U.P.B. Sci. Bull., Series A, Vol. 75, Iss. 4, 2013 ISSN 1223-7027

FINITENESS OF THE POINT SPECTRUM FOR SOME INTEGRO-DIFFERENTIAL OPERATORS

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In this paper, we find sufficient conditions on the coefficients and the kernels of a general class of nonselfadjoint integro-differential operators of any order that guarantee the finiteness of the point spectrum of these operators. The results are obtained by the direct investigation of the analyticity of the resolvent function near the essential spectrum and its holomorphic extension through the continuous spectrum. The problem is motivated by applied problems from quantum mechanics, plasma oscillation theory, and neutrons diffusion.

Keywords: Spectral theory, finiteness of the point spectrum, nonselfadjoint operators, differential and integro-differential operators, perturbation theory.

AMS Subject Classification 2010: Primary 47A55, 47A10, 47A75; Secondary 47E05, 47G20, 34L10, 34L15.

1. Introduction

In this work we study the finiteness of the point spectrum of some nonselfadjoint integro-differential operators of arbitrary order. By contrast with the case of selfadjoint operators, for which various fine methods have been elaborated thanks to fundamental theorem of spectral analysis, in the case of nonselfadjoint operators the problem of finiteness of the point spectrum is usually reduced to the uniqueness theorem of holomorphic operator-valued functions. Using this method, the finiteness of the point spectrum of various classes of nonselfadjoint operators have been investigated by many authors: differential operators of second and fourth order [1, 2]; nonselfadjoint Schrödinger operator [3-5]; perturbed differential operators with periodic coefficients see [6]; Friedrics model see [7-9]; the finite difference operators [10, 11]; perturbed integral

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Wiener-Hopf operators [12-14]; Hill operator see [15]; equation of a nonhomogeneous damped string see [16]. Some general results have been presented in [17] and [6].

In the present paper the problem of the finiteness of the point spectrum is solved by methods from perturbation theory and a direct investigation of analyticity of the resolvent function. Some results have been announced (without proofs) in [18]. In this paper, we present the detailed, more elegant and easier proofs.

It should be mentioned that similar problems for selfadjoint operators have been studied in [19-21].

The paper is organized as follows: we start Section 2 by presenting the main result and we discuss the methodology of the proof. Subsection 2.1 is dedicated to abstract results on the finiteness of the point spectrum of nonselfadjoint operators. The proof of the main theorem is split in several auxiliary lemmas that are given in Subsection 2.2. We conclude the paper with some applications of the general theorem to concrete differential operators (Section 3). All obtained results are in concordance with existing literature. For simplicity of writing we consider the case of the half line, but all results remain true for the whole real line, with appropriate adjustments.

2. Theoretical results

In the space $L_2(R_+)$ we consider the differential operator $D = i\dfrac{d}{dx}$ with its domain of definition determined by the set of all functions $u \in L_2(R_+)$, absolutely continuous on every bounded interval of the positive half-line, with generalized derivative u' (in the sense of distributions) belonging to $L_2(R_+)$, and zero boundary condition $u(0) = 0$.

Let us consider the following integro-differential operator

$$H = \sum_{\alpha,\beta=0}^{n} D^{*\alpha} M_{\alpha\beta} D^{\beta}\ , \tag{1}$$

where

$$(M_{\alpha\beta}u)(x) = a_{\alpha\beta}u(x) + q_{\alpha\beta}(x)u(x) + \int_{R_+} k_{\alpha\beta}(x,y)u(y)dy, \qquad \alpha,\beta = 0,\dots,n\,;$$

$q_{\alpha\beta}(\cdot)$ and $k_{\alpha\beta}(\cdot,\cdot)$ are some measurable functions on R_+ and $R_+ \times R_+$ respectively (generally speaking with complex values); $q_{nn}(x) = 0$, $k_{nn}(x,y) = 0$, $a_{\alpha\beta}$ are complex numbers, $a_{\alpha\beta} = \overline{a}_{\beta\alpha}$, $\alpha, \beta = 0,\dots,n;\ a_{nn} = 1$. We assume that the operator H acts on its maximal domain.

The differential operator

$$A = \sum_{\alpha,\beta=0}^{n} a_{\alpha\beta} D^{*\alpha} D^{\beta} , \tag{2}$$

of order $2n$ is a self-adjoint operator and its spectrum $\sigma(A)$ is the set of all values of the polynomial (called the symbol of the operator A)

$$A(\xi) = \sum_{\alpha,\beta=0}^{n} a_{\alpha\beta} \xi^{\alpha+\beta}, \quad -\infty < \xi < +\infty,$$

namely $\sigma(A) = [a, +\infty)$, where $a = \min\{A(\xi) | \xi \in R\}$, and the point spectrum of A is empty, where $\sigma_p(A)$ is the set of all eigenvalues of the operator A. The operator H can be viewed as a perturbation of the operator A with the following subordinated (of order less than $2n$) integro-differential operator

$$B = \sum_{\alpha,\beta=0}^{n} D^{*\alpha} (q_{\alpha\beta}(x) + K_{\alpha\beta}) D^{\beta}.$$

Thus $H = A + B$.

Note that if the functions $q_{\alpha\beta}, \alpha, \beta = 0,...,n$, vanish as $x \to +\infty$ and the integral operators $(K_{\alpha\beta} u)(x) = \int_{R_+} k_{\alpha\beta}(x, y) u(y) dy, \; x \in R_+; \alpha, \beta = 0,...,n$, are compact, then, since A is self-adjoint, due to Weyl's type theorems (see Chapter XIII, [22]), the perturbed operator H has the same essential spectrum. However, the operator H can have infinitely many eigenvalues.

The main goal of this paper is to establish sufficient conditions on the functions $q_{\alpha\beta}$ and the kernels $k_{\alpha\beta}, \alpha, \beta = 0,...,n$, that guarantee that the point spectrum $\sigma_p(H)$ of the perturbed operator H is a finite set. Next, we formulate the main result of this paper.

Theorem *If there exists a positive constant* τ *such that*

$$e^{\tau x} q_{\alpha\beta}(x) \in L_\infty(R_+), \qquad \alpha, \beta = 0,...,n, \tag{3}$$

and the integral operators with kernels

$$e^{\tau(x+y)} k_{\alpha\beta}(x, y), \qquad \alpha, \beta = 0,...,n, \tag{4}$$

are bounded in $L_2(R_+)$*, then the point spectrum of the operator* H *is finite and all eigenvalues have finite multiplicity.*

The proof of the theorem is based on the so called method of the extension of the resolvent function of unperturbed operator through its continuous spectrum (see for instance [6] for a general framework). For sake of completeness, we present here a version of this method, which will be applied consequently to the operator H of form (1).

2.1. Abstract Scheme

Let G and G_1 be two Hilbert spaces, and denote by $C(G,G_1)$ the set of all linear, closed and densely defined operators acting between G and G_1. Similarly, we denote by $B(G,G_1)$ the set of all linear and bounded operators, and by $B_\infty(G,G_1)$ the set of compact operators. If $G=G_1$, then we write $C(G)$, $B(G)$ and $B_\infty(G)$.

Throughout this subsection we assume that

$$H = A + B,$$

where A and B are linear operators acting on G, such that the following conditions hold true:

(i) The operator A is self-adjoint on G, and $\sigma_p(A)=\varnothing$;

(ii) The operator B can be represented as

$$B = RTS,$$

where

$$S \in C(G,G_1),\ R \in C(G_1,G),\ T \in B_\infty(G_1)$$

with

$$D(S)\cap D(R^*) \supset D(A)$$

and $D(S)$ denoting the domain of definition of the operator S.

(iii) There exists $z_0 \in \rho(A)$ such that the operator $S(A-z_0I)^{-1}R$ admits a bounded extension. Denote this extension by $Q(z)$.

(iv) There exists $\widetilde{Q}_\pm(\lambda) = s - \lim\limits_{\varepsilon\to 0+} Q(\lambda \pm i\varepsilon)$, for all $\lambda \in R$.

Remark. Note that by conditions (ii) and (iii) the operator $S(A-zI)^{-1}R$ admits a bounded extension for every $z \in \rho(A)$.

In what follows we denote by $Q_\pm(\lambda)$ the operator-valued function defined on $\overline{\Pi}_\pm$ that is equal to $Q(\lambda)$ if $\lambda \in \Pi_\pm$, and to $\widetilde{Q}_\pm(\lambda)$ if $\lambda \in R$, where $\Pi_+ = \{z \mid \operatorname{Im} z > 0\}$ and $\Pi_- = \{z \mid \operatorname{Im} z < 0\}$.

One can prove the following results.

Proposition 1. (see [6]) *If (i)-(iv) hold true, and if* $\lambda \in \sigma_p(H)$*, then there exists* $\varphi \in G_1, \varphi \neq 0$*, such that*

$$\left(I + Q_\pm(\lambda)T\right)\varphi = 0. \tag{5}$$

By Proposition 1, it is clear that if equality (5) holds true for a finite set of values $\lambda \in C$, then the point spectrum of the perturbed operator H is finite. Thus, the original problem is reduced to choose appropriately the operators *R,T,S* such that (i)-(iv) are satisfied and (5) holds true for a finite number of values $\lambda \in C$. To show the latter, we will use the theorem about the finiteness of the zeros of an analytic operator-valued function (see for instance Theorem XIII.13, [22], and Theorem 5.1, [23]). Of course, the function $Q_{\pm}$ is analytic on $\Pi_{\pm}$ only, and the analyticity is lost near the spectrum of the unperturbed operator *A*; however, with proper chosen *R,T,S*, and some additional assumptions on *B,* we will show that the function $Q_{\pm}(\lambda)$ can be extended analytically on a neighborhood of λ (on a Riemann surface), for any $\lambda \in \sigma(A)$. Hence the name of the method: *extension of the resolvent function of unperturbed operator through its continuous spectrum.*

2.2. Proof of the main theorem

As mentioned in the previous subsection, the task is to choose the operators *R,T,S* such that (i)-(iv) are satisfied and $Q_{\pm}(\lambda)$ admits analytic extensions, through the spectrum of unperturbed operator A, on half-plane $\overline{\Pi}_{-}$ and $\overline{\Pi}_{+}$ respectively. With this in view, we consider the following construction.

Let G_1 be the Hilbert space equal to the direct sum of $2(n+1)$ copies of $G := L_2(R_+)$, namely $G_1 = \sum_{k=0}^{n} \oplus G$. Let δ be a positive real number, and define the operators $S: G \to G_1$, $T: G_1 \to G_1$, $R: G_1 \to G$ as follows

$$Su = (S_0 u, ..., S_n u), \qquad u \in \bigcap_{\alpha=0}^{2n} Dom(S_\alpha),$$

$$Tv = \left(\sum_{\beta=0}^{2n} T_{\alpha\beta} v_\beta \right)_{\alpha=0}^{2n}, \quad v = (v_0, ..., v_{2n}),$$

$$Rv = \sum_{\alpha=0}^{2n} R_\alpha v_\alpha \ , \ v_\alpha \in \bigcap_{\beta=0}^{n} Dom(R_\beta), \ \alpha = 0,...,2n,$$

where

$$\begin{gathered} S_\beta = e^{-\delta x} D^\beta, \beta = 0,...,n \\ S_\beta = e^{-\delta x} D^n, \beta = n+1,...,2n \\ R_\alpha = D^{*\alpha-1} e^{-\delta x}, \alpha = 0,...,n, \\ R_\alpha = D^{*n} e^{-\delta x}, \alpha = n+1,...,2n, \\ T_{\alpha,\beta} = e^{\delta x} q_{\alpha-1,\beta-1} D^{-1} e^{\delta x}, \alpha, \beta = 0,...,n, \\ T_{\alpha,\alpha+n} = e^{\delta x} D^{*-1} q_{\alpha-2,n} e^{\delta x}, \alpha = 2,...,n, \\ T_{n,\alpha+n} = e^{\delta x} q_{n,\alpha-1} D^{-1} e^{\delta x}, \alpha = 0,...,n, \\ T_{n+1,n+1} = e^{\delta x} D^{*-1} q_{n-1,n} e^{\delta x}, \end{gathered} \tag{6}$$

and all other operators $T_{\alpha,\beta}$ being zero.

Under these assumptions the operator B has the following decomposition

$$B = \sum_{\alpha,\beta=0}^{2n} R_\alpha T_{\alpha\beta} S_\beta,$$

or

$$B = RTS.$$

Recall that the operator A defined by (2) is selfadjoint with empty point spectrum, and hence (i) is satisfied. Since, the operators S_α and R_β, $\alpha = 0,...,2n$, are differential operators of order at most *n*, multiplied by a bounded function, we have that $D(S) \cap D(R)$ is larger than the domain of the differential operator A of order *2n*. Note that the assumptions (3) and (4) guarantee that T is a compact operator, for any $\delta < \tau$. Hence condition (ii) from the abstract scheme is satisfied. Also, one can check directly that (iv) is satisfied (in fact for all $z \in \rho(A)$).

Due to the here-above representation of *B*, the condition (iv) and the analyticity of the function $Q_\pm(\lambda)$ are equivalent to the study of the operator-valued functions of the form

$$Q_{\beta\alpha}(\lambda) = S_\beta (A - \lambda I)^{-1} R_\alpha, \quad \alpha, \beta = 0,..., n,$$

which will be split in several proposition below.

Proposition 2. *For every $\lambda_0 \in C$, there exists a neighborhood $U(\lambda_0)$ of λ_0, such that the functions $Q_\pm(\lambda)$ have an analytic extension from $U(\lambda_0) \bigcap \Pi_\pm$ to $U(\lambda_0)$, with λ_0 being an algebraic ramification point.*

Proof. If $\lambda_0 \in \rho(A)$, the result follows immediately since the resolvent function is analytic on the resolvent set.

Let $\lambda_0 \in \sigma(A)$ and denote by ξ_k, $k = 1,2,...,2n,$ all the solutions of equation $A(\xi) - \lambda_0 = 0$. Since $A(\xi) = A(\overline{\xi})$, the number of solutions of equation $A(\xi) - \lambda_0 = 0$ from $\overline{\Pi}_+$ coincides with the number of solutions from $\overline{\Pi}_-$.

Thus, we have the representation

$$A(\xi) - \lambda_0 = \prod_{k=1}^{2n} (\xi - \xi_k),$$

where $\xi_1,...,\xi_n \in \overline{\Pi}_+$, and $\xi_{n+1},...,\xi_{2n} \in \overline{\Pi}_-$.

Case 1: assume that all roots of the polynomial $A(\xi) - \lambda_0$ are simple. By [22], there exists $U(\lambda_0)$ such that the functions $\xi_k = \xi_k(\lambda), k = 1,2,...,2n,$ are holomorphic for all $\lambda \in U(\lambda_0)$. For simplicity we will study the case when all roots belong to the real line R, that is

$$\xi_k \in \Pi_+, k = 1,...,n; \lambda \in U(\lambda_0) \bigcap \Pi_+,$$

$$\xi_k \in \Pi_-, k = n+1,...,2n; \lambda \in U(\lambda_0) \cap \Pi_+,$$
$$\xi_k(\lambda_0) \in R, \; k = 1,2,...,2n.$$

If $\lambda \in U(\lambda_0) \cap \Pi_+$ then the operator $(A-\lambda I)^{-1}$ can be represented as follows:

$$(A-\lambda I)^{-1} = (D-\xi_{2n}I)^{-1} \cdots (D-\xi_{n+1}I)^{-1}(D^* - \xi_n I)^{-1} \cdots (D^* - \xi_1 I)^{-1}. \tag{7}$$

Since $\sigma(D) = \overline{\Pi}_+$ and $\sigma(D^*) = \overline{\Pi}_-$, we conclude that every factor in (7) is a bounded operator. Thus, using (6) we conclude

$$(Q_{\beta\alpha}(\lambda)f, g) = ((D^* - \xi_1 I)^{-1} \cdots (D^* - \xi_n I)^{-1} D^{*\alpha} e^{-\delta x} f, \tag{8}$$
$$, (D^* - \overline{\xi}_{2n} I)^{-1} \cdots (D^* - \overline{\xi}_{n+1} I)^{-1} D^{*\beta+1} e^{-\delta x} g)$$

where $\alpha \le n, \beta < n, f, g \in L_2(R_+)$ and $\delta > 0.$

Note that

$$\prod_{k=1}^{n} (\xi - \xi_k)^{-1} \xi^{\alpha} = \sum_{k=1}^{n} a_k (\xi - \xi_k)^{-1} + a_0 ,$$

and

$$\prod_{k=n+1}^{2n} (\xi - \xi_k)^{-1} \xi^{\beta+1} = \sum_{k=1}^{n} \overline{b}_{k-n} (\xi - \xi_k)^{-1} + \overline{b}_0 ,$$

where $a_k = a_k(\lambda), b_k = b_k(\lambda), k = 0,1,...,n,$ are analytic functions on $U(\lambda_0)$, being linear combinations of analytic functions. This implies that (8) can be written as

$$\begin{aligned}(Q_{\beta\alpha}(\lambda)f, g) = &\sum_{j,k=1}^{n} a_j b_k ((D^* - \xi_j I)^{-1} e^{-\delta x} f, (D^* - \overline{\xi}_k I)^{-1} e^{-\delta x} g) + \\ &+ \sum_{k=1}^{n} a_0 b_k (e^{-\delta x} f, (D^* - \overline{\xi}_k I)^{-1} e^{-\delta x} g) + \\ &+ \sum_{j=1}^{n} a_j b_0 ((D^* - \xi_j I)^{-1} e^{-\delta x} f, e^{-\delta x} g),\end{aligned} \tag{9}$$

where $f, g \in L_2(R_+)$ and $\delta > 0$. Thus the analyticity of $(Q_{\alpha\beta}(\lambda)f, g)$ with respect to λ is equivalent to the analyticity of functions of the form

$$((D^* - \xi_j I)^{-1} e^{-\delta x} f, (D^* - \overline{\xi}_k I)^{-1} e^{-\delta x} g) = \tag{10}$$
$$= \int_{R_+} \left(\int_x^{\infty} e^{i\xi_j (y-x)} e^{-\delta y} f(y) dy \cdot \int_x^{\infty} e^{-i\xi_k (y-x)} e^{-\delta y} \overline{g(y)} dy \right) dx ,$$

where $f, g \in L_2(R_+)$ and $\delta > 0, \; j, k = 1,...,n$.

We choose the neighborhood $U(\lambda_0)$ such that $\left|\operatorname{Im}\xi_j\right| \le \tau < \delta$, for every $\lambda \in U(\lambda_0)$, $j = 1,2,...,2n$. We claim that the following inequality holds true

$$\int_{R_+}\left|\int_x^\infty e^{i\xi_j(y-x)}e^{-\delta y} f(y)dy\right|^2 dx \le c\int_{R_+}\left|f(y)\right|^2 dy \tag{11}$$

for every $f \in L_2(R_+)$, $j = 0,...,2n$ and $\lambda \in U(\lambda_0)$. To show this, we use a Hardy type inequality (see, for instance [24])

$$\int_{R_+}\left|\int_x^\infty e^{\mu(y-x)}\varphi(y)dy\right|^2 dx \le \frac{1}{(\operatorname{Re}\mu)^2}\int_{R_+}\left|\varphi(y)\right|^2 dy, \tag{12}$$

where $\mu \in C$ and $\operatorname{Re}\mu > 0$.

Since $\left|\operatorname{Im}\xi_j\right| \le \tau < \delta$, we have that $\operatorname{Re}(\delta - i\xi_j) > 0,$ and using (12) the left hand side of (11) can be estimated as follows

$$\int_{R_+}\left|\int_x^\infty e^{(\delta - i\xi_j)(y-x)} f(y)dy\right|^2 dx \le \frac{1}{(\delta + \left|\operatorname{Im}\xi_j\right|)^2}\int_{R_+}\left|f(y)\right|^2 dy \le \frac{1}{\delta^2}\int_{R_+}\left|f(y)\right|^2 dy.$$

Inequality (11) is proved.

Thus, the function given by (10) is holomorphic on $U(\lambda_0),$ and hence the function $((Q_{\beta\alpha})(\lambda)f, g)$ is analytic on $U(\lambda_0)$, for all $\alpha \le n$ and $\beta \le n$. Moreover, by (10) and (11), we also have that

$$\left|((Q_{\beta\alpha})(\lambda)f, g)\right| \le c\|f\|\cdot\|g\|, \quad \alpha \le n, \beta \le n.$$

This implies that the bilinear form $((Q_{\beta\alpha})(\lambda)f, g)$ generates an operator-valued function $(\widetilde{Q}_{\beta\alpha})(\lambda)$ with values in $B(G)$, and since the strong analyticity of operator-valued functions is equivalent to weak analyticity, we conclude that the operator-valued function $(\widetilde{Q}_{\beta\alpha})(\lambda)$ is analytic on $U(\lambda_0)$ and $(\widetilde{Q}_{\beta\alpha})(\lambda) = (Q_{\beta\alpha})(\lambda), \lambda \in U(\lambda_0) \bigcap \Pi_+$. Due to the uniqueness of the analytic extension, we get that $Q_{\beta\alpha}(\lambda)$ is analytic on $U(\lambda_0)$, and hence $Q_+(\lambda)$ is analytic on $U(\lambda_0)$. Thus Case 1 is finished.

Case 2: the polynomial function $A(\xi) - \lambda_0$ has real multiple roots. For simplicity we assume that ξ_1 has multiplicity m, namely

$$A(\xi) - \lambda_0 = (\xi - \xi_1)^m \cdots (\xi - \xi_{m+1}) \cdots (\xi - \xi_{2n}).$$

Similar to Case 1, we have the factorization

$$A(\xi) - \lambda_0 = (\xi - \xi_1^{(1)})(\xi - \xi_1^{(2)}) \cdots (\xi - \xi_1^{(p_1)})(\xi - \xi_2^{(1)}) \cdots (\xi - \xi_2^{(p_2)}) \cdots$$
$$\cdots (\xi - \xi_k^{(1)}) \cdots (\xi - \xi_k^{(p_k)})(\xi - \xi_s) \cdots (\xi - \xi_p),$$

where $p_1+...+p_k=m$, and $\xi_j^{(\gamma)},\ j=1,...,k;\ \gamma=1,2,...,p_j$, are the branches of multivalued function ξ_j. The following decomposition holds true

$$\xi_j^{(\gamma)}=\lambda_0+\sum_{r=1}^{\infty}\beta_r^{(j)}(\lambda-\lambda_0)^{\frac{r}{p_j}},\qquad j=1,...,k.$$

Without loss of generality, for simplicity of writing, we suppose that $m=4, k=2$ and $p_1=p_2=2$. Thus,

$$A(\xi)-\lambda_0=(\xi-\xi_1^{(1)})^2(\xi-\xi_1^{(2)})^2(\xi-\xi_3)\cdots(\xi-\xi_{2n}),$$

where $\xi_1^{(1)},\xi_1^{(2)}$ are the branches of the function $\xi_1(\lambda)$. Furthermore, we suppose that for $\lambda\in U(\lambda_0)\bigcap\Pi_+$, we have that $\xi_1^{(1)}\in\overline{\Pi}_+, \xi_1^{(2)}\in\overline{\Pi}_-, \xi_3,...,\xi_n\in\Pi_+$, $\xi_{n+3},...,\xi_{2n}\in\Pi_-$, which implies that

$$\left(A-\lambda I\right)^{-1}=(D-\xi_{2n}I)^{-1}\cdots(D-\xi_1^{(2)}I)^{-2}(D^*-\xi_nI)^{-1}\cdots(D^*-\xi_1^{(1)}I)^{-2}$$

and consequently

$$\begin{aligned}\left(Q_{\beta\alpha}(\lambda)f,g\right)=((D^*-\xi_nI)^{-1}\cdots(D^*-\xi_1^{(1)}I)^{-2}\,D^{*\alpha}e^{-\delta x}f,\\ ,(D^*-\xi_{2n}I)^{-1}\cdots(D^*-\xi_1^{(2)}I)^{-2}\,D^{*\beta+1}e^{-\delta x}g),\end{aligned}\tag{13}$$

where $f,g\in L_2(R_+), \alpha\le n$ and $\beta+1\le n$.

Since

$$\prod_{k=2}^{n}\xi^{\alpha}(\xi-\xi_1^{(1)})^{-2}(\xi-\xi_k)^{-1}=a_0+\frac{a_1}{\xi-\xi_1^{(1)}}+\frac{a_2}{(\xi-\xi_1^{(1)})^2}+\sum_{k=3}^{n}\frac{a_k}{\xi-\xi_k}$$

and

$$\prod_{k=n+1}^{2n}\xi^{\beta}(\xi-\xi_1^{(2)})^{-2}(\xi-\xi_k)^{-1}=b_0+\frac{b_1}{\xi-\xi_1^{(2)}}+\frac{b_2}{(\xi-\xi_1^{(2)})^2}+\sum_{k=n+2}^{2n}\frac{b_{n+4-k}}{\xi-\xi_k},$$

by (13) we get

$$((Q_{\beta\alpha})(\lambda)f,g)=\sum_{j,k=1}^{n}a_\alpha b_\beta((D^*-\xi_jI)^{-r}e^{-\delta x}f,(D^*-\xi_{k+n}I)^{-s}e^{-\delta x}g),$$

where $r,s=1$ or 2, $\xi_1=\xi_2=\xi_1^{(1)}$, $\xi_{n+1}=\xi_{n+2}=\xi_1^{(2)}$.

Now, let us consider the function

$$((D^*-\xi_jI)^{-2}e^{-\delta x}f,(D^*-\xi_kI)^{-2}e^{-\delta x}g),$$

where $\xi_j=\xi_j(\lambda);\ \xi_k=\xi_k(\lambda)\ \left(j,k=1,...,n\right)$ and

$$\begin{aligned}&\xi_j\in\Pi_+,\ \ j=1,...,n;\ \lambda\in U(\lambda_0)\bigcap\Pi_+\\ &\xi_k\in\Pi_-,\ \ k=1,...,n;\ \lambda\in U(\lambda_0)\bigcap\Pi_-,\\ &\left|\operatorname{Im}\xi_j\right|\le\tau<\delta,\ \ j,k=1,2,...,n;\ \lambda\in U(\lambda_0).\end{aligned}\tag{14}$$

From the above formulae, we deduce

$$((D^* - \xi_j I)^{-2} e^{-\delta x} f, (D^* - \xi_k I)^{-2} e^{-\delta x} g) =$$
$$= \int_{R_+} \left(\int_x^\infty \left(e^{i\xi_j (y-x)} e^{-\delta y} \int_y^\infty e^{i\xi_j (t-y)} e^{-\delta (t-y)} f(t) dt \right) dy \cdot \right.$$
$$\left. \cdot \int_x^\infty \left(e^{i\xi_k (y-x)} e^{-\delta y} \int_y^\infty e^{i\xi_k (t-y)} e^{-\delta (t-y)} \overline{g(t)} dt \right) dy \right) dx.$$

Similarly to Case 1, by consequently applying the inequality (11), we get the estimates

$$\left| (Q_{\beta\alpha}(\lambda) f, g) \right| \le c \cdot \|f\| \cdot \|g\| \; f, g \in L_2(R_+);\; \lambda \in U(\lambda_0);\; \alpha, \beta = 1,...,n\,.$$

This implies that the function $(Q_{\beta\alpha}(\lambda) f, g)$ can be defined on $U(\lambda_0)$ that belongs to the first Riemann surface. Moving into the second Riemann surface, the functions $\xi_1^{(k)}, k = 1,2$, also satisfy (14), so the function $(Q_{\beta\alpha}(\lambda) f, g)$ is defined on the neighborhood $U(\lambda_0)$ of a point λ_0 on the Riemann surface and it is analytic on this neighborhood, λ_0 being an algebraic point.

Thus

$$(Q_{\beta\alpha}(\lambda) f, g) = \lambda_0 + \sum_{j=1}^\infty \beta_j (\lambda - \lambda_0)^{\frac{j}{p}}\,,$$

where $\beta_j = \beta_j(f, g)$, $j = 1,2,...; \alpha \le n,\ \beta \le n.$ Therefore

$$Q_{\beta\alpha}(\lambda) = \lambda_0 + \sum_{j=1}^\infty B_j (\lambda - \lambda_0)^{\frac{j}{p}}\,,$$

where $\lambda \in U(\lambda_0)$, and $B_j \in B(G)$ $j = 1,2,...; \alpha \le n; \beta \le n$. Proposition is proved.

Proposition 3. *Let* $\widetilde{S} : L_2(R_+) \to L_2(R_+)$ *be the multiplication operator of the form* $(\widetilde{S}\varphi)(x) = e^{-\delta x}\varphi(x),$ *for some* $\delta > 0$.

Then $\left\| \widetilde{S}(D + \lambda I)^{-1}(D^* - \lambda I)^{-1}\widetilde{S} \right\| \to 0$, *as* $|\lambda| \to \infty,$ *and* $\lambda \in \Pi_+$.

Proof. For every $\lambda \in \Pi_+$ we have

$$(D + \lambda I)^{-1}(D^* - \lambda I)^{-1} = (D^* D - \lambda^2 I)^{-1}.$$

It is easy to show that

$$(D^* D - \xi^2 I)^{-1} \varphi(x) = -\frac{1}{\xi}\left[\int_0^x e^{i\xi x} \sin(\xi y)\varphi(y) dy + \int_x^\infty e^{i\xi y} \sin(\xi x)\varphi(y) dy \right], \qquad (15)$$

for all $\varphi \in L_2(R_+)$, $\xi \in C \setminus R$. Next we will prove that

$$\int_{R_+}\left|e^{-\varepsilon x}\int_0^x \varphi(y)dy\right|^2 dx \le c_1 \int_{R_+}|\varphi(y)|^2 dy \tag{16}$$

and

$$\int_{R_+}\left|\int_x^\infty e^{-\varepsilon y}\varphi(y)dy\right|^2 dx \le c_2 \int_{R_+}|\varphi(y)|^2 dy, \tag{17}$$

for every $\varphi \in L_2(R_+)$, $\xi \in \Pi_+$, $\varepsilon > 0$.

For this, we will use the following Hardy-type inequality (see [24])

$$\int_{R_+}\left|\frac{1}{x}\int_0^x \varphi(y)dy\right|^2 dx \le 4\int_{R_+}|\varphi(y)|^2 dy, \qquad \varphi \in L_2(R_+).$$

Using this, we obtain

$$\int_{R_+}\left|e^{-\varepsilon x}\int_0^x \varphi(y)dy\right|^2 dx = \int_{R_+}\left|\frac{x}{e^{\varepsilon x}\cdot x}\int_0^x \varphi(y)dy\right|^2 dx \le$$

$$\le c\int_{R_+}\left|\frac{1}{x}\int_0^x \varphi(y)dy\right|^2 dx \le 4c\int_{R_+}|\varphi(y)|^2 dy,$$

and based on inequality $\int_0^\infty\left|\int_x^\infty f(t)dt\right|^2 dx \le 4\int_0^\infty |xf(x)|^2 dx$, we get

$$\int_0^\infty\left|\int_x^\infty e^{-\varepsilon y}\varphi(y)dy\right|^2 dx \le 4\int_0^\infty\left|xe^{-\varepsilon y}\varphi(y)\right|^2 dy \le 4c\int_0^\infty|\varphi(y)|^2 dy, \quad \varphi \in L_2(R_+).$$

Therefore the relations (16) and (17) hold true.

Let $\xi \in \Pi_+^\tau$, where $\Pi_+^\tau = \{z \mid \operatorname{Im} z > -\tau, \tau > 0\}$. Then, by (16) and (17) we deduce

$$\int_{R_+}\left|\int_0^x e^{-\delta x}\cdot\sin(\xi y)\cdot e^{i\xi x}\cdot e^{-\delta y}\cdot\varphi(y)dy\right|^2 dx \le$$

$$\le \int_{R_+}\left(\left|e^{-\delta x}\cdot e^{i\xi x}\right|^2\cdot\left|\int_0^x|\sin(\xi y)|\cdot e^{-\delta y}\cdot|\varphi(y)|dy\right|^2\right)dx \le$$

$$\le \int_{R_+}\left(\left|e^{-\delta x}\cdot e^{\tau x}\right|^2\cdot\left|\int_0^x e^{\tau y}\cdot e^{-\delta y}\cdot|\varphi(y)|dy\right|^2\right)dx \le$$

$$\leq \int_{R_+}\left(\int_0^x e^{-\varepsilon x}\cdot e^{-\varepsilon y}\cdot|\varphi(y)|dy\right)^2 dx \leq c_1\int_{R_+}\left|e^{-\varepsilon y}\cdot\varphi(y)\right|^2 dy \leq c_1\int_{R_+}|\varphi(y)|^2 dy,$$

where $\xi\in\Pi_+^{\tau}$ and $\varphi\in L_2(R_+)$, with $0<\tau<\delta$ and $\varepsilon=\delta-\tau$.

Similarly

$$\int_{R_+}\left|\int_x^\infty e^{-\delta x}\cdot\sin(\xi x)\cdot e^{i\xi y}\cdot e^{-\delta y}\cdot\varphi(y)dy\right|^2 dx \leq$$

$$\leq \int_{R_+}\left(\left|e^{-\delta x}\cdot\sin(\xi x)\right|^2\cdot\left|\int_x^\infty\left|e^{i\xi y}\right|\cdot e^{-\delta y}\cdot|\varphi(y)|dy\right|^2\right)dx \leq$$

$$\leq \int_{R_+}\left(e^{-\varepsilon x}\cdot\int_x^\infty|\varphi(y)|dy\right)^2 dx \leq \int_{R_+}\left|\int_x^\infty e^{-\varepsilon y}\cdot|\varphi(y)|dy\right|^2 dx \leq c_2\int_{R_+}|\varphi(y)|^2 dy.$$

From (15), and combining the above formulae, finally, we have the estimates

$$\left\|\widetilde{S}(D^*D-\xi^2 I)^{-1}\widetilde{S}\varphi\right\| = \frac{1}{|\xi|}\left[\int_{R_+}\left(\left|\int_0^x e^{-\delta x}\cdot\sin(\xi y)\cdot e^{i\xi x}\cdot e^{-\delta y}\cdot\varphi(y)dy + \right.\right.\right.$$

$$\left.\left.\left. + \int_x^\infty e^{-\delta x}\cdot\sin(\xi x)\cdot e^{i\xi y}\cdot e^{-\delta y}\cdot\varphi(y)dy\right|^2\right)dx\right]^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{|\xi|}\left[\int_{R_+}\left|\int_0^x e^{-\delta x}\cdot e^{i\xi x}\cdot\sin(\xi y)\cdot e^{-\delta y}\cdot\varphi(y)dy\right|^2 dx\right]^{\frac{1}{2}} +$$

$$+\frac{1}{|\xi|}\left[\int_{R_+}\left|\int_0^x e^{-\delta x}\cdot\sin(\xi x)\cdot e^{i\xi y}\cdot\varphi(y)dy\right|^2 dx\right]^{\frac{1}{2}} \leq \frac{1}{|\xi|}(c_1+c_2)\|\varphi\| = \frac{c}{\xi}\|\varphi\|.$$

This implies that $\left\|\widetilde{S}(D^*D-\xi^2 I)^{-1}\widetilde{S}\right\| \leq \frac{c}{\xi}\to 0$, for all $\xi\in\overline{\Pi}_+$ and $|\xi|\to\infty$, which ends the proof.

Proposition 4. If $\lambda\in\overline{\Pi}_\pm$, then $\|Q_\pm(\lambda)\|\to 0$, when $|\lambda|\to\infty$.

Proof. The operators $(D^*-\lambda I)^{-1}\widetilde{S}$, $\widetilde{S}(D+\lambda I)^{-1}$ are bounded for $\lambda\in\overline{\Pi}_+$, and since $(\widetilde{S}(D-\lambda I)^{-1})^* = (D^*-\overline{\lambda}I)^{-1}S^*$, by Proposition 3, we conclude that

$$\left\|\widetilde{S}(D^2-\lambda^2 I)^{-1}\widetilde{S}\right\| = \left\|(D^*-\overline{\lambda}I)^{-1}\widetilde{S}\right\|^2 \to 0,$$

for $|\lambda| \to \infty, \lambda \in \overline{\Pi}_+$. This implies that $\left\|(D^* - \overline{\lambda} I)^{-1}\widetilde{S}\right\| \to 0$, as $|\lambda| \to \infty, \lambda \in \overline{\Pi}_+$.

Using the representation $A(\xi) - \lambda = \prod_{k=1}^{2n}(\xi - \xi_k), \lambda \in \Pi_+$, and the fact that $\xi_1 \cdots \xi_{2n} = a_{00} - \lambda$, we get that if $|\lambda| \to \infty$, then there exists $\xi_k \in \overline{\Pi}_+$, such that $|\xi_k| \to \infty$ (more precisely one root in $\overline{\Pi}_+$ and one root in $\overline{\Pi}_-$).

Let $|\xi_n| \to \infty$ and $|\xi_{2n}| \to \infty, |\lambda| \to \infty$, with $\xi_n \in \overline{\Pi}_+$ and $\xi_{2n} \in \overline{\Pi}_-$.

Since the function $(D^* - \lambda I)^{-1}\widetilde{S}$ is continuous on $\overline{\Pi}_+$, we have

$$\left\|(D^* - \lambda I)^{-1}\widetilde{S}\right\| \le M_+,$$

for every λ from the disk of radius $R > 0$ intersected with $\overline{\Pi}_+$, and also

$$\left\|\widetilde{S}(D^* - \lambda I)^{-1}\right\| \le M_-,$$

for every λ from the disk of radius $R > 0$ intersected with $\overline{\Pi}_-$.

Thus, we have the following estimates

$$\begin{aligned}
\|Q(\lambda)\| &= \left\|S(D - \xi_{n+1})^{-1} \cdots (D - \xi_{2n} I)^{-1} \cdot (D^* - \xi_n I)^{-1} \cdots (D^* - \xi_1 I)^{-1} R\right\| \le \\
&\le \left\|\widetilde{S} D^{\beta+1}(D - \xi_{n+1} I)^{-1} \cdots (D - \xi_{2n} I)^{-1}\right\| \cdot \\
&\cdot \left\|(D^* - \xi_1 I)^{-1} \cdots (D^* - \xi_n I)^{-1} D^{*\alpha}\widetilde{S}\right\| \le \\
&\le \sum_{k=0}^{n} b_k \left\|\widetilde{S}(D - \xi_{k+n} I)^{-1}\right\| \cdot \sum_{k=0}^{n} a_k \left\|(D^* - \xi_k I)^{-1}\widetilde{S}\right\| \le \\
&\le c\left\|\widetilde{S}(D - \xi_{2n} I)^{-1}\right\| \cdot \left\|(D^* - \xi_n I)^{-1}\widetilde{S}\right\| \to 0,
\end{aligned}$$

for $|\lambda| \to \infty$ and $\lambda \in \overline{\Pi}_+$. Similarly, one can show that the above estimates hold true also for $\lambda \in \overline{\Pi}_-$. Proposition 4 is proved.

Proposition 5. $\|Q_\pm(\lambda) T\varphi\| \to 0$, *when* $|\lambda| \to \infty$, *and* $\lambda \in \overline{\Pi}_\pm$.

Proof. Indeed, by the above proposition

$$\|Q_\pm(\lambda) T\varphi\| \le \|Q_\pm(\lambda)\| \cdot \|T\| \cdot \|\varphi\| \le \sum_{\alpha,\beta=0}^{n} \left\|S_\beta (A - \lambda I)^{-1} R_\alpha\right\| \cdot \|T\| \cdot \|\varphi\| \to 0,$$

when $|\lambda| \to \infty$, for $\lambda \in \overline{\Pi}_\pm$.

Proposition 6. *The disk of radius* $R > 0$, *centered at zero, contains a finite number of zeros of the function* $I + Q_\pm(\lambda)T$.

Proof. It follows directly from the theorem about zeros of a holomorphic operator-valued function: operator $Q_\pm(\lambda)T$ is compact for all λ from the disk of radius R, and by Proposition 2, $Q_\pm(\lambda)T$ is holomorphic.

Finally, by Proposition 5, the operator-valued function $I+Q_{\pm}(\lambda)T$ is invertible outside a disk of sufficiently large radius, and by Proposition 6, the equality (5) is satisfied for a finite number of values $\lambda \in C$.

This completes the proof of the main theorem.

3. Examples

1. Let H_1 be an operator of the form

$$(H_1 u)(x) = -\frac{d^2u}{dx^2} + q_1(x)\frac{du}{dx} + q_0(x)u(x) + \int_{R_+} k(x,y)u(y)dy,$$

where $q_i(x)$ and $k(x,y), i=0,1$, are some measurable complex-valued functions defined on R_+ and $R_+ \times R_+$ respectively. We consider the operator H_1 acting in $L_2(R_+)$ on its maximal domain.

The unperturbed operator $A = -\frac{d^2u}{dx^2}$ is a self-adjoint operator, and its spectrum $\sigma(A)$ coincides with set of all values of the polynomial $A(\xi)=\xi^2$, $-\infty<\xi<+\infty$, namely with the positive semi-axis. Moreover, the operator A has not eigenvalues.

The operator H_1 may be viewed as a perturbation of the operator A with the integro-differential operator

$$(Bu)(x) = q_1(x)\frac{du}{dx} + q_0(x)u(x) + \int_{R_+} k(x,y)u(y)dy.$$

Based on the theoretical results from the previous section, we have the following result.

Corollary 1. *If $q_i(x)e^{\delta x} \in L_\infty(R_+), i=0,1$, for some $\delta>0$, and if the integral operator with the kernel $e^{\delta(x+y)}k(x,y)$ is bounded in $L_2(R_+)$, then the operator H_1 has a finite number of eigenvalues, each of them being of finite multiplicity.*

2. Let H_2 be the operator defined in the space $L_2(R_+)$ as follows

$$(H_2 u)(x) = \frac{d^4u}{dx^4} + P(x)u(x) + \int_{R_+} k(x,y)\frac{d^4u(y)}{dy^4}dy,$$

where $P(x)$ and $k(x,y)$ are some measurable (generally speaking, with complex values) functions defined on R_+, and $R_+ \times R_+$ respectively. We consider the operator H_2 on its maximal domain. This example is similar to one considered in [2].

The unperturbed operator $A = \dfrac{d^4u}{dx^4}$ is self-adjoint, and its spectrum coincides with the positive semi-axis, with no eigenvalues.

The operator H_2 can be represented as the sum $H_2 = A + B$, where

$$(Bu)(x) = P(x)u(x) + \int_{R_+} k(x,y)\frac{d^4u(y)}{dy^4}dy.$$

The following result holds true.

Corollary 2. *If $P(x)e^{\delta x} \in L_\infty(R_+)$, for some $\delta > 0$, and if the integral operator with kernel $e^{\delta(x+y)}k(x,y)$ is bounded on $L_2(R_+)$ for some $\delta > 0$, then the operator H_2 has at most a finite set of eigenvalues. All possible eigenvalues have finite multiplicity.*

Acknowledgments

The authors would like to express their gratitude to Prof. Petru Cojuhari for suggesting this problem. The work of Igor Cialenco was partially supported by NSF grant DMS-0908099. Also, we would like to thank the anonymous referees for their helpful comments and suggestions which improved greatly the final manuscript.

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