

ON THE POINT SPECTRUM OF NONSELFADJOINT PERTURBED OPERATORS OF WIENER-HOPF TYPE

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ABSTRACT. In this paper there are obtained results on the finiteness of the point spectrum of some nonselfadjoint operators. In particular the operators of Wiener-Hopf type acting in arbitrary Hilbert space, l_2 and $L_2(\mathbb{R}_+)$ are considered.

In the present paper there is examined the problem of finiteness of the point spectrum of some nonselfadjoint operators. The similar problems have been studied for differential operators of second order [1,2] and fourth one [3], operator with finite difference of second order [4], Wiener-Hopf integral operator [5], for Friedrichs model [6] etc. Usually, in the case of nonselfadjoint perturbation, the problem of finiteness of the point spectrum is reduced to the theorem of uniqueness of analytical function.

In this paper the problem is solved by the method of holomorphic extension of resolvent of unperturbed operator through continuous spectrum and application of the theorem of holomorphic operator-valued function. The obtained results are in concordance with those established in [1–6]. Moreover, it is possible to generalize substantially the results from [4], where it is considered the operator L generated by the difference expression

$$(Ly)_j = \frac{1}{2}(y_{j-1} + y_{j+1}) + b_j y_j \quad , \quad (j = 1, 2, \dots)$$

where $(y_j) \in l_2$; $y_0 = \theta y_1$; $b_j \in \mathbb{C}$ ($j = 1, 2, \dots$); $\theta \in \mathbb{C}$. There is considered the operator with finite difference of any order and more general perturbation (not necessarily diagonal).

In the first section, we give a general scheme about finiteness of the point spectrum of perturbed operators. In section 2, we prove an abstract theorem for the case of abstract Wiener-Hopf type operators. The sections 3–5 contain various applications of the abstract theorem. Thus, it is considered the operator generated by generalized Jacobi matrix, the case when unperturbed operator is with finite difference (arbitrary order) acting in l_2 and $L_2(\mathbb{R}_+)$ respectively. The results of the present paper generalize our previous ones given in [7–8].

2. Throughout the paper, \mathfrak{H} will denote a Hilbert space, $\mathbb{B}(\mathfrak{H})$ the class of all linear and bounded operators on \mathfrak{H} , $\mathbb{B}_\infty(\mathfrak{H})$ the class of compact operators on \mathfrak{H} .

Let H_0 and B be linear and bounded operators on \mathfrak{H} such that the following assumptions are fulfilled:

- (1) The operator H_0 is selfadjoint and $\sigma_p(H_0) = \emptyset$;

- (2) The operator B can be represented in the form $B = RTS$, where $R, S \in \mathbb{B}(\mathfrak{H})$, $T \in \mathbb{B}_\infty(\mathfrak{H})$;
(3) There exists $\tilde{Q}_\pm(\lambda) = s\text{-}\lim_{\varepsilon \rightarrow 0} Q(\lambda \pm i\varepsilon)$ ($\lambda \in \mathbb{R}$), where $Q(\lambda) = S(H_0 - \lambda)^{-1}R$ ($\lambda \in \rho(H_0)$).

For simplicity of notation, let $Q_\pm(\lambda)$ be the operator-valued functions defined on $\bar{\Pi}_\pm$, which are equal to $Q(\lambda)$ if $\lambda \in \Pi_\pm = \{z \in \mathbb{C} \mid \pm \text{Im}z > 0\}$ and $\tilde{Q}_\pm(\lambda)$ if $\lambda \in \mathbb{R}$.

Proposition. *If $\lambda \in \sigma_p(H_0 + B)$ then $\text{Ker}(I + Q_\pm(\lambda)T) \neq 0$, i.e.*

$$(I + Q_\pm(\lambda)T)f = 0, \quad (f \in \mathfrak{H}, f \neq 0). \quad (1)$$

The proof of this proposition coincides essentially with those from [10] or [11], where the case of selfadjoint operator is discussed. Due to previous proposition for finiteness of the point spectrum of the operator $H_0 + B$, it is sufficient to prove that $\text{Ker}(I + Q_\pm(\lambda)T) \neq 0$ for a finite number of values $\lambda \in \mathbb{C}$.

3. Let V be an isometric operator (non-unitary) defined on \mathfrak{H} such that the operator V^* has no eigenvalues on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Hereafter, $\mathfrak{K}(V)$ denotes the linear span (nonclosed) of the operators V^n ($n = 0, \pm 1, \pm 2, \dots$), where $V^n = (V^*)^{-n}$ ($n = -1, -2, \dots$), and respectively, we will denote by $\mathfrak{K}_-(V)$ ($\mathfrak{K}_+(V)$) the linear span of the operators V^n ($n = 0, -1, \dots$) (V^n ($n = 0, 1, \dots$)). The closure of the linear sets $\mathfrak{K}(V)$, $\mathfrak{K}_\pm(V)$ in $\mathbb{B}(\mathfrak{H})$ will be denoted by $\mathfrak{A}(V)$ and $\mathfrak{A}_\pm(V)$ respectively. $\mathfrak{A}_\pm(V)$ are commutative subalgebras of the algebra $\mathbb{B}(\mathfrak{H})$. The elements of the subspace $\mathfrak{A}(V)$ are called the operators of Wiener-Hopf type (see for instance [9]). In this context, the operator $A \in \mathfrak{A}(V)$ can be regarded (in some sense) as the value of its symbol at V , i.e. $A = a(V)$. The set of all symbols constitutes the space $C(\mathbb{T})$ of all continuous functions on \mathbb{T} . Simultaneously, the set of symbols of operators from $\mathfrak{A}_+(V)$ (respectively $\mathfrak{A}_-(V)$) coincides with the set $C_+(\mathbb{T})$ (respectively $C_-(\mathbb{T})$) of all functions from $C(\mathbb{T})$ which have holomorphic extension to $W^+ = \{z \in \mathbb{C} \mid |z| < 1\}$ (respectively $W^- = \{z \in \mathbb{C} \mid |z| > 1\}$). If $a(z) \in C(\mathbb{T})$ is a real-valued function, then the operator $a(V) \in \mathfrak{A}(V)$ is selfadjoint on \mathfrak{H} . In this case, the spectrum of operator A is the set of all values of the function $a(z)$ on the unit circle \mathbb{T} , i.e. $\sigma(A) = \{a(z) \mid z \in \mathbb{T}\}$.

From now on, we will use the notations: $\mathbb{T}_{r,R} = \{z \in \mathbb{C} \mid r < |z| < R\}$ ($0 < r < R$), $W_\delta^\pm = \{z \in \mathbb{C} \mid |z| \leq \delta\}$ ($\delta > 0$), $\mathbb{T}_\delta = \mathbb{T}_{\delta, \frac{1}{\delta}}$ ($\delta \in (0, 1)$).

It is supposed that the function $a(z)$ is analytic on a ring $\mathbb{T}_{r,R}$ ($0 < r < 1 < R$) and $a(z) \in \mathbb{R}$ ($z \in \mathbb{R}$).

Let H be an operator of the form

$$H = H_0 + B, \quad (2)$$

where $H_0 = a(V)$ and $B \in \mathbb{B}(\mathfrak{H})$. The operator H_0 is selfadjoint and $\sigma(H_0) = [a, b] \subset \mathbb{R}$, where $a = \min_{z \in \mathbb{T}} a(z)$, $b = \max_{z \in \mathbb{T}} a(z)$. Moreover, $\sigma_p(H_0) = \emptyset$ (see, for example, [9]).

In the sequel, conditions on the operator B (generally speaking nonselfadjoint) will be indicated for finiteness of the point spectrum $\sigma_p(H)$ of the operator H .

Theorem 1. *Let H_0, B be operators that satisfy the following conditions:*

- (i) $H_0 = a(V)$, where $a(\cdot)$ is an analytic function on a ring $\mathbb{T}_{r,R}$ ($0 < r < 1 < R$);
- (ii) $a(z) \in \mathbb{R}$ ($z \in \mathbb{T}$);
- (iii) The operator B can be represented in the form $B = RTS$, where

$$R, S \in \mathbb{B}(\mathfrak{H}), \quad T \in \mathbb{B}_\infty(\mathfrak{H});$$
- (iv) $\lim_{n \rightarrow \infty} \sqrt[n]{\|SV^n\|} < 1$, $\lim_{n \rightarrow \infty} \sqrt[n]{\|R^*V^n\|} < 1$.

Then the set of all eigenvalues of the operator $H = H_0 + B$ is finite. Moreover, the possible eigenvalues have finite multiplicity.

Without loss of generality, we will prove this theorem in the case, when the function $a(z)$ introduced in (i) is analytic on a ring \mathbb{T}_δ ($\delta \in (0, 1)$). Now we are going to prove some auxiliary lemmas.

Lemma 2. *Suppose that the function $a(z, \lambda)$ is analytic on the set $\mathbb{T}_\delta \times U(\lambda_0)$ for an $\delta \in (0, 1)$, continuous on the boundary of \mathbb{T}_δ and $a(z, \lambda) \neq 0$ ($(z, \lambda) \in \overline{\mathbb{T}_\delta} \times U(\lambda_0)$), $\nu_+(\lambda) = \operatorname{ind}_{z \in \mathbb{T}} a(\delta z, \lambda) = 0$ ($\lambda \in U(\lambda_0)$), $\nu_-(\lambda) = \operatorname{ind}_{z \in \mathbb{T}} a(\frac{z}{\delta}, \lambda) = 0$ ($\lambda \in U(\lambda_0)$).*

Then there exists the operator $(a(V, \lambda))^{-1} \in \mathbb{B}(\mathfrak{H})$ ($\lambda \in U(\lambda_0)$) and the operator-valued function $(a(V, \lambda))^{-1}$ is analytic on $U(\lambda_0)$.

Proof. Let us consider the function $a(\frac{z}{\delta}, \lambda)$ on $\mathbb{T}_\delta \times U(\lambda_0)$. This function satisfies the following conditions: $a(\frac{z}{\delta}, \lambda)$ ($z \in \mathbb{T}, \lambda \in U(\lambda_0)$) is continuous in z , $a(\frac{z}{\delta}, \lambda) \neq 0$ ($z \in \mathbb{T}, \lambda \in U(\lambda_0)$), $\operatorname{ind}_{z \in \mathbb{T}} a(\frac{z}{\delta}, \lambda) = 0$ ($\lambda \in U(\lambda_0)$). Due to the theorem about factorization of continuous functions on the unit circle (see, for example, [12]), the function $a(\frac{z}{\delta}, \lambda)$ can be represented in the form

$$a(\frac{z}{\delta}, \lambda) = \alpha_+(z, \lambda)\alpha_-(z, \lambda), \quad (3)$$

where $\alpha_\pm(z, \lambda)$ are analytic in z on W^\pm ($\lambda \in U(\lambda_0)$), $\alpha_\pm(z, \lambda) \neq 0$ ($z \in \overline{W}^\pm, \lambda \in U(\lambda_0)$). Moreover, $\alpha_\pm(z, \lambda)$ are determined by the formulas

$$\ln \alpha_\pm(z, \lambda) = \pm \frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{\ln a(\frac{\zeta}{\delta}, \lambda)}{\zeta - z} d\zeta \quad (z \in W^\pm), \quad (4)$$

and $\alpha_\pm \in \mathfrak{R}^\pm, \alpha_\pm^{-1} \in \mathfrak{R}^\pm$ (see [9],[12]).

Since $a(\frac{z}{\delta}, \lambda) \neq 0$ ($\lambda \in U(\lambda_0)$, $z \in \mathbb{T}$) and it is analytic on $U(\lambda_0)$, it follows from (4) that $\alpha_\pm(z, \lambda)$ are analytic in λ on $U(\lambda_0)$.

So,

$$\alpha_-(z, \lambda) = a\left(\frac{z}{\delta}, \lambda\right) \alpha_+^{-1}(z, \lambda) \quad (z \in \mathbb{T}, \lambda \in U(\lambda_0)).$$

Since the function $\alpha_+^{-1}(z, \lambda)$ is analytic on $W^+ \times U(\lambda_0)$ and $a(\frac{z}{\delta}, \lambda)$ is analytic on $\mathbb{T}_{\delta^2, 1} \times U(\lambda_0)$, it may be concluded that the function $\alpha_-(z, \lambda)$ has holomorphic extension in z from W^- to $W_{\delta^2}^-$ ($\lambda \in U(\lambda_0)$).

Thus

$$a(z, \lambda) = a_+(z, \lambda)a_-(z, \lambda),$$

where $a_{\pm}(z, \lambda) = \alpha_{\pm}(\delta z, \lambda)$ ($z \in \mathbb{T}_{\delta}, \lambda \in U(\lambda_0)$).

Therefore

$$a^{-1}(z, \lambda) = a_{+}^{-1}(z, \lambda) a_{-}^{-1}(z, \lambda), \quad (z \in \mathbb{T}_{\delta}, \lambda \in U(\lambda_0)) \quad (5)$$

and as, $a_{\pm}^{-1} \in \mathfrak{R}^{\pm}$, it follows that $a_{\pm}^{-1}(z, \lambda)$ can be represented in the form

$$a_{+}^{-1}(z, \lambda) = \sum_{k=0}^{\infty} b_k(\lambda) z^k, \quad (z \in W_{\frac{1}{\delta}}^{+}, \lambda \in U(\lambda_0))$$

$$a_{-}^{-1}(z, \lambda) = \sum_{k=-\infty}^0 b_k(\lambda) z^k, \quad (z \in W_{\delta}^{-}, \lambda \in U(\lambda_0)).$$

Since $b_k(\lambda)$ ($k \in \mathbb{Z}, \lambda \in U(\lambda_0)$) are Fourier coefficients, the functions $b_k(\lambda)$ ($k \in \mathbb{Z}$) are analytic on $U(\lambda_0)$.

It follows from (5) that

$$a^{-1}(V, \lambda) = \sum_{k=0}^{\infty} b_k(\lambda) V^k \sum_{k=0}^{\infty} b_{-k}(\lambda) V^{*k} \quad (\lambda \in U(\lambda_0)).$$

□

Lemma 3. *For any $\lambda_0 \in \mathbb{R}$ there exists a neighborhood $U(\lambda_0)$ (generally speaking on the Riemann surface), such that the function $Q_{+}(\lambda)$ has holomorphic extension from $U(\lambda_0) \cap \Pi_{+}$ to $U(\lambda_0)$. Furthermore, λ_0 is an algebraic branch point. The same is true for the operator-valued function $Q_{-}(\lambda)$.*

Proof. For the case $\lambda_0 \in \rho(H_0)$ the proof is obvious, because of the analyticity of the resolvent function on the resolvent set $\rho(H_0)$.

Let $\lambda_0 \in \sigma(H_0)$. Consider the function $a(z) - \lambda$ on $\mathbb{T}_{\tau} \times U(\lambda_0)$, where $U(\lambda_0)$ is a neighborhood of point λ_0 . This function is analytic on $\mathbb{T}_{\tau} \times U(\lambda_0)$ ($\tau \in (0, 1)$).

Let z_1, z_2, \dots, z_n be the roots of the function $a(z) - \lambda_0$, m_k ($k = 1, \dots, n$) be their corresponding multiplicities. Without restriction of generality we can assume that $z_k \in \mathbb{T}$ ($k = 1, \dots, n$), $a(z) - \lambda_0 \neq 0$ ($z \in \mathbb{T}_{\tau} \setminus \mathbb{T}$). By virtue of the implicit function theorem (see for instance [13]), it follows that $a(z) - \lambda$ can be represented in the form

$$a(z) - \lambda = (z - a_1(\lambda))(z - a_2(\lambda)) \dots (z - a_k(\lambda)) a_0(z, \lambda), \quad (6)$$

where $z \in \mathbb{T}_{\tau}, \lambda \in U(\lambda_0)$, a_j ($j = 1, \dots, k$) are analytic branches of the functions z_s ($s = 1, \dots, n$), $m_1 + m_2 + \dots + m_n = k$, the $a_0(z, \lambda)$ is analytic on $\mathbb{T}_{\tau} \times U(\lambda_0)$ and $a_0(z, \lambda) \neq 0$ ($z \in \mathbb{T}_{\tau}, \lambda \in U(\lambda_0)$). Here and subsequently \mathbb{T}_{τ} and $U(\lambda_0)$ are chosen conveniently, but without loss of generality we preserve the same notations \mathbb{T}_{τ} and $U(\lambda_0)$ respectively. It should be noted that the functions a_j ($j = 1, \dots, k$) can coincide and $U(\lambda_0)$ is a neighborhood of the point λ_0 belonging to Riemann surface. In this case λ_0 is an algebraic branch point.

Suppose that a_j ($j = 1, \dots, k$) are counted such that

$$\begin{aligned} |a_j(\lambda)| < 1 & \quad (j = 1, \dots, s; \lambda \in U(\lambda_0) \cap \Pi_+) \\ |a_j(\lambda)| > 1 & \quad (j = s+1, \dots, k; \lambda \in U(\lambda_0) \cap \Pi_+) \end{aligned} \quad (7)$$

It is easily seen that $k = 2s$ and the function $z^s a_0(z, \lambda)$ satisfies the conditions of Lemma 2. It follows from (6) and (7) that the operator $(H_0 - \lambda)^{-1}$ ($\lambda \in U(\lambda_0) \cap \Pi_+$) can be represented in the form

$$\begin{aligned} (H_0 - \lambda_0)^{-1} &= (V - a_{s+1}(\lambda))^{-1} \dots (V - a_k(\lambda))^{-1} \\ & \quad b(V, \lambda)(I - a_1(\lambda)V^*)^{-1} \dots (I - a_s(\lambda)V^*)^{-1}, \end{aligned} \quad (8)$$

where $b(V, \lambda)$ is the operator from \mathfrak{R} with the symbol $(z^s a_0(z, \lambda))^{-1}$.

From (8) we have

$$\begin{aligned} (Q_+(\lambda)f, g) &= (b(V, \lambda)(I - a_1(\lambda)V^*)^{-1} \dots (I - a_s(\lambda)V^*)^{-1} Rf, \\ & \quad (V^* - \bar{a}_{s+1}(\lambda))^{-1} \dots (V^* - \bar{a}_k(\lambda))^{-1} Sg), \end{aligned} \quad (9)$$

where $\lambda \in U(\lambda_0) \cap \Pi_+$; $f, g \in \mathfrak{H}$.

Since,

$$(I - zV^*)^{-1} = \sum_{j=0}^{\infty} z^j V^{*j}, \quad (z \in W^-)$$

it follows from (9) that

$$(Q_+(\lambda)f, g) = \left(b(V, \lambda) \sum_{j=0}^{\infty} P_j(\lambda)V^{*j} Rf, \sum_{j=0}^{\infty} \bar{Q}_j(\lambda)V^{*j} Sg \right), \quad (10)$$

where $f, g \in \mathfrak{H}$, $\lambda \in U(\lambda_0) \cap \Pi_+$ and $P_j(\lambda), \bar{Q}_j(\lambda)$ ($j = 0, 1, \dots$) are polynomials of degree j in $a_1(\lambda), \dots, a_s(\lambda)$ and $\frac{1}{a_{s+1}(\lambda)}, \dots, \frac{1}{a_k(\lambda)}$ respectively.

Let $r_1 = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\|R^*V^n\|}$, $r_2 = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\|SV^n\|}$ and $\delta = \max\{\tau, r_1, r_2\}$. Choose $U(\lambda_0)$ such that $a_j(\lambda) \in \mathbb{T}_\delta$ ($\lambda \in U(\lambda_0)$; $j = 1, \dots, k$).

Thus, the series

$$\sum_{j=0}^{\infty} P_j(\lambda)V^{*j} Rf, \quad \sum_{j=0}^{\infty} \bar{Q}_j(\lambda)V^{*j} Sg \quad (11)$$

converge absolutely for $\lambda \in U(\lambda_0)$; $f, g \in \mathfrak{H}$.

Actually,

$$\begin{aligned} \sum_{j=0}^{\infty} \|P_j(\lambda)V^{*j} Rf\| &\leq \sum_{j=0}^{\infty} |P_j(\lambda)| \|V^{*j} R\| \|f\| \leq \\ &\leq \sum_{j=0}^{\infty} \frac{j!}{\delta^j} \|R^*V^j\| \|f\| \quad (\lambda \in U(\lambda_0), f \in \mathfrak{H}). \end{aligned}$$

Since $\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\frac{\|R^*V^n\|}{\delta^n} n!} < 1$ the first series from (11) converges absolutely. Similarly the second series converges absolutely too.

Thus, the right-hand of the equality (10) is equal to

$$\sum_{n,j=0}^{\infty} P_n(\lambda) Q_j(\lambda) (b(V, \lambda)V^{*n}Rf, V^{*j}Sg) , \quad (12)$$

where $\lambda \in U(\lambda_0)$; $f, g \in \mathfrak{H}$. Moreover, the series from (12) converges absolutely and

$$\left| \sum_{n,j=0}^{\infty} P_n(\lambda)Q_j(\lambda)(b(V, \lambda)V^{*n}Rf, V^{*j}Sg) \right| \leq c \|f\| \|g\| ,$$

where $c = c(\delta) \in \mathbb{R}_+$; $\lambda \in U(\lambda_0)$; $f, g \in \mathfrak{H}$.

So, the expression (12) represents a bounded biliniare functional on $\mathfrak{H} \times \mathfrak{H}$ for each $\lambda \in U(\lambda_0)$. According to Riesz theorem, this biliniare functional generates an operator $Q_1(\lambda) \in \mathbb{B}(\mathfrak{H})$ ($\lambda \in U(\lambda_0)$). Since, $a_j(\lambda)$ ($j = 1, \dots, k$; $\lambda \in U(\lambda_0)$) are analytic branches of functions $z_j(\lambda)$ ($j = 1, \dots, n$) with $\lambda = \lambda_0$ algebraic branch point, it follows that a_j ($j = 1, \dots, k$) can be expanded into a Puiseux series. This gives that

$$(Q_1(\lambda)f, g) = \lambda_0 + \sum_{j=1}^{\infty} B_j(f, g)(\lambda - \lambda_0)^{\frac{j}{p}} , \quad (13)$$

where $\lambda \in U(\lambda_0)$; $f, g \in \mathfrak{H}$; $p \in \mathbb{N}$, $B_j(f, g)$ ($j \in \mathbb{N}$) are biliniare functionals on $\mathfrak{H} \times \mathfrak{H}$ and as weak analyticity coincides with strong analyticity (see for instance [14]), the function $Q_1(\lambda)$ is analytic on $U(\lambda_0)$.

By equalities (10) and (13) the function $Q_1(\lambda)$ coincides with $Q_+(\lambda)$ on $U(\lambda_0) \cap \Pi_+$, and therefore $Q_1(\lambda)$ is a holomorphic extension of the function $Q_+(\lambda)$ from $U(\lambda_0) \cap \Pi_+$ to $U(\lambda_0)$, with $U(\lambda_0)$ being placed into Riemann surface and λ_0 algebraic branch point. \square

Since $Q_{\pm}(\lambda)$ is analytic on $\mathbb{C} \setminus [a, b]$ and $\|Q_{\pm}(\lambda)\| \rightarrow 0$ ($|\lambda| \rightarrow \infty$), by theorem about operator-valued function which is holomorphic on a domain [14,15], it follows that the set of all $\lambda \in \mathbb{C}$ which satisfies (1), is at most countable. Moreover, the possible points of accumulation belonging to $[a, b]$.

Let $\lambda_0 \in [a, b]$. By Lemma 3, $Q_{\pm}(\lambda)$ can be represented in the form

$$Q_{\pm}(\lambda) = \lambda_0 I + \sum_{j=1}^{\infty} B_j^{\pm}(\lambda - \lambda_0)^{\frac{j}{p}} , \quad (14)$$

where $\lambda \in U(\lambda_0)$, $p \in \mathbb{N}$, $B_j^{\pm} \in \mathbb{B}(\mathfrak{H})$ ($j \in \mathbb{N}$).

Take $t = (\lambda - \lambda_0)^{\frac{1}{p}}$ ($\lambda \in U(\lambda_0)$). The image $t(U(\lambda_0))$ of the neighborhood $U(\lambda_0)$ will be the neighborhood U_0 of the point $t = 0$. Let us denote $P_{\pm}(t) = Q_{\pm}(\lambda_0 + t^p)$ ($t \in U_0$). Thus, there is a bijective correspondence between the set of all $\lambda \in U(\lambda_0)$ such that satisfying (2) and the set of all $t \in U_0$ such that

$$(I + P_{\pm}(t)T)f = 0 \quad (15)$$

where $f \in \mathfrak{H}, f \neq 0$.

From (14) the operators $Q_{\pm}(t)$ can be written in the form $P_{\pm}(t) = \sum_{j=0}^{\infty} B_j^{\pm} t^j$, where $t \in U_0$, and $B_0^{\pm} = \lambda_0 I$. By the theorem about holomorphic operator-valued function, it follows that $\text{Ker}(I + P_{\pm}(t)T) \neq 0$ for a finite set of values $t_s \in U_0$ ($s = 1, \dots, m$). Moreover, $\dim \text{Ker}(I + P_{\pm}(t_s)T) < \infty$ ($s = 1, \dots, m$). Thus Theorem 1 is proved.

3. Let \mathfrak{H} be a Hilbert space and V an one-sided translation in \mathfrak{H} , i.e. $V^*V = I$, and there exists a subspace (called a wandering subspace of V) $\mathfrak{L}, \mathfrak{L} \subset \mathfrak{H}$, such that

$$V^n \mathfrak{L} \perp \mathfrak{L} \quad (n \in \mathbb{N}) \quad \mathfrak{H} = \sum_{n=0}^{\infty} \oplus V^n \mathfrak{L}. \quad (16)$$

In view of (16) every $h \in \mathfrak{H}$ can be represented in the form

$$h = \sum_{n=0}^{\infty} V^n g_n, \quad (17)$$

where $g_n \in \mathfrak{L}$ ($n = 0, 1, \dots$). Moreover,

$$\|h\|^2 = \sum_{n=0}^{\infty} \|g_n\|^2.$$

Let us consider the function $a(z)$ analytic on \mathbb{T}_{τ} for an $\tau \in (0, 1)$, real-valued on \mathbb{T} , and the operator $H = a(V) + B$ acting in the space \mathfrak{H} , where $B \in \mathbb{B}(\mathfrak{H})$.

We assume that $S = R = S_{\delta}$, where

$$S_{\delta} h = \sum_{n=0}^{\infty} e^{-\delta n} V^n g_n, \quad ,$$

h has been given by (17) and $\delta \in \mathbb{R}$. For $\delta > 0$ the operator S_{δ} is invertible, its inverse operator is not bounded and it is equal to $S_{-\delta}$. Here, we consider the situation when the operator $S_{-\delta} B S_{-\delta}$ is densely defined and bounded in \mathfrak{H} for some $\delta > 0$. Denote the extension of this operator by T_{δ} .

Theorem 2. *If the operator B is such that the operator T_{δ} is compact for some $\delta > 0$, then the operator H has at most a finite set of eigenvalues. Every eigenvalue has finite multiplicity.*

Proof. Clearly that the operator H satisfies the conditions (i)–(iii) of the Theorem 1. Since,

$$\begin{aligned} \|S V^n h\|^2 &= \left\| S \left(\sum_{k=0}^{\infty} V^{n+k} g_k \right) \right\|^2 = \left\| \sum_{k=0}^{\infty} e^{-\delta(n+k)} V^{n+k} g_k \right\|^2 = \\ &= e^{-2\delta n} \left\| \sum_{k=0}^{\infty} e^{-\delta k} V^{n+k} g_k \right\|^2 \leq e^{-2\delta n} \sum_{k=0}^{\infty} e^{-\delta k} \|V^{n+k} g_k\|^2 \leq \end{aligned}$$

$$\leq e^{-2\delta n} \sum_{k=0}^{\infty} \|g_k\|^2 = e^{-2\delta n} \|h\|^2 \quad (h \in \mathfrak{H}; n \in \mathfrak{N}),$$

we have that $\|SV^n\| \leq e^{-\delta n}$ ($\delta > 0$). Thus the condition (iv) of Theorem 1 is satisfied. \square

4. Let $\mathfrak{H} = l_2$ and V be the shift operator in \mathfrak{H} , i.e. $Vx = (0, x_1, \dots, x_{n-1}, \dots)$ ($x = (x_n) \in l_2$). We consider the unperturbed operator H_0 of the form $H_0 = \sum_{k=-\infty}^{\infty} a_k V^k$, where $a_k \in \mathbb{C}$ ($k \in \mathbb{Z}$) and $\sum_{k=-\infty}^{\infty} a_k z^k \in \mathbb{R}$ ($z \in \mathbb{T}$).

As a perturbation of H_0 we take the operator $B = [b_{jk}]_{j,k=1}^{\infty} \in \mathbb{B}(\mathfrak{H})$, where b_{jk} are complex numbers. Assume that $\sum_{k=-\infty}^{\infty} |a_k| e^{\tau|k|} < \infty$ for some $\tau > 0$. This

assumption is equivalent to analyticity of the function $a(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ on \mathbb{T}_τ . In this case Theorem 2 can be formulated as follows.

Theorem 3. *If $\sum_{k=-\infty}^{\infty} |a_k| e^{\delta|k|} < \infty$ and the operator generated by the matrix $[e^{\delta(n+k)} b_{nk}]_{n,k=1}^{\infty}$ is bounded in l_2 for some $\delta > 0$, then the operator $H = H_0 + B$ has a finite set of eigenvalues, each of them being of finite multiplicity.*

In particular, we can formulate the same result for the operator generated by generalized Jacobi matrix. Let us consider in the space l_2 the operator H generated by matrix $[a_{jk}]_{j,k=1}^{\infty}$ with complex elements such that $a_{jk} = \bar{a}_{kj}$ ($j, k = 1, 2, \dots$), and $a_{jk} = 0$ if $|j - k| > n$ for some $n \in \mathbb{N}$.

Suppose that $a_{j,j+k} \rightarrow a_k$ ($j \rightarrow \infty$; $k = 0, \pm 1, \dots, \pm n$). Let us denote $\alpha_{jk} = a_{j,j+k} - a_k$ ($k = 0, \pm 1, \dots, \pm n$). Thus, the operator H can be represented in the form $H = H_0 + B$, where

$$H_0 = \sum_{k=-n}^n a_k V^k, \quad B = [b_{jk}]_{j,k=1}^{\infty}, \quad b_{jk} = \begin{cases} \alpha_{jk} & , |j - k| \leq n \\ 0 & , |j - k| > n. \end{cases}$$

From Theorem 3 one gets at once the following

Theorem 4. *If the sequences $(e^{\delta j} \alpha_{jk})_{j=1}^{\infty}$ ($k = 0, \pm 1, \dots, \pm n$) converge for some $\delta > 0$, then the point spectrum of the operator H is at most a finite set. Furthermore, the possible eigenvalues have finite multiplicity.*

5. It should be mentioned that Theorem 1 could be applied in different cases. In this section, we consider the operator V generated in the space $L_2(\mathbb{R}_+)$ by the expression

$$(Vf)(x) = \begin{cases} f(x-1) & , x \geq 1 \\ 0 & , 0 < x < 1. \end{cases}$$

As an unperturbed operator H_0 , we choose the following selfadjoint operator

$$H_0 = \sum_{n=-\infty}^{\infty} a_n V^n,$$

where $a_n \in \mathbb{C}$ ($n \in \mathbb{Z}$) and $\sum_{n=-\infty}^{\infty} a_n z^n \in \mathbb{R}$ ($z \in \mathbb{T}$). The operator H_0 represents an operator with finite difference of arbitrary order. Let us consider a perturbation of the operator H_0 by the integral operator

$$(Bf)(x) = \int_{\mathbb{R}_+} b(x, y)f(y)dy, \quad (f \in L_2(\mathbb{R}_+)).$$

Suppose that the kernel $b(\cdot, \cdot)$ is such that the operator B (generally speaking nonselfadjoint) is bounded in $L_2(\mathbb{R}_+)$. Let S and T be the operators in $L_2(\mathbb{R}_+)$ of the form

$$(Sf)(x) = e^{-\delta x} f(x), \quad (Tf)(x) = \int_{\mathbb{R}_+} e^{\delta(x+y)} b(x, y)f(y)dy,$$

where $\delta > 0$. Then $B = STS$. It can be easily verified that the following Theorem is true

Theorem 5. *If for some $\delta > 0$, the integral operator with kernel $e^{\delta(x+y)}b(x, y)$ is bounded on $L_2(\mathbb{R}_+)$ and $\sum_{n=-\infty}^{\infty} e^{\delta|n|}|a_n| < \infty$, then the operator $H = H_0 + B$ has a finite set of eigenvalues, each of them being of finite multiplicity.*

REFERENCES

- [1] Pavlov B.S., *On a nonselfadjoint Schrödinger operator*, Problems of Mathematical Physics **1** (1966), Izdat. Leningrad. Univer., 102–132. (Russian)
- [2] Dujenkova L.I., *The holomorphic extension of the resolvent of some differential operators through continuous spectrum*, Trudi Seminara po FA (1970), Kiev, Inst. Mat., 136–206. (Russian)
- [3] Almamedov M. S., *On the spectrum of integro-differential operator of fourth order*, Dokl. Akad. Nauk SSSR **vol. 129, 3** (1988), 525–529. (Russian)
- [4] Ljance V., *A nonselfadjoint difference operator*, Dokl. Akad. Nauk SSSR **173, N 6** (1967), 1260–1263. (Russian)
- [5] Kakabadze L.V., *Finiteness of the discret spectrum of a perturbed Wiener-Hopf operator*, Funk. Anal. i Prilozhen **20, N4** (1986), 76–79. (Russian)
- [6] Ljance V., *Completely regular perturbation of a continuous spectrum*, Mat.Sb.(N.S.) **82(124)** (1970), 126–156. (Russian)
- [7] Ig.Cialenco, *Finiteness of the point spectrum of some nonselfadjoint perturbed operators of Wiener-Hopf type*, Mat.conf. Analiza Matematica și aplicații, Iași (1997), 7.
- [8] Ig.Cialenco, P.Cojuhari, *Finiteness of the point spectrum of some nonselfadjoint operators close to the operators generated by Jacobi matrix*, Bul.A.S.Rep.Moldova (1980 (to appear)).
- [9] I.Gohberg, I.Feldman, *Convolution Equations and Projection Methods for their Solution*, American Mathematical Society, Providence, 1974. (Russian)
- [10] Kuroda S.T., *An abstract stationary approach to perturbation of continuous spectra and scattering theory*, J.D'Analyse Math. **20** (1967), 57–117.
- [11] Martin Schechter, *Discreteness of the singular spectrum for Schrödinger operators*, Math. Proc. Camb. Phil. Soc. **80** (1976), 121–.
- [12] Krein M., *Integral equations on the semiaxis with kernel depending on difference of arguments*, Usp.Mat.Nauk. **XIII, 5(83)** (1958), 3–120. (Russian)
- [13] Marcushevich, *Theory of analytical functions*, vol. 1, Moskva, Nauka, 1965. (Russian)
- [14] Hille E., Phillips R., *Functional analysis and semi-groups*, Providence, 1957.

- [15] Gohberg I., Krein M., *Introduction in the theory of nonselfadjoint operators*, Moskva, Nauka, 1965. (Russian)

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