a positive number $\varepsilon > 0$ such that $-A_\theta > \varepsilon I$, for every $\theta \in \Theta$, where $I$ is the identity operator in $H^{\gamma+m}$.

It is well-known that the operator $A_\theta$ can be extended to a closed, self-adjoint operator on $L_2(M)$, the spectrum of this operator is discrete, consisting of eigenvalues of finite multiplicity (for more details see the discussion in Chapter 1 and references mentioned therein). Denote by $\{h_{k,\theta}\}$ the eigenfunctions of $A_\theta$, and suppose that $\{h_{k,\theta}\}_{k=1}^\infty$ forms an orthonormal system in $L_2(M)$. By Hilbert-Schmidt theory, this system is complete in $L_2(M)$, and $h_{k,\theta} \in H^\gamma(M) \cap C^\infty(\mathcal{M})$ for every $\gamma \in \mathbb{R}$. Generally speaking, the eigenfunctions $h_{k,\theta}$ depend on $\theta$, however, for sake of simplicity we will rule this out, and write $h_k$. From general theory of elliptic operators, under above assumptions, the operators $A_\theta : H^{\gamma+m} \rightarrow H^{\gamma-m}$ are linear bounded operators for every $\gamma \in \mathbb{R}$. Thus Assumption (H1) from Section 3.3 is satisfied.

We set $\Lambda_\theta := (\varepsilon I - A_\theta)^{1/2}$. For every $\theta \in \Theta$, the operator $\Lambda_\theta$ generates the Hilbert scale $\{H^\theta_\theta(M)\}_{s \in \mathbb{R}}$. The spaces $\{H^\theta_\theta(M)\}_{s \in \mathbb{R}}$ are equivalent for all $\theta \in \Theta$ (see for instance [76]), namely $H^\gamma_\theta(M) = H^\gamma_\theta(M)$ as sets of functions, and topologies are equivalent. Thus without loss of generality, we will consider the operator $\Lambda = \Lambda_\theta$ for some fixed value of the parameter $\theta$. Now we are under the setup of general results from section 3.3.

Assume that $\mathcal{M}_j = b_j \Lambda^{p_j}$, where $p_j \leq m$, $b_j \in \mathbb{R}$ and $j = 1, \ldots, n$. Then the operators $\mathcal{M}_j$ are pseudo-differential operators of order less than $m$, and hence are
linear bounded operators acting from $H^{\gamma+m}$ in $H^\gamma$ for every $\gamma \in \mathbb{R}$. Thus Assumption (H2) is satisfied.

We assume that initial condition $u_0$ satisfies (H3) and free term $f$ satisfies (H4). Finally, we suppose that the operators $A_0$ and $A_1$ have the same system of eigenfunctions $\{h_k\}_{k \in \mathbb{N}}$, so (H5) is satisfied too.

Let $m_i := \text{ord}(A_i)$, $i = 0, 1$, where by $\text{ord}(A)$ we denote the order of differential operator $A$. From the above assumptions obviously $2m = \max\{m_0, m_1\}$, and $\text{ord}(M_j) = p_j \leq m$.

From spectral theory of elliptic operators it follows that the asymptotics of the eigenvalues $l_{k,i}$, $\mu_{k}^j$ and $\lambda_k$ are given by

$$l_{k,i} \sim k^{m_i/2}, \mu_{k}^j \sim k^{p_j/2}, \lambda_k \sim k^{1/2}, \quad (3.26)$$

where $i = 0, 1$ and $j = 1, \ldots, n$. From here we get that $\lim_{k \to \infty} \frac{l_{k,0}^{\theta_0} + l_{k,1}^{\theta_1}}{\lambda_k^{2m}}$ is finite for every $\theta \in \Theta$. Obviously, $\mu_{k}^j = b_j \lambda_k^{p_j} \neq 0$ for every $j = 1, \ldots, n$, $k \in \mathbb{N}$. Hence, to fulfill condition (H6) it is sufficient to assume the following:

(H6.1) There exists $\varepsilon > 0$ such that $\sum_{j=1}^{n} b_j^2 \lambda_k^{2(p_j-m)} \leq \varepsilon - \frac{2(l_{k,0}^{\theta_0} + l_{k,1}^{\theta_1})}{\lambda_k^{2m}}$, where $k \in \mathbb{N}$ and $\tau = \sup(\Theta)$.

In the nutshell, Assumption (H6) requires

$$\max_j \{\text{ord}(M_j)\} \leq 2 \max\{\text{ord}(A_1), \text{ord}(A_0)\},$$

plus the coercivity condition on the coefficients. Also, (H6.1) can be replaced with the following, more restrictive but easier to check, condition

(H6.2) $\sum_{j=1}^{n} b_j^2 < -\frac{2(l_{k,0}^{\theta_0} + l_{k,1}^{\theta_1})}{\lambda_k^{2m}}$, where $\tau = \sup(\Theta)$, $k \in \mathbb{N}$.  

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Under above assumptions, the equation (3.9) has a unique solution in the space $L_2(\Omega, C((0,T); H^\gamma)) \cap L_2(\Omega \times [0,T]; H^{\gamma+m})$, and estimate (3.10) holds true.

Finally, we will discuss Assumption (H7). For the differential operators considered in this section, this condition becomes

\[(H7.1) \quad p := \max\{p_1, \ldots, p_n\} \leq m_1.\]

The equivalence of (H7) and (H7.1) follows directly from (3.26). Since all assumptions (H1)-(H7) are satisfied, we conclude

3.5.1 Proposition. Under above assumptions on differential operators $A_\theta, M_j$, the estimates $\hat{\theta}_k$ are unbiased and consistent in both time and dimension of the projection (number of Fourier coefficients), namely Theorem 3.4.1 holds true. If in addition we assume (H8), then Theorem 3.4.3 about weighted average follows. Moreover, if $p < m_1$, then $\mu = 0$, and hence the MLE’s from Theorems 3.4.1 and 3.4.3 are asymptotic efficient.

3.6 Examples and some numerical results

In this section we will proceed to some particular examples, each followed by numerical results.

3.6.1 Example. Stochastic heat equation.

Let us consider the following SPDE

\[
\begin{aligned}
\begin{align*}
\frac{du(t, x)}{dt} &= \theta \Delta u(t, x) + u(t, x) \, dW(t), \\
u(0, x) &= u_0(x), \\
u(t, 0) &= u(t, 1) = 0
\end{align*}
\end{aligned}
\]

(3.27)
where \( \Delta u = u_{xx} \), \( W \) is a standard Brownian motion, and \( u_0 \in L_2(0, T) \). For simplicity of writing we consider \( T = 1 \), but all results remain true for an arbitrary time \( T \). Using previous notations we put \( \Lambda := \sqrt{-\Delta} \) acting in \( L_2(0, 1) \) with zero boundary conditions. It is well-known that \( h_k(x) = \sqrt{2} \sin(k\pi x) \), \( k \in \mathbb{N} \), are the eigenfunctions of the operator \( \Lambda \), with corresponding eigenvalues \( \lambda_k = \pi k \), the operator \( \Lambda \) satisfies all conditions mentioned in Section 3.3 and generates the scale of Sobolev spaces \( \{ \mathcal{H}^s(0, 1) \}_{s \in \mathbb{R}} \) (compare to Example 2.2.3 - 2.3.5). The operators corresponding to the original evolution equation (3.9) are as follows:

\[
A_1 = \Delta \quad \text{and} \quad l_k^1 = -(k\pi)^2, \quad k \in \mathbb{N}; \\
A_0 = 0, \quad f \equiv 0; \\
\mathcal{M} = I, \quad n = 1 \quad \text{and} \quad \mu_k = 1, \quad k \in \mathbb{N}.
\]

Equation (3.26) makes sense only for \( \theta > 0 \). Indeed, Assumption (H6.1) leads to the following inequality \( 1 \leq \varepsilon + 2\theta(k\pi)^2 \) for every \( k \in \mathbb{N} \), which obviously is satisfied only for \( \theta > 0 \). Thus, the existence and uniqueness of the solution is guaranteed in \( \mathcal{H}^s(0, 1) \), \( s \in \mathbb{R} \) if \( \theta > 0 \).

The MLE’s (see (3.16) and (3.21)) in this case have the following form

\[
\hat{\theta}_k = -\frac{1}{(k\pi)^2} \left( \log \frac{u_k(1)}{u_k(0)} + \frac{1}{2} \right), \\
\hat{\theta}_{(N)} = -\frac{\sum_{k=1}^{N} \frac{\beta_k}{(k\pi)^2} \left( \log \frac{u_k(1)}{u_k(0)} + \frac{1}{2} \right)}{\sum_{k=1}^{N} \beta_k},
\]

(3.28)

where \( u_k \) are the Fourier coefficients of the observed solution of equation (3.26) w.r.t. \( \{h_k\}_{k \in \mathbb{N}} \), and \( \beta_k \)'s satisfy (H8).
Since \( p = 0 < m_1 = 2 \), we have that Assumptions (H7.1) is satisfied, and hence Theorems 3.4.1 and 3.4.3 hold true. Moreover, \( \mu = 0 \), which consequently implies that estimates (3.28) are consistent, and asymptotically efficient and unbiased.

We simulated the Fourier coefficients \( u_k \), using Matlab7, taking the true parameter \( \theta_0 = 1 \). Applying formulas (3.28), we evaluated the estimates \( \hat{\theta}_k \) (simple estimates), \( \hat{\theta}_{(N)} \) with \( \beta_k = k \) (weighted polynomial estimates), and \( \beta_k = \exp(k) \) (weighted exponential estimates).

In Figure 3.1 we present the graph for all three estimators and the true value of the parameter, as functions of the number of Fourier coefficients.

![Figure 3.1: Estimated Parameter. Example 3.6.1](image-url)

Figure 3.1: Estimated Parameter. Example 3.6.1
We note that \( \hat{\theta}_k \) and \( \hat{\theta}_{(N)} \) with exponential weights converge to the true parameter, basically with the same speed of convergence. However, as mentioned before, the polynomial weighted estimate converge slower.

In the Table 3.1 we present the corresponding errors, i.e. \( |\theta_0 - \hat{\theta}| \), and as we can see, for \( k = 10 \) we are within four digits precision, and taking into account the first 20 Fourier coefficients we get an error of magnitude less then \( 10^{-5} \). We see that indeed the rate of convergence is equal to \( k^{-2} \).

The numerical results are similar if the noise is driven by several independent Brownian motions, i.e. for \( n > 1 \).

3.6.2 Example. Now we will consider an example where the unknown parameter is not a factor of the leading differential operator. In our notations this means \( \text{ord}(A_0) > \text{ord}(A_1) \).

Suppose that \( u \) satisfies the following SPDE

\[
    du = (u_{xx} - \theta u) dt + \sum_{j=1}^{n} \frac{1}{j^2} (1 - \Delta)^{\alpha_j} u dW_j(t), \quad x \in [0, 1], \quad t \in [0, 1], \tag{3.29}
\]

with periodic boundary conditions, \( u(t, 0) = u(t, 1) \), with initial condition \( u(0, x) = u_0(x) \in L_2(0, 1) \) and \( q_j < 0 \). Denote by \( \Delta \) the Laplace operator \( \frac{\partial}{\partial x^2} \) on \( L_2([0, 1]) \) with periodic boundary conditions. Let us consider \( \Lambda = \sqrt{1-\Delta} \), then \( \Lambda \) generates

<table>
<thead>
<tr>
<th>( k ) \</th>
<th>Error \</th>
<th>Simple \</th>
<th>Polynomial \</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00961</td>
<td>0.0096</td>
<td>0.0096</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.00038</td>
<td>0.0008</td>
<td>0.0006</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.00009</td>
<td>0.0002</td>
<td>0.0001</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>( 4.3 \times 10^{-5} )</td>
<td>0.0001</td>
<td>( 4.7 \times 10^{-5} )</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>( 2.4 \times 10^{-5} )</td>
<td>( 6.7 \times 10^{-5} )</td>
<td>( 2.5 \times 10^{-5} )</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>( 1.6 \times 10^{-5} )</td>
<td>( 3.0 \times 10^{-5} )</td>
<td>( 1.1 \times 10^{-5} )</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Errors of estimated parameter
the Hilbert scale \( \{ \mathcal{H}^s \} \). The orthonormal system of eigenfunctions of this operator is given by: 
\[ h_0(x) = 1, \quad h_{2k}(x) = \sqrt{2} \cos(2\pi kx), \quad h_{2k-1} = \sin(2\pi kx) \] 
for \( k \in \mathbb{N} \).

Obviously, the operators \( A_1 = -I, \quad A_0 = \Delta, \quad M_j = 1/j^2(1 - \Delta)^{s_j} \) and \( \Lambda \) have a common system of eigenfunctions. Also, note that the corresponding eigenvalues satisfy the relations 
\[ l_1^k = -1, \quad l_0^2k = l_0^2k - 1 = -\left(2k\pi\right)^2, \quad \lambda_k = \sqrt{1 - (l_0^k)^2} \] 
and 
\[ \mu_j^k = \lambda_k^{2j}/j^2. \]

We see that conditions (H1)-(H5) and (H7.1) are satisfied, and now, the objective is to show that (H6.2) holds. With \( b_j = 1/j^2 \), (H6.2) becomes
\[
\sum_{j=1}^{n} \frac{1}{j^2} < -2\frac{-\tau + l_0^k}{1 - l_0^k}.
\]

Without sake of completeness we take \( \tau = 1 \), and since 
\[ \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} < 2, \]
we conclude that (H6.2) is verified. Hence, by Proposition 3.5.1 (see also Theorems 3.4.1 and representation (3.16)) there exists a unique solution of equation (3.29), and the estimators
\[
\hat{\theta}_k = \log \frac{u_k(0)}{u_k(1)} - \frac{1}{2} \sum_{j=1}^{n} \frac{1 - (2k\pi)^2 s_j}{j^2} + (2k\pi)^2 \quad (k \in \mathbb{N})
\]
are consistent and asymptotic efficient. Similarly, one may apply the Theorem 3.4.3 and get the results about the weighted average MLE’s.

Now we will present some numerical results for this example. We take \( q_j = -1 \) and \( n = 100 \). Figure 3.2 exhibits the estimated parameters evaluated by three different methods, using the first 30 Fourier coefficients. The solid horizontal line corresponds to the true value of the parameter \( \theta_0 = 1 \). As we mentioned before, Simple Estimates \( \hat{\theta}_k \) and Weighted Exponential Estimates \( \hat{\theta}_{(N)} \) for \( k, N \geq 10 \) are almost the same, while
polynomial weighted average converge slower to the true value. The error $|\hat{\theta} - \theta_0|$, $k = 25, \ldots, 45$ are presented in Figure 3.3.

![Figure 3.2: Estimated Parameter. Example 3.6.2.](image)

**3.6.3 Example.** We will conclude this section with some hypothetical examples (by indicating only the eigenvalues but not the operators themselves), that show different rates of converges of the estimates, and also, some counter-examples related to conditions (H6) and (H7). In what follows we assume (H1)-(H5) and (H8) hold true.

**3.6.3.a.** Let $A_0 = 0, A_1 = \Delta$ with zero boundary conditions. Similar to Example 1, $l_k^1 = -(k\pi)^2$. Suppose that $M$ is the operator with eigenvalues $\mu_k = (-1)^k \sqrt{k}$. It is clear that (H6) and (H7) are satisfied and $\mu = 0$. Thus, we can apply Theorems 3.4.1 and 3.4.3. The results are presented in Figure 3.4. We observe that in this case, the weighted exponential estimates perform better than simple or polynomial averaged ones. This is due to the fact that the sign of $\mu_k$ alternates.
3.6.3.b. Let $t_k^0 = -(k\pi)^2$, $l_k^1 = 1$, $\mu_k = k\pi$, then Assumption (H6) is satisfied. Since 
\[
\lim_{k \to \infty} \left( \frac{\mu_k}{k} \right)^2 = \lim_{k \to \infty} \frac{(k\pi)^2}{k} = \infty,
\]
we have that (H7) is violated. This means that there exists a unique solution, but we can not apply the above results about MLE’s. The numerical results, formally applied to this problem are shown on Figure 3.5. As we can see, all estimators diverge.

3.6.3.c. Now we take $t_k^0 = 1$, $l_k^1 = -(k\pi)^4$, $\mu_k = (k\pi)^{3.5}$, then one may check that Assumption (H7) is satisfied, while (H6) does not hold. Note that (H6) is related only to existence and uniqueness and not to the convergence of the estimators. It is known that coercivity condition (H6) is a sufficient and in some sense a necessary condition for the existence and uniqueness of the solution for parabolic SPDE. It turns out that numerical results also show that if only (H6) is violated, the estimators do not converge to the true values of the parameter $\theta_0 = 0$. The simulation are presented in Figure 3.6. We see that the error is more than .1 for k=100.
Figure 3.4: Estimators vs number of Fourier coefficients. Example 3.6.3.a

Figure 3.5: Estimators vs number of Fourier coefficients. Example 3.6.3.b
Figure 3.6: Estimators vs number of Fourier coefficients. Example 3.6.3.e
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