On time consistency of dynamic risk and performance measures in discrete time

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Abstract

In this paper we provide a unified and flexible framework for study of the time consistency of risk and performance measures. The proposed framework integrates existing forms of time consistency as well as various connections between them. In our approach the time consistency is studied for a large class of maps that are postulated to satisfy only two properties – monotonicity and locality. This makes our framework fairly general. The time consistency is defined in terms of an update rule – a novel notion introduced in this paper. We design various updates rules that allow to recover several known forms of time consistency, and to study some new forms of time consistency.

Keywords: time consistency, update rule, dynamic LM-measure, dynamic risk measure, dynamic acceptability index, measure of performance.

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1 Introduction

In the seminal paper by Artzner et al. [3], the authors proposed an axiomatic approach to defining risk measures that are meant to give a numerical value of the riskiness of a given financial contract or portfolio. Also in [3], the authors coined the notion of coherent risk measure - a real valued function acting on the space of random variables (such as, payments at some future fixed time or terminal cashflows) that is monotone, subaditive, positive homogenous and cash additive[1]. Each of these properties have clear financial interpretation, and the values of these measures of risk can be interpreted as the capital requirement for the purpose of regulating the risk assumed by market participants (typically, by banks). In particular, the risk measures are generalizations of the well-known Value-At-Risk (V@R).

Alternatively, one can view the risk measures as a tool that allows to establish preference orders on the set of cashflows according to their riskiness. Assuming single time period market (static case), and finite probability space, the authors in [3] also gave a numerical representation of such measures in terms of sets of probability measures (or generalized scenarios). In the current literature this type of representation is referred to as robust or dual representation.

[1]Precise definitions of all these notions are given in Section 2. We note that in the original paper Artzner et al. [3], the term translation invariance corresponds to cash additivity in the present manuscript.

Following [3], an extensive body of work was devoted to exploration of the axiomatic approach to risk measures. One line of research is to consider a larger class of risk measures by imposing weaker properties. For example, in the definition of coherent risk measures, subaditivity and positive homogeneity can be replaced by convexity, which leads to a larger class of functions, called convex risk measures (cf. [36, 40, 47, 6, 45]). Subsequently, Cheridito and Li [20], Biagini and Frittelli [8] studied risk measures on Orlicz Hearts, law-invariant risk measures have been investigated by Kusuoka [49] and Frittelli and Rosazza Gianin [42], general quasi-concave measures were studied in Drapeau and Kupper [33]; for a systematic discussion of static risk measures we refer the reader to the monographs by Delbaen [29] and Föllmer and Schied [37, Chapter 4].

Following a similar axiomatic approach, Cherny and Madan [26] introduced the notion of coherent acceptability index – function defined on a set of random variables that takes positive values and that are monotone, quasi-concave, and scale invariant. Coherent acceptability indices can be viewed as generalizations of performance measures such as Sharpe Ratio, Gain to Loss Ratio, Risk Adjusted Return on Capital. Coherent acceptability indices appear to be a tool very well tailored to assessing both risk and reward of a given cashflow.

Another line of research that branched out from [3] was dedicated to extension of the theory of risk and performance measure to the dynamical, multi-period setup, where the flow of information is modeled by a filtration, say $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$, that is a component of the underlying probability space $(\Omega, \mathcal{F}_T, \mathbb{F}, P)$. In this case the risk measures are defined on the set of $\mathcal{F}_T$-measurable random variables that correspond to terminal cashflows, or, more generally, on the set of adapted stochastic processes that correspond to dividend streams or to cumulative cashflows. Most of the axioms from static case are transferred to the dynamic setup in a natural way, with addition of requirement that the measures are $\mathbb{F}$-adapted and, frequently, that they are independent of the past. From another point of view, an extension from one period to multi-period models can be realized through robust representation theorems, essentially by replacing expectations with conditional expectations. Dynamic risk measures obtained by this procedure are referred to as conditional risk measures (cf. [53, 32, 14]).

As shown in one of the first papers that studied dynamic coherent risk measures, Riedel [52], if one is concerned about making noncontradictory decisions (from the risk point of view) over the time, then an additional axiom, called time consistency, is needed. Over the past decade significant progress has been made towards expanding the theory of dynamic risk measures and their time consistency. For example, so called cocycle condition (for convex risk measures) was studied in [15, 35], recursive construction was exploited in [19], relation to acceptance and rejection sets was studied in [30], the concept of prudence was introduced in [51], connections to g-expectations were studied in [54], and the relation to Bellman's principle of optimalty was shown in [4]. For more details on dynamic cash-additive measures also called dynamic monetary risk measures, we also refer the reader to [21, 22, 23, 28, 32, 44, 52, 43, 41, 53, 59, 31, 58, 39, 9], as well as to a comprehensive survey paper [1] and the references therein.

Let us briefly recall the concept of strong time consistency of monetary risk measures, which is one of the most recognized forms of time consistency. Assume that $\rho_t(X)$ is the value of a dynamic monetary risk measure at time $t \in [0, T]$, that corresponds to the riskiness, at time t, of the terminal cashflow X, with X being an $\mathcal{F}_T$-measurable random variable. The monetary risk measure is said to be strongly time consistent if for any $t < s \leq T$, and any $\mathcal{F}_T$-measurable random variables $X, Y \in \mathcal{X}$ we have that

$$\rho_s(X) = \rho_s(Y) \quad \Rightarrow \quad \rho_t(X) = \rho_t(Y). \tag{1.1}$$

The financial interpretation of strong time consistency is clear – if X is as risky as Y at some future time s, then today, at time t, X is also as risky as Y. One of the main features of the strong time consistency is its connection to dynamic programming principle. It is not hard to show that in the L^∞ framework, a monetary risk measure is strongly time consistent if and only if

$$\rho_t = \rho_t(-\rho_s), \quad 0 \leq t < s \leq T. \tag{1.2}$$

All other forms of time consistency for monetary risk measures, such as weak, acceptance consistent, rejection consistent, are tied to this connection as well. In [58], the author proposed a general approach to time consistency for cash-additive risk measures by introducing so called 'test sets' or 'benchmark sets.' Each form of time consistency was associated to a benchmark set of random variables, and larger benchmark sets correspond to stronger forms of time consistency.

The first study of time consistency of scale invariance measures is due to Bielecki et al. [13], where the authors elevated the theory of coherent acceptability indices to dynamic setup in discrete time. It was pointed out that none of the forms of time consistency for risk measures is suitable for scale invariant maps. Recursive property similar to (1.2) or benchmark sets approach essentially can not be applied to scale invariant maps. Consequently, one of the main challenge was to find an appropriate form of time consistency of acceptability indices, that would be both financially reasonable and mathematically tractable. For the case of random variables (terminal cashflows), the proposed form of time consistency for a dynamic coherent acceptability index α reads as follows: for any $\mathcal{F}_t$-measurable random variables $m_t,\ n_t$, and any $t < T$, the following implications hold

$$\begin{aligned} \alpha_{t+1}(X) \geq m_t \quad &\Rightarrow \quad \alpha_t(X) \geq m_t, \\ \alpha_{t+1}(X) \leq n_t \quad &\Rightarrow \quad \alpha_t(X) \leq n_t. \end{aligned} \tag{1.3}$$

The financial interpretation is also clear – if tomorrow X is acceptable at least at level m_t, then today X is also acceptable at least at level m_t; similar interpretation holds true for the second part (1.3). It is fair to say, we think, that dynamic acceptability indices and their time consistency properties play a critical role in so called conic approach to valuation and hedging of financial contracts [12, 10, 55].

We recall that both risk measures and performance measures, in the nutshell, put preferences on the set of cashflows. While the corresponding forms of time consistency (1.1) and (1.3) for these classes of maps, as argued above, are different, we note that generally speaking both forms of time consistency are linking preferences between different times. The aim of this paper is to present a unified and flexible framework for time consistency of risk and performance measures, that integrates existing forms of time consistency as well as various connections between them. We consider a (large) class of maps that are postulated to satisfy only two properties - monotonicity and locality[2] - and we study time consistency of such maps. These two properties, in our opinion, have to be satisfied by any reasonable dynamic risk or performance measure. We introduce the notion of an update rule that is meant to link preferences between different times.[3] The time consistency is defined in terms of an update rule. We provide various update rules that allow to recover several known forms of time consistency, and also such that allow to study some new forms of time consistency. When appropriate, for each form of time consistency we consider separately the case of terminal cashflows, referred in this paper as the case of random variables, and the case of dividend

[2]See Section 2 for rigorous definitions along with a detailed discussion of each property.

[3]In needs to be stressed that our notion of the update rule is different from the notion of update rule used in [58].

streams, referred to as the case of stochastic processes. For each type of time consistency we provide different equivalent formulations along with a discussion regarding financial interpretation and suitability of each rule. We also provide a comprehensive analysis of the connections between considered forms of time consistency, followed by a set of examples that illustrate diverse forms of time consistency. We should note that this paper is the first step that we made towards a unified theory of time consistency of risk/performance measures, and some relevant questions such as robust or dual representations are beyond the scope of this work and they will be addressed in the future.

The paper is organized as follows. In Section 2 we introduce some necessary notations, and we provide discussion of extension of the notion of conditional expectation and of conditional essential infimum/supremum to the case of random variables that take values in $[-\infty, \infty]$. Also here we introduce all other basic notions used throughout the paper, and we present the main object of our study – the Dynamic LM-measure. In Section 3 we set forth the main concepts of the paper – the notion of an updated rule and the definition of time consistency of a dynamic LM-measure. We prove a general result about time consistency, that can be viewed as counterpart of dynamic programming principle (1.2), and that is used conveniently in the sequel. Section 3 is devoted to various types of time consistency. Each type of time consistency is discussed in a separate subsection, within which, whenever relevant, we consider, respectively, the case of random variables or the case of random processes. We start with the weakest form of time consistency – the weak time consistency, and we conclude with the notion of super/submartingale time consistency. We present some fundamental properties for each type of time consistency, and we establish some relationships between them. A number of examples are presented in Section 5. Concluding remarks are summarized in Section 6, where we also provide convenient flowcharts depicting relationships between various forms of time consistency. To ease the exposition of the main concepts, all technical proofs are deferred to the Appendix, unless stated otherwise directly below the theorem or proposition.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{T}}, P)$ be a filtered probability space, with $\mathcal{F}_0 = \{\Omega, \emptyset\}$, and $\mathbb{T} = \{0, 1, \dots, T\}$, for fixed and finite time horizon $T \in \mathbb{N}$.[4]

For $\mathcal{G} \subseteq \mathcal{F}$ we denote by $L^0(\Omega, \mathcal{G}, P)$, and $\bar{L}^0(\Omega, \mathcal{G}, P)$ the sets of all $\mathcal{G}$-measurable random variables with values in $(-\infty, \infty)$, and $[-\infty, \infty]$, respectively. In addition, we will use the notation $L^p(\mathcal{G}) := L^p(\Omega, \mathcal{G}, P)$ $L^p_t := L^p(\mathcal{F}_t)$, and $L^p := L^p_T$, for $p \in \{0, 1, \infty\}$. Analogous definitions will apply to $\bar{L}^0$. We will also use notation $\mathbb{V}^p := \{(V_t)_{t\in\mathbb{T}} : V_t \in L^p_t\}$, for $p \in \{0, 1, \infty\}$.

Throughout this paper, $\mathcal{X}$ will denote either the space of random variables L^p, or the space of adapted processes $\mathbb{V}^p$, for $p \in \{0, 1, \infty\}$. If $\mathcal{X} = L^p$, for $p \in \{0, 1, \infty\}$, then the elements $X \in \mathcal{X}$ are interpreted as discounted terminal cash-flows. On the other hand, if $\mathcal{X} = \mathbb{V}^p$, for $p \in \{0, 1, \infty\}$, then the elements of $\mathcal{X}$, are interpreted as discounted dividend processes. It needs to be remarked, that all concepts developed for $\mathcal{X} = \mathbb{V}^p$ can be easily adapted to the case of cumulative discounted value processes. The case of random variables can be viewed as a particular case of stochastic processes by considering cash-flows with only the terminal payoff, i.e. stochastic processes such that $V = (0, \dots, 0, V_T)$. Nevertheless, we treat this case separately for transparency. For both

[4]For most of the results hold true or can be adjusted respectively, to the case of infinite time horizon. For sake of brevity, we will omit the discussion of this case here.

cases we will consider standard pointwise order, understood in the almost sure sense. In what follows, we will also make use of the multiplication operator denoted as $\cdot_t$ and defined by:

$$
\begin{aligned}
&m \cdot_t V := (V_0, \dots, V_{t-1}, mV_t, mV_{t+1}, \dots),\\
&m \cdot_t X := mX,
\end{aligned}
\tag{2.1}
$$

for $V \in \{(V_t)_{t\in\mathbb{T}} \mid V_t \in L^0_t\}$, $X \in L^0$ and $m \in L^\infty_t$. In order to ease the notation, if no confusion arises, we will drop $\cdot_t$ from the above product, and we will simply write mV and mX instead of $m \cdot_t V$ and $m \cdot_t X$, respectively.

Remark 2.1. We note that the space $\mathbb{V}^p$, $p \in \{0, 1, \infty\}$, endowed with multiplication $(\cdot_t,)$ does not define a proper L^0–module [34] (e.g. $0 \cdot_t V = 0$ for any $V \in \mathbb{V}^p$). However, in what follows, we will adopt some concepts from L^0-module theory which naturally fit into our study. Moreover, as in many cases we consider, if one additionally assume *independence of the past*, and replaces $V_0, \dots, V_{t-1}$ with 0s in (2.1), then $\mathcal{X}$ becomes an L^0–module. We refer the reader to [9, 11] for a thorough discussion on this matter.

We will also make use of stochastic process $1_{\{t\}} \in \mathbb{V}^p$ defined by

$$
1_{\{t\}} = (\underbrace{0, 0, \dots, 0}_{t}, 1, 0, 0, \dots), \quad t \in \mathbb{T}.
$$

Throughout, we will use the convention that $\infty - \infty = -\infty$ and $0 \cdot \pm\infty = 0$. Note that the distributive does not hold true in general: $(-1)(\infty - \infty) = \infty \neq -\infty + \infty = -\infty$.

For $t \in \mathbb{T}$ and $X \in \bar{L}^0$ we define the (generalized) $\mathcal{F}_t$-conditional expectation of X by

$$
E[X|\mathcal{F}_t] := \lim_{n\to\infty} E[(X^+ \wedge n)|\mathcal{F}_t] - \lim_{n\to\infty} E[(X^- \wedge n)|\mathcal{F}_t],
$$

where $X^+ = (X \vee 0)$ and $X^- = (-X \vee 0)$.

Next we will present some elementary properties of the generalized expectation.

Proposition 2.2. *For any $X, Y \in \bar{L}^0$ and $s, t \in \mathbb{T}$, $s > t$ we get*

1) $E[\lambda X|\mathcal{F}_t] \leq \lambda E[X|\mathcal{F}_t]$ for $\lambda \in L^0_t$, and $E[\lambda X|\mathcal{F}_t] = \lambda E[X|\mathcal{F}_t]$ for $\lambda \in L^0_t$, $\lambda \geq 0$;

2) $E[X|\mathcal{F}_t] \leq E[E[X|\mathcal{F}_s]|\mathcal{F}_t]$, and $E[X|\mathcal{F}_t] = E[E[X|\mathcal{F}_s]|\mathcal{F}_t]$ for $X \geq 0$;

3) $E[X|\mathcal{F}_t] + E[Y|\mathcal{F}_t] \leq E[X + Y|\mathcal{F}_t]$, and $E[X|\mathcal{F}_t] + E[Y|\mathcal{F}_t] = E[X + Y|\mathcal{F}_t]$ if $X, Y \geq 0$;

Remark 2.3. All inequalities in Proposition 2.2 can be strict. Assume that $t = 0$ and $k, s \in \mathbb{T}$, $k > s > 0$, and let $\xi \in L^0_k$ be such that $\xi = \pm 1$, ξ is independent of $\mathcal{F}_s$, and $P(\xi = 1) = P(\xi = -1) = 1/2$. We consider $Z \in L^0_s$ such that $Z \geq 0$, and $E[Z] = \infty$. By taking $\lambda = -1$, $X = \xi Z$ and $Y = -X$, we get strict inequalities in 1), 2) and 3).

For $X \in L^\infty$ and $t \in \mathbb{T}$, we will denote by $\operatorname{ess\,inf}_t X$ a unique (up to a set of measure zero), $\mathcal{F}_t$-measurable random variable, such that for any $A \in \mathcal{F}_t$, the following equality holds true

$$
\operatorname*{ess\,inf}_{\omega\in A} X = \operatorname*{ess\,inf}_{\omega\in A} (\operatorname{ess\,inf}_t X).
\tag{2.2}
$$

We will call this random variable the *$\mathcal{F}_t$-conditional essential infimum of X*. We refer the reader to [5] for a detailed proof of the existence and uniqueness of the conditional essential infimum. We will call $\operatorname{ess\,sup}_t(X) := -\operatorname{ess\,inf}_t(-X)$ the $\mathcal{F}_t$-*conditional essential supremum of* $X \in L^\infty$. Through standard limit procedure, we extend these to notions to the space $\bar{L}^0$. For any $t \in \mathbb{T}$ and $X \in \bar{L}^0$, we define the $\mathcal{F}_t$-conditional essential infimum by[5]

$$\operatorname{ess\,inf}_t X := \lim_{n\to\infty} \Big[\operatorname{ess\,inf}_t(X^+ \wedge n)\Big] - \lim_{n\to\infty} \Big[\operatorname{ess\,sup}_t(X^- \wedge n)\Big], \tag{2.3}$$

and respectively we put $\operatorname{ess\,sup}_t(X) := -\operatorname{ess\,inf}_t(-X)$.

For convenience, we present some fundamental properties of conditional essential infimum and supremum, for $\bar{L}^0$ setup, that will be used throughout the paper.

Proposition 2.4. *For any* $X, Y \in \bar{L}^0$, $s, t \in \mathbb{T}, s \geq t$, *and* $A \in \mathcal{F}_t$ *we have*

1) $\operatorname{ess\,inf}_{\omega\in A} X = \operatorname{ess\,inf}_{\omega\in A}(\operatorname{ess\,inf}_t X)$;

2) If $\operatorname{ess\,inf}_{\omega\in A} X = \operatorname{ess\,inf}_{\omega\in A} U$ *for some* $U \in \bar{L}^0_t$, *then* $U = \operatorname{ess\,inf}_t X$;

3) $X \geq \operatorname{ess\,inf}_t X$;

4) If $Z \in \bar{L}^0_t$, *is such that* $X \geq Z$, *then* $\operatorname{ess\,inf}_t X \geq Z$;

5) If $X \geq Y$, *then* $\operatorname{ess\,inf}_t X \geq \operatorname{ess\,inf}_t Y$;

6) $\mathbb{1}_A \operatorname{ess\,inf}_t X = \mathbb{1}_A \operatorname{ess\,inf}_t(\mathbb{1}_A X)$;

7) $\operatorname{ess\,inf}_s X \geq \operatorname{ess\,inf}_t X$;

The analogous results are true for $\{\operatorname{ess\,sup}_t\}_{t\in\mathbb{T}}$.

The proof for the case $X, Y \in L^\infty$ can be found in [5]. Since for any $n \in \mathbb{N}$ and $X, Y \in \bar{L}^0$ we get $X^+ \wedge n \in L^\infty$, $X^- \wedge n \in L^\infty$ and $X^+ \wedge X^- = 0$, the extension of the proof to the case $X, Y \in \bar{L}^0$ is straightforward, and we omit it here.

Remark 2.5. Similarly to [5], the conditional essential infimum $\operatorname{ess\,inf}_t(X)$ can be alternatively defined as the largest $\mathcal{F}_t$-measurable random variable, which is smaller than X, i.e. properties 3) and 4) from Proposition 2.4 are characteristic properties for conditional essential infimum.

Next, we define the generalized versions of ess inf and ess sup of a family (possible uncountable) of random variables: For $\{X_i\}_{i\in I}$, where $X_i \in \bar{L}^0$, we let

$$\operatorname*{ess\,inf}_{i\in I} X_i := \lim_{n\to\infty} \Big[\operatorname{ess\,inf}_{i\in I}(X_i^+ \wedge n)\Big] - \lim_{n\to\infty} \Big[\operatorname{ess\,sup}_{i\in I}(X_i^- \wedge n)\Big].$$

Note that, in view of [46, Appendix A], $\operatorname{ess\,inf}_{i\in I} X_i \wedge n$ and $\operatorname{ess\,sup}_{i\in I} X_i \wedge n$ are well defined, so that $\operatorname{ess\,inf}_{i\in I} X_i$ is well defined.

Furthermore, if for any $i, j \in I$, there exists $k \in I$, such that $X_k \leq X_i \wedge X_j$, then there exists a sequence $i_n \in I, n \in \mathbb{N}$, such that $\{X_{i_n}\}_{n\in\mathbb{N}}$ is nonincreasing and $\operatorname{ess\,inf}_{i\in I} X_i = \inf_{n\in\mathbb{N}} X_{i_n} = \lim_{n\to\infty} X_{i_n}$. Analogous results hold true for $\operatorname{ess\,sup}_{i\in I} X_i$.

We also recall here the notion of a determining family of sets (see for instance [24] for more details). Towards this end we first let

$$P_t := \{Z \in L^1 \mid Z \geq 0,\ E[Z|\mathcal{F}_t] = 1\}.$$

[5]Since both sequences $\operatorname{ess\,inf}_t(X^+ \wedge n)$ and $\operatorname{ess\,sup}_t(X^- \wedge n)$ are monotone, the corresponding limits exist.

Alternatively, one can view the elements $Z \in P_t$, as Radon-Nikodym derivatives of probability measures Q such that $Q \ll P$ and $Q|_{\mathcal{F}_t} = P|_{\mathcal{F}_t}$. We will call a family of sets $\{\mathcal{D}_t\}_{t\in\mathbb{T}}$, a *determining family* if for any $t \in \mathbb{T}$, $\mathcal{D}_t \subseteq P_t$, $\mathcal{D}_t \neq \emptyset$, and $\mathcal{D}_t$ is uniformly integrable, L^1-closed and $\mathcal{F}_t$-convex[6].

We will say that the map $f : \mathcal{X} \to \bar{L}^0$ is:

- *Normalized* if $f(0) = 0$;
- *Monotone* if for any $X, Y \in \mathcal{X}$, $X \leq Y \Rightarrow f(X) \leq f(Y)$;

Additionally, if $\mathcal{X} = L^p$, then for a fixed $t \in \mathbb{T}$, we will say that a map $f_t : \mathcal{X} \to \bar{L}^0_t$ is

- *Local* if $\mathbb{1}_A f_t(X) = \mathbb{1}_A f_t(\mathbb{1}_A X)$;
- *Cash additive* if $f_t(X + m) = f_t(X) + m$;
- *Quasi-concave* if $f_t(\lambda X + (1-\lambda)Y) \geq f_t(X) \wedge f_t(Y)$;
- *Scale invariant* if $f_t(\beta X) = f_t(X)$;

for any $A \in \mathcal{F}_t$, $X \in \mathcal{X}$, $m, \lambda, \beta \in L^p_t$, such that $0 \leq \lambda \leq 1$, $\beta > 0$.

On the other hand if $\mathcal{X} = \mathbb{V}^p$, for a fixed $t \in \mathbb{T}$, we say that a map $f_t : \mathcal{X} \to \bar{L}^0_t$ is

- *Local* if $\mathbb{1}_A f_t(V) = \mathbb{1}_A f_t(\mathbb{1}_A \cdot_t V)$;
- *Cash additive* if $f_t(V + m1_{\{s\}}) = f_t(V) + m$;
- *Quasi-concave* if $f_t(\lambda \cdot_t V + (1-\lambda) \cdot_t V') \geq f_t(V) \wedge f_t(V')$;
- *Scale invariant* if $f_t(\beta \cdot_t V) = f_t(V')$;
- *Translation invariant* if $f_t(V + m1_{\{t\}}) = f_t(V + m1_{\{s\}})$;
- *Independent of the past* if $f_t(V) = f_t(V - 0 \cdot_t V)$;

for any $s \in \mathbb{T}$, $A \in \mathcal{F}_t$, $V, V' \in \mathcal{X}$, $m, \lambda, \beta \in L^p_t$, such that $0 \leq \lambda \leq 1$, $\beta > 0$, $\|\beta\|_\infty < \infty$ and $s \geq t$.

Note that if $\mathcal{X} = L^p$, then, recalling that we can interpret random variables as stochastic processes (cf. Section 2), *translation invariance* and *independence of the past* are automatically satisfied; this is the very reason why we presented these notions only for the case of stochastic processes.

We will say that a family $\{f_t\}_{t\in\mathbb{T}}$ of maps $f_t : \mathcal{X} \to \bar{L}^0_t$ is monotone, local, cash additive, etc., if it has the corresponding property for any $t \in \mathbb{T}$. A family, indexed by $x \in \mathbb{R}_+$, of maps $\{f^x_t\}_{t\in\mathbb{T}}$, will be called *decreasing*, if $f^x_t(X) \leq f^y_t(X)$ for all $X \in \mathcal{X}$, $t \in \mathbb{T}$ and $x, y \in \mathbb{R}_+$, such that $x \geq y$.

Next, we introduce the main object of this study.

Definition 2.6. A family $\{f_t\}_{t\in\mathbb{T}}$ of maps $f_t : \mathcal{X} \to \bar{L}^0_t$ is a *Dynamic LM-measure* if $\{f_t\}_{t\in\mathbb{T}}$ is local and monotone.

[6] By $\mathcal{F}_t$-convex we mean that for any $Z_1, Z_2 \in \mathcal{D}_t$ and $\lambda \in L^0_t$ such that $0 \leq \lambda \leq 1$ we get $\lambda Z_1 + (1-\lambda) Z_2 \in \mathcal{D}_t$.

We believe that locality and monotonicity are two properties that must be satisfied by any reasonable dynamic measure of performance and/or measure of risk. Monotonicity property is natural for any numerical representation of an order between elements of $\mathcal{X}$. The locality property essentially means that the values of the LM-measure restricted to a set $A \in \mathcal{F}$ remain invariant with respect to the values of the arguments outside of the same set $A \in \mathcal{F}$; in particular, the events that will not happen in the future do not change the value of the measure today.

Throughout the paper we will use $\{f_t\}_{t\in\mathbb{T}}$ to denote (a general) family of maps $f_t : \mathcal{X} \to \bar{L}^0_t$, while $\{\varphi_t\}_{t\in\mathbb{T}}$ will be reserved for dynamic LM-measures. Also, it will be clear from the context which space $\mathcal{X}$ we have in mind.

Next, we recall several important subclasses of dynamic LM-measures. A family $\{\varphi_t\}_{t\in\mathbb{T}}$ of maps $\varphi_t : \mathcal{X} \to \bar{L}^0_t$ is a

- *Dynamic monetary risk measure* if $\{-\varphi_t\}_{t\in\mathbb{T}}$ is independent of the past, local, monotone, normalized and cash additive;
- *Dynamic convex risk measure* if $\{\varphi_t\}_{t\in\mathbb{T}}$ is a convex dynamic monetary risk measure;
- *Dynamic performance index* if $\{\varphi_t\}_{t\in\mathbb{T}}$ is independent of the past, translation invariant, local, monotone and scale invariant;
- *Acceptability Index* if $\{\varphi_t\}_{t\in\mathbb{T}}$ is a quasi-concave dynamic performance index.

All these classes of measures have been extensively studied in the literature over the past decade. Usually, it is postulated that a risk or performance measure is independent of the past – the current measurement of risk/performance of a cashflow only accounts for future payoffs. For a discussion about past dependent measures we refer to [9]. Convexity, concavity, quasi-convexity and quasi-concavity are related to the fact that diversification reduces the risk and increases the performance - diversification helps. Cash additivity is the key property that distinguishes risk measures from all other measures. This property means that adding \$$m$ to a portfolio today reduces the overall risk by the same amount m. From the regulatory perspective, the value of a risk measure is typically interpreted as the minimal capital requirement for a bank. For more details on coherent/covex/monearty risk measures we refer the reader to the survey papers [38, 1]. The distinctive property of performance measures is scale invariance - a rescaled portfolio or cashflow is accepted at the same level. Performance and acceptability indices were studied in [26, 13, 9, 18, 11], and they are meant to provide assessment of how good a financial position is. In particular, [18] gives examples of performance indices that are not acceptability indices. It needs to be noted that the theory developed in this paper can also be applied to sub-scale invariant dynamic performance indices studied in [55, 12].

3 Definition of time consistency

In this section we introduce the main concept of this paper - the time consistency of dynamic risk and performance measures, or more generally, the time consistency of dynamic LM-measures introduced in the previous section.

We recall that these dynamic LM-measures are defined on $\mathcal{X}$, where $\mathcal{X}$ either denotes the space L^p of random variables or the space $\mathbb{V}^p$ of stochastic processes, for $p \in \{0, 1, \infty\}$, so, our study of time consistency is done relative to such spaces. Nevertheless, the definition of time consistency can be easily adapted to more general spaces, such as Orlicz hearts (as studied in [20]) or topological L^0-modules (see for instance [9]). Usually, the need to consider spaces smaller than L^0 or $\mathbb{V}^0$ is motivated by the aim to obtain so called robust representation of such measures. For this, a certain topological structure is required (cf. Remark 4.17). On the other hand, 'time consistency' refers only to consistency of measurements in time, where no particular topological structure is needed, and thus most of the results obtained here hold true for $p = 0$.

Assume that $\{\varphi_t\}_{t\in\mathbb{T}}$ is a dynamic LM-measure on $\mathcal{X}$. For an arbitrary fixed $X \in \mathcal{X}$ and $u \in \mathbb{T}$ the value $\varphi_u(X)$ represents a quantification (measurement) of preferences about X at time u. Clearly, it is reasonable to require that any such quantification (measurement) methodology should be coherent as time passes. This is precisely the motivation behind the concepts of time consistency of dynamic LM-measures.

There are various forms of time consistency proposed in the literature, some of them suitable for one class of measures, other for a different class of measures, without a unified approach to fit them all. For example, for dynamic convex (or coherent) risk measures various version of time consistency surveyed in [1] can be seem as versions of the celebrated dynamic programming principle. On the other hand, as shown in [13], dynamic programming principle essentially is not suited for scale invariant measures such as dynamic acceptability indices, and the authors introduce a new type of time consistency tailored for these measures and provide a robust representation of them. Nevertheless, in all these cases the time consistency property connects, in a coherent way, the measurements at different times.

Next, we will introduce the notion of update rule that serves as the main tool in relating the measurements of preferences at different times, and also, it is the main building block of our unified theory of time consistency property.

Definition 3.1. We will call a family $\mu = \{\mu_{t,s}\}_{s>t}$, $s, t \in \mathbb{T}$, of maps $\mu_{t,s} : \bar{L}^0_s \times \mathcal{X} \to \bar{L}^0_t$ an *update rule* if for any $s > t$, the map $\mu_{t,s}$ satisfies the following conditions:

1) (Locality) $\mathbb{1}_A \mu_{t,s}(m, X) = \mathbb{1}_A \mu_{t,s}(\mathbb{1}_A m, X)$;

2) (Monotonicity) if $m \geq m'$, then $\mu_{t,s}(m, X) \geq \mu_{t,s}(m', X)$;

for any $X \in \mathcal{X}$, $A \in \mathcal{F}_t$ and $m, m' \in \bar{L}^0_s$.

We are now ready to introduce the general definition of time consistency.

Definition 3.2.[7] Let μ be an update rule. We will say that the dynamic LM-measure $\{\varphi_t\}_{t\in\mathbb{T}}$ is *μ-acceptance time consistent* if

$$\varphi_s(X) \geq m_s \quad \Longrightarrow \quad \varphi_t(X) \geq \mu_{t,s}(m_s, X), \tag{3.1}$$

for all $s, t \in \mathbb{T}$, $s > t$, $X \in \mathcal{X}$ and $m_s \in \bar{L}^0_s$. Respectively, we will say that $\{\varphi_t\}_{t\in\mathbb{T}}$ is *μ-rejection time consistent* if

$$\varphi_s(X) \leq m_s \quad \Longrightarrow \quad \varphi_t(X) \leq \mu_{t,s}(m_s, X), \tag{3.2}$$

[7] We introduce the concept of time consistency only for LM-measures, as this is the only class of measures used in this paper. However, the definition itself is suitable for any map acting from $\mathcal{X}$ to $\bar{L}^0$. For example, traditionally in the literature, the time consistency is defined for dynamic risk measures (negatives of LM-measures), and the above definition of time consistency will be appropriate, although one has to flip 'acceptance' with 'rejection'.

for all $s, t \in \mathbb{T}$, $s > t$, $X \in \mathcal{X}$ and $m_s \in \bar{L}^0_s$. If properties (3.1) and (3.2) are satisfied only for $s, t \in \mathbb{T}$, such that $s = t + 1$, then we will say that $\{f_t\}_{t\in\mathbb{T}}$ is *one step μ-acceptance time consistent* and *one step μ-rejection time consistent*, respectively.

Since LM-measures are local and monotone, properties with clear financial interpretations, the update rules are naturally assumed to be local and monotone too.

We see that the first argument $m \in \bar{L}^0_s$ in $\mu_{t,s}$ serves as a benchmark to which the measurement $\varphi_s(X)$ is compared. The presence of the second argument, $X \in \mathcal{X}$, in $\mu_{t,s}$, allows the update rule to depend on the objects (the Xs), which the preferences are applied to. However, as we will see in next section, there are natural situations when the update rules are independent of $X \in \mathcal{X}$, and sometimes they do not even depend on the future times $s \in \mathbb{T}$.

Next, we define several particular classes of update rules, suited for our needs.

Definition 3.3. Let μ be an update rule. We will say that μ is:

1) *X-invariant*, if $\mu_{t,s}(m, X) = \mu_{t,s}(m, 0)$;

2) *sX-invariant*, if there exists a family $\{\mu_t\}_{t\in\mathbb{T}}$ of maps $\mu_t : \bar{L}^0 \to \bar{L}^0_t$, such that $\mu_{t,s}(m, X) = \mu_t(m)$;

3) *Projective*, if it is sX-invariant and $\mu_t(m_t) = m_t$;

for any $s, t \in \mathbb{T}$, $s > t$, $X \in \mathcal{X}$, $m \in \bar{L}^0_s$ and $m_t \in \bar{L}^0_t$.

Remark 3.4. If an update rule $\mu = \{\mu_{t,s}\}_{s>t}$ is sX-invariant, then it is enough to consider only the corresponding family $\{\mu_t\}_{t\in\mathbb{T}}$. Hence, with slight abuse of notation we will write $\mu = \{\mu_t\}_{t\in\mathbb{T}}$ and call it an update rule as well.

The financial interpretation of acceptance time consistency is straightforward: if $X \in \mathcal{X}$ is accepted at some future time $s \in \mathbb{T}$, at least at level m, then today, at time $t \in \mathbb{T}$, it is accepted at least at level $\mu_{t,s}(m, X)$. Similarly for rejection time consistency. Essentially, the update rule μ translates the preference levels at time s to preference levels at time t. As it turns out, this simple and intuitive definition of time consistency, with appropriately chosen μ, will cover various cases of time consistency for risk and performance measures that can be found in the existing literature. Moreover, it will allow us to establish some fundamental properties of the LM-measures and some important connections between different versions of time consistency.

Next, we will give an equivalent formulation of time consistency. While the proof of the equivalence is simple, the result itself will be conveniently used in the sequel. Moreover, it can be viewed as a counterpart of dynamic programming principle, which is an equivalent formulation of dynamic consistency for convex risk measures.

Proposition 3.5. *Let μ be an update rule and let $\{\varphi_t\}_t$ be an LM-measure. Then,*

1) $\{\varphi_t\}_{t\in\mathbb{T}}$ is μ-acceptance time consistent if and only if

$$\varphi_t(X) \geq \mu_{t,s}(\varphi_s(X), X), \tag{3.3}$$

for any $X \in \mathcal{X}$ and $s, t \in \mathbb{T}$, such that $s > t$.

2) $\{\varphi_t\}_{t\in\mathbb{T}}$ is μ-rejection time consistent if and only if

$$\varphi_t(X) \leq \mu_{t,s}(\varphi_s(X), X), \tag{3.4}$$

for any $X \in \mathcal{X}$ and $s, t \in \mathbb{T}$, such that $s > t$.

The financial interpretation of (3.3) is similar to that of (3.1): if in the future, at time s, we accept the cash-flow X at level $\varphi_s(X)$, then today, at time t, we should accept the same cash-flow at least at level $\mu_{t,s}(\varphi_t(X),X)$ – the update of the acceptance level of X from time s to time t. Analogous interpretation applies to rejection time consistency.

Remark 3.6. It is clear, and also naturally desired, that a monotone transformation of an LM-measure will not change the preference order of the underlying elements. We want to emphasize that a monotone transformation will also preserve the time consistency. In other words, the preference orders will be also preserved in time. Indeed, if $\{\varphi_t\}_{t\in\mathbb{T}}$ is μ-acceptance time consistent, and $g:\bar{\mathbb{R}}\to\bar{\mathbb{R}}$ is a strictly monotone function, then the family $\{g\circ\varphi_t\}_{t\in\mathbb{T}}$ is $\widetilde{\mu}$-acceptance time consistent, where the update rule $\widetilde{\mu}$ is defined by $\widetilde{\mu}_{t,s}(m,X)=g(\mu_{t,s}(g^{-1}(m),X))$, for $t,s\in\mathbb{T}$, $s>t$, $X\in\mathcal{X}$ and $m\in\bar{L}^0_s$.

4 Selected types of time consistency

In this section we will analyze various types of time consistency, including some of those that have been studied in the literature, using the framework developed earlier in this paper.

If $\mathcal{X}=L^p$, for $p\in\{0,1,\infty\}$, then the elements $X\in\mathcal{X}$ are interpreted as discounted terminal cash-flows. On the other hand, if $\mathcal{X}=\mathbb{V}^p$, for $p\in\{0,1,\infty\}$, then the elements of $\mathcal{X}$, are interpreted as discounted dividend processes. It needs to be remarked, that all concepts developed for $\mathcal{X}=\mathbb{V}^p$ can be easily adapted to the case of cumulative discounted value processes (cf. Example 5.8).

While we preserve the same name for time consistency for both random variables and stochastic processes, the update rules may differ significantly. Usually, the case of stochastic processes is more intricate. If φ is an LM-measure, and $V\in\mathbb{V}^p$, then in order to compare $\varphi_t(V)$ and $\varphi_s(V)$, for $s>t$, one also needs to take into account the cash-flows between times t and s.

Before moving to the concrete definitions of time consistency, we will give some general remarks about relationship between time consistency for random variables and time consistency for random processes.

In what follows, for the case of random variables, $\mathcal{X}=L^p$, we we will only consider update rules that are X-invariant. Hence, as it will be clear later, the case of random variables can be viewed as a particular case of stochastic processes by considering cash-flows with only the terminal payoff, i.e. stochastic processes such that $V=1_{\{T\}}V_T$. Nevertheless, we treat this case separately for transparency.

In the present work, in the case of stochastic processes, we will focus on one step update rules, such that

$$\mu_{t,t+1}(m,V)=\mu_{t,t+1}(m,0)+f(V_t), \tag{4.1}$$

where $f:\bar{\mathbb{R}}\to\bar{\mathbb{R}}$ is a Borel measurable function, such that $f(0)=0$. We do this primarily to allow for a direct link between our results and the existing literature. We note, that any such one step update rule μ can be easily adapted to the case of random variables. Indeed, upon setting $\widetilde{\mu}_{t,t+1}(m):=\mu_{t,t+1}(m,0)$ we get a one step X-invariant update rule $\widetilde{\mu}$, which is suitable for random variables. Moreover, $\widetilde{\mu}$ will define the corresponding type of one step time consistency for random variables. Of course, this correspondence between update rule for processes and random variables is valid only for 'one step' setup.

Finally, we note that for update rules, which admit the so called nested composition property (cf.

[57, 56] and references therein),

$$\mu_{t,s}(m,V) = \mu_{t,t+1}(\mu_{t+1,t+2}(\dots \mu_{s-2,s-1}(\mu_{s-1,s}(m,V),V)\dots V),V), \tag{4.2}$$

we have that μ-acceptance (resp. μ-rejection) time consistency is equivalent to one step μ-acceptance (resp. μ-rejection) time consistency.

This is another reason why we consider only one step update rules for stochastic processes, however one can consider more exotic forms of time consistency, within proposed framework, and derive numerous properties and relationships between them, a task that we will leave for further studies.

We will now proceed with our discussion of various types of time consistency. Usually, whenever relevant, we consider in separate subsections the case of random variables and the case random processes. We start with the weakest form of time consistency - the weak time consistency, and we conclude with the notion super/submartingale time consistency.

4.1 Weak time consistency

The notion of weak time consistency was introduced in [58], and subsequently studied in [1, 4, 23, 32, 2, 23]. The idea is that if 'tomorrow', say at time s, we accept $X \in \mathcal{X}$ at level $m_s \in \mathcal{F}_s$, then 'today', say at time t, we would accept X at least at any level smaller or equal than m_s, adjusted by the information $\mathcal{F}_t$ available at time t (cf. (4.5)). Similarly, if tomorrow we reject X at level smaller than $m_s \in \mathcal{F}_s$, then today, we should also reject X at any level bigger than m_s, adapted to the flow of information $\mathcal{F}_t$. This suggests that the update rules should be taken as $\mathcal{F}_t$-conditional essential infimum and supremum, respectively. First, we will show that $\mathcal{F}_t$-conditional essential infimum and supremum are projective update rules.

Proposition 4.1. *The family $\mu^{\inf} := \{\mu_t^{\inf}\}_{t\in\mathbb{T}}$ of maps $\mu_t^{\inf} : \bar{L}^0 \to \bar{L}_t^0$ given by $\mu_t^{\inf}(m) = \operatorname{ess\,inf}_t m$, is a projective*[8] *update rule. Moreover,*

$$\mu_t^{\inf}(m) = \operatorname*{ess\,inf}_{Z\in P_t} E[Zm|\mathcal{F}_t]. \tag{4.3}$$

Similar result is true for family $\mu^{\sup} := \{\mu_t^{\sup}\}_{t\in\mathbb{T}}$, defined by $\mu_t^{\sup}(m) = \operatorname{ess\,sup}_t m$.

4.1.1 Random variables

Recall that the case of random variables corresponds to $\mathcal{X} = L^p$, for a fixed $p \in \{0, 1, \infty\}$. We proceed with the definition of weak acceptance and weak rejection time consistency (for random variables).

Definition 4.2. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure. Then φ is said to be

- *Weakly acceptance time consistent* if it is $\mu^{\inf}$-acceptance time consistent,
- *Weakly rejection time consistent*, if it is $\mu^{\sup}$-rejection time consistent.

[8] See Remark 3.4 for the comment about notation.

Definition 4.2 of time consistency is equivalent to many forms of time consistency studied in the current literature. Usually, the weak time consistency is considered for dynamic monetary risk measures on L^∞ (cf. [1] and references therein), to which we refer to as 'classical weak time consistency.' It was proved in [1] that in the classical weak time consistency framework, weak acceptance (respectively weak rejection) time consistency is equivalent to the statement that for any $X \in \mathcal{X}$ and $s > t$, we get

$$\varphi_s(X) \geq 0 \Rightarrow \varphi_t(X) \geq 0 \qquad \text{(resp. } \leq\text{)}. \tag{4.4}$$

This was the very starting point for our definition of weak acceptance (respectively weak rejection) time consistency, and the next proposition explains why so.

Proposition 4.3. *Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure. The following conditions are equivalent*

1) φ is weakly acceptance time consistent, i.e. for any $X \in \mathcal{X}$, $t, s \in \mathbb{T}$, $s > t$, and $m_s \in \bar{L}^0_s$,

$$\varphi_s(X) \geq m_s \Rightarrow \varphi_t(X) \geq \operatorname{ess\,inf}_t(m_s). \tag{4.5}$$

2) For any $X \in \mathcal{X}$, $s, t \in \mathbb{T}$, $s > t$, $\varphi_t(X) \geq \operatorname{ess\,inf}_t \varphi_s(X)$.

3) For any $X \in \mathcal{X}$, $s, t \in \mathbb{T}$, $s > t$, and $m_t \in \bar{L}^0_t$,

$$\varphi_s(X) \geq m_t \Rightarrow \varphi_t(X) \geq m_t.$$

If additionally $\{-\varphi_t\}_{t\in\mathbb{T}}$ is a dynamic monetary risk measure, then the above conditions are equivalent to

4) For any $X \in \mathcal{X}$ and $s, t \in \mathbb{T}$, $s > t$,

$$\varphi_s(X) \geq 0 \Rightarrow \varphi_t(X) \geq 0.$$

Similar result holds true for weak rejection time consistency.

Property 3) in Proposition 4.3 was also suggested as the notion of (weak) acceptance and (weak) rejection time consistency in the context of scale invariant measures, called acceptability indices (cf. [7, 13]).

As next result shows, the weak time consistency is indeed one of the weakest forms of time consistency, being implied by any time consistency generated by a projective rule.

Proposition 4.4. *Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure and let μ be a projective update rule. If $\{\varphi_t\}_{t\in\mathbb{T}}$ is μ-acceptance (resp. μ-rejection) time consistent, then $\{\varphi_t\}_{t\in\mathbb{T}}$ is weakly acceptance (resp. weakly rejection) time consistent.*

Remark 4.5. Recall that time consistency is preserved under monotone transformations, Remark 3.6. Thus, for any strictly monotone function $g : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$, if $\{\varphi_t\}_{t\in\mathbb{T}}$ is weakly acceptance (resp. weakly rejection) time consistent, then $\{g \circ \varphi_t\}_{t\in\mathbb{T}}$ also is weakly acceptance (resp. weakly rejection) time consistent.

4.1.2 Stochastic processes

In this subsection we assume that $\mathcal{X} = \mathbb{V}^p$, for a fixed $p \in \{0, 1, \infty\}$, i.e. we consider the case of adapted stochastic processes.

Definition 4.6. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure (for stochastic processes). We say that φ is

- *Weakly acceptance time consistent* if it is one step μ-acceptance time consistent, where the update rule is given by
$$\mu_{t,t+1}(m, V) = \operatorname{ess\,inf}_t(m) + V_t.$$
- *Weakly rejection time consistent*, if it is one step μ-acceptance time consistent, where
$$\mu_{t,t+1}(m, V) = \operatorname{ess\,sup}_t(m) + V_t.$$

Similarly to Proposition 4.3, we have the following result.

Proposition 4.7. *Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure on $\mathbb{V}^p$. The following conditions are equivalent*

1) φ is weakly acceptance time consistent, i.e. for any $V \in \mathcal{X}$, $t < T$ ($t \in \mathbb{T}$) and $m_{t+1} \in \bar{L}^0_{t+1}$,
$$\varphi_{t+1}(V) \geq m_{t+1} \Rightarrow \varphi_t(X) \geq \operatorname{ess\,inf}_t(m_{t+1}) + V_t.$$

2) For all $V \in \mathcal{X}$, $t \in \mathbb{T}$, $t < T$,
$$\varphi_t(V) \geq \operatorname{ess\,inf}_t(\varphi_{t+1}(V)) + V_t.$$

3) For all $V \in \mathcal{X}$, $t \in \mathbb{T}$, $t < T$ and $m_t \in \bar{L}^0_t$,
$$\varphi_{t+1}(V) \geq m_t \Rightarrow \varphi_t(V) - V_t \geq m_t.$$

If additionally $\{-\varphi_t\}_{t\in\mathbb{T}}$ is a dynamic monetary risk measure, then the above conditions are equivalent to

4) For all $V \in \mathcal{X}$ and $t \in \mathbb{T}$, $t < T$,
$$\varphi_{t+1}(V) \geq 0 \Rightarrow \varphi_t(V) - V_t \geq 0.$$

Analogous results hold true for weak rejection time consistency.

As mentioned earlier, the update rule, and consequently weak time consistency for stochastic processes, depends also on the value of the process (the dividend paid) at time t. If tomorrow, at time $t+1$, we accept $X \in \mathcal{X}$ at level greater than $m_{t+1} \in \mathcal{F}_{t+1}$, then today at time t, we will accept X at least at level $\operatorname{ess\,inf}_t m_{t+1}$ (i.e. the worst level of m_{t+1} adapted to the information $\mathcal{F}_t$) plus the dividend V_t received today.

Finally, we present the counterpart of Proposition 4.4 for the case of stochastic processes.

Proposition 4.8. *Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure on $\mathbb{V}^p$ and let ϕ be a projective update rule. Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be one step μ-acceptance (resp. one step μ-rejection) time consistent, where μ is given by*
$$\mu_{t,t+1}(m, V) = \phi_t(m + V_t), \quad m \in \bar{L}^0_{t+1},\ V \in \mathcal{X}.$$
Then, $\{\varphi_t\}_{t\in\mathbb{T}}$ is weakly acceptance (resp. weakly rejection) time consistent.

The proof of Proposition 4.8 is analogous to the proof of Proposition 4.4, and we omit it.

4.2 Semi-weak time consistency (for stochastic processes)

In this section we introduce the concept of semi-weak time consistency for stochastic processes. As it turns out, for the case of random variables semi-weak time consistency coincides with the definition of weak time consistency, hence omitted here. Thus, we take $\mathcal{X} = \mathbb{V}^p$, for a fixed $p \in \{0, 1, \infty\}$. As it was shown [13], none of the existing, at that time, forms of time consistency were suitable for scale-invariant maps, such as acceptability indices. In fact, even the weak acceptance and the weak rejection time consistency for stochastic processes (as defined in the present paper) are too strong in case of scale-invariant maps. Because of that we need even a weaker notion of time consistency, which we will refer to as semi-weak acceptance and semi-weak rejection time consistency. The notion of semi-weak time consistency for stochastic processes, introduced next, is suited precisely for such maps, and we refer the reader to [13] for a detailed discussion on time consistency for scale invariant measures and their dual representations[9].

Definition 4.9. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure (for processes). Then φ is said to be:

- *Semi-weakly acceptance time consistent* if it is one step μ-acceptance time consistent, where the update rule is given by

$$\mu_{t,t+1}(m, V) = 1_{\{V_t\geq 0\}}\mu_t^{\inf}(m) + 1_{\{V_t<0\}}(-\infty).$$

- *Semi-weakly rejection time consistent* if it is one step μ'-rejection time consistent, where the update rule is given by

$$\mu'_{t,t+1}(m, V) = 1_{\{V_t\leq 0\}}\mu_t^{\sup}(m) + 1_{\{V_t>0\}}(+\infty).$$

It is straightforward to check that weak acceptance/rejection time consistency for stochastic processes always implies semi-weak acceptance/rejection time consistency.

Next, we will show that the definition of semi-weak time consistency is indeed equivalent to time consistency introduced in [13], that was later studied in [7, 12].

Proposition 4.10. *Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure on $\mathbb{V}^p$. The following conditions are equivalent*

1) φ is semi-weakly acceptance time consistent, i.e. for all $V \in \mathcal{X}$, $t \in \mathbb{T}$, $t < T$, and $m_t \in \bar{L}^0_t$,

$$\varphi_{t+1}(V) \geq m_{t+1} \Rightarrow \varphi_t(V) \geq \mathbb{1}_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t(m_{t+1}) + 1_{\{V_t<0\}}(-\infty).$$

2) For all $V \in \mathcal{X}$ and $t \in \mathbb{T}$, $t < T$, $\varphi_t(V) \geq \mathbb{1}_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t(\varphi_{t+1}(V)) + 1_{\{V_t<0\}}(-\infty)$.

3) For all $V \in \mathcal{X}$, $t \in \mathbb{T}$, $t < T$, and $m_t \in \bar{L}^0_t$, such that $V_t \geq 0$ and $\varphi_{t+1}(V) \geq m_t$, then $\varphi_t(V) \geq m_t$.

Similar result is true for semi-weak rejection time consistency.

[9] In [13] the authors combined both semi-weak acceptance and rejection time consistency into one single definition and call it time consistency.

Property 3) in Proposition 4.10 illustrates best the financial meaning of semi-weak acceptance time consistency: if tomorrow we accept the dividend stream $V \in \mathcal{X}$ at level m_t, and if we get a positive dividend V_t paid today at time t, then today we accept the cash-flow V at least at level m_t as well. Similar interpretation is valid for semi-weak rejection time consistency.

The next two results will give an important (dual) connection between cash additive measures and scale invariant measures.

Proposition 4.11. *Let $\{\varphi_t^x\}_{t\in\mathbb{T}}$, $x \in \mathbb{R}_+$, be a decreasing family of dynamic LM-measures. Assume that for each $x \in \mathbb{R}_+$, $\{\varphi_t^x\}_{t\in\mathbb{T}}$ is weakly acceptance (resp. weakly rejection) time consistent. Then, the family $\{\alpha_t\}_{t\in\mathbb{T}}$ of maps $\alpha_t : \mathcal{X} \to \bar{L}_t^0$ defined by*

$$\alpha_t(V) = \sup\{x \in \mathbb{R}_+ : \varphi_t^x(V) \geq 0\}, \tag{4.6}$$

is a semi-weakly acceptance (resp. semi-weakly rejection) time consistent dynamic LM-measure.

Proposition 4.12. *Let $\{\alpha_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure, which is independent of the past and translation invariant. Assume that $\{\alpha_t\}_{t\in\mathbb{T}}$ is semi-weakly acceptance (resp. semi-weakly rejection) time consistent. Then, for any $x \in \mathbb{R}_+$, the family $\{\varphi_t^x\}_{t\in\mathbb{T}}$ defined by*

$$\varphi_t^x(V) = \inf\{c \in \mathbb{R} : \alpha_t(V - c1_{\{t\}}) \leq x\}, \tag{4.7}$$

is a weakly acceptance (resp. weakly rejection) time consistent dynamic LM-measure.

This type of dual representation, i.e. (4.6)–(4.7), first appeared in [26] where the authors studied static (one period of time) scale invariant measures. Subsequently, in [13], the authors extended these results to the case of stochastic processes with special emphasis on time consistency property. In contrast to [13], we consider an arbitrary probability space, not just a finite one.

4.3 Middle time consistency

Before we give the definition of middle acceptance/rejection time consistency, we need to introduce the concept of LM-extension of an LM-measure for random variables.

Definition 4.13. Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure (for random variables). We will call a family $\widehat{\varphi} = \{\widehat{\varphi}_t\}_{t\in\mathbb{T}}$ of maps $\widehat{\varphi}_t : \bar{L}^0 \to \bar{L}_t^0$ *an LM-extension of* $\{\varphi_t\}_{t\in\mathbb{T}}$, if for any $t \in \mathbb{T}$, $\widehat{\varphi}_t|_{\mathcal{X}} \equiv \varphi_t$, and $\widehat{\varphi}_t$ is local and monotone on $\bar{L}_0$.[10]

We will show below that such extensions exist, for which we will make use of the following auxiliary sets:

$$\mathcal{Y}_A^+(X) := \{Y \in \mathcal{X} \mid \ \mathbb{1}_A Y \geq \mathbb{1}_A X\}, \qquad \mathcal{Y}_A^-(X) := \{Y \in \mathcal{X} \mid \ \mathbb{1}_A Y \leq \mathbb{1}_A X\},$$

defined for any $X \in \bar{L}^0$ and $A \in \mathcal{F}$.

Definition 4.14. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure. The collection of functions $\varphi^+ = \{\varphi_t^+\}_{t\in\mathbb{T}}$, where $\varphi_t^+ : \bar{L}^0 \to \bar{L}_t^0$ is defined as[11]

$$\varphi_t^+(X) := \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}_A^+(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(+\infty)\Big], \tag{4.8}$$

[10] That is, it satisfies monotonicity and locality on $\bar{L}_0$, as in 5) and 6) in Proposition 2.4.

[11] We will use the convention $\operatorname{ess\,sup} \emptyset = -\infty$ and $\operatorname{ess\,inf} \emptyset = \infty$.

is called the *upper LM-extension of* φ.

Respectively, the collection of functions $\varphi^- = \{\varphi_t^-\}_{t\in\mathbb{T}}$, where $\varphi_t^- : \bar{L}^0 \to \bar{L}_t^0$, and

$$\varphi_t^-(X) := \operatorname*{ess\,sup}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_A \operatorname*{ess\,sup}_{Y\in\mathcal{Y}_A^-(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(-\infty) \Big], \tag{4.9}$$

is called the *lower LM-extension of* φ.

The next result shows that $\varphi^\pm$ are two 'extreme' extensions, and any other extension is sandwiched between them.

Proposition 4.15. *Let* $\{\varphi_t\}_{t\in\mathbb{T}}$ *be a dynamic LM-measure. Then,* φ^- *and* φ^+ *are LM-extensions of* φ*. Moreover, let* $\widehat{\varphi}$ *be an LM-extension of* φ*. Then, for any* $X \in \bar{L}^0$ *and* $t \in \mathbb{T}$ *we get*

$$\varphi_t^-(X) \leq \widehat{\varphi}_t(X) \leq \varphi_t^+(X). \tag{4.10}$$

Clearly, generally speaking the maps (4.8) and (4.9) are not equal, and thus the extensions of an LM-measure are not unique.

Remark 4.16. Let $t \in \mathbb{T}$ and $\mathcal{B} \subseteq \bar{L}^0$ be such that, for any $A \in \mathcal{F}_t$, $\mathbb{1}_A\mathcal{B} \subseteq \mathcal{B}$ and $\mathbb{1}_A\mathcal{B} + \mathbb{1}_{A^c}\mathcal{B} \subseteq \mathcal{B}$. As a generalization of Proposition 4.15, one can show that for any $\mathcal{F}_t$-local and monotone[12] mapping $f : \mathcal{B} \to \bar{L}_t^0$, the maps $f^\pm$ defined analogously as in (4.8) and (4.9) will be extensions of f to $\bar{L}^0$, preserving locality and monotonicity. We omit the detailed proof here.

Remark 4.17. For large classes of LM-measures, as mentioned earlier, there exists a 'robust representation' type theorem - essentially a representation, via convex duality, as a function of conditional expectation. We refer the reader to [9] and references therein, where the authors present a general result on robust representation for dynamic quasi-concave upper semi-continuous LM-measures. Hence, an alternative construction of extensions can be obtained through the robust representations of LM-measures, by considering conditional expectations defined on extended real line, etc.

Also, if an LM-measure admits some 'Lebesgue type of continuity', the extension can be traditionally obtained by

$$\bar{f}(X) := \liminf_{m\to\infty} \liminf_{n\to-\infty} f\Big(n \vee X \wedge m\Big), \quad m, n \in \mathbb{Z}. \tag{4.11}$$

Finally, we show that any extension of an LM–measure is an sX-invariant update rule, and we give necessary and sufficient conditions when this update rule is also projective.

Proposition 4.18. *Any LM-extension* $\widehat{\varphi}$ *of a dynamic LM–measure* φ *is an sX-invariant update rule. Moreover,* $\widehat{\varphi}$ *is projective if and only if* $\varphi_t(X) = X$*, for* $t \in \mathbb{T}$ *and* $X \in \mathcal{X} \cap \bar{L}_t^0$*.*

4.3.1 Random variables

Let us start with the definition of middle acceptance and middle rejection time consistency.

Definition 4.19. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure. Then φ is said to be

[12] That is, $\mathcal{F}_t$-local and monotone on $\mathcal{B}$.

- *Middle acceptance time consistent* if it is φ^{-}-acceptance time consistent.
- *Middle rejection time consistent*, if it is φ^{+}-rejection time consistent.

As in the case of weak time consistency, the notion of middle time consistency is usually presented for functions $\{-\varphi_t\}_{t\in\mathbb{T}}$ being dynamic monetary risk measures on L^∞ (cf. [1] and references therein). It is not difficult to prove (cf. [1]), that in L^∞ framework the middle acceptance (resp. middle rejection) time consistency is equivalent to the statement that

$$\varphi_t(X) \geq \varphi_t(\varphi_s(X)) \qquad (\text{resp. } \leq), \quad X \in \mathcal{X},\ s > t. \tag{4.12}$$

However, in case of a general domain of definition $\mathcal{X}$ of φ, we may have that $\varphi_s(X) \notin \mathcal{X}$ and, consequently, (4.12) cannot be used directly for time consistency. This is precisely the reason why we have introduced the LM-extensions. On the other hand, due to the fact that in Definition 4.19 the update rules are extensions, our concept of middle time consistency is stronger than the classical approach to middle time consistency, as shown in the next result.

Proposition 4.20. *Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure. The following two conditions are equivalent*

1) φ is middle acceptance time consistent, i.e. for any $X \in \mathcal{X}$, $s,t \in \mathbb{T}$, $s > t$, and $m_s \in \bar{L}^0_s$,

$$\varphi_s(X) \geq m_s \Rightarrow \varphi_t(X) \geq \varphi_t^{-}(m_s).$$

2) For any $X \in \mathcal{X}$, $s,t \in \mathbb{T}$, $s > t$,

$$\varphi_t(X) \geq \varphi_t^{-}(\varphi_s(X)).$$

If additionally $\{-\varphi_t\}_{t\in\mathbb{T}}$ is a dynamic monetary risk measure, then 1) or 2) implies

3) For any $X \in \mathcal{X}$, $s,t \in \mathbb{T}$, $s > t$, and $Y \in \mathcal{X} \cap \bar{L}^0_s$, we get

$$\varphi_s(X) \geq \varphi_s(Y) \Rightarrow \varphi_t(X) \geq \varphi_t(Y).$$

Analogous results are true for middle rejection time consistency,

The proof of the equivalence of 1) and 2) in Proposition 4.20 follows immediately from Proposition 3.5, and the proof that 1) implies 3) is straightforward upon taking $m_s = \varphi_s(Y)$.

Next, we will show that, in principle, middle acceptance time consistency is not suited for acceptability indices [13, 26].

Proposition 4.21. *Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure such that*

1) $\varphi_t(X) = \infty$, for any $t \in \mathbb{T}$ and $X \in \mathcal{X}$, such that $X \geq 0$ and $P[X > 0] > 0$;

2) there exists $X_0 \in \mathcal{X}$ and $t_1, t_2 \in \mathbb{T}$, $t_1 \neq t_2$, such that $0 < \varphi_{t_i}(X_0) < \infty$, for $i = 1, 2$.

Then, $\{\varphi_t\}_{t\in\mathbb{T}}$ is not middle acceptance time consistent.

Remark 4.22. Properties 1) and 2) in Proposition 4.21 are characteristic for acceptability indices: the first property is related to 'arbitrage consistency' proposed in [26]; the second property is a technical assumption that eliminates degenerate cases. Thus, the concept of middle acceptance time consistency, and therefore (as seen in next section) the concept of strong time consistency, is not proper for such maps.

Remark 4.23. In general, middle acceptance/rejection time consistency does not imply weak acceptance/rejection time consistency. Indeed, let us consider $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$, such that $\varphi_t(X) = t$ (resp. $\varphi_t(X) = -t$) for all $X \in L^0$. Since $\varphi_t(0) = t \not\geq \operatorname{ess\,inf}_t \varphi_s(0) = s$ (resp. $-t \not\leq -s$), for $s > t$, we conclude that φ is not weakly acceptance (resp. weakly rejection) time consistent. On the other hand $\varphi_t(X) = \varphi_t(\varphi_s(X))$ for any $X \in L^0$, and hence φ is both middle acceptance and middle rejection time consistent.

4.3.2 Stochastic processes

In this section we will adapt the middle time consistency to the case of stochastic processes, and we start with the definition of one step LM-extensions.

As before, for the case of stochastic processes we take $\mathcal{X} = \mathbb{V}^p$, for a fixed $p \in \{0, 1, \infty\}$. In what follows we will also make use of notation $\mathbb{T}' = \{0, 1, \dots, T-1\}$.

For a dynamic LM-measure $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$, we denote by $\widetilde{\varphi} = \{\widetilde{\varphi}_t\}_{t\in\mathbb{T}'}$ a family of maps $\widetilde{\varphi}_t : L^p_{t+1} \to \bar{L}^0_t$ given by

$$\widetilde{\varphi}_t(X) := \varphi_t(1_{\{t+1\}}X). \tag{4.13}$$

Since φ is monotone and local on $\mathbb{V}^p$, then, clearly, $\widetilde{\varphi}_t$ is local and monotone on L^p_{t+1}. Next, similar to the previous section, for any $t \in \mathbb{T}'$, we consider the extension of $\widetilde{\varphi}_t$ to $\bar{L}^0_{t+1}$, preserving locality and monotonicity (see Remark 4.16). Note that formally $\widetilde{\varphi}$ is not an LM-measure, since the domain of the definition depends on $t \in \mathbb{T}'$, however, with slight abuse of notation, we will call such extension *one step LM-extension* of $\widetilde{\varphi}$. For any $\widetilde{\varphi}_t$ and $t \in \mathbb{T}'$, we consider the maps $\widetilde{\varphi}^+_t : \bar{L}^0_{t+1} \to \bar{L}^0_t$ and $\widetilde{\varphi}^-_t : \bar{L}^0_{t+1} \to \bar{L}^0_t$ defined as in (4.8) and (4.9), with the sets $\mathcal{Y}^+_A(X)$ and $\mathcal{Y}^-_A(X)$ there replaced by

$$\mathcal{Y}^+_{t,A}(X) := \{Y \in L^p_{t+1} \mid \ \mathbb{1}_A Y \geq \mathbb{1}_A X\}, \qquad \mathcal{Y}^-_{t,A}(X) := \{Y \in L^p_{t+1} \mid \ \mathbb{1}_A Y \leq \mathbb{1}_A X\},$$

for any $X \in \bar{L}^0_{t+1}$, We will call $\widetilde{\varphi}^+$ and $\widetilde{\varphi}^-$ *upper and lower one step LM-extensions* of $\widetilde{\varphi}$, respectively. Now, we are ready to present the definition of middle acceptance and middle rejection time consistency for processes.

Definition 4.24. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure (for stochastic processes). Then φ is said to be

- *Middle acceptance time consistent* if it is one step μ-acceptance time consistent, where the update rule is given by
$$\mu_{t,t+1}(m, V) = \widetilde{\varphi}^-_t(m + V_t).$$
- *Middle rejection time consistent* if it is one step μ-rejection time consistent, where the update rule is given by
$$\mu_{t,t+1}(m, V) = \widetilde{\varphi}^+_t(m + V_t).$$

Proposition 4.25. *Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure on $\mathbb{V}^p$. The following conditions are equivalent*

1) φ is middle acceptance time consistent, i.e. for any $V \in \mathcal{X}$, $t \in \mathbb{T}'$ and $m_{t+1} \in \bar{L}^0_{t+1}$,

$$\varphi_{t+1}(V) \geq m_{t+1} \Rightarrow \varphi_t(V) \geq \widetilde{\varphi}_t^-(m_{t+1} + V_t).$$

2) For any $V \in \mathcal{X}$ and $t \in \mathbb{T}'$, $\varphi_t(V) \geq \widetilde{\varphi}_t^-(\varphi_{t+1}(V) + V_t)$.

If additionally $\{-\varphi_t\}_{t\in\mathbb{T}}$ is a dynamic monetary risk measure, then 1) or 2) implies

3) For any $V, V' \in \mathcal{X}$, and $t \in \mathbb{T}'$, we get

$$\varphi_{t+1}(V) \geq \varphi_{t+1}(1_{\{t+1\}}V'_{t+1}) \Rightarrow \varphi_t(V) \geq \widetilde{\varphi}_t^-(V'_{t+1} + V_t).$$

Analogous results are true for middle rejection time consistency.

The first part of Proposition 4.25 is a straightforward implication of Proposition 3.5. Since for cash additive measures $\varphi_{t+1}(1_{\{t+1\}}V'_{t+1}) = V'_{t+1}$, then, by taking $m_{t+1} = V'_{t+1}$ in 1), the second part follows immediately.

4.4 Strong time consistency

The strong version of time consistency was one of the first concepts of time consistency studied in the literature. This has been primarily done for dynamic coherent risk measures and subsequently for dynamic convex risk measures, and there exists an extensive literature on this subject (cf. [1, 4, 23, 32, 2, 23]. The key feature of strong time consistency is its relationship with the dynamic programming type principle [4]. The definition that we will propose here will be slightly stronger (see Proposition 4.27), but nevertheless, the main idea will remain the same.

4.4.1 Random variables

In this subsection we assume that $\mathcal{X} = L^p$, for a fixed $p \in \{0, 1, \infty\}$, i.e. we consider the case of random variables. Let us start with the definition of strong time consistency.

Definition 4.26. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure. Then φ is said to be *strongly time consistent* if there exists $\widehat{\varphi}$, LM-extension of φ, such that the family φ is both $\widehat{\varphi}$-acceptance and $\widehat{\varphi}$-rejection time consistent.

Using (4.12), we conclude if $\{-\varphi_t\}_{t\in\mathbb{T}}$ is dynamic monetary risk measure on L^∞ (see also [1] and references therein), then its strong time consistency property is equivalent to the following property

$$\varphi_t(X) = \varphi_t(\varphi_s(X)), \quad \text{for any } \; X \in \mathcal{X},\; s > t, \tag{4.14}$$

known as Bellman's principle or dynamic programming principle. As mentioned in previous section, once the LM-measure is defined on a larger space than L^∞, to make sense of dynamic programming principle, and thus to make sense of strong time consistency, one needs to work with proper extensions of φ. The next key result shows an alternative formulation for strong time consistency, that also has a clear financial interpretation.

Proposition 4.27. *Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure, so that for any $t \in \mathbb{T}$ there exists $X \in \mathcal{X}$ such that $\varphi_t(X) = 0$. The following conditions are equivalent:*

1) There exists an update rule μ, such that μ is X-invariant and the family φ is both μ-acceptance and μ-rejection time consistent.

2) For any $X, Y \in \mathcal{X}$, $s, t \in \mathbb{T}$, $s > t$,

$$\varphi_s(X) = \varphi_s(Y) \Rightarrow \varphi_t(X) = \varphi_t(Y).$$

In particular 1) and 2) are satisfied if one of the following (equivalent) conditions hold

3) φ is strongly time consistent.

4) There exists $\widehat{\varphi}$, LM-extension of φ, such that for any $X \in \mathcal{X}$, $s, t \in \mathbb{T}$, $s > t$ we get $\varphi_t(X) = \widehat{\varphi}_t(\varphi_s(X))$.

Remark 4.28. Property 2) in Proposition 4.27 is called, in the existing literature, the strong time consistency (for risk or monetary measures). Note that strong time consistency introduced in Definition 4.26 is stronger than property 2) in Proposition 4.27. In particular, the update rule considered in Definition 4.26 is sX-invariant (cf. Proposition 4.18), while property 2) guarantees existence of update rule, which is just X-invariant.

4.4.2 Stochastic processes

In the next definition we will use family $\widetilde{\varphi}$ defined in (4.13).

Definition 4.29. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure. Then φ is called *strongly time consistent* if there exists $\widehat{\varphi}$, a one step LM-extension of $\widetilde{\varphi}$, such that φ is both one step μ-acceptance and one step μ-rejection time consistent with respect to

$$\mu_{t,t+1}(m, V) = \widehat{\varphi}_t(m + V_t).$$

Proposition 4.30. *Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure on $\mathbb{V}^p$. Assume that φ is independent of the past, and that for any $t \in \mathbb{T}$, there exists $V \in \mathcal{X}$ such that $\varphi_t(V) = 0$. The following two conditions are equivalent:*

1) There exists an update rule μ, such that: for all $t \in \mathbb{T}'$, $m \in \bar{L}^0_t$, and $V, V' \in \mathcal{X}$, satisfying $V_t = V'_t$, we have $\mu_{t,t+1}(m, V) = \mu_{t,t+1}(m, V')$; the family φ is both one step μ-acceptance and one step μ-rejection time consistent.

2) For any $V, V' \in \mathcal{X}$, and $t \in \mathbb{T}'$,

$$V_t = V'_t \text{ and } \varphi_{t+1}(V) = \varphi_{t+1}(V') \quad \Rightarrow \quad \varphi_t(V) = \varphi_t(V').$$

In particular 1) and 2) are satisfied if one of the following (equivalent) conditions hold

3) φ is strongly time consistent.

4) There exists $\widehat{\varphi}$, one step LM-extension of $\widetilde{\varphi}$, such that for any $V \in \mathcal{X}$ and $t \in \mathbb{T}'$, we get $\varphi_t(V) = \widehat{\varphi}_t(\varphi_{t+1}(V) + V_t)$.

4.5 Submartingales, supermartingales and robust expectations

The definition of projective update rule is strictly connected to the definition of so called (conditional) non-linear expectation (see for instance [27] for definition and properties of non-linear expectation). In [54, 50], the authors made an important connections between non-linear expectations and dynamic risk measures. It was also shown (see, for instance, [17, 16] for details) that among dynamic convex risk measures, the dynamic coherent risk measures are the only ones which satisfy Jensen's inequality for dynamic maps, a property critically important in our framework, as it leads to projective update rules for which time consistency is invariant under concave transformations (see Proposition 4.32). One particularly important case is obtained by using as an update rule the standard expectation operator. Finally, we want to mention that this type of time consistency in L^∞ framework, was studied in [32, Section 5] and is related to the definition of supermartingale and submartingale property.

4.5.1 Random variables

Definition 4.31. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure and let $\mu = \{\mu_t\}_{t\in\mathbb{T}}$ be given by $\mu_t(m) = E[m|\mathcal{F}_t]$ (for $m \in \bar{L}^0$). Then φ is said to be

- *Supermartingale time consistent* if it is μ-acceptance time consistent, i.e. for any $X \in \mathcal{X}$, and $m_s \in \mathcal{F}_s$, we have
$$\varphi_s(X) \geq m_s \;\Rightarrow\; \varphi_t(X) \geq E[m_s|\mathcal{F}_t].$$
- *Submartingale time consistent* if it is μ-rejection time consistent, i.e. for any $X \in \mathcal{X}$, and $m_s \in \mathcal{F}_s$, we have
$$\varphi_s(X) \leq m_s \;\Rightarrow\; \varphi_t(X) \leq E[m_s|\mathcal{F}_t].$$

Next result is devoted to a more general class of updates rules, and hence concepts of time consistency, for which we do not give a specific name. The case of super/sub-martingale time consistency will correspond to the particular case of determining sets $D_t = \{1\}$.

Proposition 4.32. *Let $\{D_t\}_{t\in\mathbb{T}}$ be a determining family of sets, and let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure. Consider the family of maps $\phi = \{\phi_t\}_{t\in\mathbb{T}}$ and $\phi' = \{\phi'_t\}_{t\in\mathbb{T}}$, $\phi_t, \phi'_t : \bar{L}^0 \to \bar{L}^0_t$, given by the following robust expectations*[13]

$$\phi_t(m) = \operatorname*{ess\,inf}_{Z\in D_t} E[Zm|\mathcal{F}_t], \qquad \phi'_t(m) = \operatorname*{ess\,sup}_{Z\in D_t} E[Zm|\mathcal{F}_t]. \tag{4.15}$$

Then, the following statements hold true:

1) the families ϕ and ϕ' are projective update rules;

2) if $\{\varphi_t\}_{t\in\mathbb{T}}$ is ϕ-acceptance time consistent, then $\{g\circ\varphi_t\}_{t\in\mathbb{T}}$ is also ϕ-acceptance time consistent, for any increasing, and concave function $g : \bar{\mathbb{R}} \to \mathbb{R}$.

3) if $\{\varphi_t\}_{t\in\mathbb{T}}$ is ϕ'-rejection time consistent, then $\{g\circ\varphi_t\}_{t\in\mathbb{T}}$ is also ϕ'-rejection time consistent, for any increasing, and convex function $g : \bar{\mathbb{R}} \to \mathbb{R}$.

[13] The term robust is inspired by robust representations of risk measures.

Remark 4.33. Note that since any determining family of sets $\mathcal{D}$ is subset of $\mathcal{P}$ (see definition of determining sets in Section 2), we have that $\phi_t(m) \geq \operatorname{ess\,inf}_{Z\in\mathcal{P}_t} E[m|\mathcal{F}_t]$. Thus, an LM-measure that is acceptance time consistent with respect to the update rule ϕ_t is also weakly acceptance time consistent. In particular, any supermartingale consistent LM-measure is also weakly acceptance time consistent. Similar statement holds true for rejection consistency.

4.5.2 Stochastic Processes

The sub/super-martingale time consistency is defined similarly, by considering one step update rules of the form $\mu_{t,t+1}(m, V) = E[m|\mathcal{F}_t] + V_t$. Similar to Proposition 4.32, we have that time consistency property generated by updates rules of the form $\mu_{t,t+1}(m, V) = \phi_t(m+V_t)$ are invariant under concave/convex transformations.

Proposition 4.34. *Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure (for processes). Let a one step update rule $\mu = \{\mu_t\}_{t\in\mathbb{T}}$ be given by $\mu_{t,t+1}(m, V) = \phi_t(m+V_t)$, for $\{\phi_t\}_{t\in\mathbb{T}}$ defined in (4.15). Then*

1) if φ is μ-acceptance time consistent, then $g\circ\varphi = \{g\circ\varphi_t\}_{t\in\mathbb{T}}$ also is μ-acceptance time consistent, for any increasing, and concave function $g : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$.

2) if φ is μ-rejection time consistent, then $g \circ \varphi = \{g \circ \varphi_t\}_{t\in\mathbb{T}}$ also is μ-rejection time consistent, for any increasing, and convex function $g : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$.

The proof of Proposition 4.34 is analogous to the proof of Proposition 4.32.

5 Examples

This section is devoted to various examples, known in the literature, that illustrate the different concepts of time consistency for risk and performance measures introduced above, as well as some relationships between them.

Example 5.1 (Negative of Conditional Weighted Value at Risk)**.** Let $\mathcal{X} = L^0$. For $\alpha \in (0, 1)$ we consider the family of sets $\{D^\alpha_t\}_{t\in\mathbb{T}}$ defined by

$$D^\alpha_t := \{Z \in L^1 : 0 \leq Z \leq \alpha^{-1},\ E[Z|\mathcal{F}_t] = 1\}, \tag{5.1}$$

and we let $\{\varphi^\alpha_t\}_{t\in\mathbb{T}}$ be given by

$$\varphi^\alpha_t(X) = \operatorname*{ess\,inf}_{Z\in D^\alpha_t} E[ZX|\mathcal{F}_t], \qquad t \in \mathbb{T},\ X \in L^0. \tag{5.2}$$

The map φ^α, defined in (5.2), is a dynamic coherent risk measure, for any fixed parameter $\alpha \in (0, 1)$ (see [24] for details). Moreover, it is supermartingale time-consistent, and thus, by Remark 4.33, it is also weakly acceptance time consistent. Indeed, let $t, s \in \mathbb{T}$ be such that $s > t$. It is easy to check that $D^\alpha_s \subseteq D^\alpha_t$. Hence,

$$\varphi^\alpha_t(X) = \operatorname*{ess\,inf}_{Z\in D^\alpha_t} E[ZX|\mathcal{F}_t] \leq \operatorname*{ess\,inf}_{Z\in D^\alpha_s} E[ZX|\mathcal{F}_t] = \operatorname*{ess\,inf}_{Z\in D^\alpha_s} E[E[ZX|\mathcal{F}_s]|\mathcal{F}_t]. \tag{5.3}$$

Now, using the fact that D_s^α is L^1-closed (see [24] for details), for any $X \in L^0$, there exist $Z_X^* \in D_s^\alpha$ such that $\varphi_s^\alpha(X) = E[Z_X^* X|\mathcal{F}_s]$. This implies that

$$\operatorname*{ess\,inf}_{Z\in D_s} E[E[ZX|\mathcal{F}_s]|\mathcal{F}_t] \leq E[E[Z_X^* X|\mathcal{F}_s]|\mathcal{F}_t] = E[\operatorname*{ess\,inf}_{Z\in D_s} E[ZX|\mathcal{F}_s]|\mathcal{F}_t] = E[\varphi_s^\alpha(X)|\mathcal{F}_t]. \tag{5.4}$$

Combining (5.3) and (5.4), we conclude that φ^α is supermartingale time-consistent. On the other hand, for any $\alpha \in (0,1)$, the map φ^α is not middle acceptance time consistent and nor weakly rejection time consistent (see [4] for counterexamples).

Example 5.2 (Dynamic TV@R Acceptability Index for Processes)**.** Tail Value at Risk Acceptability Index was introduced in [26], as a scale invariance measure of performance for the case of random variables. Along the lines of [13], here we extend this notion to the case of stochastic processes. Let $\mathcal{X} = \mathbb{V}^0$, and for a fixed $\alpha \in (0,1]$ we consider the sets $\{\mathcal{D}_t^\alpha\}_{t\in\mathbb{T}}$ defined as in (5.1). We consider the distortion function $g(x) = \frac{1}{1+x}$, $x \in \mathbb{R}^+$, and we define $\rho^x = \{\rho_t^x\}_{t\in\mathbb{T}}$, $x \in \mathbb{R}_+$, as follows

$$\rho_t^x(V) = \operatorname*{ess\,inf}_{Z\in D_t^{g(x)}} E[Z\sum_{i=t}^T V_i|\mathcal{F}_t], \quad V \in \mathbb{V},\ t \in \mathcal{T}. \tag{5.5}$$

One could easily show that ρ^x is an increasing (with respect to x) family of dynamic coherent risk measures for processes (see [26] and [13] for details). Hence, the map $\alpha = \{\alpha_t\}_{t\in\mathbb{T}}$ given by

$$\alpha_t(V) = \sup\{x \in \mathbb{R}_+ : \rho_t^x(V) \geq 0\}, \tag{5.6}$$

is an acceptability index for processes (see [26] and [13]).

Using similar arguments as in Example 5.1, we conclude that ρ^x is weakly acceptance time consistent, for any fixed $x \in \mathbb{R}_+$. Hence, by Proposition 4.11 we obtain that α is semi-weakly acceptance time consistent.

On the other hand α is not semi-weakly rejection time consistent. Indeed, following similar reasoning as in the proof of duality from [13] and using Proposition 4.12, we get that if α is semi-weakly rejection time consistent, then $\{\rho_t^x\}_{t\in\mathbb{T}}$ is weakly rejection time consistency, for any $x \in \mathbb{R}_+$. This leads to a contradiction, since the maps $\{\rho_t^x\}_{t\in\mathbb{T}}$ are not weakly rejection time consistent, as stated in Example 5.1.

Example 5.3 (Dynamic RAROC for processes)**.** Risk Adjusted Return On Capital (RAROC) is a popular scale invariant measure of performance; we refer the reader to [26] for study of static RAROC, and to [13] for its extension to dynamic setup. We consider the space $\mathcal{X} = \mathbb{V}^1$ and we fix $\alpha \in (0,1)$. Dynamic RAROC, at level α, is defined as follows

$$\varphi_t(V) := \begin{cases} \frac{E[\sum_{i=t}^T V_i|\mathcal{F}_t]}{-\rho_t^\alpha(V)} & \text{if } E[\sum_{i=t}^T V_i|\mathcal{F}_t] > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5.7}$$

where $\rho_t^\alpha(V) = \operatorname*{ess\,inf}_{Z\in\mathcal{D}_t^\alpha} E[Z\sum_{i=t}^T V_i|\mathcal{F}_t]$, and $\{\mathcal{D}_t^\alpha\}_{t\in\mathbb{T}}$ given by (5.1). We use the convention $\varphi_t(V) = +\infty$, if $\rho_t(V) \geq 0$. In [13] it was shown that the map (5.7) is a dynamic acceptability index for processes. Moreover, for any fixed $t \in \mathbb{T}$,

$$\varphi_t(V) = \sup\{x \in \mathbb{R}_+ : \phi_t^x(V) \geq 0\},$$

where $\phi_t^x(V) = \operatorname*{ess\,inf}_{Z\in\mathcal{B}_t^x} E[Z(\sum_{i=t}^T V_i)|\mathcal{F}_t]$ with

$$\mathcal{B}_t^x = \{Z \in L^1 : Z = \frac{1}{1+x} + \frac{x}{1+x} Z_1, \text{ for some } Z_1 \in \mathcal{D}_t^\alpha\}.$$

It is easy to check, that the family $\{\varphi_t^x\}_{t\in\mathbb{T}}$ is a dynamic coherent risk measure for processes, and by similar arguments as in Example 5.1, we get that φ_t^x is weakly acceptance time consistent but not weakly rejection time consistent, for any fixed $x \in \mathbb{R}_+$. Since $1 \in \mathcal{D}_t^\alpha$, we also get that $\{\phi_t^x\}_{t\in\mathbb{T}}$ is increasing with $x \in \mathbb{R}_+$, and by using similar arguments as in Example 5.2, we conclude that φ is semi-weakly acceptance time consistent and not semi-weakly rejection time consistent.

Example 5.4 (Dynamic Gain Loss Ratio)**.** Dynamic Gain Loss Ratio (dGLR) is another popular measure of performance, which essentially improves on some drawbacks of Sharpe Ratio (such as penalizing for positive returns), and it is equal to the ratio of expected return over expected losses. Formally, for $\mathcal{X} = \mathbb{V}^1$, dGLR is defined as

$$\varphi_t(V) := \begin{cases} \frac{E[\sum_{i=t}^T V_i|\mathcal{F}_t]}{E[(\sum_{i=t}^T V_i)^-|\mathcal{F}_t]}, & \text{if } E[\sum_{i=t}^T V_i|\mathcal{F}_t] > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.8}$$

For various properties and dual representations of dGLR see for instance [13, 9]. In [13], the authors showed that dGLR is both semi-weakly acceptance and semi-weakly rejection time consistent, although assuming that Ω is finite. For sake of completeness we will show here that dGLR is semi-weakly acceptance time consistency; semi-weakly rejection time consistency is left to an interested reader as an exercise.

Assume that $t \in \mathbb{T}'$, and $V \in \mathcal{X}$. In view of Proposition 3.5, it is enough to show that

$$\varphi_t(V) \geq \mathbb{1}_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t(\varphi_{t+1}(V)) + \mathbb{1}_{\{V_t<0\}}(-\infty). \tag{5.9}$$

On the set $\{V_t < 0\}$ the inequality (5.9) is trivial. Since φ_t is non-negative and local, without loss of generality, we may assume that $\operatorname{ess\,inf}_t(\varphi_{t+1}(V)) > 0$. Moreover, $\varphi_{t+1}(V) \geq \operatorname{ess\,inf}_t(\varphi_{t+1}(V))$, which implies

$$E[\sum_{i=t+1}^T V_i|\mathcal{F}_{t+1}] \geq \operatorname{ess\,inf}_t(\varphi_{t+1}(V)) \cdot E[(\sum_{i=t+1}^T V_i)^-|\mathcal{F}_{t+1}]. \tag{5.10}$$

Using (5.10) we obtain

$$\begin{aligned}
\mathbb{1}_{\{V_t\geq 0\}} E[\sum_{i=t}^T V_i|\mathcal{F}_t] &\geq \mathbb{1}_{\{V_t\geq 0\}} E[E[\sum_{i=t+1}^T V_i|\mathcal{F}_{t+1}]|\mathcal{F}_t] \\
&\geq \mathbb{1}_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t(\varphi_{t+1}(V)) \cdot E[\mathbb{1}_{\{V_t\geq 0\}} E[(\sum_{i=t+1}^T V_i)^-|\mathcal{F}_{t+1}]|\mathcal{F}_t] \\
&\geq \mathbb{1}_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t(\varphi_{t+1}(V)) \cdot E[(\sum_{i=t}^T V_i)^-|\mathcal{F}_t].
\end{aligned} \tag{5.11}$$

Note that $\operatorname{ess\,inf}_t(\varphi_{t+1}(V)) > 0$ implies that $\varphi_{t+1}(V) > 0$, and thus $E[\sum_{i=t+1}^T V_i|\mathcal{F}_{t+1}] > 0$. Hence, on set $\{V_t \geq 0\}$, we have

$$\mathbb{1}_{\{V_t\geq 1\}}E[\sum_{i=t}^{T} V_i|\mathcal{F}_t] \geq \mathbb{1}_{\{V_t\geq 1\}}E[E[\sum_{i=t+1}^{T} V_i|\mathcal{F}_{t+1}]|\mathcal{F}_t] > 0.$$

Combining this and (5.11), we conclude the proof.

Example 5.5 (Negative of Dynamic Entropic Risk Measure)**.** Entropic Risk Measure is a classical convex risk measure. The dynamic version of it (more precisely, up to a negative sign) is defined as follows

$$\varphi_t^\gamma(X) = \begin{cases} \frac{1}{\gamma}\ln E[\exp(\gamma X)|\mathcal{F}_t] & \text{if } \gamma \neq 0, \\ E[X|\mathcal{F}_t] & \text{if } \gamma = 0. \end{cases} \tag{5.12}$$

where $X \in \mathcal{X} = L^1$, $t \in \mathbb{T}$. It can be proved that for $\gamma \leq 0$, the map $-\varphi_t^\gamma$ is a dynamic convex risk measure, and that for any $\gamma \in \mathbb{R}$, the map φ^γ is strongly time consistent (cf. [48]). Since it is also cash-additive, strong time consistency implies both weak rejection and weak acceptance time consistency. Moreover (see [48, 11] for details) $\{\varphi_t^\gamma\}_{t\in\mathbb{T}}$ is supermartingale time consistent if and only if $\gamma \geq 0$, and submartingale time consistent if and only if $\gamma \leq 0$.

Example 5.6 (Negative of Dynamic Entropic Risk Measure with non-constant risk aversion)**.** One can generalise the Dynamic Entropic Risk Measure (5.12) by taking time dependent risk aversion parameters. Towards this end, it is enough to consider the dynamic risk measure on $\mathcal{X} = L^\infty$, given by

$$\varphi_t^{\gamma_t}(X) = \begin{cases} \frac{1}{\gamma_t}\ln E[\exp(\gamma_t X)|\mathcal{F}_t] & \text{if } \gamma_t \neq 0, \\ E[X|\mathcal{F}_t] & \text{if } \gamma_t = 0. \end{cases} \tag{5.13}$$

where $\{\gamma_t\}_{t\in\mathbb{T}}$ is such that $\gamma_t \in L_t^\infty$, $t \in \mathbb{T}$. Noting that the map introduced in (5.12) is increasing in γ [48], it could be easily shown (see [1] for the idea of the proof) that $\{\varphi_t^{\gamma_t}\}_{t\in\mathbb{T}}$ is strongly time consistent if and only if $\{\gamma_t\}_{t\in\mathbb{T}}$ is a constant process, and that it is middle acceptance time consistent if and only if $\{\gamma_t\}_{t\in\mathbb{T}}$ is a non-increasing process, and that it is middle rejection time consistent if and only if $\{\gamma_t\}_{t\in\mathbb{T}}$ is non-decreasing.

Example 5.7 (Dynamic Certainty Equivalent)**.** Dynamic Certainty Equivalents is a large class of dynamic risk measures, with Dynamic Entropic Risk Measure being a particular case of them. For this example, consistently with [48], we consider an infinite time horizon, and we take $\mathbb{T} = \mathbb{N}$ and $\mathcal{X} = L^\infty$. We let $U : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ be a strictly increasing and continuous function on $\bar{\mathbb{R}}$, i.e. strictly increasing and continuous on $\mathbb{R}$, with $U(\pm\infty) = \lim_{n\to\pm\infty} U(n)$. Let $\varphi = \{\varphi_t\}_{t\in\mathbb{T}}$ be defined by

$$\varphi_t(X) = U^{-1}(E[U(X)|\mathcal{F}_t]), \qquad X \in \mathcal{X},\ t \in \mathbb{T}. \tag{5.14}$$

It is easy to check that φ is a strongly time consistent dynamic LM-measure. It belongs to the class of so called dynamic certainty equivalents [48]. In [48], the authors showed that any dynamic LM-measure, which is finite, normalized, strictly monotone, continuous, law invariant, admits Fatou property and is strongly time consistent, can be represented as (5.14) for some U. We also refer to [7] for a more general approach to dynamic certainty equivalents (e.g. by using stochastic utility functions U), and to [9] for the definition of certainty equivalents for processes.

Example 5.8 (Dynamic Risk Sensitive Criterion)**.** In recent paper [11] the authors introduce the notion of Dynamic Limit Growth Index (DLGI) that is designed to measure the long-term performance of a financial portfolio in discrete time. The dynamic analog of Risk Sensitive Criterion (cf. [60] and references therein) is a particular case of DLGI. We consider an infinite time horizon setup, $\mathbb{T} = \mathbb{N}$, and the following space suitable for our needs $\mathbb{V}^p_{\ln} := \{(W_t)_{t\in\mathbb{T}} : W_t > 0, \ln W_t \in L^p_t\}$. To be consistent with [11], we view the elements of $\mathcal{X}$ as (cumulative) value processes of portfolios of some financial securities, which have integrable growth expressed as cumulative log-return (note that everywhere else in the present paper, the stochastic processes represent dividend streams). On this space, let $\varphi^\gamma = \{\varphi^\gamma_t\}_{t\in\mathbb{T}}$ be defined by

$$\varphi^\gamma_t(W) = \begin{cases} \liminf_{T\to\infty} \frac{1}{T}\frac{1}{\gamma} \ln E[W^\gamma_T|\mathcal{F}_t], & \text{if } \gamma \neq 0, \\ \liminf_{T\to\infty} \frac{1}{T} E[\ln W_T|\mathcal{F}_t], & \text{if } \gamma = 0. \end{cases} \tag{5.15}$$

where γ is a fixed real number. It was proved in [11] that φ^γ defined by (5.15) is a dynamic measure of performance; moreover, φ^γ is μ-acceptance time consistent, w.r.t. $\mu = \{\mu_t\}_{t\in\mathbb{T}}$ given by $\mu_t(m) = E[m|\mathcal{F}_t]$, if and only if $\gamma > 0$, and μ-rejection time consistent (w.r.t. the same μ) if and only if $\gamma < 0$.

The following table is meant to help the reader to navigate through the examples presented above relative to various types of time consistency studied in this paper. We will use the following abbreviations for time consistency: WA - weak acceptance; WR - weak rejection; sWA - semi weak acceptance; sWR - semi weak rejection; MA - middle acceptance; MR - middle rejection; STR - strong Sub - submartinagle; Sup - supermartinagle.

If a cell is marked with ✓ mark, that means that the corresponding property of time consistency is satisfied; otherwise the property is not satisfied in general.

We note that Example 5.8 is not represented in the table due to the "dichotomous" nature of the example. The DGLI evaluates a process V, but it does it through a limiting procedure, which really amounts to evaluating the process through its "values at $T = \infty$." We refer the reader to [11] for a detailed discussion on various properties of this measure.

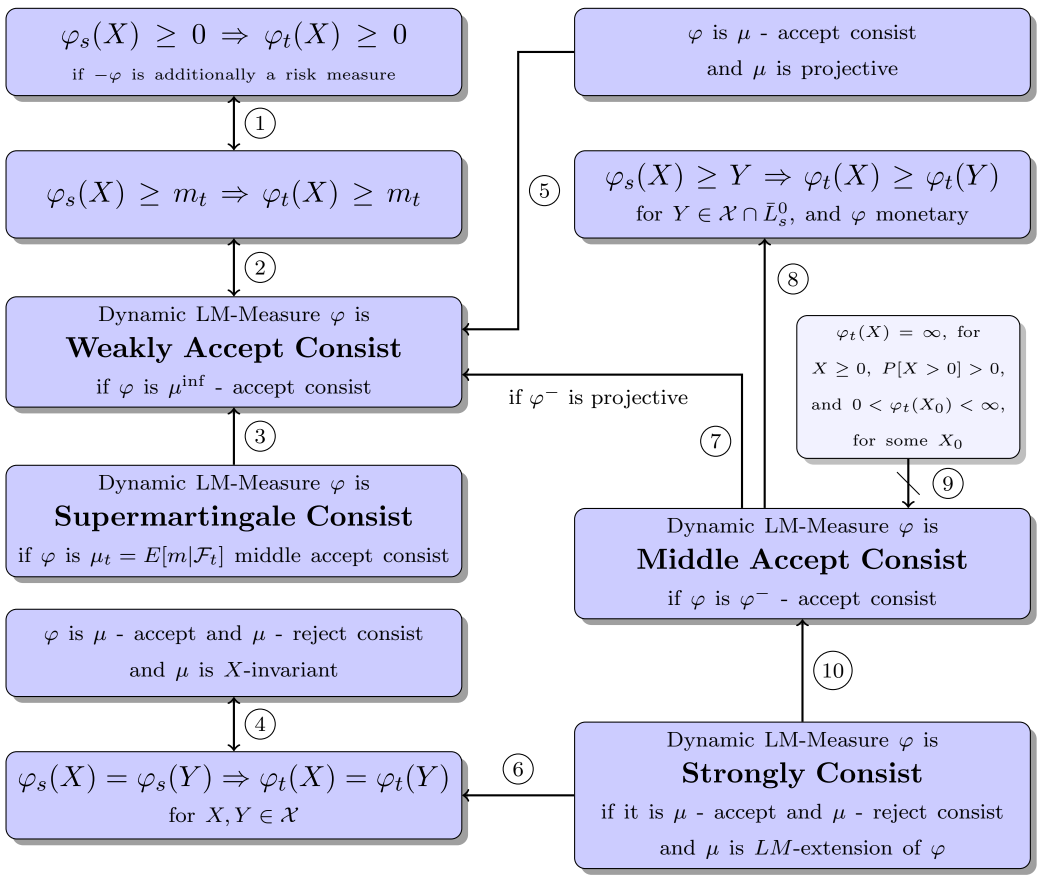
		$\mathcal{X}$	WA	WR	sWA	sWR	MA	MR	STR	Sub	Sup
Example 5.1		L^p	✓		✓						✓
Example 5.2		$\mathbb{V}^p$			✓						
Example 5.3		$\mathbb{V}^p$			✓						
Example 5.4		$\mathbb{V}^p$			✓	✓					
Example 5.5	$\gamma \geq 0$	L^p	✓	✓	✓	✓	✓	✓	✓		✓
	$\gamma \leq 0$		✓	✓	✓	✓	✓	✓	✓	✓	
Example 5.6	$\gamma_t \downarrow$	L^p	✓		✓		✓				✓*
	$\gamma_t \uparrow$			✓		✓		✓		✓**	
Example 5.7		L^p	✓	✓	✓	✓	✓	✓	✓		

*if $\gamma_t \geq 0$, **if $\gamma_t \leq 0$

6 Concluding remarks

The main goal of this paper was to develop a unified framework for time consistency of LM-measures that, in particular, comprises various types of time consistency for dynamic risk measures

Figure 1: *Summary of results for acceptance time consistency for random variables*



and dynamic performance measures known in the existing literature. The obtained results are summarised in the Chartflows 1 and 2. For convenience, we label (by circled or rectangled numbers) each arrow (implication or equivalence) in the flowcharts, and we relate the labels to the relevant result from the paper, along with comments on converse implications whenever appropriate.

1. Proposition 4.3, 4)

1. Proposition 4.3, 3)

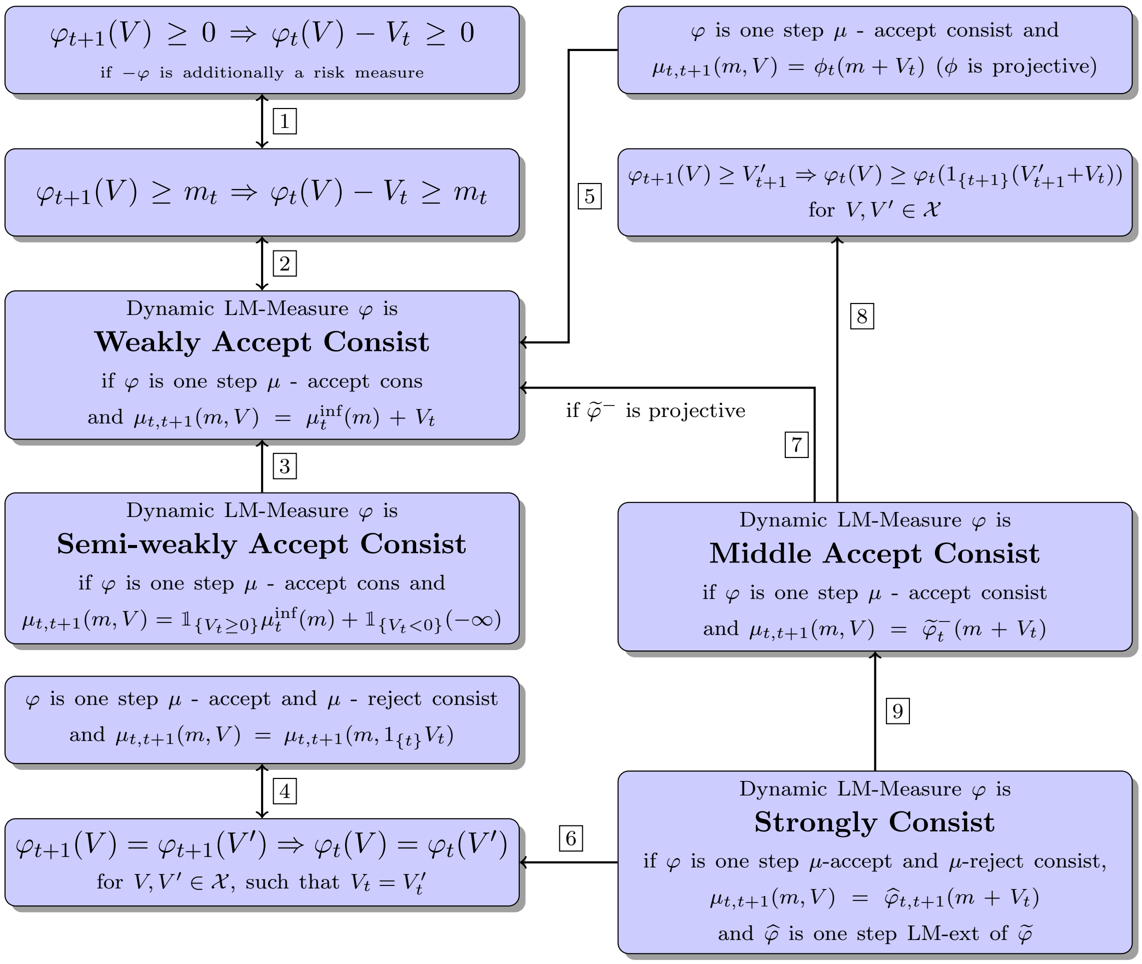
3. Remark 4.33 and Proposition 4.4. The converse implication is not true in general, see Example 5.5.

4. Proposition 4.27, 1), 2)

5. Proposition 4.4. Generally speaking the converse implication is not true. See for instance Example 5.5: negative of Dynamic Entropic Risk Measure with $\gamma < 0$ is weakly acceptance time consistent, but it is not supermaringale time consistent, i.e. it is not acceptance time consistent with respect to the projective update rule $\mu_t = E_t[m|\mathcal{F}_t]$.

6. Proposition 4.27, 3), 4). The converse implication is not true in general. As a counterexample, consider $\varphi_t(X) = tE[X]$.

Figure 2: *Summary of results for acceptance time consistency for stochastic processes*



(7) Proposition 4.4, and see also (5). In general, middle acceptance time consistency does not imply weak acceptance time consistency, see Remark 4.23.

(8) Proposition 4.20, 3)

(9) Proposition 4.21

(10) Proposition 4.15. The converse implication is not true in general, see Example 5.6.

$\boxed{1}$ Proposition 4.7, 4)

$\boxed{2}$ Proposition 4.7, 3)

$\boxed{3}$ Proposition 4.10, 3)

$\boxed{4}$ Proposition 4.30, 1), 2)

$\boxed{5}$ Proposition 4.8

$\boxed{6}$ Proposition 4.30, 3), 4)

$\boxed{7}$ Proposition 4.8, and see also 5.

$\boxed{8}$ Proposition 4.25, 3)

$\boxed{9}$ Proposition 4.15.

Remark 6.1. The converse implications in Flowchart 2 do not hold true in general, and one can use the same counterexamples as in the case of random variables.

A Appendix

Proof of Proposition 2.2.

Proof. First note that for any $X, Y \in \bar{L}^0$, $\lambda \in L^0_t$ such that $X, Y, \lambda \geq 0$, and for any $s, t \in \mathbb{T}$, $s > t$, by Monotone Convergence Theorem, and using the convention $0 \cdot \pm\infty = 0$ we get

$$E[\lambda X|\mathcal{F}_t] = \lambda E[X|\mathcal{F}_t]; \tag{A.1}$$
$$E[X|\mathcal{F}_t] = E[E[X|\mathcal{F}_s]|\mathcal{F}_t]; \tag{A.2}$$
$$E[X|\mathcal{F}_t] + E[Y|\mathcal{F}_t] = E[X+Y|\mathcal{F}_t]. \tag{A.3}$$

Moreover, for $X \in \bar{L}^0$, we also have

$$E[-X|\mathcal{F}_t] \leq -E[X|\mathcal{F}_t]. \tag{A.4}$$

For the last inequality we used the convention $\infty - \infty = -\infty$.

Next, using (A.1)-(A.4), we will prove the announced results. Assume that $X, Y \in \bar{L}^0$.
1) If $\lambda \in L^0_t$, and $\lambda \geq 0$, then, by (A.1) we get

$$\begin{aligned} E[\lambda X|\mathcal{F}_t] &= E[(\lambda X)^+|\mathcal{F}_t] - E[(\lambda X)^-|\mathcal{F}_t] = E[\lambda X^+|\mathcal{F}_t] - E[\lambda X^-|\mathcal{F}_t] = \\ &= \lambda E[X^+|\mathcal{F}_t] - \lambda E[X^-|\mathcal{F}_t] = \lambda E[X|\mathcal{F}_t]. \end{aligned}$$

From here, and using (A.4), for a general $\lambda \in L^0_t$, we deduce

$$\begin{aligned} E[\lambda X|\mathcal{F}_t] &= E[1_{\{\lambda\geq 0\}}\lambda X + 1_{\{\lambda<0\}}\lambda X|\mathcal{F}_t] = 1_{\{\lambda\geq 0\}}\lambda E[X|\mathcal{F}_t] + 1_{\{\lambda<0\}}(-\lambda)E[-X|\mathcal{F}_t] \leq \\ &\leq 1_{\{\lambda\geq 0\}}\lambda E[X|\mathcal{F}_t] + 1_{\{\lambda<0\}}\lambda E[X|\mathcal{F}_t] = \lambda E[X|\mathcal{F}_t]. \end{aligned}$$

2) The proof of 2) follows from (A.2) and (A.4); for $X \in L^0$ see also the proof in [25, Lemma 3.4].
3) On the set $\{E[X|\mathcal{F}_t] = -\infty\} \cup \{E[Y|\mathcal{F}_t] = -\infty\}$ the inequality is trivial due to the convention $\infty - \infty = -\infty$. On the other hand the set $\{E[X|\mathcal{F}_t] > -\infty\} \cap \{E[Y|\mathcal{F}_t] > -\infty\}$ could be represented as the union of the sets $\{E[X|\mathcal{F}_t] > n\} \cap \{E[Y|\mathcal{F}_t] > n\}$ for $n \in \mathbb{Z}$ on which the inequality becomes the equality, due to (A.3). $\square$

Proof of Proposition 3.5.

Proof. Let μ be an update rule.

1) The implication ($\Rightarrow$) follows immediately, by taking in the definition of acceptance time consistency $m_s = \varphi_s(X)$.

($\Leftarrow$) Assume that $\varphi_t(X) \geq \mu_{t,s}(\varphi_s(X), X)$, for any $s, t \in \mathbb{T}, s > t$, and $X \in \mathcal{X}$. Let $m_s \in \bar{L}^0_s$ be such that $\varphi_s(X) \geq m_s$. Using monotonicity of μ, we get $\varphi_t(X) \geq \mu_{t,s}(\varphi_s(X), X) \geq \mu_{t,s}(m_s, X)$.

2) The proof is similar to 1). $\square$

Proof of Proposition 4.1.

Proof. Monotonicity and locality of $\mu^{\inf}$ is a straightforward implication of Proposition 2.4. Thus, $\mu^{\inf}$ is sX-invariant update rule. The projectivity comes straight from the definition (see Remark 2.5). Now, let a family $\mu = \{\mu_t\}_{t\in\mathbb{T}}$ of maps $\mu_t : \bar{L}^0 \to \bar{L}^0_t$ be given by

$$\mu_t(m) = \operatorname*{ess\,inf}_{Z\in P_t} E[Zm|\mathcal{F}_t] \tag{A.5}$$

Before proving (4.3), we will need to prove some facts about μ.

First, let us show that μ is sX-invariant update rule. Let $t \in \mathbb{T}$. Monotonicity is straightforward. Indeed, let $m, m' \in \bar{L}^0$ be such that $m \geq m'$. For any $Z \in P_t$, using the fact that $Z \geq 0$, we get $Zm \geq Zm'$. Thus, $E[Zm|\mathcal{F}_t] \geq E[Zm'|\mathcal{F}_t]$ and consequently $\operatorname{ess\,inf}_{Z\in P_t} E[Zm|\mathcal{F}_t] \geq \operatorname{ess\,inf}_{Z\in P_t} E[Zm'|\mathcal{F}_t]$. Locality follows from the fact, for any $A \in \mathcal{F}_t$ and $m \in \bar{L}^0$, using Proposition 2.2, convention $0 \cdot \pm\infty = 0$, and the fact that for any $Z_1, Z_2 \in P_t$ we have $\mathbb{1}_A Z_1 + \mathbb{1}_{A^c} Z_2 \in P_t$, we get

$$\begin{aligned}
\mathbb{1}_A \mu_t(m) &= \mathbb{1}_A \operatorname*{ess\,inf}_{Z\in P_t} E[Zm|\mathcal{F}_t] \\
&= \mathbb{1}_A \operatorname*{ess\,inf}_{Z\in P_t} (E[(\mathbb{1}_A Z)m|\mathcal{F}_t] + E[(\mathbb{1}_{A^c} Z)m|\mathcal{F}_t]) \\
&= \mathbb{1}_A \operatorname*{ess\,inf}_{Z\in P_t} E[(\mathbb{1}_A Z)m|\mathcal{F}_t] + \mathbb{1}_A \operatorname*{ess\,inf}_{Z\in P_t} E[(\mathbb{1}_{A^c} Z)m|\mathcal{F}_t] \\
&= \mathbb{1}_A \operatorname*{ess\,inf}_{Z\in P_t} E[Z(\mathbb{1}_A m)|\mathcal{F}_t] + \mathbb{1}_A \operatorname*{ess\,inf}_{Z\in P_t} \mathbb{1}_{A^c} E[Zm|\mathcal{F}_t] \\
&= \mathbb{1}_A \mu_t(\mathbb{1}_A m).
\end{aligned}$$

Thus, μ is sX-invariant update rule.

Secondly, let us prove that we get

$$m \geq \mu_t(m), \tag{A.6}$$

for any $m \in \bar{L}^0$. Let $m \in L^0$. For $\alpha \in (0,1)$ let[14]

$$Z_\alpha := \mathbb{1}_{\{m\leq q_t^+(\alpha)\}} E[\mathbb{1}_{\{m\leq q_t^+(\alpha)\}}|\mathcal{F}_t]^{-1}. \tag{A.7}$$

where $q_t^+(\alpha)$ is $\mathcal{F}_t$-conditional (upper) α quantile of m, defined as

$$q_t^+(\alpha) := \operatorname{ess\,sup}\{Y \in L^0_t \mid E[\mathbb{1}_{\{m\leq Y\}}|\mathcal{F}_t] \leq \alpha\}.$$

[14]In the risk measure framework, it might be seen as the risk minimazing scenario for conditional $CV@R_\alpha$.

For $\alpha \in (0,1)$, noticing that $Z_\alpha < \infty$, due to convention $0 \cdot \infty = 0$ and the fact that

$$\{E[\mathbb{1}_{\{m \le q_t^+(\alpha)\}} | \mathcal{F}_t] = 0\} \subseteq \{\mathbb{1}_{\{m \le q_t^+(\alpha)\}} = 0\} \cup B,$$

for some B, such that $P[B] = 0$, we conclude that $Z_\alpha \in P_t$. Moreover, by the definition of $q_t^+(\alpha)$, there exists a sequence $Y_n \in L_t^0$, such that $Y_n \nearrow q_t^+(\alpha)$, and

$$E[\mathbb{1}_{\{m < Y_n\}} \mid \mathcal{F}_t] \le \alpha.$$

Consequently, by monotone convergence theorem, we have

$$E[\mathbb{1}_{\{m < q_t^+(\alpha)\}} \mid \mathcal{F}_t] \le \alpha.$$

Hence, we deduce

$$P[m < q_t^+(\alpha)] = E[\mathbb{1}_{\{m < q_t^+(\alpha)\}}] \le E[E[\mathbb{1}_{\{m < q_t^+(\alpha)\}} | \mathcal{F}_t]] \le E[\alpha] = \alpha,$$

which implies that

$$P[m \ge q_t^+(\alpha)] \ge (1 - \alpha). \tag{A.8}$$

On the other hand

$$\begin{aligned} \mathbb{1}_{\{m \ge q_t^+(\alpha)\}} m \ge \mathbb{1}_{\{m \ge q_t^+(\alpha)\}} q_t^+(\alpha) &= \mathbb{1}_{\{m \ge q_t^+(\alpha)\}} q_t^+(\alpha) E[Z_\alpha | \mathcal{F}_t] \\ &\ge \mathbb{1}_{\{m \ge q_t^+(\alpha)\}} E[Z_\alpha q_t^+(\alpha) | \mathcal{F}_t] \ge \mathbb{1}_{\{m \ge q_t^+(\alpha)\}} E[Z_\alpha m | \mathcal{F}_t], \end{aligned}$$

which combined with (A.8), implies that

$$P\Big[m \ge E[Z_\alpha m | \mathcal{F}_t]\Big] \ge 1 - \alpha. \tag{A.9}$$

Hence, using (A.9), and the fact that

$$E[Z_\alpha m | \mathcal{F}_t] \ge \mu_t(m), \quad \alpha \in (0,1),$$

we get that

$$P[m \ge \mu_t(m)] \ge 1 - \alpha.$$

Letting $\alpha \to 0$, we conclude that (A.6) holds true for $m \in L^0$.

Now, assume that $m \in \bar{L}^0$, and let $A := \{E[\mathbb{1}_{\{m = -\infty\}} | \mathcal{F}_t] = 0\}$. Similar to the arguments above, we get

$$\mathbb{1}_A m \ge \mu_t(\mathbb{1}_A m).$$

Since $\mu_t(0) = 0$, and due to locality of μ_t, we deduce

$$\mathbb{1}_A m \ge \mu_t(\mathbb{1}_A m) = \mathbb{1}_A \mu_t(\mathbb{1}_A m) = \mathbb{1}_A \mu_t(m). \tag{A.10}$$

Moreover, taking $Z = 1$ in (A.5), we get

$$\mathbb{1}_{A^c} m \ge \mathbb{1}_{A^c}(-\infty) = \mathbb{1}_{A^c} E[m | \mathcal{F}_t] \ge \mathbb{1}_{A^c} \mu_t(m). \tag{A.11}$$

Combining (A.10) and (A.11), we concludes the proof of (A.6) for all $m \in \bar{L}^0$.

Finally, we will show that μ_t defined as in (A.5) satisfies property 1) from Proposition 2.4, which will consequently imply equality (4.3). Let $m \in \bar{L}^0$ and $A \in \mathcal{F}_t$. From the fact that $m \geq \mu_t(m)$ we get

$$\operatorname*{ess\,inf}_{\omega\in A} m \geq \operatorname*{ess\,inf}_{\omega\in A} \mu_t(m).$$

On the other hand we know that $\mathbb{1}_A \operatorname{ess\,inf}_{\omega\in A} m \leq \mathbb{1}_A m$ and $\mathbb{1}_A \operatorname{ess\,inf}_{\omega\in A} m \in \bar{L}^0_t$, so

$$\begin{aligned}\operatorname*{ess\,inf}_{\omega\in A} m = \operatorname*{ess\,inf}_{\omega\in A}(\mathbb{1}_A \operatorname*{ess\,inf}_{\omega\in A} m) = \operatorname*{ess\,inf}_{\omega\in A}(\mathbb{1}_A\mu_t(\mathbb{1}_A \operatorname*{ess\,inf}_{\omega\in A} m)) \leq \\ \leq \operatorname*{ess\,inf}_{\omega\in A}(\mathbb{1}_A\mu_t(\mathbb{1}_A m)) = \operatorname*{ess\,inf}_{\omega\in A}(\mathbb{1}_A\mu_t(m)) = \operatorname*{ess\,inf}_{\omega\in A} \mu_t(m)\end{aligned}$$

which proves the equality. The proof for $\operatorname{ess\,sup}_t$ is similar and we omit it here. This concludes the proof. □

Proof of Proposition 4.3.

Proof. We will only show the proof for acceptance consistency. The proof for rejection consistency is similar. Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure.

1) ⇔ 2). This is a direct application of Proposition 3.5.

1) ⇒ 3). Assume that φ is weakly acceptance consistent, and let $m_t \in \bar{L}^0_t$ be such that $\varphi_s(X) \geq m_t$. Then, using Proposition 3.5, we get $\varphi_t(X) \geq \operatorname{ess\,inf}_t(\varphi_s(X)) \geq \operatorname{ess\,inf}_t(m_t) = m_t$, and hence 3) is proved.

3) ⇒ 1). By the definition of conditional essential infimum, $\operatorname{ess\,inf}_t(\varphi_s(X)) \in \bar{L}^0_t$, for any $X \in \mathcal{X}$, and $t, s \in \mathcal{T}$. Moreover, by Proposition 2.4.(3), we have that $\varphi_s(X) \geq \operatorname{ess\,inf}_t(\varphi_s(X))$. Using assumption 3) with $m_t = \operatorname{ess\,inf}_t(\varphi_s(X))$, we immediately obtain $\varphi_t(X) \geq \operatorname{ess\,inf}_t(\varphi_s(X))$. Due to Proposition 3.5 this concludes the proof.

3) ⇔ 4). Clearly 3) ⇒ 4). If additionally φ is a monetary risk measure, then in particular $-\varphi$ is cash-additive. Hence, for any $m_t \in \bar{L}^0_t$ such that $\varphi_s(X) \geq m_t$, we have that $\varphi_s(X - m_t) \geq 0$, and since 4) holds true, we get that $\varphi_t(X - m_t) \geq 0$. Invoking one more time cash-additivity, we complete the proof. □

Proof of Proposition 4.4.

Proof. Then, using Proposition 2.4, for any $t, s \in \mathbb{T}$, $s > t$, and any $X \in \mathcal{X}$, we get

$$\varphi_t(X) \geq \mu_t(\varphi_s(X)) \geq \mu_t(\operatorname{ess\,inf}_s(\varphi_s(X))) \geq \mu_t(\operatorname{ess\,inf}_t(\varphi_s(X))) = \operatorname{ess\,inf}_t(\varphi_s(X)).$$

The proof for rejection time consistency is similar. □

Proof of Proposition 4.7.

Proof. We will only show the proof for weakly acceptance consistency. The proof for rejection consistency is similar. Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure.

1) ⇔ 2). This is a direct implication of Proposition 3.5.

1) ⇒ 3). Let $m_t \in \bar{L}^0_t$ be such that $\varphi_{t+1}(V) \geq m_t$. Using the monotonicity of $\operatorname{ess\,inf}_t$, we have

$$\varphi_t(V) \geq \operatorname{ess\,inf}_t(\varphi_{t+1}(V)) + V_t \geq \operatorname{ess\,inf}_t(m_t) + V_t = m_t + V_t,$$

which concludes the proof.
3) $\Rightarrow$ 1). By Proposition 2.4, we get $\varphi_{t+1}(V) \geq \operatorname{ess\,inf}_t(\varphi_{t+1}(V))$ for any $V \in \mathcal{X}$, and $\operatorname{ess\,inf}_t(\varphi_{t+1}(X)) \in \bar{L}^0_t$. Using assumption 3) with $m_t = \operatorname{ess\,inf}_t(\varphi_{t+1}(X))$ we immediately obtain

$$\varphi_t(V) \geq \operatorname{ess\,inf}_t(\varphi_{t+1}(V)) + V_t$$

and using 2) the weakly acceptance time consistency of φ follows.
3) $\Rightarrow$ 4) is obvious (take $m_t = 0$).
4) $\Rightarrow$ 3) Let us now assume that $\{\varphi_t\}_{t\in\mathbb{T}}$ is a negative of dynamic risk measure. For given $m_t \in \bar{L}^0_t$ it is enough to apply 3) to the process $V \pm 1_{t+1} m_t$, and 4) follows.

□

Proof of Proposition 4.10.

Proof. We will only show the proof for acceptance consistency. The proof for rejection consistency is similar. Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure.
1) $\Leftrightarrow$ 2). This is a direct implication of Proposition 3.5.
2) $\Rightarrow$ 3). Assume that φ is semi-weakly acceptance consistent. Let $V \in \mathcal{X}$ and $m_t \in \bar{L}^0_t$ be such that $\varphi_{t+1}(V) \geq m_t$ and $V_t \geq 0$. Then, by monotonicity of $\mu^{\inf}_t$, we have

$$\varphi_t(V) \geq \mathbb{1}_{\{V_t\geq 0\}} \mu^{\inf}_t(\varphi_{t+1}(V)) \geq \mu^{\inf}_t(m_t) = \operatorname{ess\,inf}_t(m_t) = m_t,$$

and hence 3) is proved.
3) $\Rightarrow$ 2). Let $V \in \mathcal{X}$. We need to show that

$$\varphi_t(V) \geq 1_{\{V_t\geq 0\}} \mu^{\inf}_t(\varphi_{t+1}(V)) + 1_{\{V_t<0\}}(-\infty). \tag{A.12}$$

On the set $\{V_t < 0\}$ inequality (A.12) is trivial. We know that

$$(\mathbb{1}_{\{V_t\geq 0\}} \cdot_t V)_t \geq 0 \quad \text{and} \quad \varphi_{t+1}(\mathbb{1}_{\{V_t\geq 0\}} \cdot_t V) \geq \operatorname{ess\,inf}_t \varphi_{t+1}(\mathbb{1}_{\{V_t\geq 0\}} \cdot_t V).$$

Thus, for $m_t = \operatorname{ess\,inf}_t \varphi_{t+1}(\mathbb{1}_{\{V_t\geq 0\}} \cdot_t V)$, using locality of φ and $\mu^{\inf}$ as well as 3), we get

$$\mathbb{1}_{\{V_t\geq 0\}}\varphi_t(V) = \mathbb{1}_{\{V_t\geq 0\}}\varphi_t(\mathbb{1}_{\{V_t\geq 0\}} \cdot_t V) \geq \mathbb{1}_{\{V_t\geq 0\}} m_t = \mathbb{1}_{\{V_t\geq 0\}} \mu^{\inf}_t(\varphi_{t+1}(V)).$$

and hence (A.12) is proved on the set $\{V_t \geq 0\}$. This conclude the proof of 2). □

Proof of Proposition 4.11

Proof. The proof of locality and monotonicity of (4.6) is straightforward (see [13] for details). Let us assume that $\{\varphi^x_t\}_{t\in\mathbb{T}}$ is weakly acceptance time consistent. Using Proposition 4.7 we get

$$\begin{aligned}
1_{\{V_t\geq 0\}}\alpha_t(V) &= 1_{\{V_t\geq 0\}}\Big(\sup\{x \in \mathbb{R}_+ : 1_{\{V_t\geq 0\}}\varphi^x_t(V) \geq 0\}\Big) \\
&\geq 1_{\{V_t\geq 0\}}\Big(\sup\{x \in \mathbb{R}_+ : 1_{\{V_t\geq 0\}}[\operatorname{ess\,inf}_t \varphi^x_{t+1}(V) + V_t] \geq 0\}\Big) \\
&\geq 1_{\{V_t\geq 0\}}\Big(\sup\{x \in \mathbb{R}_+ : 1_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t \varphi^x_{t+1}(V) \geq 0\}\Big) \\
&= 1_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t \Big(\sup\{x \in \mathbb{R}_+ : 1_{\{V_t\geq 0\}}\varphi^x_{t+1}(V) \geq 0\}\Big) \\
&= 1_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t \alpha_{t+1}(V).
\end{aligned}$$

This leads to inequality

$$\alpha_t(V) \geq 1_{\{V_t\geq 0\}} \operatorname{ess\,inf}_t \alpha_{t+1}(V) + 1_{\{V_t<0\}}(-\infty),$$

which, by Proposition 4.10, is equivalent to semi-weak rejection time consistency. The proof of weak acceptance time consistency is similar. □

Proof of Proposition 4.12

Proof of Proposition 4.12. The proof of locality and monotonicity of (4.7) is straightforward (see [13] for details). Let us prove weak acceptance time consistency. Let us assume that $\{\alpha_t\}_{t\in\mathbb{T}}$ is semi-weakly acceptance time consistent. Using Proposition 3.5 we get

$$\begin{aligned}
\varphi_t^x(V) &= \inf\{c\in\mathbb{R} : \alpha_t(V - c1_{\{t\}}) \leq x\} \\
&= \inf\{c\in\mathbb{R} : \alpha_t(V - c1_{\{t+1\}}) \leq x\} \\
&= \inf\{c\in\mathbb{R} : \alpha_t(V - c1_{\{t+1\}} - V_t 1_{\{t\}}) \leq x\} + V_t \\
&\geq \inf\{c\in\mathbb{R} : 1_{\{0\geq 0\}} \operatorname{ess\,inf}_t \alpha_{t+1}(V - c1_{\{t+1\}} - V_t 1_{\{t\}}) + 1_{\{0<0\}}(-\infty) \leq x\} + V_t \\
&= \inf\{c\in\mathbb{R} : \operatorname{ess\,inf}_t \alpha_{t+1}(V - c1_{\{t+1\}}) \leq x\} + V_t \\
&= \operatorname{ess\,inf}_t \big(\inf\{c\in\mathbb{R} : \alpha_{t+1}(V - c1_{\{t+1\}}) \leq x\}\big) + V_t \\
&= \operatorname{ess\,inf}_t \varphi_{t+1}^x(V) + V_t,
\end{aligned}$$

which, is equivalent to weak acceptance time consistency of φ. The proof of rejection time consistency is similar. □

Proof of Proposition 4.15.

Proof. We will show the proof for φ^+ only; the proof for φ^+ is similar. Consider a fixed $t\in\mathbb{T}$. (Adaptivity) It is easy to note that for any $X\in\bar{L}^0$, and $A\in\mathcal{F}_t$, we get

$$\Big[\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}_A^+(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(\infty)\Big] \in \bar{L}_t^0. \tag{A.13}$$

Indeed, for any $X\in\bar{L}^0$, ess inf of the set of $\mathcal{F}_t$-measurable random variables $\{\varphi_t(Y)\}_{Y\in\mathcal{Y}_A^+(X)}$ is $\mathcal{F}_t$-measurable (see [46], Appendix A), which implies (A.13) for any $A\in\mathcal{F}_t$. Thus, $\varphi_t^+(X)\in\bar{L}_t^0$.
(Monotonicity) If $X\geq X'$ then for any $A\in\mathcal{F}_t$ we get $\mathcal{Y}_A^+(X)\subseteq\mathcal{Y}_A^+(X')$, and consequently, for any $A\in\mathcal{F}_t$,

$$\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}_A^+(X)} \varphi_t(Y) \geq \mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}_A^+(X')} \varphi_t(Y),$$

which implies $\varphi_t^+(X)\geq\varphi_t^+(X')$.
(Locality) Let $B\in\mathcal{F}_t$ and $X\in\bar{L}^0$. It is enough to consider $A\in\mathcal{F}_t$, such that $\mathcal{Y}_A^+(X)\neq\emptyset$, as otherwise we get $\varphi_t^+(X)\equiv\infty$. For any such $A\in\mathcal{F}_t$, we get

$$\mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}_A^+(X)} \varphi_t(Y) = \mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}_{A\cap B}^+(X)} \varphi_t(Y). \tag{A.14}$$

Indeed, let us assume that $\mathcal{Y}^+_A(X) \neq \emptyset$. As $\mathcal{Y}^+_A(X) \subseteq \mathcal{Y}^+_{A\cap B}(X)$, we have

$$\mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \varphi_t(Y) \geq \mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_{A\cap B}(X)} \varphi_t(Y).$$

On the other hand, for any $Y \in \mathcal{Y}^+_{A\cap B}(X)$, and any fixed $Z \in \mathcal{Y}^+_A(X)$ (note that $\mathcal{Y}^+_A(X) \neq \emptyset$), we get

$$\mathbb{1}_B Y + \mathbb{1}_{B^c} Z \in \mathcal{Y}^+_A(X).$$

Thus, using locality of φ_t, we deduce

$$\mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_{A\cap B}(X)} \varphi_t(Y) = \mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_{A\cap B}(X)} \mathbb{1}_B \varphi_t(\mathbb{1}_B Y + \mathbb{1}_{B^c} Z) \geq \mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \varphi_t(Y),$$

which proves (A.14). It is easy to see that $\mathcal{Y}^+_{A\cap B}(X) = \mathcal{Y}^+_{A\cap B}(\mathbb{1}_B X)$, and thus

$$\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_{A\cap B}(X)} \varphi_t(Y) = \mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_{A\cap B}(\mathbb{1}_B X)} \varphi_t(Y). \tag{A.15}$$

Combining (A.14), (A.15), and the fact that $\mathcal{Y}^+_A(X) \neq \emptyset$ implies $\mathcal{Y}^+_A(\mathbb{1}_B X) \neq \emptyset$, we continue

$$\begin{aligned}
\mathbb{1}_B \varphi^+_t(X) &= \mathbb{1}_B \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(\infty)\Big] \\
&= \mathbb{1}_B \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \varphi_t(Y) + \mathbb{1}_{A^c\cap B}(\infty)\Big] \\
&= \mathbb{1}_B \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_{A\cap B}(X)} \varphi_t(Y) + \mathbb{1}_{A^c\cap B}(\infty)\Big] \\
&= \mathbb{1}_B \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_{A\cap B} \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_{A\cap B}(\mathbb{1}_B X)} \varphi_t(Y) + \mathbb{1}_{A^c\cap B}(\infty)\Big] \\
&= \mathbb{1}_B \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(\mathbb{1}_B X)} \varphi_t(Y) + \mathbb{1}_{A^c}(\infty)\Big] \\
&= \mathbb{1}_B \varphi^+_t(\mathbb{1}_B X).
\end{aligned}$$

(Extension) If $X \in \mathcal{X}$, then for any $A \in \mathcal{F}_t$, we get $X \in \mathcal{Y}^+_A(X)$. Thus,

$$\varphi^+_t(X) = \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(\infty)\Big] = \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_A \varphi_t(X) + \mathbb{1}_{A^c}(\infty)\Big] = \varphi_t(X).$$

As the above results are true for any $t \in \mathbb{T}$, thus we have proved that φ^+ is an extension of φ. Let us now show (4.10) for φ^+.

Let $\widehat{\varphi}$ be an extension of φ, and let $X \in \bar{L}^0$ and $t \in \mathbb{T}$. Due to monotonicity and locality of $\widehat{\varphi}_t$, for any $A \in \mathcal{F}_t$ and $Y \in \mathcal{Y}^+_A(X)$, we get $\mathbb{1}_A \widehat{\varphi}_t(X) \leq \mathbb{1}_A \widehat{\varphi}_t(Y)$. Thus, recalling that $\operatorname{ess\,inf} \emptyset = \infty$, we have

$$\widehat{\varphi}_t(X) \leq \mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \widehat{\varphi}_t(Y) + \mathbb{1}_{A^c}(\infty) = \mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(\infty). \tag{A.16}$$

Since (A.16) holds true for any $A \in \mathcal{F}_t$, we conclude that

$$\widehat{\varphi}_t(X) \leq \operatorname*{ess\,inf}_{A\in\mathcal{F}_t} \Big[\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(\infty)\Big] = \varphi^+_t(X).$$

The proof of the second inequality is analogous. $\square$

Proof of Proposition 4.18.

Proof. The first part follows immediately from the definition of LM-extension. Clearly, projectivity of $\widehat{\varphi}$ implies that $\varphi_t(X) = X$, for $X \in \mathcal{X}_t$. To prove the opposite implication, it is enough to prove that φ^+ and φ^- are projective. Assume that φ is such that $\varphi_t(X) = X$, for $t \in \mathbb{T}$ and $X \in \mathcal{X}_t$. Let $X \in \bar{L}^0_t$. For any $n \in \mathbb{N}$, we get

$$\mathbb{1}_{\{n \geq X \geq -n\}}\varphi^+_t(X) = \mathbb{1}_{\{n \geq X \geq -n\}}\varphi^+_t(\mathbb{1}_{\{n \geq X \geq -n\}}X) = \mathbb{1}_{\{n \geq X \geq -n\}}\varphi_t(\mathbb{1}_{\{n \geq X \geq -n\}}X) = \mathbb{1}_{\{n \geq X \geq -n\}}X.$$

Thus, on set $\bigcup_{n\in\mathbb{N}}\{-n \leq X \leq n\} = \{-\infty < X < \infty\}$, we have

$$\varphi^+_t(X) = X, \qquad \text{for } X \in \bar{L}^0_t. \tag{A.17}$$

Next, for any $A \in \mathcal{F}_t$, such that $A \subseteq \{X = \infty\}$, we get $\mathcal{Y}^+_A(X) = \emptyset$, which implies $\mathbb{1}_{\{X=\infty\}}\varphi^+(X) = \infty$. Finally, for any $n \in \mathbb{R}$, using locality of φ^+_t and the fact that $n \in \mathcal{X}_t$, we get

$$\mathbb{1}_{\{X=-\infty\}}\varphi^+_t(X) \leq \mathbb{1}_{\{X=-\infty\}}\varphi^+_t(\mathbb{1}_{\{X=-\infty\}}n) = \mathbb{1}_{\{X=-\infty\}}\varphi_t(n) = \mathbb{1}_{\{X=-\infty\}}n,$$

which implies $\mathbb{1}_{\{X=-\infty\}}\varphi^+(X) = -\infty$. Hence (A.17) holds true on entire space. The proof for φ^- is analogous. □

Proof of Proposition 4.21.

Proof. Let us assume that φ satisfies 1), 2) and it is φ^--acceptance time consistent. Using Proposition 3.5 and the monotonicity of φ^-, we get

$$\infty > \varphi_{t_1}(X_0) \geq \varphi^-_{t_1}(\varphi_{t_2}(X_0)) \geq \varphi^-_{t_1}(\varphi_{t_2}(X_0) \wedge 1) = \varphi_{t_1}(\varphi_{t_2}(X_0) \wedge 1) = \infty,$$

which leads to contradiction. □

Proof of Proposition 4.27.

Proof. Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure.
1) ⇒ 2). Assume that μ is an X-invariant update rule, such that φ is both μ-acceptance and μ-rejection consistent. Then, by Theorem 3.5, $\varphi_t(X) = \mu_{t,s}(\varphi_s(X), 0)$, for any $t \in \mathbb{T}$ and $X \in \mathcal{X}$. Let $s, t \in \mathbb{T}$ and $X, Y \in \mathcal{X}$ be such that $s > t$ and $\varphi_s(X) \geq \varphi_s(Y)$. From the above, and by monotonicity of μ, we have

$$\varphi_t(X) = \mu_{t,s}(\varphi_s(X), 0) \geq \mu_{t,s}(\varphi_s(Y), 0) = \varphi_t(Y).$$

2) ⇒ 1). Let $t, s \in \mathbb{T}$ be such that $s > t$, and consider the following set

$$\mathcal{X}_{\varphi_s} = \{X \in \bar{L}^0 \mid X = \varphi_s(Y) \text{ for some } Y \in \mathcal{X}\}.$$

From 2), for any $X, Y \in \mathcal{X}$, such that $\varphi_s(X) = \varphi_s(Y)$, we get $\varphi_t(X) = \varphi_t(Y)$. Thus, there exists a map $\phi_{t,s} : \mathcal{X}_{\varphi_s} \to \bar{L}^0_t$ such that

$$\phi_{t,s}(\varphi_s(X)) = \varphi_t(X), \quad X \in \mathcal{X}.$$

Next, since there exists $Z \in \mathcal{X}$, such that $\varphi_s(Z) = 0$, using locality of φ, we get that for any $X \in \mathcal{X}_{\varphi_s}$, $A \in \mathcal{F}_t$, there exist $Y \in \mathcal{X}$, so that

$$\mathbb{1}_A X = \mathbb{1}_A \varphi_t(Y) = \mathbb{1}_A \mathbb{1}_A \varphi_t(\mathbb{1}_A Y) + \mathbb{1}_{A^c} \varphi_t(\mathbb{1}_{A^c} Z) = \varphi(\mathbb{1}_A Y + \mathbb{1}_{A^c} Z).$$

Thus, $\mathbb{1}_A X \in \mathcal{X}_{\varphi_s}$, for any $A \in \mathcal{F}_t$, $X \in \mathcal{X}_{\varphi_s}$. Hence, from 2) and locality of φ, for any $X, Y \in \mathcal{X}_{\varphi_s}$, $A \in \mathcal{F}_t$, we get

(A) $X \geq Y \Rightarrow \phi_{t,s}(X) \geq \phi_{t,s}(Y)$;

(B) $\mathbb{1}_A \phi_{t,s}(X) = \mathbb{1}_A \phi_{t,s}(\mathbb{1}_A X)$.

In other words, $\phi_{t,s}$ is local and monotone on $\mathcal{X}_{\varphi_s} \subseteq \bar{L}^0_s$. In view of Remark 4.16), there exists an extension of $\phi_{t,s}$, say $\widehat{\phi}_{t,s} : \bar{L}^0_s \to \bar{L}^0_t$, which is local and monotone on $\bar{L}^0_s$. Finally, we take $\mu_{t,s} : \bar{L}^0_s \times \mathcal{X} \to \bar{L}^0_t$ defined by

$$\mu_{t,s}(m, X) := \widehat{\phi}_{t,s}(m), \qquad X \in \mathcal{X}, m \in \bar{L}^0_s.$$

Clearly the family $\mu_{t,s}$ is an X-invariant update rule, and thus, by Proposition 3.5, φ is both μ-acceptance and μ-rejection time consistent.

The proof of the second part of Proposition 4.27 is immediate. Clearly, 3) $\Rightarrow$ 1) and 3) $\Leftrightarrow$ 4), due to Proposition 3.5. □

Proof of Proposition 4.30.

Proof. Let $\{\varphi_t\}_{t \in \mathbb{T}}$ be a dynamic LM-measure, which is independent of the past.

1) $\Rightarrow$ 2). Assume that μ is an update rule, fulfilling condition from 1), such that φ is both μ-acceptance and μ-rejection consistent. Then, by Proposition 3.5, $\varphi_t(X) = \mu_{t,t+1}(\varphi_{t+1}(X), Y)$, for any $t \in \mathbb{T}'$, $X \in \mathcal{X}$ and $Y \in \mathcal{X}$, such that $X_t = Y_t$. Let $t \in \mathbb{T}'$ and $X, Y \in \mathcal{X}$ be such that $X_t = Y_t$ and $\varphi_{t+1}(X) \geq \varphi_{t+1}(Y)$. From the above, and by monotonicity of μ, we have

$$\varphi_t(X) = \mu_{t,t+1}(\varphi_{t+1}(X), X) = \mu_{t,t+1}(\varphi_{t+1}(X), Y) \geq \mu_{t,t+1}(\varphi_{t+1}(Y), Y) = \varphi_t(Y).$$

2) $\Rightarrow$ 1). Let $t \in \mathbb{T}'$ and consider the following set

$$\mathcal{X}_{\varphi_{t+1}} = \{X \in \bar{L}^0 \mid X = \varphi_{t+1}(Y) \text{ for some } Y \in \mathcal{X}\}.$$

From 2), for any $X, Y \in \mathcal{X}$, such that $\varphi_{t+1}(X) = \varphi_{t+1}(Y)$ and $X_t = Y_t$, we get $\varphi_t(X) = \varphi_t(Y)$. Thus, using independence of the past of φ, there exists a map $\phi_{t,t+1} : \mathcal{X}_{\varphi_{t+1}} \times L^p_t \to \bar{L}^0_t$ such that

$$\phi_{t,t+1}(\varphi_{t+1}(X), Y_t) = \varphi_t(X - 1_{\{t\}}(X_t - Y_t)), \quad X \in \mathcal{X}.$$

Next, since there exists $Z \in \mathcal{X}$, such that $\varphi_{t+1}(Z) = 0$, using locality of φ, we get that for any $X \in \mathcal{X}_{\varphi_{t+1}}$, $A \in \mathcal{F}_t$, there exist $Y \in \mathcal{X}$, so that

$$\mathbb{1}_A X = \mathbb{1}_A \varphi_{t+1}(Y) = \mathbb{1}_A \varphi_{t+1}(\mathbb{1}_A \cdot_{t+1} Y) + \mathbb{1}_{A^c} \varphi_{t+1}(\mathbb{1}_{A^c} \cdot_{t+1} Z) = \varphi_{t+1}(\mathbb{1}_A \cdot_{t+1} Y + \mathbb{1}_{A^c} \cdot_{t+1} Z).$$

Thus, $\mathbb{1}_A X \in \mathcal{X}_{\varphi_{t+1}}$, for any $A \in \mathcal{F}_t$, $X \in \mathcal{X}_{\varphi_{t+1}}$. Hence, from 2) and locality of φ, for any $X, X' \in \mathcal{X}_{\varphi_{t+1}}$, $Y_t \in L^p_t$ and $A \in \mathcal{F}_t$, we get

(A) $X \geq X' \Rightarrow \phi_{t,t+1}(X, Y_t) \geq \phi_{t,t+1}(X', Y_t)$;

(B) $\mathbb{1}_A \phi_{t,t+1}(X, Y_t) = \mathbb{1}_A \phi_{t,t+1}(\mathbb{1}_A X, Y_t)$.

In other words, for any fixed $Y_t \in L^p_t$, $\phi_{t,t+1}(\cdot, Y_t)$ is local and monotone on $\mathcal{X}_{\varphi_{t+1}} \subseteq \bar{L}^0_{t+1}$. In view of Remark 4.16, for any fixed $Y_t \in L^p_t$ there exists an extension (to $\bar{L}^0_{t+1}$) of $\phi_{t,t+1}(\cdot, Y_t)$, say $\widehat{\phi}_{t,t+1}(\cdot, Y_t)$, which is local and monotone on $\bar{L}^0_{t+1}$. Finally, we take $\mu_{t,t+1} : \bar{L}^0_{t+1} \times \mathcal{X} \to \bar{L}^0_t$ defined by

$$\mu_{t,t+1}(m, X) := \widehat{\phi}_{t,t+1}(m, X_t), \qquad X \in \mathcal{X}, m \in \bar{L}^0_{t+1}.$$

Clearly the family $\mu_{t,t+1}$ is a (one step) update rule. Moreover, we get

$$\mu_{t,t+1}(m, X) = \mu_{t,t+1}(m, X'),$$

for $m \in \bar{L}^0_{t+1}$ and $X, X' \in \mathcal{X}$, such that $X_t = X'_t$. Finally, by Proposition 3.5, φ is both μ-acceptance and μ-rejection time consistent, as

$$\varphi_t(X) = \varphi_t(X - 1_{\{t\}}(X_t - X_t)) = \phi_{t,t+1}(\varphi_{t+1}(X), X_t) = \mu_{t,t+1}(\varphi_{t+1}(X), X).$$

The proof of the second part of Proposition 4.30 is immediate. Clearly, 3) $\Rightarrow$ 1) and 3) $\Leftrightarrow$ 4), due to Proposition 3.5. □

Proof of Proposition 4.32.

Proof. Let us consider $\{\phi_t\}_{t\in\mathbb{T}}$ and $\{\phi'_t\}_{t\in\mathbb{T}}$ as given in (4.15).

1) The proof of monotonicity and locality is similar to the one for conditional essential infimum and supremum, Proposition 2.4. Finally, for any $t \in \mathbb{T}$, $Z \in D_t$ and $m \in \bar{L}^0_t$, since $E[Z|\mathcal{F}_t] = 1$, we immediately get

$$E[Zm|\mathcal{F}_t] = 1_{\{m\geq 0\}} m E[Z|\mathcal{F}_t] + 1_{\{m<0\}}(-m)E[-Z|\mathcal{F}_t] = m,$$

and thus, $\phi_t(m) = \phi'_t(m) = m$, for any $m \in \bar{L}^0_t$. Hence, $\{\phi_t\}_{t\in\mathbb{T}}$ is projective.

2) Let $\{\varphi_t\}_{t\in\mathbb{T}}$ be a dynamic LM-measure which is ϕ-rejection time consistent, and $g : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ be an increasing, concave function. Then, for any $X \in \mathcal{X}$, we get

$$g(\varphi_t(X)) \geq g(\phi_t(\varphi_s(X)) = g(\operatorname*{ess\,inf}_{Z\in D_t} E[Z\varphi_s(X)|\mathcal{F}_t]) = \operatorname*{ess\,inf}_{Z\in D_t} g(E[Z\varphi_s(X)|\mathcal{F}_t]. \tag{A.18}$$

Recall that any $Z \in D_t$ is a Radon-Nikodym derivative of some measure Q with respect to P, and thus we have $E[ZX|\mathcal{F}_t] = E_Q[X|\mathcal{F}_t]$. Hence, by Jensen's inequality, we deduce

$$\operatorname*{ess\,inf}_{Z\in D_t} g(E[Z\varphi_t(X)|\mathcal{F}_t]) \geq \operatorname*{ess\,inf}_{Z\in D_t} E[Zg(\varphi_t(X))|\mathcal{F}_t] = \phi_t(g(\varphi_s(X))). \tag{A.19}$$

Combining (A.18) and (A.19), ϕ-acceptance time consistency of $\{g \circ \varphi_t\}_{t\in\mathbb{T}}$ follows.

3) The proof is analogues to 2). □

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