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ON THE NONSELFADJOINT PERTURBATIONS OF THE WIENER-HOPF INTEGRAL OPERATORS

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ABSTRACT. Finiteness criteria are established for the point spectrum of nonself-adjoint perturbed Wiener-Hopf integral operators.

KEYWORDS: *Specral theory, nonselfadjoint operator, perturbation theory, Wiener-Hopf type operator.*

AMS SUBJECT CLASSIFICATION: Primary 47A55; Secondary 47A10, 47A75, 47B35.

1. In the present paper we discuss the spectrum of the integral operator of the form

$$(H\varphi)(x) = \int\limits_{\mathbb{R}_+} a(x-y)\varphi(y)\,\mathrm{d}y + \int\limits_{\mathbb{R}_+} b(x,y)\varphi(y)\,\mathrm{d}y, \tag{1}$$

where $a \in L_1(\mathbb{R})$, $b(\cdot,\cdot)$ is a measurable function with respect to both variables $x, y \in \mathbb{R}_+$. In general, the operator H is supposed to be a nonselfadjoint operator acting on the space $L_2(\mathbb{R}_+)$.

In the works [5] and [6], it is considered the case of the self-adjoint operator H and there are obtained results on the finiteness of its point spectrum $\sigma_{\mathrm{p}}(H)$ (i.e. the set of eigenvalues including those contained in the essential spectrum $\sigma_{\mathrm{ess}}(H)$). The conditions on the kernel $b(x,y)$ of perturbation turned out to depend on the maximal multiplicity of the zeros of the symbol of the unperturbed operator (which is a Wiener-Hopf operator).

In this paper it is considered the case in which H is a nonselfadjoint operator and our main purpose is to give conditions under which the set $\sigma_{\mathrm{p}}(H)$ is finite. The spectral analysis of a quadratic pencil of ordinary differential operators ([10]) leads to the problem of the finiteness of the discrete spectrum for nonselfadjoint perturbed Wiener-Hopf operators. In the work [11] the problem is treated by supposing that the boundary of the essential spectrum of H does not contain the point $\lambda = 0$ and it is shown that the conditions $\exp(\tau|x|)a(x) \in L_1(\mathbb{R})$, $b(x,y) \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$, $\exp(\tau(x+y))\frac{\partial^{j+k}}{\partial x^j \partial y^k} b(x,y) \in L_1(\mathbb{R}_+ \times \mathbb{R}_+)$, $j,k = 0,1,2$, $\tau > 0$, guarantee the finiteness of the discrete spectrum of H (i.e. the set of eigenvalues which lie outside of $\sigma_{\mathrm{ess}}(H)$). Our approach allows us to obtain more refined conditions without any restrictions concerned the situation of the origin point. Moreover, we prove the finiteness of the perturbed eigenvalues including those which belong to $\sigma_{\mathrm{ess}}(H)$. So, if the integral operator with the kernel $\exp(\tau(x+y))b(x,y)$ $(\tau > 0)$ is bounded in $L_2(\mathbb{R}_+)$, then each λ, $\lambda \neq 0$, is not an accumulation point of the perturbed eigenvalues. However, the point zero has the same property if it is required the smoothness of kernel $b(x,y)$ in the sense that the integral operators with the kernels $\frac{\partial^{j+k}}{\partial x^j \partial y^k} b(x,y)$ $(k,j = 0,1,\ldots)$ must be compact on $L_2(\mathbb{R}_+)$ (more precisely, see Theorem 2).

To derive our results, we use the methods developed by us in [3] (see also [2] and [4]). Our approach is based on the theory of analytic operator-valued functions, namely, the method of analytic continuation on Riemann surfaces is applied. It should be noted that similar results for perturbed discrete Wiener-Hopf operators (and more general ones) were obtained in [2] and [3]. The corresponding results for some integro-differential operators are contained in [4]. We stress that our methods, in a sense, differ from those developed in [1], [11], [13]–[16]. The results from [16] deal to the study of differential operators of second order. In [14] and [15], it is studied the operators connected with the Friedrichs model.

2. Let H_0 be a Wiener-Hopf integral operator

$$(H_0\varphi)(x) = \int\limits_{\mathbb{R}_+} a(x-y)\varphi(y)\,\mathrm{d}y,$$

acting on the space $L_2(\mathbb{R}_+)$. Here it is considered $a \in L_1(\mathbb{R})$. Throughout the paper the condition $a(-x) = \overline{a(x)}$, a.e. $x \in \mathbb{R}_+$, is supposed. This means that the operator H_0 is self-adjoint in $L_2(\mathbb{R}_+)$. Namely, in this case, the origin point necessarily belongs to the boundary of the essential spectrum of H_0. It is well-known ([8], [12]) that the spectrum of the operator H_0 coincides with the set of all points λ of the curve $\lambda = \widehat{a}(\xi)$, $-\infty \leqslant \xi \leqslant \infty$, where $\widehat{a}(\xi)$ $(\xi \in \mathbb{R})$ denotes the Fourier transform of the function $a(x)$, i.e.

$$\widehat{a}(\xi) = \int_{\mathbb{R}} \exp(\mathrm{i}\xi x)a(x)\,\mathrm{d}x \quad (\xi \in \mathbb{R}).$$

The function $\widehat{a}(\xi)$ is called the symbol of the operator H_0. Thus, $\sigma(H_0) = [a, b]$, where $a = \min\{\widehat{a}(\xi) \mid \xi \in \overline{\mathbb{R}}\}$, $b = \max\{\widehat{a}(\xi) \mid \xi \in \overline{\mathbb{R}}\}$. We note that the point spectrum $\sigma_{\mathrm{p}}(H_0)$ is absent. It is well-known (see for instance [9] and [12]) that the operator H_0 can be viewed (in a certain sense) as the value of the function $A(z)$, $A(z) = \widehat{a}(\mathrm{i}\frac{1+z}{1-z})$, of the operator V defined in $L_2(\mathbb{R}_+)$ by the following formula

$$(V\varphi)(x) = \varphi(x) - 2\int_0^x \exp(y - x)\varphi(y)\,\mathrm{d}y \quad (\text{a.e. } x \in \mathbb{R}_+).$$

It should be mentioned that the operator V is an isometric (nonunitary) operator, i.e. $V^*V = I$. The function $A(z)$, $z \in \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, is also called the symbol of the operator H_0 (see for instance [8], p. 66).

In the sequel we suppose to be fulfilled the following assumption.

A1. The function $A(z)$ is analytic on a ring $\mathbb{T}_{r,R} := \{z \in \mathbb{C} \mid r < |z| < R\}$ with $0 < r < 1$, $R > 1$.

Further, let B denote the integral operator in $L_2(\mathbb{R}_+)$ defined by

$$(B\varphi)(x) = \int_{\mathbb{R}_+} b(x, y)\varphi(y)\,\mathrm{d}y,$$

where b is a complex-valued function on $\mathbb{R}_+^2$.

Let $\lambda_0 \in \sigma(H_0)$ and let us consider the function $A(z) - \lambda_0$ on the domain $\mathbb{T}_\tau \times U(\lambda_0)$, where $U(\lambda_0)$ is a neighborhood of the point λ_0, $\mathbb{T}_\tau := \mathbb{T}_{\tau,\frac{1}{\tau}} \subset \mathbb{T}_{r,R}$ $(\tau \in (0, 1))$. By A1, the function $A(z) - \lambda_0$, $z \in \mathbb{T}$, has only a finite set of zeros on the unit circle $\mathbb{T}$ and their corresponding multiplicities are integer numbers. Denote the roots of the function $A(z) - \lambda_0$ by $z_1, \ldots, z_n$ (considered pairwise distinct numbers) and let $m_1, \ldots, m_n$ be their multiplicities. Without loss of generality, we assume that $A(z) - \lambda_0 \neq 0$, $z \in \mathbb{T}_\tau \setminus \mathbb{T}$. By virtue of the theorem of implicit function, there exists a neighborhood $U(\lambda_0)$ and a ring $\mathbb{T}_\tau$ (here and subsequently we preserve the same notations for $U(\lambda_0)$ and $\mathbb{T}_\tau$ respectively) such that $A(z) - \lambda_0$ can be represented in the form

$$A(z) - \lambda = (z - \alpha_1(\lambda))(z - \alpha_2(\lambda)) \cdots (z - \alpha_k(\lambda))\alpha_0(z, \lambda), \tag{2}$$

where $z \in \mathbb{T}_\tau$, $\lambda \in U(\lambda_0)$, α_j $(j = 1, \ldots, k)$ are analytic branches of the functions z_s $(s = 1, \ldots, n)$, $m_1 + \cdots + m_n = k$, $\alpha_0(z, \lambda)$ is an analytic function on $\mathbb{T}_\tau \times U(\lambda_0)$, and $\alpha_0(z, \lambda) \neq 0$, $(z, \lambda) \in \mathbb{T}_\tau \times U(\lambda_0)$. It should be mentioned that some of the functions α_j can coincide and $U(\lambda_0)$ (generally speaking) is a neighborhood of the point λ_0 belonging to a Riemann surface, with the algebraical branch point λ_0.

Let α_j $(j = 1, \ldots, k)$ be enumerated such that

$$
\begin{aligned}
&|\alpha_j(\lambda)| < 1 \quad (j = 1, \ldots, s;\ \lambda \in U(\lambda_0) \cap \Pi_+), \\
&|\alpha_j(\lambda)| > 1 \quad (j = s+1, \ldots, k;\ \lambda \in U(\lambda_0) \cap \Pi_+),
\end{aligned}
\tag{3}
$$

where $\Pi_+ := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. It may be proved that the Wiener-Hopf operator $A_0(\lambda)$, $\lambda \in U(\lambda_0)$, which corresponds to the symbol $z^s \alpha_0(z, \lambda)$ is invertible for every $\lambda \in U(\lambda_0)$, and the inverse operator is an analytic operator-valued function relative to λ, $\lambda \in U(\lambda_0)$ (cf. [12]). It follows from (2) that the operator $H_0 - \lambda$ can be represented as follows:

$$
H_0 - \lambda = A_-(\lambda) A_0(\lambda) A_+(\lambda),
\tag{4}
$$

where $A_-(\lambda) = (I - \alpha_1(\lambda)V^*) \cdots (I - \alpha_s(\lambda)V^*)$, $A_+(\lambda) = (V - \alpha_{s+1}(\lambda)I) \cdots (V - \alpha_k(\lambda)I)$, $\lambda \in U(\lambda_0)$.

The operator $V^* - \lambda I$ $(|\lambda| > 1)$ is invertible and the inverse operator is defined by the formula

$$
(V^* - \alpha I)^{-1}\varphi(x) = \frac{\varphi(x)}{1 - \alpha} + \frac{2}{(1 - \alpha)^2} \int_x^\infty \exp(\mathrm{i}\xi(x - y))\varphi(y)\,\mathrm{d}y,
\tag{5}
$$

where $\xi = \mathrm{i}(1 + \alpha)^{-1}(1 - \alpha)$. Moreover, the operator $V^* - \alpha I$ for $|\alpha| = 1$ is one-to-one and there exists a closed but unbounded operator $(V^* - \lambda I)^{-1}$. It is easy to see that the operator $(V^* - \lambda I)^{-1}$ for $|\lambda| = 1$, $\lambda \neq 1$, is integral and it is calculated by (5). We note that the operator $(V^* - I)^{-1}$ is a differential one, namely $(V^* - I)^{-1} = -\frac{1}{2}(I + \mathrm{i}D^*)$, where D denotes the differential operator with the domain consisting of all the functions $\varphi \in L_2(\mathbb{R}_+)$, which are absolutely continuous on every bounded interval of the positive semi-axis, $\varphi(0) = 0$, and whose derivative $\varphi\prime$ (in the sense of distributions) belongs to $L_2(\mathbb{R}_+)$. Thus, by virtue of (3) and (4), we can write

$$
(H_0 - \lambda)^{-1} = A_+^{-1}(\lambda) A_0^{-1}(\lambda) A_-^{-1}(\lambda) \quad (\lambda \in U(\lambda_0) \cap \overline{\Pi}_+).
$$

Let $\lambda \in \sigma_{\mathrm{p}}(H) \cap \overline{\Pi}_+$, i.e. $Hv = \lambda v$ for some $v \in L_2(\mathbb{R}_+)$, $v \neq 0$. Taking into account the factorization (4) and denoting $v = A_+^{-1}(\lambda)u$ one obtains

$$
(I + Q(\lambda))u = 0,
\tag{6}
$$

where

$$Q(\lambda) = A_0^{-1}(\lambda)A_-^{-1}(\lambda)BA_+^{-1}(\lambda) \quad (\lambda \in \overline{\Pi}_+).$$

Here, the operator $Q(\lambda)$ is considered with the maximum domain.

In order to prove the finiteness of the set of eigenvalues of the operator H we will use the following considerations. Suppose that for each $\lambda_0 \in \sigma(H_0)$ there exists a neighborhood $U(\lambda_0)$ and an operator-valued function $Q_+(\lambda) : U(\lambda_0) \to \mathbb{B}_\infty(L_2(\mathbb{R}_+))$ (($\mathbb{B}_\infty(L_2(\mathbb{R}_+))$ stands for the class of all compact operators on $L_2(\mathbb{R}_+)$) such that Q_+ is analytic on $U(\lambda_0)$ and $Q_+(\lambda) \supset Q(\lambda)$, $\lambda \in U(\lambda_0) \cap \overline{\Pi}_+$. Then the set of all $\lambda \in \overline{\Pi}_+$ which satisfy (6) is finite by the theorem of the analytic operator-valued function. Hence, the set of the eigenvalues of the perturbed operator H which belong to $\overline{\Pi}_+$ is finite, and each of the eigenvalues has finite multiplicity. The same conclusion can be obtained for $\overline{\Pi}_-$ ($\Pi_- := \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$). In this way we have obtained that the point spectrum of the operator H is a finite set.

Returning to our operator (1) the operator-valued function $Q(\lambda)$ ($\lambda \in U(\lambda_0) \cap \overline{\Pi}_+$) can be represented as a sum of terms of the form

$$\gamma(\lambda)(V^* - \alpha(\lambda))^{-m}B(V - \beta(\lambda))^{-n}, \tag{7}$$

where $\lambda \in U(\lambda_0) \cap \overline{\Pi}_+$, $m, n \in \mathbb{N}$, $\alpha(\lambda) = \alpha_j(\lambda)^{-1}$ $(j = 1, \dots, s)$, $\beta(\lambda) = \alpha_j(\lambda)$ $(j = s + 1, \dots, k)$, $\gamma(\lambda)$ is a combination of the functions α_j $(j = 1, \dots, k)$, so, $\gamma(\lambda)$ is analytic on $U(\lambda_0)$.

As it has been mentioned above, the problem of finiteness of the set of eigenvalues of the operator H will be proved if there exists an analytic continuation from $U(\lambda_0) \cap \overline{\Pi}_+$ on $U(\lambda_0)$ of the functions determined by (7).

For convenience let

$$(R_\alpha\varphi)(x) = \int_x^\infty \exp(\mathrm{i}\xi(x - y))\varphi(y)\,\mathrm{d}y, \quad (S_\delta\varphi)(x) = \exp(-\delta x)\varphi(x),$$

where $\varphi \in L_2(\mathbb{R}_+)$, $\xi = \mathrm{i}\frac{1+\alpha}{1-\alpha}$, $\delta \in \mathbb{R}$.

In the sequel we need the following auxiliary assertions

Lemma 1. *For each $\alpha_0 \in \mathbb{T} \setminus \{1\}$ there exists a neighborhood $V(\alpha_0)$ of the point α_0, such that the operator-valued function $F(\alpha) = R_\alpha^n S_\delta$ ($n \in \mathbb{N}$, $\delta > 0$) has an analytic continuation from $V(\lambda_0) \cap D_-$ on $V(\alpha_0)$ ($D_- := \{z \in \mathbb{C} \mid |z| > 1\}$).*

Proof. Let $F_1(\cdot)$ be a function of the form

$$(F_1(\alpha)\varphi)(x) = \frac{1}{n!}\int_x^\infty \exp(\mathrm{i}\xi(x - y))(y - x)^{n-1}\exp(-\delta y)\varphi(y)\,\mathrm{d}y,$$

where $\varphi \in L_2(\mathbb{R}_+)$, $\alpha \in V(\alpha_0)$, $V(\alpha_0)$ is a neighborhood of the point α_0 such that $1 \notin V(\alpha_0)$.

By the inequality

$$\int_{\mathbb{R}_+} \left| \int_x^\infty \exp(\mu(x-y))\varphi(y)\,\mathrm{d}y \right|^2 \mathrm{d}x \leqslant \frac{1}{\operatorname{Re}\mu} \int_{\mathbb{R}_+} |\varphi(y)|^2\,\mathrm{d}y \quad (\operatorname{Re}\mu > 0,\ \varphi \in L_2(\mathbb{R}_+)),$$

we can choose a neighborhood $V(\alpha_0)$ such that $\|F_1(\alpha)\varphi(x)\|^2 \leqslant c\|\varphi\|$, where $\alpha \in V(\alpha_0)$ and c is a positive constant depending only on n and δ. It is easy to see that $F_1(\alpha) = F(\alpha)$, $\alpha \in V(\alpha_0) \cap \overline{D}_-$. Since the function $\xi = \mathrm{i}(1+\alpha)^{-1}(1-\alpha)$ is analytic on $V(\alpha_0)$ and according to the foregoing stated we get that $F_1(\alpha)$ is analytic on $V(\alpha_0)$. ∎

Let $\lambda_0 \neq 0$. In this case, without loss of generality, we can suppose that the functions α_j from decomposition (2) are such that $\alpha_j(\lambda) \neq 1$, $j = 1, \ldots, s$, $\lambda \in U(\lambda_0)$.

Hence, the operator $(V^* - \alpha_j(\lambda))^{-m}$ $(m \in \mathbb{N};\ \lambda \in U(\lambda_0) \cap \Pi_+;\ j = 1\ldots, s)$ can be represented as a linear combination of the operators R_α^j $(j = 0, 1, \ldots, m)$. Thus, by (7), it is sufficient to establish the existence of analytic continuation of the functions of the form $(R_{\alpha(\lambda)})^j B (R^*_{\beta(\lambda)})^s$ $(j, s \in \mathbb{N})$ from $U(\lambda_0) \cap \overline{\Pi}_+$ on the whole neighborhood $U(\lambda_0)$. To this end we suppose that the kernel $b(\,\cdot\,,\cdot\,)$ satisfies the following assumption.

A2. For some $\tau > 0$ the integral operator with the kernel $\exp(\tau(x+y))b(x,y)$ is bounded in $L_2(\mathbb{R}_+)$.

Let T be an integral operator of the form

$$(T\varphi)(x) = \int_{\mathbb{R}_+} \exp(\delta(x+y))b(x,y)\varphi(y)\,\mathrm{d}y \quad (\varphi \in L_2(\mathbb{R}_+);\ 0 < \delta < \tau).$$

It is obvious that $B = S_\delta T S_\delta$ and $S_\delta \in \mathbb{B}(L_2(\mathbb{R}_+))$, $T \in \mathbb{B}_\infty(L_2(\mathbb{R}_+))$. By Lemma 1, we deduce that the functions $(R_{\alpha(\lambda)})^j S_\delta T S_\delta (R^*_{\beta(\lambda)})^s$ $(j, s \in \mathbb{N})$ have analytic continuation from $U(\lambda_0) \cap \overline{\Pi}_+$ on $U(\lambda_0)$. Thus, one obtains the following result.

THEOREM 1. *If the operators H_0 and B satisfy the conditions* A1–A2*, then each λ, $\lambda \neq 0$, is not an accumulation point of the set of eigenvalues of the perturbed operator $H = H_0 + B$.*

Let us consider the case $\lambda_0 = 0$. In this case, the symbol $A(z)$ of the operator H_0 is vanishing at the point $z = 1$. Thus, at least one of the functions $\alpha_j(\lambda)$

($j = 1, \dots, k$) from representation (2) has the property that $\alpha_j(\lambda) \to 1$ ($\lambda \to 0$). We consider that at least one of the terms $(V^* - \alpha(\lambda))^{-m} B (V - \beta(\lambda))^{-n}$ is such that $\alpha(\lambda) \to 1$, $\beta(\lambda) \to 1$ ($\lambda \to 0$).

In the sequel, we need the following additional assumption.

A3. The integral operators $B_{j,k}$ with the kernels $\frac{\partial^{j+k}}{\partial x^j \partial y^k} b(x,y)$ $(k, j = 0, 1, \dots)$ are compact on $L_2(\mathbb{R}_+)$ and uniformly bounded with respect to $j, k \in \mathbb{N}$, i.e. $\|B_{j,k}\| < M$, where $j, k \in \mathbb{N}$, M is a positive constant.

Let $Q_1(\lambda)$ denote the following operator-valued function

$$Q_1(\lambda) = \sum_{j,k=0}^{\infty} q_{j,k} (\alpha(\lambda) - 1)^j (\beta(\lambda) - 1)^k (I + \mathrm{i}D^*)^{j+m} B (I - \mathrm{i}D)^{k+m},$$

where $\lambda \in U(0)$, $q_{j,k}$ are some real numbers $m, n \in \mathbb{N}$ and it is considered that $\operatorname{Dom} Q_1(\lambda) := \bigcap_{k=0}^{\infty} \operatorname{Dom} D^k$. In what follows this domain is denoted by $\mathfrak{L}_\infty$.

Lemma 2. *There exists a neighborhood $U(0)$ of the point $\lambda = 0$ such that the operator $Q_1(\lambda)$ has a bounded extension for every $\lambda \in U(0)$. Moreover, this extension is analytic of λ, $\lambda \in U(0)$.*

Proof. Let $Q_{j,k} = (I + \mathrm{i}D^*)^{j+m} B (I - \mathrm{i}D)^{k+n}$ with the domain $\mathfrak{L}_\infty$. We have

$$Q_1(\lambda) = \sum_{j,k=0}^{\infty} q_{j,k} (\alpha(\lambda) - 1)^j (\beta(\lambda) - 1)^k Q_{j,k}. \tag{8}$$

Estimating directly the norm of $Q_{j,k}\varphi$ ($\varphi \in \mathfrak{L}_\infty$), it follows that $\|Q_{j,k}\varphi\| \leqslant c\|\varphi\|$, where $\varphi \in \mathfrak{L}_\infty$, $j, k \in \mathbb{N}$, and c is a positive constant such that $c = O(2^{j+k})$.

Therefore, the series (8) converges uniformly on a neighborhood $U(0)$ of the point $\lambda = 0$. Since $\overline{\mathfrak{L}_\infty} = L_2(\mathbb{R}_+)$, it follows that $Q_1(\lambda)$ is densely defined and bounded in $L_2(\mathbb{R}_+)$. Consequently, the function $Q_1(\lambda)$ is analytic on $U(0)$. ∎

Take the neighborhood $U(0)$ as in Lemma 2 (diminished if it is necessary) such that $(V^* - \alpha(\lambda))^{-m} B (V - \beta(\lambda))^{-n} \varphi = Q_1(\lambda)\varphi$ for every $\lambda \in U(0) \cap \overline{\Pi}_+$ and $\varphi \in L_2(\mathbb{R}_+)$.

The following theorem summarizes above discussion.

Theorem 2. *If the operators H_0 and B satisfy the conditions* A1–A3*, then the point spectrum of the perturbed operator H is at most a finite set. Moreover, the possible eigenvalues have finite multiplicity.*

Example 1. Let H_0 be the Wiener-Hopf operator with the kernel $a(x) = \exp(-|x|)$, $x \in \mathbb{R}$. In this case $\widehat{a}(\xi) = 2(\xi^2 + 1)^{-1}$, $A(z) = 1/2(z^{-1} - 1)(z - 1)$

and $\sigma(H_0) = [0,2]$. The function $A(z) - \lambda$ satisfies the condition A1 and can be represented in the form $A(z)-\lambda = 1/2\,\alpha^{-1}(\lambda)(z^{-1}-\alpha(\lambda))(z-\alpha(\lambda))$, where $\lambda \in \mathbb{C}$, $\alpha(\lambda) = 1 - \lambda \pm \sqrt{\lambda(\lambda-2)}$ and α are such that $|\alpha(\lambda)| > 1$, $\lambda \in \Pi_+$.

Let $A_+(\lambda) = V - \alpha(\lambda)$, $A_-(\lambda) = 1/2\,\alpha^{-1}(\lambda)(V^* - \alpha(\lambda))$ and $A_0(\lambda) = I$ ($\lambda \in \mathbb{C}$). Thus, the operator $H_0 - \lambda$ has the representation of the form (4). From Theorem 1 it follows that if the kernel $b(\,\cdot\,,\cdot\,)$ of the integral operator B satisfies the conditions A2 and A3, then the perturbed operator $H = H_0 + B$ has a finite set of eigenvalues.

In some cases, the condition A3 can be replaced with weaker conditions. This is realized, if some of the functions $\alpha_j(\lambda)$ $(j = 1,\dots,s;\ \lambda \in U(0))$ from the representation (2) tend to 1 as $\lambda \to 0$ and $\alpha_j(\lambda) \neq 1$ $(j = s+1,\dots,k;\ \lambda \in U(0))$ then it is sufficient that the kernel b verify the condition

A4. The integral operators with the kernels $\exp(\delta x)\frac{\partial^j}{\partial y^j}b(x,y)$ $(j = 0,1,\dots)$ are compact (viewed as operators acting on $L_2(\mathbb{R}_+)$) and uniformly bounded with respect to $j \in \mathbb{N}$.

Similarly, if $\alpha_j(\lambda) \neq 1$ $(j = 1,\dots,s;\ \lambda \in U(0))$ and $\alpha_j(\lambda) \to 1$ $(\lambda \to 0)$ for some indexes $j = s+1,\dots,k$ the condition A3 can be replaced with the following one.

A5. The integral operators with the kernels $\frac{\partial^j}{\partial x^j}\exp(\delta y)b(x,y)$ $(j = 0,1,\dots)$ are compact in $L_2(\mathbb{R}_+)$ for some $\delta > 0$ and uniformly bounded with respect to $j \in \mathbb{N}$.

EXAMPLE 2. Let us consider the unperturbed operator H_0 with the kernel $a(x) = \frac{\mathrm{i}}{2}\exp(x)$ if $x < 0$ and $a(x) = -\frac{\mathrm{i}}{2}\exp(-x)$ if $x > 0$. The Fourier transform of the function $a(x)$, $x \in \mathbb{R}$, is $\widehat{a}(x) = \xi(\xi^2+1)^{-1}$, $\xi \in \mathbb{R}$. By this, $A(z) = \mathrm{i}/4(z^{-1}+1)(z-1)$ and $\sigma(H_0) = [-1/2, 1/2]$. In this case the function $A(z)$ satisfies the assumption A1 and the function $A(z) - \lambda$ can be represented as follows

$$A(z) - \lambda = \mathrm{i}(4\alpha(\lambda))^{-1}(z^{-1} + \alpha(\lambda))(z - \alpha(\lambda)),$$

where $\lambda \in \mathbb{C}$, $\alpha(\lambda) = -2\mathrm{i}\lambda + 2\sqrt{1/4 - \lambda^2}$ and $\alpha(\lambda)$ is considered such that $|\alpha(\lambda)| > 1$ $(\lambda \in \mathbb{C} \setminus [-1/2, 1/2])$. Therefore, the following result holds.

If the integral operator B satisfies the conditions A2, A4 and A5, then the point spectrum of the perturbed operator $H = H_0 + B$ is finite.

3. In this subsection we consider the case of the perturbed operator B of the following form

$$(B\varphi)(x) = \int\limits_{\mathbb{R}_+} b(x+y)\varphi(y)\,\mathrm{d}y \quad (\varphi \in L_2(\mathbb{R}_+)). \tag{9}$$

It is well-known ([9]) that the integral operator B of the form (9) is compact if the function b belongs to $L_1(\mathbb{R}_+)$. Therefore, in this case, by Theorems 1 and 2, the following results are true.

THEOREM 3. *If for each $\tau > 0$, the function $\exp(\tau x)b(x)$, $x \in \mathbb{R}_+$, belongs to $L_1(\mathbb{R}_+)$, then each λ, $\lambda \neq 0$, is not an accumulation point of the set of eigenvalues of the perturbed operator $H = H_0 + B$.*

THEOREM 4. *If the kernel b satisfies the conditions of Theorem 3 and, in addition, $b^{(k)} \in L_1(\mathbb{R}_+)$, $k = 0, 1, \ldots$ and there exists a constant c, $c > 0$, such that $\|b^{(k)}\|_{L_1(\mathbb{R}_+)} < c$, $k = 0, 1, \ldots$, then the perturbed operator $H_0 + B$ has a finite set of eigenvalues, each of them being of finite multiplicity.*

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