DYNAMIC COHERENT ACCEPTABILITY INDICES AND THEIR APPLICATIONS IN FINANCE

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ABSTRACT

This thesis presents a unified framework for studying coherent acceptability indices in a dynamic setup.

We study dynamic coherent acceptability indices and dynamic coherent risk measures. In particular, we establish a duality between them. We derive representation theorems for both dynamic coherent acceptability indices and dynamic coherent risk measures in terms of so called dynamically consistent sequence of sets of probability measures.

In addition, we present an alternative approach to study dynamic coherent acceptability indices and the representation theorem.

Finally, we provide examples and counterexamples of dynamic coherent acceptability indices, and their applications in portfolio management.

CHAPTER 1
INTRODUCTION

In this introductory chapter we provide the motivation for our research. We were inspired by some problems pertaining to financial risk management, where one needs to assess potential rewards vis-a-vis potential losses associated with any financial investment process, or financial portfolio management process.

A financial portfolio is a collection of investment assets held by professional institutions or individuals. The assets in the portfolio may include money market accounts, stocks, bonds, forwards, futures, options, swaps and other synthetic or hybrid financial products. In modern finance, a portfolio is becoming more and more complex, consisting of various classes of assets, and the complexity requires advanced mathematical tools to manage it successfully.

Any financial portfolio gives rise to various cash flows that are typically distributed in time. Such cash flows will be referred to as cash flow stream. In general, there are two important concepts concerned with a cash flow stream: “return” and “risk”.

Portfolio managers are responsible for selecting investment assets for their portfolios. The composition of a portfolio typically changes in time. It is important to assess quality of a portfolio mix as time progresses not only ex–post, but also ex–ante. Our research will contribute to handle this problem by providing precise mathematical measures to differentiate or rank various portfolios.

Traditionally, in the area of optimal portfolio selection, utility functions are used to discriminate between portfolios, or, in other words, to assess quality of the portfolio vis-a-vis the chosen utility funtion. Utility functions however suffer from certain disadvantages, that make their use of limited value, given one wants to assess performance of a portfolio looking at portfolio's return versus portfolio's risk. In particular, it does not seem to be possible to interpret classical measures of financial performance, such as Sharpe ratio or Gain-to-Loss ratio, in terms of utility functions. The study presented in this thesis, is in fact meant to begin mathematical investigation of abstract form of classical financial measures of performance in the context of dynamic investment processes.

Return on investment (ROI), relative to a given portfolio, is a measure of how profitable a cash flow stream is. One classical ROI is *simple return*, which is defined as the relative change in the value of a portfolio over a specified time horizon. For a comprehensive discussion of classical ROIs we refer to [26].

Typical ROIs do not account for the riskiness of the portfolio. Consequently, an investment manager does not assess the quality of her/his investment strategy solely based on analysis of a ROI. An appropriate measure of riskiness of the portfolio, or of portfolio risk, needs to be accounted for as well.

Developing good and useful measures of portfolio risk has become an important research activity over the past several years, both in academia and in financial industry (cf. [17, 23, 27, 36, 10, 48]). Advanced mathematical tools are usually needed to handle the complex distribution of the cash flow stream, and hence to measure the risk. An overview of the mathematics of (static) risk measures will be given in chapter 2.

Generally speaking, higher return implies higher risk. The risk tolerance for a portfolio is usually limited. Hence, portfolio managers are typically concerned with finding satisfactory balance between return and risk associated with an investment process when making decisions. Various measures have been developed to quantify this balance. Such measures are typically referred to as *performance measures* or *measures of performance* (MOP).

A classical MOP, widely used in the financial industry, is the Sharpe Ratio (SR) introduced by Sharpe in [46]. It is defined as $\mathrm{SR} := \frac{\mathbb{E}[R-R_f]}{\sigma}$ if $\mathbb{E}[R-R_f] > 0$ and 0 otherwise, where R is the portfolio return, R_f is the risk-free rate, $\mathbb{E}[R-R_f]$ is the expected value of the excess return $R - R_f$, and σ is the standard deviation of the excess return.

SR is expressed as a ratio of expected excess return to standard deviation, and thus in financial applications it measures expected excess return of a portfolio in units of portfolio's standard deviation. SR is therefore used to characterize how well the return of an asset compensates the investor for the risk taken, and as a classical tool to rank portfolios. The higher the SR is, the better the portfolio performs.

However, SR has some well-documented weaknesses. The major drawback of SR is that it uses standard deviation to quantify risk. The reason of course is that positive returns also contribute to this measure of risk. To eliminate this unwanted feature, other ratio-types MOPs were proposed after SR, such as Sortino Ratio (SOR) (cf. [47]), Gain Loss Ratio (GLR) (cf. [5]) and Risk Adjusted Return on Capital (RAROC) (cf. [34]):

Definition 1.0.1. *Sortino Ratio is defined as follows:*

$$\mathrm{SOR}(X) := \begin{cases} \frac{\mathbb{E}[X]}{\sqrt{\mathbb{E}[(X^-)^2]}}, & \text{if} \quad \mathbb{E}[X] > 0\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

where X denotes the portfolio excess return $R - R_f$ and $X^- = \max\{-X, 0\}$. By convention, $\mathrm{SOR}(0) = +\infty$*, where* 0 *stands for no cash flows.*

Definition 1.0.2. *Gain Loss Ratio is defined as the ratio of the mean excess return to the expectation of the negative excess return:*

$$\mathrm{GLR}(X) := \begin{cases} \frac{\mathbb{E}[X]}{\mathbb{E}[X^-]}, & \text{if} \quad \mathbb{E}[X] > 0\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

where X denotes the portfolio excess return $R - R_f$ and $X^- = \max\{-X, 0\}$. By convention, $\mathrm{GLR}(0) = +\infty$*, where* 0 *stands for no cash flows.*

Definition 1.0.3. *Risk-Adjusted Return on Capital is defined as the ratio of the mean excess return to some selected risk measure ρ:*

$$\mathrm{RAROC}(X) := \begin{cases} \frac{\mathbb{E}[X]}{\rho[X]}, & \text{if} \quad \mathbb{E}[X] > 0\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

where X denotes the portfolio excess return $R - R_f$. By convention, $\mathrm{RAROC}(X) = +\infty$ *if $\rho(X) \leq 0$.*

We note that the above MOPs focus on downside risk, which makes them more attractive as compared with the Sharpe ratio.

All the MOPs mentioned above share some common desirable features: they are unit-less, they are increasing functions of return and decreasing functions of risk; moreover, according to these MOPs diversification of a portfolio improves its performance[1]. This observation inspires a natural study of MOPs in a unified mathematical framework.

Recently, Cherny and Madan [14] originated an effort to provide a mathematical framework to study these measures in a unified way. The study of [14] was done in static, one-time period setup. Cherny and Madan coined the term Acceptability

[1]See Proposition 2.1.1 for a strict formulation

Index (AI) as a mathematical terminology for MOPs. Our research goal is to elevate the mathematical framework for studying AIs to dynamical, multi-period setup, where cash flows are considered as random processes, and one needs to assess their acceptability consistently in time. In particular, we are concerned not just with the total cumulative terminal value of the cash flow stream as seen from the initial time of the investment process, but also with all remaining cumulative cash flows between each intermediate time and the terminal time of the investment process.

As a parallel research, we also contributed to the extension of static risk measures to dynamic setup. It will be seen that there is a strong duality relationship between acceptability indices and risk measures in both static and dynamic frameworks. It should be also mentioned that the theory we developed here covers dynamic versions of some classical MOPs, and it is not just a theory in itself.

The rest of the thesis is organized as follows: A survey of framework for studying *static coherent acceptability indices* (SCAIs) will be presented in Chapter 2 following Cherny and Madan [14]. In Chapter 2 we shall also provide a brief survey of these aspects of the theory of *static coherent risk measures* (SCRMs) that are relevant for this thesis. Next, in Chapter 3, we shall extend the static theory of Chapter 2 to dynamic framework; in particular, we shall present the duality between *dynamic coherent acceptability indices* (DCAIs) and *dynamic coherent risk measures* (DCRMs). The major result, a representation theorem for DCAIs in terms of dynamically consistent sequence of sets of probability measures, will be derived in Chapter 4. Another important result – the representation theorem for DCRMs – will also be presented in Chapter 4. In Chapter 5 we introduce an alternative way to define dynamic coherent acceptability indices; indices derived in this way are termed alternative dynamic coherent acceptability indices (ADCAIs), and we provide the corresponding representation theorem. The study done in this chapter is based on theory of DCRMs developed by Riedel in [42]. Finally, in Chapter 6, we discuss some examples of DCAIs, and their applications. In particular, we show that dGLR, which will be defined in Chapter 6, is a DCAI but is not an ADCAI.

The results of Chapter 3 and Chapter 4 are to be published in a paper (cf. [6]) that is in revision for Mathematical Finance.

CHAPTER 2

STATIC THEORY: COHERENT ACCEPTABILITY INDICES AND RISK MEASURES

In this chapter, we review the theory of static coherent acceptability indices (SCAIs), developed in [14], and we review the theory of static coherent risk measures (SCRMs) originated in [3].

2.1 SCAIs and SCRMs

We assume a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As a matter of fact, for risk measure theory, many results can be extended to an arbitrary probability space, but the proofs become very technical. For acceptability index theory, we work within finite probability space. The extension to general probability space is part of future work.

We denote by $\mathcal{G}$ the space of all bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. The random variable $X \in \mathcal{G}$ can be regarded as the total cumulative terminal value of a cash flow stream. We denote by $\mathcal{P}$ the set of all probability measures absolutely continuous to the reference probability measure $\mathbb{P}$. In addition, throughout this thesis, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{N}$ denotes the set of all natural numbers without number 0.

2.1.1 Definition. A risk measure is a function $\rho : \mathcal{G} \rightarrow \mathbb{R}$, whereas an acceptability index is a function $\alpha : \mathcal{G} \rightarrow [0, +\infty]$. Risk measure is used to determine the amount of possible loss at the end of investment period. Hence, it is in dollar unit and should be valued in $\mathbb{R}$. However, acceptability index is understood as the degree of acceptability of a cash flow; in a sense, it represents a measure of efficiency of the cash flow. A larger index indicates better performance, with $\alpha(X) = +\infty$ for X being an 'arbitrage opportunity' and $\alpha(X) = 0$ for X being a 'surely loss' portfolio. Acceptability index is therefore an ordinal and unitless concept, which can be valued on the extended positive half of the real line $[0, +\infty]$.

Acceptability index and risk measure as such are too broad concepts, and they may not fulfill certain practically desirable properties. That is why researchers focused their attention on more specific concepts of the coherent acceptability index and coherent risk measure.

Definition 2.1.1. *A function $\alpha : \mathcal{G} \to [0, +\infty]$ is called static coherent acceptability index if the following properties are satisfied:*

(S1) Monotonicity. If $X \leq Y$, then $\alpha(X) \leq \alpha(Y)$;

(S2) Scale invariance. For every $X \in \mathcal{G}$ and $\lambda > 0$, $\alpha(\lambda X) = \alpha(X)$;

(S3) Quasi-concavity. If $\alpha(X) \geq x$, $\alpha(Y) \geq x$ for some $x \in (0, +\infty]$, then $\alpha(\lambda X + (1 - \lambda)Y) \geq x$ for all $\lambda \in [0, 1]$;

(S4) Fatou Property. If $|X_n| \leq 1$, $\alpha(X_n) \geq x$ for all $n \geq 1$, and $X_n \to X$, as $n \to \infty$, in probability, then $\alpha(X) \geq x$.

The above properties have natural financial interpretation. For example, (S1) states that if Y dominates X at every state $\omega \in \Omega$, then Y is acceptable at least at the same level as X is; (S2) implies that cash flows with the same direction of trade have the same level of acceptance. Quasi-concavity (S3) implies that a diversified portfolio improves the performance of its components; to see this, it is enough to take $x = \min\{\alpha(X), \alpha(Y)\}$. Fatou Property (S4) is a technical continuity property, which is used for constructing the duality between coherent acceptability indices and coherent risk measures.

Proposition 2.1.1. *SR, SOR, GLR and RAROC all satisfy (S3)-quasi-concavity.*

Proof. We only show that SR and SOR are quasi-concave. GLR and RAROC have been verified to be quasi-concave in [14]. For convenience, denote by X and Y excess returns for two arbitrary portfolios.

First, we show that SR is quasi-concave. If $\mathrm{SR}(X) \geq x$ and $\mathrm{SR}(Y) \geq x$ for some $x > 0$, by the definition of SR, we can see that $\mathbb{E}[X] > 0$, $\mathbb{E}[Y] > 0$, and

$$\mathrm{SR}(X) = \frac{\mathbb{E}[X]}{\sigma(X)} \geq x\,, \quad \mathrm{SR}(Y) = \frac{\mathbb{E}[Y]}{\sigma(Y)} \geq x\,.$$

If $\sigma(X) = 0$ or $\sigma(Y) = 0$, then X or Y will be a positive constant. In this case, by the properties of standard deviation, we can verify that $\mathrm{SR}(\lambda X + (1-\lambda)Y) = \frac{\mathbb{E}[\lambda X+(1-\lambda)Y]}{\sigma(\lambda X+(1-\lambda)Y)} \geq x$ for all $\lambda \in [0,1]$.

If both $\sigma(X) > 0$ and $\sigma(Y) > 0$, then for all $\lambda \in [0,1]$,

$$\mathbb{E}\big[\lambda X + (1-\lambda)Y\big] = \lambda\mathbb{E}[X] + (1-\lambda)\mathbb{E}[Y] \geq x\lambda\sigma(X) + x(1-\lambda)\sigma(Y)\,. \tag{2.1}$$

Let ρ be the correlation between X and Y, then $-1 \leq \rho \leq 1$ and

$$\begin{aligned}
\sigma^2(\lambda X + (1-\lambda)Y) &= \sigma^2(\lambda X) + \sigma^2\big((1-\lambda)Y\big) + 2\rho\sigma(\lambda X)\sigma\big((1-\lambda)Y\big) \\
&\leq \sigma^2(\lambda X) + \sigma^2\big((1-\lambda)Y\big) + 2\sigma(\lambda X)\sigma\big((1-\lambda)Y\big) \\
&= \Big(\sigma(\lambda X) + \sigma\big((1-\lambda)Y\big)\Big)^2\,,
\end{aligned}$$

which implies that $\sigma\big(\lambda X + (1-\lambda)Y\big) \leq \sigma(\lambda X) + \sigma\big((1-\lambda)Y\big) = \lambda\sigma(X) + (1-\lambda)\sigma(Y)$. Then, (2.1) gives

$$\mathbb{E}\big[\lambda X + (1-\lambda)Y\big] \geq x\sigma(\lambda X + (1-\lambda)Y)\,.$$

In addition, we have $\mathbb{E}[\lambda X + (1-\lambda)Y] = \lambda\mathbb{E}[X] + (1-\lambda)\mathbb{E}[Y] > 0$. By the definition of SR,

$$\mathrm{SR}\big(\lambda X + (1-\lambda)Y\big) = \frac{\mathbb{E}\big[\lambda X + (1-\lambda)Y\big]}{\sigma\big(\lambda X + (1-\lambda)Y\big)} \geq x\,.$$

This concludes the proof that SR is quasi-concave.

By Cauchy-Schwarz inequality, we have

$$\mathbb{E}[X^-Y^-] \leq \sqrt{\mathbb{E}[(X^-)^2]\mathbb{E}[(Y^-)^2]}\,.$$

Then, by the similar way for SR, SOR can be proved quasi-concave with the following fact:

$$\begin{aligned}&\left(\lambda\sqrt{\mathbb{E}[(X^-)^2]}+(1-\lambda)\sqrt{\mathbb{E}[(Y^-)^2]}\right)^2\\ =&\lambda^2\mathbb{E}[(X^-)^2]+(1-\lambda)^2\mathbb{E}[(Y^-)^2]+2\lambda(1-\lambda)\sqrt{\mathbb{E}[(X^-)^2]\mathbb{E}[(Y^-)^2]}\\ \geq&\lambda^2\mathbb{E}[(X^-)^2]+(1-\lambda)^2\mathbb{E}[(Y^-)^2]+2\lambda(1-\lambda)\mathbb{E}[X^-Y^-]\\ =&\mathbb{E}\left[\left(\lambda X^-+(1-\lambda)Y^-\right)^2\right]\geq\mathbb{E}\left[\left((\lambda X+(1-\lambda)Y)^-\right)^2\right].\end{aligned}$$

□

Now we shall introduce the definition of static coherent risk measures originated in [3].

Definition 2.1.2. *A function $\rho : \mathcal{G} \to \mathbb{R}$ is called static coherent risk measure if the following properties are satisfied:*

(R1) Monotonicity. If $X \leq Y$, then $\rho(X) \geq \rho(Y)$;

(R2) Positive homogeneity. For every $X \in \mathcal{G}$ and $\lambda \geq 0$, $\rho(\lambda X) = \lambda\rho(X)$;

(R3) Translation property. For every $X \in \mathcal{G}$ and $c \in \mathbb{R}$, $\rho(X + c) = \rho(X) - c$;

(R4) Subadditivity. For every $X \in \mathcal{G}$ and $Y \in \mathcal{G}$, $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Monotonicity (R1) implies that higher cash flow stream Y should have lower risk. Positive homogeneity (R2) indicates that scaling a portfolio will also scale its risk. (R3) means that if an investor adds or subtracts a deterministic amount of cash into the portfolio, the risk will be reduced or increased by the same amount.

Subadditivity (R4) is the diversification property, which agrees with the well-known investment principle that a diversified portfolio has lower risk. Note that, (R2) and (R3) together imply that a portfolio with a deterministic future cash flow $c \in \mathbb{R}$ has risk $-c$.

The theory of static risk measures has been explored and extended by many researchers; to mention just a few of them: Fölmer and Schied [29, 30] generalized the concept of SCRMs to static convex; law-invariant risk measures have been investigated by Kusuoka [39]; for a systematic discussion on static risk measures we refer reader to the monographs by Delbaen [20] and Fölmer and Schied [31].

Many researchers have also contributed to the extension of risk measure theory to dynamic framework, see for instance [4, 9, 11, 12, 13, 18, 32, 33, 37, 42, 44, 50]. Our first research result, which is named *alternative dynamic coherent acceptability indices* (ADCAIs), is established on the DCRMs theory by Riedel in [42]. Riedel's theory will be shown in Appendix A, and ADCAIs will be presented in Chapter 5.

Now we show the duality between SCRMs and SCAIs. This is one of the main contribution by Cherny and Madan in [14].

Theorem 2.1.1. *An unbounded above function $\alpha : \mathcal{G} \to [0, +\infty]$ is a SCAI if and only if there exists an increasing family of SCRMs $(\rho^x)_{x\in(0,+\infty)}$, such that $\rho^x(X) \le \rho^y(X)$ for all $X \in \mathcal{G}$ with $x \le y$, and*

$$\alpha(X) = \sup \{x \in (0, +\infty) \ : \ \rho^x(X) \le 0\} , \tag{2.2}$$

where $\inf \emptyset = \infty$ *and* $\sup \emptyset = 0$.

This theorem indicates that every SCAI can be characterized in terms of an increasing family of SCRMs $(\rho^x)_{x\in(0,+\infty)}$, and vice versa.

2.1.2 Examples.

We will introduce some examples of static risk measures. A traditional and popular risk measure being used in financial industry is Value at Risk (VaR) (cf. [24, 35]), defined as $\mathrm{VaR}_\alpha(X) := \inf\{c \in \mathbb{R} \mid \mathbb{P}[X + c < 0] \leq \alpha\}$ for $\alpha \in (0, 1)$. It can be verified that VaR does not satisfy subadditivity (R4), and therefore is not a SCRM.

Based on VaR, researchers proposed Tail Value at Risk (TVaR) (cf. [2]), defined as $\mathrm{TVaR}_\alpha(X) := -\inf_{\mathbb{Q}\in\mathcal{Q}_\alpha} \mathbb{E}_\mathbb{Q}[X]$, for a given $\alpha \in (0, 1]$, where $\mathcal{Q}_\alpha$ is the set of probability measures absolutely continuous with respect to $\mathbb{P}$ such that $d\mathbb{Q}/d\mathbb{P} \leq \alpha^{-1}$.

It is well-known that TVaR is a SCRM. Intuitively, VaR differs from TVaR that it only indicates in what probability the loss will exceed a certain amount without knowing how bad it is, whereas TVaR measures how bad the loss will be.

Next, we discuss several examples of static coherent acceptability indices. First, note that Sharpe Ratio does not satisfy the monotonicity (S1), and hence it is not a static coherent acceptability index. Gain Loss Ratio, however, is verified to be a SCAI by Cherny and Madan in [14]. They also showed that RAROC is a static coherent acceptability index, if coherent risk measure is selected to define RAROC.

The Duality Theorem 2.1.1 provides another way to construct examples for static coherent acceptability indices. For example, since TVaR is a static coherent risk measure, we define AIT as $\mathrm{AIT}(X) := \sup\{x \in (0, +\infty), \mathrm{TVaR}_{\frac{1}{1+x}}(X) \leq 0\}$. By the Theorem 2.1.1, it is a static coherent acceptability index. More examples such as AIW, AIMIN, AIMAX, AIMINMAX, AIMAXMIN etc, have been also presented in [14].

2.2 Set of Probability Measures and Representation Theorems

In this section, we shall discuss how to represent both static coherent risk measure and static coherent acceptability index in terms of set of probability measures. Mathematically, it provides abstract and uniform formulas to study SCRMs and SCAIs; practically, it also endows them with a straightforward financial meaning by regarding each *probability measure* as a market *scenario* or *condition*.

The *Standard Portfolio Analysis of Risk* (SPAN) system is a popular measure for assessing portfolio risk. It was developed and implemented by Chicago Mercantile Exchange (CME) in 1988. Since then, it has become the industry standard and has been adopted by most options and futures exchanges over the world.

SPAN evaluates overall portfolio risk by calculating the worst possible loss that a portfolio may have, given sixteen different scenarios or market conditions. Artzner et al in [3] provides a detailed example regarding the SPAN computation, and shows that the calculation can be viewed as producing the maximum of the expected loss under each of sixteen probability measures.

This methodology, however, allows users to extend to any number of scenarios to meet their particular needs. In the mathematical model, a collection of scenarios is understood as a set of probability measures. Therefore, we can extend SPAN to define a risk measure given any specific set of probability measures (scenarios).

Definition 2.2.1. *Given any non-empty set of probability measures (scenarios) $\mathcal{Q}$, define $\rho_{\mathcal{Q}}$ as follows*

$$\rho_{\mathcal{Q}}(X) := -\inf\{\mathbb{E}_{\mathbb{Q}}[X] \ : \ \mathbb{Q} \in \mathcal{Q}\}\,.$$

It can be shown that $\rho_{\mathcal{Q}}$ is a SCRM. It is called risk measure with respect to the set $\mathcal{Q}$ of probability measures.

The risk defined above has a straightforward financial meaning, which claims that given a set of probability measures (market scenarios), the risk is simply the negative value of minimum expectation under the specified set. It is actually the worst condition a portfolio manager can expect under a certain set of scenarios regarding market change.

One of the greatest results in modeling risk measure theory is that any coherent risk measure can be represented by a set of probability measures (scenarios), which leads to the following representation theorem in [3]:

Theorem 2.2.1. *A function* $\rho : \mathcal{G} \to \mathbb{R}$ *is a coherent risk measure if and only if*

$$\rho(X) = -\inf\{\, \mathbb{E}_{\mathbb{Q}}[X] \;:\; \mathbb{Q} \in \mathcal{Q}\}\,, \tag{2.3}$$

for a certain set $\mathcal{Q}$ *of probability measures absolutely continuous with respect to* $\mathbb{P}$.

The above representation theorem is established in [3] for finite Ω, and generalized to a general probability space in [21].

It is worth to mention that representation theorem does not imply a one-to-one map. That is, there may exist two different sets of probability measures, which give rise to the same coherent risk measure by (2.3). As a matter of fact, it can be verified that any set of probability measures and its closed convex hull generate the same coherent risk measure.

Definition 2.2.2. *In a real vector space* V*, for any subset* $X \subset V$*, the convex hull* X^C *of* X *is defined as follows:*

$$X^C := \Big\{ \sum_{i=1}^{n} \lambda_i x_i \;\Big|\; x_i \in X, \lambda_i \in \mathbb{R}, \lambda > 0, \sum_{i=1}^{n} \lambda_i = 1 \Big\}\,,$$

where n *can be an arbitrary natural number.*

Proposition 2.2.1. *Let* $\mathcal{Q}$ *be a subset of* $\mathcal{P}$*. Denote the closed convex hull of* $\mathcal{Q}$ *by* $\bar{\mathcal{Q}}^C$*. We have*

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X] = \inf_{\mathbb{Q}\in\bar{\mathcal{Q}}^C} \mathbb{E}_{\mathbb{Q}}[X]\,,$$

for all $X \in \mathcal{G}$*. Hence,* $\mathcal{Q}$ *and* $\bar{\mathcal{Q}}^C$ *generate the same coherent risk measure defined by* (2.3).

Proof. We observe that $\mathcal{Q}$ is a subset of its closed convex hull $\bar{\mathcal{Q}}^C$, and thus

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X] \geq \inf_{\mathbb{Q}\in\bar{\mathcal{Q}}^C} \mathbb{E}_{\mathbb{Q}}[X] \,.$$

The converse inequality will be verified in two steps. First, we consider $\mathcal{Q}^C$ the convex hull of $\mathcal{Q}$. For any $\mathbb{Q}_0 \in \mathcal{Q}^C$, by the Definition 2.2.2, there exists $\mathbb{Q}_1, \mathbb{Q}_2, \cdots, \mathbb{Q}_n$ such that $\mathbb{Q}_0 = \sum_{i=1}^n \lambda_i \mathbb{Q}_i$ with $\lambda_i > 0$, and $\sum_{i=1}^n \lambda_i = 1$. Hence, by the linearity of expectation,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_0}[X] &= \sum_{i=1}^n \big(\lambda_i \mathbb{E}_{\mathbb{Q}_i}[X]\big) \\
&\geq \sum_{i=1}^n \big(\lambda_i \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X]\big) \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X] \sum_{i=1}^n \lambda_i = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X] \,,
\end{aligned}$$

for any $X \in \mathcal{G}$. Then, by Lemma B.0.1, we conclude that

$$\inf_{\mathbb{Q}\in\mathcal{Q}^C} \mathbb{E}_{\mathbb{Q}}[X] \geq \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X] \,. \tag{2.4}$$

Next step, note that $\bar{\mathcal{Q}}^C$ is the closure of $\mathcal{Q}^C$, then for any $\mathbb{Q}_0 \in \bar{\mathcal{Q}}^C$, there exists a sequence $(\mathbb{Q}_1, \mathbb{Q}_2, \cdots)$ with each $\mathbb{Q}_n \in \mathcal{Q}^C$ such that

$$\lim_{n\to\infty} \mathbb{Q}_n = \mathbb{Q}_0 \,.$$

Since we are in finite probability space, the above limit is state-wise. By the linearity of expectation and finiteness of the probability space,

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[X] = \mathbb{E}_{\mathbb{Q}_0}[X] \,,$$

for all $X \in \mathcal{G}$. Then, $\mathbb{E}_{\mathbb{Q}_0}[X] \geq \inf_{\mathbb{Q}\in\mathcal{Q}^C} \mathbb{E}_{\mathbb{Q}}[X]$. By Lemma B.0.1,

$$\inf_{\mathbb{Q}\in\bar{\mathcal{Q}}^C} \mathbb{E}_{\mathbb{Q}}[X] \geq \inf_{\mathbb{Q}\in\mathcal{Q}^C} \mathbb{E}_{\mathbb{Q}}[X] \,.$$

Together with (2.4), we get

$$\inf_{\mathbb{Q}\in\bar{\mathcal{Q}}^C} \mathbb{E}_{\mathbb{Q}}[X] \geq \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X]\,.$$

Finally, we conclude that, for any $X \in \mathcal{G}$,

$$\inf_{\mathbb{Q}\in\bar{\mathcal{Q}}^C} \mathbb{E}_{\mathbb{Q}}[X] = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X]\,.$$

By the Definition 2.2.1, $\mathcal{Q}$ and $\bar{\mathcal{Q}}^C$ must generate the same coherent risk measure. □

We will extend this proposition to dynamic framework in Chapter 4, which is essential to verify the representation theorem for dynamic coherent acceptability indices.

We conclude this chapter by presenting the representation theorem for SCAIs (cf [14]). This is a direct result of Duality Theorem 2.1.1 and Representation Theorem 2.2.1 for SCRMs.

Theorem 2.2.2. *An unbounded above function $\alpha : \mathcal{G} \to [0, +\infty]$ is a SCAI if and only if there exists an increasing family $(\mathcal{D}_x)_{x\in(0,+\infty]}$ of sets of probability measures, such that $\mathcal{D}_x \subset \mathcal{D}_y$ for $x \leq y$, and α admits the following representation*

$$\alpha(X) = \sup\left\{x \in (0, +\infty) \ : \ \inf_{Q\in\mathcal{D}_x} \mathbb{E}_{\mathbb{Q}}[X] \geq 0\right\}\ , \tag{2.5}$$

where $\inf\emptyset = \infty$ *and* $\sup\emptyset = 0$.

The representation theorem indicates that any SCAI can be characterized by an increasing family of sets of probability measures. On the other hand, given any increasing family of sets of probability measures, we can define a SCAI through (2.5).

CHAPTER 3

DYNAMIC COHERENT ACCEPTABILITY INDICES AND DYNAMIC COHERENT RISK MEASURES

In this chapter, we introduce the theory of dynamic coherent acceptability indices (DCAIs). We start with presenting the mathematical framework, and then proceed to the properties used to define DCAIs. As a parallel research, we also study dynamic coherent risk measures (DCRMs), as well as the duality between DCAIs and DCRMs.

3.1 Mathematical Preliminaries

Typically, in dynamic framework, new market information is updated as time moves forward. The new information may include underlying assets price movement, new economic policies or political events etc. The dynamic acceptability indices should be able to assess performance of the cash flow stream accounting for the newly acquired information.

Note that one may attempt to use a sequence of static (one-period) acceptability indices. However, by doing this one may end up with a sequence of measurements that are not consistent in time and contradict the updated information, in the sense to be explained below (cf. Property D7). The motivation for developing a theory of DCAIs was to provide performance measurements consistently in time and compatible with the information process.

To avoid technical problems, we consider a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and finite time horizon $\mathcal{T} = \{0, 1, 2, \ldots, T\}$. The finiteness of probability and time space is a good starting point to carry out research on DCAIs. Analogous to static theory, the proofs will become very technical if we extend to an arbitrary probability space and continuous time.

We assume that the reference probability measure $\mathbb{P}$ is of full support. In the finite probability space, this assumption is used to eliminate the irrelevant states. Throughout the rest of the thesis, we adopt the usual[2] convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

To facilitate the proofs in later sections, we introduce the concept of partition and give some basic results of partition in finite probability space.

Definition 3.1.1. *A partition of a probability space Ω is a collection of exhaustive and mutually exclusive subsets,*

$$\{P_1, \ldots, P_n\}, \text{ such that } P_i \cap P_j = \emptyset, \forall\ i \neq j \text{ and } \cup_{k=1}^{n} P_k = \Omega\,.$$

In a finite probability space, the algebra of events generated by the partition is the collection of all unions of P_j's. These sets $(P_i)_{i=1,\ldots,n}$ are the fundamental building blocks for the algebra. In fact, it has been shown in [38] that in finite probability space Ω, any algebra is generated by a unique partition of Ω.

Definition 3.1.2. *Algebra $\mathcal{F}_1$ is said to be included in algebra $\mathcal{F}_2$ if $\mathcal{F}_1 \subset \mathcal{F}_2$.*

If $\mathcal{F}_1$ is included in $\mathcal{F}_2$, the partition that generates $\mathcal{F}_2$ has finer sets than the ones that generate $\mathcal{F}_1$.

In dynamic framework, we endow the underlying probability space Ω with the sequence of algebras, called filtration which models the flow of information.

Definition 3.1.3. *A filtration $\mathbb{F}$ is the collection of algebras,*
$\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_t, \ldots, \mathcal{F}_T\}$ *with* $\mathcal{F}_t \subset \mathcal{F}_{t+1}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, *and* $\mathcal{F}_T = \mathcal{F}$.

As time passes, an observer knows more and more detailed information, that is, finer and finer partitions of Ω, as illustrated in the following corollary that will be

[2]Same as the convention in Theorem 2.2.2

applied in the later proofs.

Proposition 3.1.1. *For $\mathcal{F}_t \in \mathbb{F}$, there exists a unique partition of Ω, say $\{P_1^t, P_2^t, \ldots, P_{n_t}^t\}$, that generates $\mathcal{F}_t$. And for each $P_i^t, i = 1, \ldots, n_t$, there exists $\{P_{i,1}^{t+1}, P_{i,2}^{t+1}, \ldots, P_{i,j_i}^{t+1}\}$, which is a subset of the partition that uniquely generates $\mathcal{F}_{t+1}$, such that $P_i^t = P_{i,1}^{t+1} \cup P_{i,2}^{t+1} \cdots \cup P_{i,j_i}^{t+1}$, where $j_i \in \mathbb{N}$.*

Denote by $\Upsilon^t := \{P_1^t, P_2^t, \ldots, P_{n_t}^t\}$ the unique partition of Ω at time t that generates $\mathcal{F}_t$. Thus, the number of the elements in Υ^t is n_t. A cash flow stream is modeled as a stochastic process instead of random variable. We denote such processes by $D = \{D_t(\omega)\}_{t=0}^T$, which are adapted to the filtration $\mathbb{F}$. We also denote by $\mathcal{D}$ the set of all bounded stochastic processes. In addition, c will denote a generic constant, and m will denote a generic random variable. Let X still be a bounded random variable and $\mathcal{G}$ be the space of all bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Finally, a standing (financial type) assumption, which we make without loss of generality, is that the interest rates are zero.

3.2 Definition and Properties of DCAIs

Analogous to static theory for SCAIs, we define DCAIs through a set of properties.

Definition 3.2.1. *A dynamic coherent acceptability index is a function $\alpha : \mathcal{T} \times \mathcal{D} \times \Omega \to [0, +\infty]$ that satisfies the following set of properties:*

(D1) Adaptiveness. *For any $t \in \mathcal{T}$ and $D \in \mathcal{D}$, $\alpha_t(D)$ is $\mathcal{F}_t$-measurable;*

(D2) Independence of the past. *For any $t \in \mathcal{T}$ and $D, D' \in \mathcal{D}$, if there exists $A \in \mathcal{F}_t$ such that $1_A D_s = 1_A D'_s$ for all $s \geq t$, then $1_A \alpha_t(D) = 1_A \alpha_t(D')$;*

(D3) Monotonicity. *For any $t \in \mathcal{T}$ and $D, D' \in \mathcal{D}$, if $D_s(\omega) \geq D'_s(\omega)$ for all $s \geq t$ and $\omega \in \Omega$, then $\alpha_t(D, \omega) \geq \alpha_t(D', \omega)$ for all $\omega \in \Omega$;*

(D4) Scale invariance. *$\alpha_t(\lambda D, \omega) = \alpha_t(D, \omega)$ for all $\lambda > 0$, $D \in \mathcal{D}$, $t \in \mathcal{T}$, and $\omega \in \Omega$;*

(D5) Quasi-concavity. *If $\alpha_t(D, \omega) \geq x$ and $\alpha_t(D', \omega) \geq x$ for some $t \in \mathcal{T}$, $\omega \in \Omega$, $D, D' \in \mathcal{D}$, and $x \in (0, +\infty]$, then $\alpha_t(\lambda D + (1-\lambda)D', \omega) \geq x$ for all $\lambda \in [0,1]$;*

(D6) Translation invariance. *$\alpha_t(D + m1_{\{t\}}, \omega) = \alpha_t(D + m1_{\{s\}}, \omega)$ for every $t \in \mathcal{T}$, $D \in \mathcal{D}$, $\omega \in \Omega$, $s \geq t$ and every $\mathcal{F}_t$-measurable random variable m;*

(D7) Dynamic consistency. *For any $t \in \{0, 1, \ldots, T-1\}$ and $D, D' \in \mathcal{D}$, if $D_t(\omega) \geq 0 \geq D'_t(\omega)$ for all $\omega \in \Omega$, and there exists a non-negative $\mathcal{F}_t$-measurable random variable m such that $\alpha_{t+1}(D, \omega) \geq m(\omega) \geq \alpha_{t+1}(D', \omega)$ for all $\omega \in \Omega$, then $\alpha_t(D, \omega) \geq m(\omega) \geq \alpha_t(D', \omega)$ for all $\omega \in \Omega$.*

Property (D1) is a natural property in a dynamic setup and it assumes that a DCAI is adapted to the same information flow $\{\mathcal{F}_t\}_{t=0}^T$ as is any cash flow $D \in \mathcal{D}$.

Property (D2) postulates that in the dynamic context the current measurement of performance of a cash flow D only accounts for future payoffs. To decide, at any given point of time, whether one should hold on to a position generating the cash flow D, one may want to compare the measurement of the performance of the future payoffs (provided by DCAI at this point of time) to already known past payoffs.

Properties (D3)-(D5) are naturally inherited from the static case (cf. Definition 2.1.1). Translation invariance (D6) implies that if a known dividend m is added to D at time t (today), or at any future time $s \geq t$, then all such adjusted cash flows are accepted today at the same level.

Dynamic consistency (D7) is the key property in the dynamic setup which relates the values of the index between two consecutive days in a consistent manner. It can be interpreted from financial point of view as follows: if a portfolio has a nonnegative cashflow today, then we accept this portfolio today at least at the same level as we would accept it tomorrow; similarly, if today's cashflow is nonpositive the acceptance level today can not be larger than the level of acceptance tomorrow.

3.2.1 Normality of DCAIs. For both technical and practical purposes, we introduce another property for DCAIs: normalization. We start from an intuitive example to illustrate its importance.

Example 3.2.1. *Given a non-negative constant c on the extended positive half of the real line $[0, +\infty]$, define a function $\alpha : \mathcal{T} \times \mathcal{D} \times \Omega \to [0, +\infty]$ as follows:*

$$\alpha_t(D, \omega) := c\,,$$

for all $t \in \mathcal{T}, D \in \mathcal{D}$ and $\omega \in \Omega$. The function α is a constant function. It can be verified that α is a dynamic coherent acceptability index.

However, from practical point of view, α from Example 3.2.1 is not a good candidate for portfolio performance measurement since it gives a constant value for all portfolios and thus can not be used as a ranking tool. This example shows the necessity to normalize DCAIs by requiring that an AI reaches the boundaries 0 and $+\infty$, which in a sense allows to differentiate portfolios.

Definition 3.2.2. *A dynamic coherent acceptability index α is called normalized if for all $t \in \mathcal{T}$ and $\omega \in \Omega$, there exist two portfolios $D, D' \in \mathcal{D}$ such that*

$$\alpha_t(D, \omega) = +\infty \quad \text{and} \quad \alpha_t(D', \omega) = 0\,.$$

Note that normalization will exclude the degenerate examples of acceptability indices such as a constant index over all states, times, and portfolios (Example 3.2.1).

Also, it is reasonable to have that "an arbitrage portfolio" is acceptable at highest level, whereas a sure loss portfolio should be acceptable at lowest level. It can be shown that a normalized index reaches $+\infty$ for every strictly positive cash flow and 0 if the cash flow is strictly negative:

Proposition 3.2.1. *If a dynamic coherent acceptability index α is normalized, then we have,*

$$\alpha_t(D^{c,s}) = +\infty \quad \text{for } c > 0, \text{ and } \alpha_t(D^{c,s}) = 0 \quad \text{for } c < 0, \quad \text{for all } t \in \mathcal{T},$$

where, for any $\omega \in \Omega$ and $s \geq t$, $D^{c,s}(r,\omega) = c$ for $r = s$ and zero otherwise.

Proof. First, we show that $\alpha_t(D^{c,s}) = +\infty$ for $c > 0$ and $s \geq t$. If not, there exist $\bar{c} > 0$, $\bar{t} \in \mathcal{T}$, $\bar{s} \geq \bar{t}$ and $\bar{\omega} \in \Omega$ such that $\alpha_{\bar{t}}(D^{\bar{c},\bar{s}}, \bar{\omega}) < +\infty$. By (D6) – translation invariance of α,

$$\alpha_{\bar{t}}(D^{\bar{c},\bar{t}}, \bar{\omega}) = \alpha_{\bar{t}}(D^{\bar{c},\bar{s}}, \bar{\omega}) < +\infty\,.$$

For any positive constant $c_+ > 0$, we have $\frac{c_+}{\bar{c}} > 0$. Hence, by $(D4)$ – scale invariance of α,

$$\alpha_{\bar{t}}(D^{c_+,\bar{t}}, \bar{\omega}) = \alpha_{\bar{t}}(\frac{c_+}{\bar{c}} D^{\bar{c},\bar{t}}, \bar{\omega}) = \alpha_{\bar{t}}(D^{\bar{c},\bar{t}}, \bar{\omega}) < +\infty\,. \tag{3.1}$$

Since α is normalized, there exists $\bar{D} \in \mathcal{D}$ such that $\alpha_{\bar{t}}(\bar{D}, \bar{\omega}) = +\infty$. Since $\bar{D}$ is a bounded process, there exists a positive finite constant c_0 such that for all $t \in \mathcal{T}$ and $\omega \in \Omega$,

$$\bar{D}_t(\omega) \leq c_0\,.$$

Define a new process $D^{\text{new}} = c_0 1_{\{\bar{t},\bar{t}+1,\ldots,T\}}$, we get $\bar{D}_s(\omega) \leq D^{\text{new}}_s(\omega)$ for all $s \geq \bar{t}$ and $\omega \in \Omega$. By (D3) – monotonicity of α,

$$\alpha_{\bar{t}}(\bar{D}, \bar{\omega}) \leq \alpha_{\bar{t}}(D^{new}, \bar{\omega})\,,$$

for $\omega \in \Omega$. Then, (D6), (D4) and (3.1) together imply

$$\alpha_{\bar{t}}(D^{new}, \bar{\omega}) = \alpha_{\bar{t}}((T - t + 1)D^{c_0,\bar{t}}, \bar{\omega}) = \alpha_{\bar{t}}(D^{c_0,\bar{t}}, \bar{\omega}) < +\infty\,,$$

which indicates $\alpha_{\bar{t}}(\bar{D},\bar{\omega}) < +\infty$. It contradicts the fact $\alpha_{\bar{t}}(\bar{D},\bar{\omega}) = +\infty$. Hence, for all $c > 0$, $t \in \mathcal{T}$, $s \geq t$ and $\omega \in \Omega$,

$$\alpha_t(D^{c,s},\omega) = +\infty\,.$$

Next, we show that $\alpha_t(D^{c,s}) = 0$ for $c < 0$ and $s \geq t$. If not, there exists a $\bar{c} < 0$, $\bar{t} \in \mathcal{T}$, $\bar{s} \geq \bar{t}$ and $\bar{\omega} \in \Omega$ such that

$$\alpha_{\bar{t}}(D^{\bar{c},\bar{s}},\bar{\omega}) > 0\,.$$

By (D6) – translation invariance of α, $\alpha_{\bar{t}}(D^{\bar{c},\bar{t}},\bar{\omega}) = \alpha_{\bar{t}}(D^{\bar{c},\bar{s}},\bar{\omega}) > 0$. For any negative constant $c_- < 0$, we have $\frac{c_-}{\bar{c}} > 0$. Hence, by $(D4)$ – scale invariance of α,

$$\alpha_{\bar{t}}(D^{c_-,\bar{t}},\bar{\omega}) = \alpha_{\bar{t}}(\frac{c_-}{\bar{c}}D^{\bar{c},\bar{t}},\bar{\omega}) = \alpha_{\bar{t}}(D^{\bar{c},\bar{t}},\bar{\omega}) > 0\,. \tag{3.2}$$

Since α is normalized, there exists $\bar{D} \in \mathcal{D}$ such that $\alpha_{\bar{t}}(\bar{D},\bar{\omega}) = 0$. Since $\bar{D}$ is a bounded process, there exists a negative finite constant c_0 such that for all $t \in \mathcal{T}$ and $\omega \in \Omega$,

$$\bar{D}_t(\omega) > c_0\,.$$

Define a new process $D^{\text{new}} = c_0 1_{\{\bar{t},\bar{t}+1,\ldots,T\}}$, we get $\bar{D}_s(\omega) \geq D^{\text{new}}_s(\omega)$ for all $s \geq \bar{t}$ and $\omega \in \Omega$. Then, by (D3) – monotonicity of α,

$$\alpha_{\bar{t}}(\bar{D},\bar{\omega}) \geq \alpha_{\bar{t}}(D^{new},\bar{\omega})\,,$$

for $\omega \in \Omega$. Then, (D6), (D4) and (3.2) together imply

$$\alpha_{\bar{t}}(D^{new},\bar{\omega}) = \alpha_{\bar{t}}((T-t+1)D^{c_0,\bar{t}},\bar{\omega}) = \alpha_{\bar{t}}(D^{c_0,\bar{t}},\bar{\omega}) > 0\,,$$

which indicates $\alpha_{\bar{t}}(\bar{D},\bar{\omega}) > 0$. It contradicts the fact $\alpha_{\bar{t}}(\bar{D},\bar{\omega}) = 0$. Hence, for all $c < 0$, $t \in \mathcal{T}$, $s \geq t$ and $\omega \in \Omega$,

$$\alpha_t(D^{c,s},\omega) = 0\,.$$

□

Since we are working with a bounded stochastic processes, when adding (withdrawing) enough large cash to (from) a specific portfolio, the performance level will reach infinity (zero) index level. The following corollary shows this fact.

Corollary 3.2.1. *If a dynamic coherent acceptability index α is normalized, then for any given $D \in \mathcal{D}$ and $t \in \mathcal{T}$, there exist two finite constants c_u and c_l such that,*

$$\alpha_t(D + c_u 1_{\{t\}}, \omega) = +\infty \quad \textit{and} \quad \alpha_t(D + c_l 1_{\{t\}}, \omega) = 0\,,$$

for all $\omega \in \Omega$.

Proof. Given any $D \in \mathcal{D}$ and $t \in \mathcal{T}$, since D is a bounded process, there exists a finite positive number $c_{\max}$ such that

$$|D_s(\omega)| + 1 \leq c_{\max}\,, \tag{3.3}$$

for all $s \geq t$ and $\omega \in \Omega$. Let us define $c_u := (T-t+1)c_{\max}$ and $c_l := -(T-t+1)c_{\max}$. Then, by (D6) – translation invariance of α,

$$\begin{aligned}\alpha_t(D + c_u 1_{\{t\}}, \omega) &= \alpha_t(D + (T-t+1)c_{\max} 1_{\{t\}}, \omega) \\ &= \alpha_t(D + \sum_{s=t}^{T} c_{\max} 1_{\{s\}}, \omega)\,.\end{aligned}$$

Note that for each $s \in \{t, t+1, \ldots, T\}$, (3.3) gives $D_s + c_{\max} 1_{\{s\}} \geq 1$. By (D3) – monotonicity of α,

$$\alpha_t(D + c_u 1_{\{t\}}, \omega) \geq \alpha_t(1_{\{t\}}, \omega)\,, \tag{3.4}$$

for all $\omega \in \Omega$. Since α is normalized, by Proposition 3.2.1,

$$\alpha_t(1_{\{t\}}, \omega) = +\infty\,.$$

Hence, together with (3.4), we get

$$\alpha_t(D + c_u 1_{\{t\}}, \omega) = +\infty\,.$$

By (D6) – translation invariance of α,

$$\begin{aligned}\alpha_t(D + c_l 1_{\{t\}}, \omega) &= \alpha_t(D - (T - t + 1)c_{\max} 1_{\{t\}}, \omega) \\ &= \alpha_t(D - \sum_{s=t}^{T} c_{\max} 1_{\{s\}}, \omega)\,.\end{aligned}$$

Note that for each $s \in \{t, t+1, \ldots, T\}$, (3.3) implies that $D_s - c_{\max} 1_{\{s\}} \leq -1$. By (D3) – monotonicity of α,

$$\alpha_t(D + c_l 1_{\{t\}}, \omega) \leq \alpha_t(-1_{\{t\}}, \omega)\,, \tag{3.5}$$

for all $\omega \in \Omega$. Since α is normalized, by Proposition 3.2.1,

$$\alpha_t(-1_{\{t\}}, \omega) = 0\,.$$

Hence, together with (3.5), we have

$$\alpha_t(D + c_l 1_{\{t\}}, \omega) = 0\,.$$

□

Next, we will introduce a desired technical property for dynamic setting.

Definition 3.2.3. *A dynamic acceptability index α is called right-continuous if* $\lim_{c \to 0^+} \alpha_t(D + c1_{\{t\}}, \omega) = \alpha_t(D, \omega)$, *for all $t \in \mathcal{T}$, $D \in \mathcal{D}$, and $\omega \in \Omega$.*

It should be noted that Proposition 3.2.1 does not imply the value of $\alpha_t(0)$. In [14], (S4) – Fatou Property would conclude that an unbounded above SCAI has the property $\alpha(0) = +\infty$. Fatou Property is a continuous-type property for SCAIs.

Similar to the Fatou Property, the right-continuous property for DCAIs gives rise to $\alpha_t(0) = +\infty$.

Proposition 3.2.2. *If α is a normalized and right-continuous dynamic coherent acceptability index, then $\alpha_t(0) = +\infty$ for all $t \in \mathcal{T}$.*

Proof. If α is normalized, by Proposition 3.2.1,

$$\alpha_t(c1_t) = +\infty\,,$$

for any positive $c > 0$ and $t \in \mathcal{T}$. Then, since α is right-continuous,

$$\alpha_t(0) = \lim_{c\to 0^+} \alpha_t(c1_{\{t\}}) = +\infty\,.$$

□

3.2.2 Dynamic Consistency of DCAIs. Dynamic consistency property plays a central role in the dynamic theory of both acceptability indices and risk measures. Generally speaking, a simple generalization of a static measurement into multiple periods framework, may not satisfy dynamic consistency. Such an example will be presented in the Chapter 6.

In the following, we define two properties that will be verified to be 'equivalent' to (D7) – dynamic consistency.

Definition 3.2.4. *For a function $\alpha : \mathcal{T}\times\mathcal{D}\times\Omega \to [0, +\infty]$, two properties are defined as follows:*

(D7-I) *For a given $t \in \{0, 1, \ldots, T-1\}$ and $D, D' \in \mathcal{D}$, if $D_t(\omega) = D'_t(\omega) = 0$ for all $\omega \in \Omega$, and there exists a non-negative $\mathcal{F}_t$-measurable random variable m such that $\alpha_{t+1}(D, \omega) \geq m(\omega) \geq \alpha_{t+1}(D', \omega)$ for all $\omega \in \Omega$, then $\alpha_t(D, \omega) \geq m(\omega) \geq \alpha_t(D', \omega)$ for all $\omega \in \Omega$.*

(D7-II) *For a given $t \in \{0, 1, \ldots, T-1\}$ and $D \in \mathcal{D}$, if $D_t(\omega) = 0$ for all $\omega \in \Omega$, then*

$$1_A \min_{\omega\in A} \alpha_{t+1}(D, \omega) \leq 1_A\alpha_t(D) \leq 1_A \max_{\omega\in A} \alpha_{t+1}(D, \omega)\,,$$

for all $A \in \mathcal{F}_t$.

We observe that (D7-I) is more restrictive than (D7), that is if a dynamic acceptability index α satisfies (D7), it also satisfies (D7-I).

Property (D7-II) shows that if a portfolio has a zero value today, then this portfolio is acceptable at most the maximum level of tomorrow and at least the minimum level of tomorrow. From mathematical point of view, it gives both maximum and minimum of α.

Proposition 3.2.3. *If a function $\alpha : \mathcal{T} \times \mathcal{D} \times \Omega \to [0, +\infty]$ satisfies properties (D2) – independence of the past, and (D3) – monotonicity, then (D7) and (D7-I) are equivalent.*

Proof. (D7-I) is stronger than (D7), then (D7) implies (D7-I).

Assume α satisfies (D2), (D3) and (D7-I), we will show that α also satisfy (D7). For $D, D' \in \mathcal{D}$, if $D_t(\omega) \geq 0 \geq D'_t(\omega)$ for all $\omega \in \Omega$, and there exists a non-negative $\mathcal{F}_t$-measurable random variable m such that $\alpha_{t+1}(D, \omega) \geq m(\omega) \geq \alpha_{t+1}(D', \omega)$ for all $\omega \in \Omega$, define $\hat{D} := 1_{\{t+1,\dots,T\}} D$ and $\hat{D}' := 1_{\{t+1,\dots,T\}} D'$. By (D2) – independence of the past of α,

$$\alpha_{t+1}(\hat{D}, \omega) = \alpha_{t+1}(D, \omega) \quad \text{and} \quad \alpha_{t+1}(\hat{D}', \omega) = \alpha_{t+1}(D', \omega)\,,$$

for all $\omega \in \Omega$. Then,

$$\alpha_{t+1}(\hat{D}, \omega) = \alpha_{t+1}(D, \omega) \geq m(\omega) \geq \alpha_{t+1}(D', \omega) = \alpha_{t+1}(\hat{D}', \omega)\,.$$

Since $\hat{D}_t(\omega) = \hat{D}'{}_t(\omega) = 0$ for all $\omega \in \Omega$, by (D7-I),

$$\alpha_t(\hat{D}, \omega) \geq m(\omega) \geq \alpha_t(\hat{D}', \omega)\,. \tag{3.6}$$

Note that $D_t(\omega) \geq 0 \geq D'_t(\omega)$ for all $\omega \in \Omega$, then by the definition of $\hat{D}$ and $\hat{D}'$,

$$D_s(\omega) \geq \hat{D}_s(\omega) \quad \text{and} \quad \hat{D}'{}_s(\omega) \geq D'_s(\omega)\,,$$

for all $s \geq t$ and $\omega \in \Omega$. By (D3) and (3.6),

$$\alpha_t(D,\omega) \geq \alpha_t(\hat{D},\omega) \geq m(\omega) \geq \alpha_t(\hat{D}',\omega) \geq \alpha_t(D',\omega)\,,$$

for all $\omega \in \Omega$. Therefore, (D7) holds true for α. □

Corollary 3.2.2. *As a direct result of Proposition 3.2.3, we conclude that the set of properties (D1)-(D7) is equivalent to (D1)-(D6) and (D7-I).*

The next proposition shows the equivalence between (D7-I) and (D7-II).

Proposition 3.2.4. *If α is normalized, then (D7-I) and (D7-II) are equivalent.*

Proof. First, we show that (D7-II) implies (D7-I). For $D, D' \in \mathcal{D}$, if $D_t(\omega) = D'_t(\omega) = 0$ for all $\omega \in \Omega$, and there exists a non-negative $\mathcal{F}_t$-measurable random variable m such that $\alpha_{t+1}(D,\omega) \geq m(\omega) \geq \alpha_{t+1}(D',\omega)$ for all $\omega \in \Omega$, then we can derive, for each $P_i^t \in \Upsilon^t$,

$$1_{P_i^t}\alpha_{t+1}(D,\omega) \geq 1_{P_i^t}m(\omega) \geq 1_{P_i^t}\alpha_{t+1}(D',\omega)\,.$$

Denote $c^i := m(\omega)$ where $\omega \in P_i^t$, then c^i is a constant and

$$1_{P_i^t}\min_{\omega\in P_i^t}\alpha_{t+1}(D,\omega) \geq 1_{P_i^t}c^i \geq 1_{P_i^t}\max_{\omega\in P_i^t}\alpha_{t+1}(D',\omega)\,.$$

By (D7-II),

$$1_{P_i^t}\alpha_t(D) \geq 1_{P_i^t}\min_{\omega\in P_i^t}\alpha_{t+1}(D,\omega) \geq 1_{P_i^t}c^i \geq 1_{P_i^t}\max_{\omega\in P_i^t}\alpha_{t+1}(D',\omega) \geq 1_{P_i^t}\alpha_t(D')\,.$$

Since the above inequality holds true for all $P_i^t \in \Upsilon^t$, we get

$$\alpha_t(D,\omega) \geq m(\omega) \geq \alpha_t(D',\omega),\ \forall\, \omega \in \Omega\,.$$

Next, we show that (D7-I) implies (D7-II). For any $D \in \mathcal{D}$, if $D_t(\omega) = 0$ for all $\omega \in \Omega$, define D' and m as follows:

$$D' := -1_{\{t+1\}}1_\Omega\,,$$

$$m := \sum_{P_i^t\in\Upsilon^t} 1_{P_i^t}\min_{\omega\in P_i^t}\alpha_{t+1}(D,\omega)\,.$$

Then, m is a non-negative, $\mathcal{F}_t$-measurable random variable, and $D_t(\omega) = D'_t(\omega) = 0$ for all $\omega \in \Omega$. In addition, since α is normalized, by Proposition 3.2.1,

$$1_{P_i^t}\alpha_{t+1}(D) \geq 1_{P_i^t} \min_{\omega \in P_i^t} \alpha_{t+1}(D,\omega) \geq 0 = 1_{P_i^t}\alpha_{t+1}(D')\,,$$

for all $P_i^t \in \Upsilon^t$. Thus, $\alpha_{t+1}(D,\omega) \geq m(\omega) \geq \alpha_{t+1}(D',\omega)$ for all $\omega \in \Omega$. By (D7-I), since $D_t(\omega) = D'_t(\omega) = 0$ for all $\omega \in \Omega$,

$$\alpha_t(D,\omega) \geq m(\omega) \geq \alpha_t(D',\omega)\,.$$

Then, for all $A \in \mathcal{F}_t$, by the definition of m,

$$1_A\alpha_t(D) \geq 1_A \min_{\omega \in A} \alpha_{t+1}(D,\omega)\,.$$

On the other hand, we define D' and m as follows:

$$D' := 1_{\{t+1\}}1_\Omega\,,$$
$$m := \sum_{P_i^t \in \Upsilon^t} 1_{P_i^t} \max_{\omega \in P_i^t} \alpha_{t+1}(D,\omega)\,.$$

By analogous argument, for all $A \in \mathcal{F}_t$, we can verify that

$$1_A \max_{\omega \in A} \alpha_{t+1}(D,\omega) \geq 1_A\alpha_t(D)\,.$$

□

Corollary 3.2.3. *Using Proposition 3.2.3 and Proposition 3.2.4 we conclude that, if a function $\alpha : \mathcal{T} \times \mathcal{D} \times \Omega \to [0,+\infty]$ is normalized and satisfies (D2) – independence of the past, and (D3) – monotonicity, then (D7) and (D7-II) are equivalent for function α.*

3.2.3 Relevancy Property of DCAIs. In dynamic framework, as time passes, more and more possible states are excluded from happening in the future. Relevancy implies that the index level at current time t should be irrelevant with those excluded states.

We provide alternative properties for both (D3) and (D7) in terms of relevancy.

Definition 3.2.5. *For a function $\alpha : \mathcal{T} \times \mathcal{D} \times \Omega \to [0, +\infty]$, two properties are defined as follows:*

(D3-I) *For any $t \in \mathcal{T}$, $D, D' \in \mathcal{D}$, if there exists $A \in \mathcal{F}_t$ such that $1_A D_s \geq 1_A D'_s$ for all $s \geq t$, then $1_A \alpha_t(D) \geq 1_A \alpha_t(D')$;*

(D7-III) *For any $t \in \{0, 1, \ldots, T-1\}$ and $D, D' \in \mathcal{D}$, if there exists $A \in \mathcal{F}_t$ and a non-negative $\mathcal{F}_t$-measurable random variable m, such that $1_A D_t \geq 0 \geq 1_A D'_t$ and $1_A \alpha_{t+1}(D) \geq 1_A m \geq 1_A \alpha_{t+1}(D')$, then $1_A \alpha_t(D) \geq 1_A m \geq 1_A \alpha_t(D')$.*

Proposition 3.2.5. *A normalized $\alpha : \mathcal{T} \times \mathcal{D} \times \Omega$ is a dynamic coherent acceptability index if and only if it satisfies (D1),(D2),(D3-I),(D4),(D5),(D6) and (D7-III).*

Proof. **Sufficiency.** It is obvious if $A = \Omega$ in (D3-I) and (D7-III).

Necessity. All we need to prove is that a dynamic coherent acceptability index α satisfies (D3-I) and (D7-III).

First, we show that α satisfies (D3-I). For $D, D' \in \mathcal{D}$, if there exists $A \in \mathcal{F}_t$ such that $1_A D_s \geq 1_A D'_s$ for $s \geq t$, we can define two new portfolios $\hat{D}$ and $\hat{D}'$ such that, $\hat{D} := 1_{\{t,\ldots,T\}} 1_A D$ and $\hat{D}' := 1_{\{t,\ldots,T\}} 1_A D'$. By (D2) – independence of the past of α,

$$1_A \alpha_t(\hat{D}) = 1_A \alpha_t(D) \quad \text{and} \quad 1_A \alpha_t(\hat{D}') = 1_A \alpha_t(D').$$

We observe that $1_A \hat{D}_s \geq 1_A \hat{D'}_s$ and $1_{\{\Omega \setminus A\}} \hat{D}_s = 1_{\{\Omega \setminus A\}} \hat{D'}_s = 0$ for $s \geq t$. It implies $\hat{D}_s(\omega) \geq \hat{D'}_s(\omega)$ for all $\omega \in \Omega, s \geq t$. By (D3), we have $\alpha_t(\hat{D}) \geq \alpha_t(\hat{D}')$. Thus,

$$1_A \alpha_t(D) = 1_A \alpha_t(\hat{D}) \geq 1_A \alpha_t(\hat{D}') = 1_A \alpha_t(D').$$

Second, we show that α satisfies (D7-III). For $D, D' \in \mathcal{D}$, if there exists $A \in \mathcal{F}_t$ such that $1_A D_t \geq 0 \geq 1_A D'_t$, and a non-negative $\mathcal{F}_t$-measurable random variable m

such that $1_A\alpha_{t+1}(D) \geq 1_A m \geq 1_A\alpha_{t+1}(D')$, define $\tilde{m} := \alpha_t(1_{t+1}1_\Omega)$, and two new portfolios $\hat{D} := 1_{\{t,\ldots,T\}}1_A D - 1_{\{t+1\}}1_{\{\Omega\setminus A\}}$ and $\hat{D}' := 1_{\{t,\ldots,T\}}1_A D' - 1_{\{t+1\}}1_{\{\Omega\setminus A\}}$. By (D2) – independence of the past of α,

$$1_A\alpha_t(\hat{D}) = 1_A\alpha_t(D)\,, \quad 1_A\alpha_{t+1}(\hat{D}) = 1_A\alpha_{t+1}(D)\,.$$
$$1_A\alpha_t(\hat{D}') = 1_A\alpha_t(D')\,, \quad 1_A\alpha_{t+1}(\hat{D}') = 1_A\alpha_{t+1}(D')\,.$$

Then,

$$1_A\alpha_{t+1}(\hat{D}) = 1_A\alpha_{t+1}(D) \geq 1_A m \geq 1_A\alpha_{t+1}(D') = 1_A\alpha_{t+1}(\hat{D}')\,.$$

By the definition of $\hat{D}$ and $\hat{D}'$, together with (D2) and Proposition 3.2.1, we get

$$1_{\{\Omega\setminus A\}}\alpha_{t+1}(\hat{D}) = 1_{\{\Omega\setminus A\}}\alpha_{t+1}(\hat{D}') = 0\,.$$

Thus,

$$\alpha_{t+1}(\hat{D}) \geq 1_A m \geq \alpha_{t+1}(\hat{D}')\,.$$

Note that $\hat{D}_t \geq 0 \geq \hat{D}'_t$. By (D7), we have $\alpha_t(\hat{D}) \geq 1_A m \geq \alpha_t(\hat{D}')$. Finally, we conclude that

$$1_A\alpha_t(D) = 1_A\alpha_t(\hat{D}) \geq 1_A m \geq 1_A\alpha_t(\hat{D}') = 1_A\alpha_t(D')\,.$$

□

As a conclusion of this section, we stress that normality for DCAIs is required for Proposition 3.2.1, Proposition 3.2.4, Proposition 3.2.5 and their related corollaries. Later on, we will show that normality is also an important property necessary for most major results for DCAIs.

3.3 Definition and Properties of DCRMs

As mentioned in Chapter 2, there is a strong relationship between coherent acceptability indices and coherent risk measures. In fact, as seen from Theorem

2.1.1, any SCAI α can be represented in terms of a family of coherent risk measures $\rho^x, x > 0$:

$$\alpha(D) = \sup\{x \in [0, +\infty) : \rho^x(D) \leq 0\} . \tag{3.7}$$

Looking at (3.7) one might think that a natural approach to constructing a DCAI would be to use this representation but to replace the static coherent risk measures in (3.7) by their dynamic counterpart. The representation (3.15) that we derive below shows that this is indeed the case. The delicate issue however is, what family of dynamic coherent risk measures should be used. It turns out that in order to produce a DCAI satisfying a financially acceptable set of dynamic properties, one needs to use a carefully crafted family of dynamic coherent risk measures. In this section we introduce such a family of dynamic coherent risk measures and we compare our definition of coherent dynamic risk measures with an analogous one that has been studied in other literature.

Definition 3.3.1. *Dynamic coherent risk measure is a function $\rho : \mathcal{T} \times \mathcal{D} \times \Omega \to \mathbb{R}$ that satisfies the following properties:*

(A1) Adaptiveness. *$\rho_t(D)$ is $\mathcal{F}_t$-measurable for all $t \in \mathcal{T}$ and $D \in \mathcal{D}$;*

(A2) Independence of the past. *If $1_A D_s = 1_A D'_s$ for some $t \in \mathcal{T}$, $D, D' \in \mathcal{D}$, and $A \in \mathcal{F}_t$ and for all $s \geq t$, then $1_A \rho_t(D) = 1_A \rho_t(D')$;*

(A3) Monotonicity. *If $D_s(\omega) \geq D'_s(\omega)$ for some $t \in \mathcal{T}$ and $D, D' \in \mathcal{D}$, and for all $s \geq t$ and $\omega \in \Omega$, then $\rho_t(D, \omega) \leq \rho_t(D', \omega)$ for all $\omega \in \Omega$;*

(A4) Homogeneity. *$\rho_t(\lambda D, \omega) = \lambda \rho_t(D, \omega)$ for all $\lambda > 0$, $D \in \mathcal{D}$, $t \in \mathcal{T}$, and $\omega \in \Omega$;*

(A5) Subadditivity. *$\rho_t(D + D', \omega) \leq \rho_t(D, \omega) + \rho_t(D', \omega)$ for all $t \in \mathcal{T}$, $D, D' \in \mathcal{D}$, and $\omega \in \Omega$;*

(A6) Translation invariance. *$\rho_t(D+m1_{\{s\}}) = \rho_t(D) - m$ for every $t \in \mathcal{T}$, $D \in \mathcal{D}$, $\mathcal{F}_t$-measurable random variable m, and all $s \geq t$;*

(A7) Dynamic consistency.

$$1_A(\min_{\omega \in A} \rho_{t+1}(D,\omega) - D_t) \leq 1_A \rho_t(D) \leq 1_A(\max_{\omega \in A} \rho_{t+1}(D,\omega) - D_t)\,,$$

for every $t \in \{0,1,\ldots,T-1\}$, $D \in \mathcal{D}$ and $A \in \mathcal{F}_t$.

In the previous section, we introduced the normality of DCAIs. We now present the normality of DCRMs.

Proposition 3.3.1. *If ρ is a dynamic coherent risk measure, then $\rho_t(c1_{\{s\}},\omega) = -c$, for all $c \in \mathbb{R}$, $t \in \mathcal{T}$, $\omega \in \Omega$ and $s \geq t$.*

Proof. Given some fixed $t \in \mathcal{T}$ and $\omega \in \Omega$, denote by $\lambda := \rho_t(0,\omega)$. Then, by (A6) – translation invariance of ρ, we deduce

$$\rho_t(c1_{\{s\}},\omega) = \rho_t(0,\omega) - c = \lambda - c\,, \tag{3.8}$$

for all $c \in \mathbb{R}$. In particular, for $c = 1$, we have $\rho_t(1_{\{s\}},\omega) = \lambda - 1$. Hence, by (A4) – homogeneity of ρ, it follows that

$$\begin{aligned}\rho_t(c_u 1_{\{s\}},\omega) &= c_u \rho_t(1_{\{s\}},\omega)\\ &= c_u(\lambda - 1)\,, \quad \text{for all } \; c_u > 0.\end{aligned}$$

Combining this with (3.8) we get $\lambda - c_u = c_u\lambda - c_u$, and consequently $\lambda(1-c_u) = 0$. Since the last equality holds true for arbitrary positive c_u, we have that $\lambda = 0$, and thus $\rho_t(0,\omega) = 0$. Thus, by (3.8), $\rho_t(c1_{\{s\}},\omega) = \rho_t(0,\omega) - c = -c$. □

Note that, in particular, $\rho_t(0) = 0$, for all $t \in \mathcal{T}$.

We want to mention that our definition of DCRM differs from the definition given in previous studies essentially only by the dynamic consistency property. For sake of completeness, we will present here how property (A7) relates to two other forms of dynamic consistency found in [13] and [42].

(A7-I) For all times $t = 0, \ldots, T-1$ and positions $D, D' \in \mathcal{D}$ with $D_t = D'_t$ the following holds true: $\rho_{t+1}(D, \omega) = \rho_{t+1}(D', \omega)$ for all $\omega \in \Omega$ implies $\rho_t(D, \omega) = \rho_t(D', \omega)$ for all $\omega \in \Omega$;

(A7-II) $\rho_t(D) = \rho_t(D_t 1_{\{t\}} - \rho_{t+1}(D) 1_{\{t+1\}})$ for all times $t = 0, 1, \ldots, T-1$ and positions $D \in \mathcal{D}$.

(A7-I) is the dynamic consistency property for DCRM defined by Riedel [42]. Property (A7-II) can be viewed as a dynamic programming principle similar to the dynamic consistency defined by, for example, Cheridito, Delbaen and Kupper [13]. However, it should be mentioned that in [13] the set of objects for which the risk is measured is different from ours, and hence the comparison is rather formal. The objects we are working with are dividend processes, whereas value processes are considered in [13]. Our approach is closest to the DCRM defined by Riedel [42].

Other researchers also proposed various dynamic consistency properties, see for instance [4, 7, 8, 22, 25, 28, 43, 49].

Proposition 3.3.2. *If a function $\rho : \mathcal{T} \times \mathcal{D} \times \Omega \to \mathbb{R}$ satisfies (A1)-(A6), then (A7-I) is equivalent to (A7-II).*

Proof. First, we show that (A7-I) implies (A7-II). Given a position $D \in \mathcal{D}$, we define another position $\tilde{D} := 1_{\{t\}} D - 1_{\{t+1\}} \rho_{t+1}(D)$. By (A1) – independence of the past and (A6) – translation invariance,

$$\rho_{t+1}(\tilde{D}) = \rho_{t+1}(-1_{\{t+1\}} \rho_{t+1}(D)) = \rho_{t+1}(0) + \rho_{t+1}(D) = \rho_{t+1}(D)\,.$$

By the definition of $\tilde{D}$, we have $\tilde{D}_t = D_t$. Therefore, by (A7-I),

$$\rho_t(D) = \rho_t(\tilde{D}) = \rho_t(D_t 1_{\{t\}} - \rho_{t+1}(D) 1_{\{t+1\}})\,.$$

Second, we show that (A7-II) implies (A7-I). If $D_t = D'_t$ and $\rho_{t+1}(D) = \rho_{t+1}(D')$, we define two new positions $\tilde{D} := D_t 1_{\{t\}} - \rho_{t+1}(D)1_{\{t+1\}}$ and $\bar{D} := D'_t 1_{\{t\}} - \rho_{t+1}(D')1_{\{t+1\}}$. We can observe that $\tilde{D} = \bar{D}$. By (A2) – independent of past, we have $\rho_t(\tilde{D}) = \rho_t(\bar{D})$. Then, (A7-II) implies,

$$\rho_t(D) = \rho_t(\tilde{D}) = \rho_t(\bar{D}) = \rho_t(D')\,.$$

Finally, (A7-I) holds true. □

Given the risk at time $t+1$, we are able to apply dynamic programming principle to compute the risk at time t. The next proposition will show the relationship of our dynamic consistency and dynamic programming principle.

Proposition 3.3.3. *For a function $\rho : \mathcal{T} \times \mathcal{D} \times \Omega \to \mathbb{R}$, if (A1)-(A6) hold true, (A7-II) – dynamic programming principle implies (A7). Hence, (A7) is more general than (A7-II).*

Proof. For all $t \in \{0, 1, \ldots, T-1\}$, $D \in \mathcal{D}$ and $A \in \mathcal{F}_t$, by (A7-II) – dynamic programming principle, (A6) – translation invariance and (A2) – independence of the past,

$$\begin{aligned} 1_A \rho_t(D) &= 1_A \rho_t(D_t 1_{\{t\}} - \rho_{t+1}(D)1_{\{t+1\}}) \\ &= 1_A \rho_t(-\rho_{t+1}(D)1_{\{t+1\}}) - 1_A D_t \\ &= 1_A \rho_t(-1_A \rho_{t+1}(D)1_{\{t+1\}}) - 1_A D_t\,. \end{aligned}$$

Define two new positions $\tilde{D}_{\min}$ and $\tilde{D}_{\max}$ as follows:

$$\begin{aligned} \tilde{D}_{\min} &:= -1_{t+1} 1_A \min_{\omega \in A} \rho_{t+1}(D, \omega)\,, \\ \tilde{D}_{\max} &:= -1_{t+1} 1_A \max_{\omega \in A} \rho_{t+1}(D, \omega)\,. \end{aligned}$$

Note that

$$-1_A \min_{\omega\in A} \rho_{t+1}(D,\omega) \geq -1_A \rho_{t+1}(D,\omega)\,,$$

$$-1_A \max_{\omega\in A} \rho_{t+1}(D,\omega) \leq -1_A \rho_{t+1}(D,\omega)\,.$$

Together with (A3) – monotonicity, we get

$$\rho_t(\tilde{D}_{\min}) \leq \rho_t(-1_{\{t+1\}} 1_A \rho_{t+1}(D))\,,$$

$$\rho_t(\tilde{D}_{\max}) \geq \rho_t(-1_{\{t+1\}} 1_A \rho_{t+1}(D))\,.$$

Since $A \in \mathcal{F}_t$, we observe that $-1_A \min_{\omega\in A} \rho_{t+1}(D,\omega)$ is $\mathcal{F}_t$-measurable. Thus, by (A6) – translation invariance,

$$\rho_t(\tilde{D}_{\min}) = \rho_t(0) + 1_A \min_{\omega\in A} \rho_{t+1}(D,\omega) = 1_A \min_{\omega\in A} \rho_{t+1}(D,\omega)$$

$$\rho_t(\tilde{D}_{\max}) = \rho_t(0) + 1_A \max_{\omega\in A} \rho_{t+1}(D,\omega) = 1_A \max_{\omega\in A} \rho_{t+1}(D,\omega)\,.$$

From all the above, we can derive that

$$\begin{aligned}
1_A \rho_t(D) &= 1_A \rho_t(-1_A \rho_{t+1}(D) 1_{\{t+1\}}) - 1_A D_t \\
&\geq 1_A \rho_t(\tilde{D}_{\min}) - 1_A D_t \\
&= 1_A \min_{\omega\in A} \rho_{t+1}(D,\omega) - 1_A D_t \\
&= 1_A (\min_{\omega\in A} \rho_{t+1}(D,\omega) - D_t)\,,
\end{aligned}$$

and

$$\begin{aligned}
1_A \rho_t(D) &= 1_A \rho_t(-1_A \rho_{t+1}(D) 1_{\{t+1\}}) - 1_A D_t \\
&\leq 1_A \rho_t(\tilde{D}_{\max}) - 1_A D_t \\
&= 1_A \max_{\omega\in A} \rho_{t+1}(D,\omega) - 1_A D_t \\
&= 1_A (\max_{\omega\in A} \rho_{t+1}(D,\omega) - D_t)\,.
\end{aligned}$$

Finally, (A7) holds true. Hence, we conclude that (A7) is more general than (A7-II). $\square$

Corollary 3.3.1. *Using Proposition 3.3.2 and Proposition 3.3.3, we conclude that if (A1)-(A6) hold true, (A7) is more general than (A7-I).*

3.4 Duality between DCAIs and DCRMs

We start this section with several definitions that will be used in the main results derived in this section.

Definition 3.4.1. *A family of dynamic coherent risk measures* $(\rho^x)_{x\in(0,+\infty)}$ *is called increasing if* $\rho^x_t(D,\omega) \geq \rho^y_t(D,\omega)$, *for all* $x \geq y > 0$, $t \in \mathcal{T}$, $D \in \mathcal{D}$ *and* $\omega \in \Omega$.

Definition 3.4.2. *A family of dynamic coherent risk measures* $(\rho^x)_{x\in(0,+\infty)}$ *is called left-continuous if* $\lim_{x\to x_0^-} \rho^x_t(D,\omega) = \rho^{x_0}_t(D,\omega)$, *for all* $t \in \mathcal{T}$, $D \in \mathcal{D}$, *and* $\omega \in \Omega$.

Theorem 3.4.1. *Assume that* α *is a normalized dynamic coherent acceptability index. Then, the set of functions* $\rho^x, x \in \mathbb{R}$, *defined by*

$$\rho^x_t(D,\omega) := \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}},\omega) \geq x\}\,, \tag{3.9}$$

for all $t \in \mathcal{T}$, $D \in \mathcal{D}$ *and* $\omega \in \Omega$, *is an increasing, left-continuous family of dynamic coherent risk measures.*

Proof. First we will show that ρ^x defined by (3.9) is well-defined. Since α is normalized, by Corollary 3.2.1, for all $t \in \mathcal{T}$, $D \in \mathcal{D}$, there exist two finite constants c_u and c_l such that

$$\alpha_t(D + c_u 1_{\{t\}},\omega) = +\infty \quad \text{and} \quad \alpha_t(D + c_l 1_{\{t\}},\omega) = 0\,,$$

for all $\omega \in \Omega$. Hence, for every $x \in (0,+\infty)$, the set $\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}},\omega) \geq x\}$ is not empty, and $c_l \leq \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}},\omega) \geq x\}$. From here we conclude that infimum from (3.9) is finite, and hence ρ^x is well-defined.

Next we will show that $\rho^x, x \in (0,+\infty)$, satisfies the properties (A1)-(A7).

Fix any $P_i^t \in \Upsilon^t$, by (D1)-adaptiveness of α,

$$\alpha_t(D + c1_{\{t\}}, \omega_1) = \alpha_t(D + c1_{\{t\}}, \omega_2)\,,$$

for every $t \in \mathcal{T},\ D \in \mathcal{D},\ c \in \mathbb{R}$ and every $\omega_1, \omega_2 \in P_i^t$. Hence,

$$\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega_1) \geq x\} = \{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega_2) \geq x\}\,,$$

for all $x \in (0, +\infty)$. Taking the infimum of both sides, we get

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega_1) \geq x\} = \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega_2) \geq x\}\,,$$

which consequently implies that $\rho_t^x(D, \omega_1) = \rho_t^x(D, \omega_2)$ for every $\omega_1, \omega_2 \in P_i^t$. Since the above argument holds true for all $P_i^t \in \Upsilon^t$, we know ρ_t^x is $\mathcal{F}_t$-measurable and (A1) is verified.

By (D2) – independence of the past of α, we have that

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega^0) \geq x\} = \inf\{c \in \mathbb{R} : \alpha_t(D' + c1_{\{t\}}, \omega^0) \geq x\}\,,$$

for any $t \in \mathcal{T},\ D, D' \in \mathcal{D}$ such that $1_A D_s = 1_A D'_s$, for all $s \geq t$, and for every $\omega^0 \in A \in \mathcal{F}_t$. From here, by (3.9), we have that $\rho_t^x(D, \omega^0) = \rho_t^x(D', \omega^0)$, hence (A2) is satisfied for all $x \in (0, +\infty)$.

Next we will prove that ρ^x satisfies (A3). Assume that $t \in \mathcal{T}$ and $D, D' \in \mathcal{D}$ are such that $D_s(\omega) \geq D'_s(\omega)$ for all $s \geq t$ and $\omega \in \Omega$. Then, for all $c \in \mathbb{R}$,

$$(D + c1_{\{t\}})_s(\omega) \geq (D' + c1_{\{t\}})_s(\omega) \ \text{ for } s \geq t \text{ and } \omega \in \Omega\,.$$

By (D3) – monotonicity of α,

$$\alpha_t(D + c1_{\{t\}}, \omega) \geq \alpha_t(D' + c1_{\{t\}}, \omega)\,, \tag{3.10}$$

for all $c \in \mathbb{R}$ and $\omega \in \Omega$. From here, we deduce the following inclusion

$$\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\} \supseteq \{c \in \mathbb{R} : \alpha_t(D' + c1_{\{t\}}, \omega) \geq x\}\,.$$

Taking infimum of both sets, by the definition of ρ^x, the monotonicity $(A3)$ follows. Similarly, the homogeneity of ρ^x follows from the scale invariance of α.

Next we show that ρ^x satisfies $(A5)$. Let $t \in \mathcal{T}$, $D, D' \in \mathcal{D}$ and $\omega \in \Omega$, and let us take $c_1, c_2 \in \mathbb{R}$ such that

$$\alpha_t(D + c_i 1_{\{t\}}, \omega) \geq x\,, \quad i = 1, 2.$$

Then, by (D5) – quasi-concavity of α,

$$\alpha_t(\frac{1}{2}D + \frac{1}{2}c_1 1_{\{t\}} + \frac{1}{2}D' + \frac{1}{2}c_2 1_{\{t\}}, \omega) \geq x\,,$$

and therefore by (D4) – scale invariance of α, we get $\alpha_t(D + D' + (c_1 + c_2)1_{\{t\}}, \omega) \geq x$. This implies that $c_1 + c_2 \in \{c \in \mathbb{R} : \alpha_t(D + D' + c1_{\{t\}}, \omega) \geq x\}$. Hence,

$$\begin{aligned} c_1 + c_2 &\geq \inf\{c \in \mathbb{R} : \alpha_t(D + D' + c1_{\{t\}}, \omega) \geq x\} \\ &= \rho_t^x(D + D', \omega)\,. \end{aligned} \tag{3.11}$$

Note that the above inequality holds true for all $c_1 \in \{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}$ and $c_2 \in \{c \in \mathbb{R} : \alpha_t(D' + c1_{\{t\}}, \omega) \geq x\}$. By taking infimum in (3.11), first with respect to c_1, and then with respect to c_2, we get the following inequality,

$$\begin{aligned} &\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\} + \inf\{c \in \mathbb{R} : \alpha_t(D' + c1_{\{t\}}, \omega) \geq x\} \\ &\geq \rho_t^x(D + D', \omega)\,. \end{aligned}$$

By the definition (3.9) of ρ^x, we have, $\rho_t^x(D, \omega) + \rho_t^x(D', \omega) \geq \rho_t^x(D + D', \omega)$, and hence $(A5)$ is checked.

Now we will check that ρ^x satisfies (A6), translation invariance. Fix an $\omega^0 \in \Omega$, $t \in \mathcal{T}$, $D \in \mathcal{D}$ and an $\mathcal{F}_t$-measurable random variable m. Denote by P_i^t the unique element of partition of $\mathcal{F}_t$ such that $\omega^0 \in P_i^t$. This yields that the cash-flows $m1_{\{t\}}$ and $m(\omega^0)1_{\{t\}}$ agree on the set P_i^t, and for all times $s \geq t$. Then, for any constant $c \in \mathbb{R}$, we have

$$1_{P_i^t}(D + m1_t + c1_{\{t\}})_s = 1_{P_i^t}(D + m(\omega^0)1_{\{t\}} + c1_{\{t\}})_s\,, \text{ for } s \geq t\,.$$

By (D2) – independence of the past of α,

$$1_{P_i^t}\alpha_t(D + m1_t + c1_{\{t\}}) = 1_{P_i^t}\alpha_t(D + m(\omega^0)1_{\{t\}} + c1_{\{t\}})\,.$$

Since m is $\mathcal{F}_t$-measurable, by (D6) – translation invariance of α,

$$\alpha_t(D + m1_s + c1_{\{t\}}, \omega^0) = \alpha_t(D + m1_t + c1_{\{t\}}, \omega^0)\,, \quad \text{for all } s \geq t.$$

Combining the above with (3.9), we deduce

$$\begin{aligned}
\rho_t^x(D + m1_{\{s\}}, \omega^0) &= \inf\{c \in \mathbb{R} : \alpha_t(D + m1_{\{s\}} + c1_{\{t\}}, \omega^0) \geq x\} \\
&= \inf\{c \in \mathbb{R} : \alpha_t(D + m1_{\{t\}} + c1_{\{t\}}, \omega^0) \geq x\} \\
&= \inf\{c \in \mathbb{R} : \alpha_t(D + m(\omega^0)1_{\{t\}} + c1_{\{t\}}, \omega^0) \geq x\} \\
&= \inf\{m(\omega^0) + c \in \mathbb{R} : \alpha_t(D + (m(\omega^0) + c)1_{\{t\}}, \omega^0) \geq x\} - m(\omega^0) \\
&= \rho_t^x(D, \omega) - m(\omega^0)\,.
\end{aligned}$$

Since ω^0 is arbitrarily chosen in Ω, we obtain $\rho_t^x(D + m1_{\{s\}}) = \rho_t^x(D) - m$, for all $s \geq t$, and (D6) is checked.

Next we will show that ρ^x satisfies $(A7)$, dynamic consistency. Assume that $t \in \mathcal{T}$, $D \in \mathcal{D}$ and $A \in \mathcal{F}_t$ are fixed, and denote by $c_{\min}^{t,D,A} := \min_{\omega \in A} \rho_{t+1}^x(D, \omega)$ and $c_{\max}^{t,D,A} := \max_{\omega \in A} \rho_{t+1}^x(D, \omega)$. By the definition (3.9) of ρ^x,

$$c_0 < \inf\{c \in \mathbb{R} : \alpha_{t+1}(D + c1_{\{t+1\}}, \omega) \geq x\}\,,$$

for any $c_0 < c_{\min}^{t,D,A}$ and $\omega \in A$. Thus,

$$\alpha_{t+1}(D + c_0 1_{\{t+1\}}, \omega) < x\,,$$

for all $\omega \in A$. Due to the finiteness of the probability space Ω, there exists a number $\epsilon_{A,c_0} > 0$, such that $\alpha_{t+1}(D + c_0 1_{\{t+1\}}, \omega) \leq x - \epsilon_{A,c_0}$, for all $\omega \in A$. By (D2) – independent of the past of α,

$$\alpha_{t+1}(D - D_t 1_{\{t\}} + c_0 1_{\{t+1\}}, \omega) = \alpha_{t+1}(D + c_0 1_{\{t+1\}}, \omega) \leq x - \epsilon_{A,c_0}\,,$$

for all $\omega \in A$. Note that $1_A(D - D_t 1_{\{t\}} + c_0 1_{\{t+1\}})_t = 1_A(D_t - D_t) = 0$. Since α is a normalized DCAI, by Corollary 3.2.3, we know that α also satisfied (D7-II). Then,

$$\alpha_t(D - D_t 1_{\{t\}} + c_0 1_{\{t+1\}}, \omega) \leq 1_A \max_{\bar{\omega} \in A} \alpha_{t+1}(D - D_t 1_{\{t\}} + c_0 1_{\{t+1\}}, \bar{\omega}) \leq x - \epsilon_{A,c_0}\,,$$

for all $\omega \in A$. Consequently, since c_0 is a constant, by (D6)

$$\begin{aligned} \alpha_t(D + (c_0 - D_t) 1_{\{t\}}, \omega) &= \alpha_t(D - D_t 1_{\{t\}} + c_0 1_{\{t\}}, \omega) \\ &= \alpha_t(D - D_t 1_{\{t\}} + c_0 1_{\{t+1\}}, \omega) \\ &\leq x - \epsilon_{A,c_0} < x\,, \end{aligned}$$

for all $\omega \in A$ and $c_0 < c_{\min}^{t,D,A}$. By the definition of ρ^x

$$\rho_t^x(D, \omega) = \inf\{c \in \mathbb{R} : \alpha_t(D + c 1_{\{t\}}, \omega) \geq x\} \geq c_0 - D_t(\omega)\,,$$

for all $\omega \in A$ and $c_0 < c_{\min}^{t,D,A}$. Hence, $\rho_t^x(D, \omega) \geq c_{\min}^{t,D,A} - D_t(\omega)$, or equivalently $1_A \rho_t^x(D) \geq 1_A(\min_{\omega \in A} \rho_{t+1}^x(D, \omega) - D_t)$. Similarly, one can show that $1_A \rho_t^x(D) \leq 1_A(\max_{\omega \in A} \rho_{t+1}(D, \omega) - D_t)$, and thus (A7) is established. All the above imply that ρ^x is a DCRM for every $x > 0$.

Assume that $x \geq y > 0$. Then

$$\{c \in \mathbb{R} : \alpha_t(D + c 1_{\{t\}}, \omega) \geq x\} \subseteq \{c \in \mathbb{R} : \alpha_t(D + c 1_{\{t\}}, \omega) \geq y\}\,,$$

which implies implies that

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c 1_{\{t\}}, \omega) \geq x\} \geq \inf\{c \in \mathbb{R} : \alpha_t(D + c 1_{\{t\}}, \omega) \geq y\}\,.$$

Therefore, by definition (3.9) of ρ^x, we have

$$\rho_t^x(D, \omega) \geq \rho_t^y(D, \omega)\,,$$

for all $t \in \mathcal{T}, D \in \mathcal{D},\ \omega \in \Omega$ and $x \geq y > 0$. Hence, the family of dynamic coherent risk measures $(\rho^x)_{x \in (0, +\infty)}$ is non-decreasing.

Finally, we will show that $(\rho^x)_{x\in(0,+\infty)}$ is left-continuous. Let x_0 be any positive number. Then,

$$\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} \subseteq \{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}\,,$$

for all $x < x_0$. Taking infimum of both sides, we get

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} \geq \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}\,. \tag{3.12}$$

By taking the left limit of the right hand side, we have,

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} \geq \lim_{x\to x_0^-} \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}\,. \tag{3.13}$$

If the above inequality holds strictly, then there exists a constant c_0 such that,

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} > c_0 > \lim_{x\to x_0^-} \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}\,. \tag{3.14}$$

Note that, by (3.12), $\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}$ is an non-decreasing function with respect to x. Therefore, the second inequality in (3.14) implies that,

$$c_0 > \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}\,,$$

for all $x < x_0$. Hence, by (D3) – monotonicity of α, $\alpha_t(D + c_0 1_{\{t\}}, \omega) \geq x$, for all $x < x_0$, and thus

$$\alpha_t(D + c_0 1_{\{t\}}, \omega) \geq \lim_{x\to x_0^-} x = x_0\,.$$

On the other hand, by the first inequality in (3.14), we deduce that,

$$\alpha_t(D + c_0 1_{\{t\}}, \omega) < x_0\,.$$

Contradiction. Therefore, we should have strict equality in (3.13). □

Next theorem shows the representation of a DCAI in terms of a family of DCRMs.

Theorem 3.4.2. *Assume that $(\rho^x)_{x\in(0,+\infty)}$ is an increasing family of dynamic coherent risk measures. Then the function α defined as follows,*

$$\alpha_t(D,\omega) := \sup\{x \in (0,+\infty) : \rho_t^x(D,\omega) \leq 0\}\,, \tag{3.15}$$

for $t \in \mathcal{T}$, $D \in \mathcal{D}$ and $\omega \in \Omega$, is a normalized, right-continuous, dynamic coherent acceptability index.

Proof. Note that the assumption $\sup\emptyset = 0$ guarantees that α from (3.15) is well-defined and takes values in $[0,+\infty]$.

In the following, we will prove that α defined in (3.15) satisfies the properties (D1)–(D7).

(D1) - adaptiveness, and (D2) - independence of the past, follow immediately from the definition of α, and from adaptiveness (A1) and independence of the past (A2) of ρ^x.

Let $t \in \mathcal{T}$, $D, D' \in \mathcal{D}$, and assume that $D_s(\omega) \geq D'_s(\omega)$ for all $s \geq t$, and $\omega \in \Omega$. By (A3) – monotonicity of ρ^x,

$$\rho_t^x(D) \leq \rho_t^x(D')\,, \qquad \text{for all } x > 0\,. \tag{3.16}$$

Note that, for any $x_0 \in \{x \in (0,+\infty) : \rho_t^x(D',\omega) \leq 0\}$, we have $\rho_t^{x_0}(D',\omega) \leq 0$, which combined with (3.16) implies $\rho_t^{x_0}(D,\omega) \leq \rho_t^{x_0}(D',\omega) \leq 0\,, \quad \omega \in \Omega$. Therefore,

$$\{x \in (0,+\infty) : \rho_t^x(D,\omega) \leq 0\} \supseteq \{x \in (0,+\infty) : \rho_t^x(D',\omega) \leq 0\}$$

By taking supremum of both sides, we get

$$\sup\{x \in (0,+\infty) : \rho_t^x(D,\omega) \leq 0\} \geq \sup\{x \in (0,+\infty) : \rho_t^x(D',\omega) \leq 0\}\,,$$

and hence, by the definition (3.15) of α, property (D3) follows.

By homogeneity (A4) of ρ^x, for every $\lambda > 0, D \in \mathcal{D}, t \in \mathcal{T}$ and $\omega \in \Omega$, we have,

$$\begin{aligned}\alpha_t(\lambda D, \omega) &= \sup\{x \in (0, +\infty) : \lambda \rho_t^x(D, \omega) \leq 0\} \\ &= \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \leq 0\} \\ &= \alpha_t(D, \omega)\,.\end{aligned}$$

Therefore, α is scale invariant and satisfies (D4).

Next we will prove that α is quasi-concave. For given $t \in \mathcal{T}$, and $x^0 \in (0, +\infty]$, if $D, D' \in \mathcal{D}$ are such that $\alpha_t(D, \omega) \geq x^0, \alpha_t(D', \omega) \geq x^0$, then, by definition (3.15) of α, we have

$$\begin{aligned}\sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \leq 0\} \geq x^0\,, \\ \sup\{x \in (0, +\infty) : \rho_t^x(D', \omega) \leq 0\} \geq x^0\,.\end{aligned}$$

Using these, and monotonicity of ρ^x in x, we conclude that for any $x < x^0$,

$$\rho_t^x(D, \omega) \leq 0\,, \quad \rho_t^x(D, \omega) \leq 0\,.$$

By (A4), homogeneity of ρ^x, we note that for any $\lambda \in [0, 1]$ and $x < x^0$,

$$\begin{aligned}\rho_t^x(\lambda D, \omega) = \lambda \rho_t^x(D, \omega) \leq 0\,, \\ \rho_t^x((1-\lambda)D', \omega) = (1-\lambda)\rho_t^x(D', \omega) \leq 0\,.\end{aligned}$$

From here, by (A5), subadditivity of ρ^x, we get

$$\rho_t^x(\lambda D + (1-\lambda)D', \omega) \leq \rho_t^x(\lambda D, \omega) + \rho_t^x((1-\lambda)D', \omega) \leq 0\,,$$

for any $x < x^0$. Hence $\sup\{x \in (0, +\infty) : \rho_t^x(\lambda D + (1-\lambda)D', \omega) \leq 0\} \geq x^0$, and thus, by definition (3.15) of α, we have, $\alpha(\lambda D + (1-\lambda)D', \omega) \geq x_0$. This yields quasi-concavity of α.

Assume that $D \in \mathcal{D}$, and m is an $\mathcal{F}_t$-measurable random variable. By (3.15) and (A6), we get

$$\begin{aligned}\alpha_t(D + m1_{\{s\}}, \omega) &= \sup\{x \in (0, +\infty) : \rho_t^x(D + m1_{\{s\}}, \omega) \leq 0\} \\ &= \sup\{x \in (0, +\infty) : \rho_t^x(D + m1_{\{t\}}, \omega) \leq 0\} \\ &= \alpha_t(D + m1_{\{t\}}, \omega)\,,\end{aligned}$$

for all $s \geq t$ and $\omega \in \Omega$. Hence, α satisfies property (A6).

Now, let us show that α satisfies dynamic consistency property (D7). Assume that $D, D' \in \mathcal{D}$, and $t \in \mathcal{T}$ are such that $D_t(\omega) \geq 0 \geq D'_t(\omega)$ for all $\omega \in \Omega$, and there exists a non-negative $\mathcal{F}_t$-measurable random variable m such that $\alpha_{t+1}(D, \omega) \geq m(\omega) \geq \alpha_{t+1}(D', \omega)$ for all $\omega \in \Omega$. By definition (3.15),

$$\sup\{x \in (0, +\infty) : \rho_{t+1}^x(D, \omega) \leq 0\} \geq m(\omega) \geq \sup\{x \in (0, +\infty) : \rho_{t+1}^x(D', \omega) \leq 0\}\,,$$

for all $\omega \in \Omega$. Let us fix an $\bar{\omega} \in \Omega$, and denote by $\bar{c} := m(\bar{\omega})$. There exists a $P_i^t \in \Upsilon^t$ such that $\bar{\omega} \in P_i^t$. From the above inequality, we conclude that for all $\omega \in P_i^t$,

$$\sup\{x \in (0, +\infty) : \rho_{t+1}^x(D, \omega) \leq 0\} \geq \bar{c} \geq \sup\{x \in (0, +\infty) : \rho_{t+1}^x(D', \omega) \leq 0\}\,.$$

Then, for all $c' > \bar{c}$ and $\omega \in P_i^t$, $c' > \sup\{x \in (0, +\infty) : \rho_{t+1}^x(D', \omega) \leq 0\}$, which consequently implies that

$$\rho_{t+1}^{c'}(D', \omega) > 0\,. \tag{3.17}$$

Also note that $\sup\{x \in (0, +\infty) : \rho_{t+1}^x(D, \omega) \leq 0\} > c$, for any $c < \bar{c}$. By monotonicity of ρ^x with respect to x, we have $\rho_{t+1}^c(D, \omega) \leq 0,\ \omega \in P_i^t$. Due to the finiteness of Ω, (3.17) implies that

$$\min_{\omega \in P_i^t} \rho_{t+1}^{c'}(D', \omega) > 0\,,$$

for all $c' > \bar{c}$. Using (A7), dynamic consistency of ρ^x, we get the following

$$1_{P_i^t}\rho_t^{c'}(D') \geq 1_{P_i^t}(\min_{\omega \in P_i^t} \rho_{t+1}^{c'}(D',\omega) - D_t')$$

$$= 1_{P_i^t} \min_{\omega \in P_i^t} \rho_{t+1}^{c'}(D',\omega) - 1_{P_i^t} D_t', \quad c' > \bar{c}\,.$$

Equivalently,

$$\rho_t^{c'}(D',\omega) \geq \min_{\omega \in P_i^t} \rho_{t+1}^{c'}(D',\omega) - D_t'(\omega) > -D_t'(\omega) \geq 0\,, \tag{3.18}$$

for all $\omega \in P_i^t$, and $c' > \bar{c}$.

If

$$\bar{c} < \sup\{x \in (0,+\infty) : \rho_t^x(D',\omega') \leq 0\}\,,$$

for some $\omega' \in P_i^t$, then there exists a constant c^0 such that

$$\bar{c} < c^0 < \sup\{x \in (0,+\infty) : \rho_t^x(D',\omega') \leq 0\}\,.$$

This implies that $\rho_t^{c^0}(D',\omega') \leq 0$, that contradicts (3.18). Therefore,

$$\bar{c} \geq \sup\{x \in (0,+\infty) : \rho_t^x(D',\omega) \leq 0\}\,,$$

and by (3.15), we have

$$\bar{c} \geq \alpha_t(D',\omega)\,, \quad \omega \in P_i^t. \tag{3.19}$$

By similar arguments, one can show that

$$\bar{c} \leq \alpha_t(D,\omega)\,, \quad \omega \in P_i^t. \tag{3.20}$$

Since $\bar{\omega}$ was arbitrarily chosen, by (3.19) and (3.20), we finally conclude that,

$$\alpha_t(D,\omega) \geq m(\omega) \geq \alpha_t(D',\omega), \quad \text{for all } \ \omega \in \Omega\,.$$

Thus (A7) is checked.

Let us show that α is right-continuous. Given $t \in \mathcal{T}, D \in \mathcal{D}$ and $\omega \in \Omega$, we have

$$\{x \in (0,+\infty) : \rho_t^x(D,\omega) \leq 0\} \subseteq \{x \in (0,+\infty) : \rho_t^x(D,\omega) \leq c\}\,,$$

for any constant $c > 0$. Taking the supremum of both sides, and then the limit of the right hand side as $c \to 0+$, we get

$$\sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\} \le \lim_{c \to 0^+} \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le c\}\,. \quad (3.21)$$

If the above inequality holds strictly, then there exists $x^0 \in (0, +\infty)$ such that

$$\sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\} < x^0 < \lim_{c \to 0^+} \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le c\}\,. \quad (3.22)$$

The second inequality implies that

$$x^0 < \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le c\}, \quad \text{for all } c > 0.$$

By monotonicity of ρ^x, we deduce that $\rho_t^{x^0}(D, \omega) \le c$. Since the last inequality holds true for all $c > 0$, we have that

$$\rho_t^{x^0}(D, \omega) \le \lim_{c \to 0^+} c = 0\,,$$

that contradicts first strict inequality in (3.22). Therefore, we have equality in (3.21). Using this equality, and (A6), translation invariance of ρ^x, we write

$$\begin{aligned}
\alpha_t(D, \omega) &= \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\} \\
&= \lim_{c \to 0^+} \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le c\} \\
&= \lim_{c \to 0^+} \sup\{x \in (0, +\infty) : \rho_t^x(D + c1_{\{t\}}, \omega) \le 0\} \\
&= \lim_{c \to 0^+} \alpha_t(D + c1_{\{t\}}, \omega)\,,
\end{aligned}$$

and continuity of α is established.

Finally, we will prove that α is normalized. Given a fixed $t \in \mathcal{T}$, consider the following cash-positions

$$D_{\text{pos}} := 1_{\{t\}}, \quad D_{\text{neg}} := -1_{\{t\}}\,.$$

Recall that $\rho_t(0) = 0$. By (3.15) and (A6), we have

$$\begin{aligned}\alpha_t(D_{\text{pos}}, \omega) &= \sup\{x \in (0, +\infty) : \rho_t^x(1_{\{t\}}, \omega) \le 0\} \\ &= \sup\{x \in (0, +\infty) : \rho_t^x(0, \omega) - 1 \le 0\} \\ &= \sup\{x \in (0, +\infty) : -1 \le 0\} = +\infty\,.\end{aligned}$$

Similarly, one can show that $\alpha_t(D_{\text{neg}}, \omega) = 0$.

The proof is complete. □

We conclude this section with two results: one provides a representation of a DCAI in terms of a family of DCRMs; the other one gives a representation of DCRM in terms of a DCAI.

Theorem 3.4.3. *If α is a normalized, right-continuous, dynamic coherent acceptability index, then there exists a left-continuous and increasing family of dynamic coherent risk measures $(\rho^x)_{x \in (0,+\infty)}$, such that*

$$\alpha_t(D, \omega) = \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}\,. \tag{3.23}$$

Proof. For every $x \in (0, +\infty)$, define $\rho^x = (\rho_t^x)_{t=0}^T$ as follows,

$$\rho_t^x(D, \omega) := \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \ge x\}\,, \tag{3.24}$$

for all $t \in \mathcal{T}$, $D \in \mathcal{D}$ and $\omega \in \Omega$. By Theorem 3.4.1, $(\rho^x)_{x \in (0,+\infty)}$ is an increasing, left-continuous, family of dynamic coherent risk measures. We will show that

$$\alpha_t(D, \omega) = \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}\,,$$

for all $t \in \mathcal{T}, D \in \mathcal{D}$ and $\omega \in \Omega$.

Fix $t \in \mathcal{T}, D \in \mathcal{D}$, $\omega \in \Omega$. For all $y_0 > \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}$, we have $\rho_t^{y_0}(D, \omega) > 0$. By (3.24), $\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \ge y_0\} > 0$, and hence,

$$\alpha_t(D, \omega) = \alpha_t(D + 01_{\{t\}}, \omega) < y_0\,.$$

Since the above inequality holds true for all $y_0 > \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}$, we conclude that

$$\alpha_t(D, \omega) \le \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}\,.$$

On the other hand, for all $y_0 < \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}$, since ρ is an increasing function with respect to x, we have $\rho_t^{y_0}(D, \omega) \le 0$. By (3.24), $\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \ge y_0\} \le 0$, and hence, for all $\eta > 0$,

$$\alpha_t(D, \omega) = \alpha_t(D + \eta 1_{\{t\}}, \omega) \ge y_0\,.$$

Since α is right-continuous,

$$\alpha_t(D, \omega) = \lim_{\eta \to 0^+} \alpha_t(D + \eta 1_{\{t\}}, \omega) \ge y_0$$

Note that the above inequality holds true for all $y_0 < \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}$, we conclude that

$$\alpha_t(D, \omega) \ge \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}\,.$$

Finally, we have that $\alpha_t(D, \omega) = \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}$. □

Theorem 3.4.4. *If $(\rho^x)_{x \in (0, +\infty)}$ is a left-continuous and increasing family of dynamic coherent risk measures, then there exists a right-continuous and normalized dynamic coherent acceptability index α such that,*

$$\rho_t^x(D, \omega) = \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \ge x\}\,,$$

Proof. Define the function α as follows,

$$\alpha_t(D, \omega) := \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \le 0\}\,, \tag{3.25}$$

for all $t \in \mathcal{T}$, $D \in \mathcal{D}$ and $\omega \in \Omega$. By Theorem 3.4.2, α is a right-continuous and normalized dynamic coherent acceptability index.

We will show that

$$\rho_t^x(D,\omega) = \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\},$$

for all $x \in (0, +\infty)$, $t \in \mathcal{T}, D \in \mathcal{D}$ and $\omega \in \Omega$.

Fix $t \in \mathcal{T}, D \in \mathcal{D}$, $\omega \in \Omega$ and $x_0 \in (0, +\infty)$. For all $y_0 > \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\}$, we have $\alpha_t(D + y_0 1_{\{t\}}, \omega) \geq x_0$. By (3.25), $\sup\{x \in (0, +\infty) : \rho_t^x(D + y_0 1_{\{t\}}, \omega) \leq 0\} \geq x_0$, and hence for $\eta > 0$,

$$\rho_t^{x_0 - \eta}(D + y_0 1_{\{t\}}, \omega) \leq 0.$$

Since ρ^x is a left-continuous function, we have that $\rho_t^{x_0}(D + y_0 1_{\{t\}}, \omega) \leq 0$. By (A6), $\rho_t^{x_0}(D, \omega) \leq y_0$. The above inequality holds for all $y_0 > \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\}$. Hence,

$$\rho_t^{x_0}(D,\omega) \leq \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\}.$$

On the other hand, for all $y_0 < \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\}$, we have $\alpha_t(D + y_0 1_{\{t\}}, \omega) < x_0$. By (3.25), $\sup\{x \in (0, +\infty) : \rho_t^x(D + y_0 1_{\{t\}}, \omega) \leq 0\} < x_0$, and hence,

$$\rho_t^x(D + y_0 1_{\{t\}}, \omega) > 0.$$

Then, by (A6), $\rho_t^x(D, \omega) > y_0$. The above inequality holds for all $y_0 < \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\}$. Hence,

$$\rho_t^{x_0}(D,\omega) \geq \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\}.$$

Finally, we have that $\rho_t^{x_0}(D, \omega) = \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\}$. □

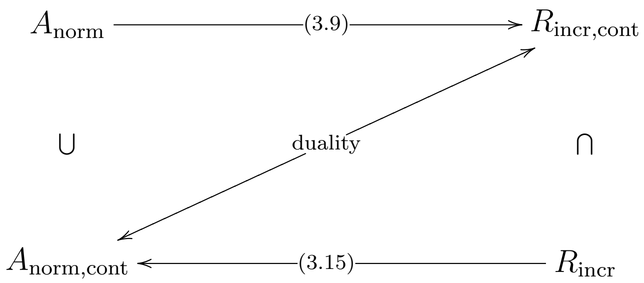
We conclude this chapter by summarizing the above four theorems in diagram. Let us define

$A_{\text{norm}} :=$ the set of normalized DCAIs

$A_{\text{norm,cont}} :=$ the set of normalized and right-continuous DCAIs

$R_{\text{incr}} :=$ the set of increasing families of DCRMs

$R_{\text{incr,cont}} :=$ the set of increasing and left-continuous families of DCRMs



Remark 3.4.1.

1. $A_{\text{norm,cont}} \subset A_{\text{norm}}$ *and* $R_{\text{incr,cont}} \subset R_{\text{incr}}$.

2. *A left-continuous, increasing family of DCRMs can be represented by a normalized DCAI through* (3.9).

3. *A right-continuous, normalized DCAI can be represented by an increasing family of DCRMs through* (3.15).

4. *There is duality between right-continuous, normalized DCAIs and left-continuous, increasing families of DCRMs.*

CHAPTER 4

CONSISTENT SETS OF PROBABILITY MEASURES AND REPRESENTATION THEOREMS

In Section 2.2 we examined the set of probability measures (scenarios) from both mathematical and financial point of view. We showed that both SCRMs and SCAIs can be represented in terms of sets of probability measures. In this chapter we will discuss sets of probability measures in a dynamic setup.

Theorem 2.2.1 indicates that every set of probability measures generates a SCRM. However, due to dynamic consistency property for DCRM, the set of probability measures that can generate a DCRM has to possess some additional features. A set of probability measures having such additional features is referred to as a dynamic consistent set of probability measures. For a thorough discussion of various definitions of dynamic consistent sets of probability measures and their relationship with dynamic consistency property for dynamic risk measures we refer the reader to [1, 12, 37] and references therein.

4.1 Dynamically Consistent Sequence of Sets of Probability Measures

In this section we shall discuss the concept of dynamically consistent sequence of sets of probability measures, or, for short, consistent sets of probability measures. Note that in dynamic setup, traditional researchers usually consider a fixed individual set of probability measures over time, whereas our research will focus on a sequence of sets of probability measures.

4.1.1 Definitions. Suppose we have the same mathematical setup and notations as in Section 3.1. In what follows we denote by $\mathcal{P}$ the set of all absolutely continuous probability measures with respect to the underlying probability $\mathbb{P}$, and $\mathcal{P}^e$ the set of all equivalent probability measures with respect to $\mathbb{P}$. Recall that our standing assumption is that $\mathbb{P}$ has full support. Note that in this case, due to the finiteness of Ω, the set $\mathcal{P}$ consists of all probability measures on Ω, and also $\mathcal{P}^e$ coincides with the set of all probability measures on Ω of full support.

Definition 4.1.1. *For any set of probability measures $\mathcal{Q} \subset \mathcal{P}$, its effective subset $\mathrm{eff}^{t,i}(\mathcal{Q})$ with respect to $P_i^t \in \Upsilon^t$ is defined as follows:*

$$\mathrm{eff}^{t,i}(\mathcal{Q}) := \{\mathbb{Q} \in \mathcal{Q} \ : \ \mathbb{Q}(P_i^t) > 0\}.$$

Definition 4.1.2. *An individual set of probability measures $\mathcal{Q} \subset \mathcal{P}$ is called full-support with respect to filtration $\mathbb{F}$ if $\mathrm{eff}^{t,i}(\mathcal{Q}) \neq \emptyset$ for all $t \in \mathcal{T}$ and $i \in \{1, 2, \cdots, n_t\}$.*

Definition 4.1.3. *A sequence of sets of probability measures $\{\mathcal{Q}_t\}_{t=0}^T$ is called full-support with respect to filtration $\mathbb{F}$ if $\mathrm{eff}^{t,i}(\mathcal{Q}_t) \neq \emptyset$ for all $t \in \mathcal{T}$ and $i \in \{1, 2, \cdots, n_t\}$.*

Note that if an individual set of probability measures $\mathcal{Q} \subset \mathcal{P}$ is full-support, then a sequence of sets of probability measures $\{\mathcal{Q}_t\}_{t=0}^T$ defined as $\mathcal{Q}_t := \mathcal{Q}$ for all $t \in \mathcal{T}$ is full-support as well.

Definition 4.1.4. *For any $P_i^t \in \Upsilon^t$, let $\mathcal{Q}$ be a subset of $\mathcal{P}$ with $\mathrm{eff}^{t,i}(\mathcal{Q}) \neq \emptyset$. The infimum conditional expectation $\inf\limits_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|P_i^t]$ is defined as follows:*

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] := \inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] = \inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \left(\sum_{\omega\in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} X(\omega) \right)$$

for all $X \in \mathcal{G}$.

Definition 4.1.5. *For any $P_i^t \in \Upsilon^t$, let $\mathcal{Q}$ be a subset of $\mathcal{P}$ with $\mathrm{eff}^{t,i}(\mathcal{Q}) \neq \emptyset$. The supremum conditional expectation $\sup\limits_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|P_i^t]$ is defined as follows:*

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] := \sup_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] = \sup_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \left(\sum_{\omega\in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} X(\omega) \right)$$

for all $X \in \mathcal{G}$.

We denote by $\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ a random variable such that for all $P_i^t \in \Upsilon^t$ and $\omega \in P_i^t$, $\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t](\omega) := \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|P_i^t]$. In addition, we denote by $\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ a random variable such that for all $P_i^t \in \Upsilon^t$ and $\omega \in P_i^t$, $\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t](\omega) := \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|P_i^t]$.

Note that both $\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ and $\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ are $\mathcal{F}_t$-measurable.

Definition 4.1.6. *Let $\mathcal{Q}$ be a subset of $\mathcal{P}$. $\mathcal{Q}$ is called strongly consistent with respect to filtration $\mathbb{F}$, if it is full-support and the following equality holds true*

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\Big[X \mid \mathcal{F}_t\Big] = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\Big[\inf_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}\big[X \mid \mathcal{F}_{t+1}\big] \mid \mathcal{F}_t\Big],$$

for every $t \in \{0, \dots, T-1\}$, and $X \in \mathcal{G}$.

Definition 4.1.7. *Let $\mathcal{Q}$ be a subset of $\mathcal{P}$. $\mathcal{Q}$ is called weakly consistent with respect to filtration $\mathbb{F}$, if it is full-support and the following inequality holds true*

$$1_A \max_{\omega\in A}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|\mathcal{F}_{t+1}\big](\omega)\right\} \geq 1_A \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|\mathcal{F}_t\big],$$

for every $t \in \{0, \dots, T-1\}$, $A \in \mathcal{F}_t$, and $X \in \mathcal{G}$.

The following proposition shows that a strongly consistent set of probability measures is also weakly consistent.

Proposition 4.1.1. *If a set of probability measures $\mathcal{Q} \subseteq \mathcal{P}$ is strongly consistent, then $\mathcal{Q}$ is also weakly consistent.*

Proof. $\mathcal{Q}$ is strongly consistent indicates that $\mathcal{Q}$ is full-support with respect to $\mathbb{F}$. By Definition 4.1.7, it is enough to show that for each $P_i^t \in \Upsilon^t$,

$$\max_{\omega\in P_i^t}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|\mathcal{F}_{t+1}\big](\omega)\right\} \geq \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|P_i^t\big].$$

Since $\mathcal{Q}$ is strongly consistent, Definitions 4.1.6 and 4.1.4 implies

$$\begin{aligned}\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|P_i^t\big] &= \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\Big[\inf_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}\big[X \mid \mathcal{F}_{t+1}\big] \mid P_i^t\Big] \\ &= \inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}\Big[1_{P_i^t} \inf_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}\big[X \mid \mathcal{F}_{t+1}\big] \mid P_i^t\Big] \\ &\leq \inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}\left[\max_{\omega\in P_i^t}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|\mathcal{F}_{t+1}\big](\omega)\right\} \,\middle|\, P_i^t\right]. \qquad (4.1)\end{aligned}$$

Since $\max_{\omega\in P_i^t}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|\mathcal{F}_{t+1}\big](\omega)\right\}$ is a constant, for each $\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})$,

$$\mathbb{E}_{\mathbb{Q}}\left[\max_{\omega\in P_i^t}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|\mathcal{F}_{t+1}\big](\omega)\right\} \,\middle|\, P_i^t\right] = \max_{\omega\in P_i^t}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|\mathcal{F}_{t+1}\big](\omega)\right\}.$$

Therefore, (4.1) gives

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|P_i^t\big] \leq \max_{\omega\in P_i^t}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[X|\mathcal{F}_{t+1}\big](\omega)\right\}.$$

Hence, $\mathcal{Q}$ is weakly consistent, and the proof is complete. □

Next, we define the consistency on a sequence of sets of probability measures.

Definition 4.1.8. *A sequence of sets of probability measures $\{\mathcal{Q}_t\}_{t=0}^T$, with $\mathcal{Q}_t \subseteq \mathcal{P}$, is called dynamically consistent with respect to the filtration $\mathbb{F}$, if the sequence is full-support and the following inequality holds true*

$$\begin{aligned}1_A \min_{\omega\in A}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}](\omega)\right\} &\leq 1_A \inf_{\mathbb{Q}\in\mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \\ &\leq 1_A \max_{\omega\in A}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}](\omega)\right\},\end{aligned}$$

for every $t\in\{0,\ldots,T-1\}$, $A\in\mathcal{F}_t$, and $X\in\mathcal{G}$.

We will discuss how to construct dynamically consistent sequences of sets of probability measures from dynamic consistent individual sets of probability measures.

Proposition 4.1.2. *For any full-support set of probability measures $\mathcal{Q} \subseteq \mathcal{P}$, the following inequalities hold true*

$$1_A \min_{\omega \in A} \left\{ \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}](\omega) \right\} \leq 1_A \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \,, \tag{4.2}$$

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\inf_{\mathbb{M} \in \mathcal{Q}} \mathbb{E}_{\mathbb{M}}[X|\mathcal{F}_{t+1}]|\mathcal{F}_t] \leq \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \,, \tag{4.3}$$

for every $t \in \{0, \ldots, T-1\}$, $A \in \mathcal{F}_t$, and $X \in \mathcal{G}$.

Proof. Fix $t \in \{0, \ldots, T-1\}$ and $P_i^t \in \Upsilon^t$, by Proposition 3.1.1, we can write $P_i^t = \cup_{j=1}^{k_i} P_{i,j}^{t+1}$. Since $\mathcal{Q}$ is full-support, $\mathrm{eff}^{t,i}(\mathcal{Q}) \neq \emptyset$. For any $\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q})$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}[X|P_i^t] \\ &= \sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} X(\omega) = \sum_{j=1}^{k_i} \sum_{\omega \in P_{i,j}^{t+1}} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} X(\omega) \\ &= \sum_{\mathbb{Q}(P_{i,j}^{t+1}) \neq 0} \frac{\mathbb{Q}(P_{i,j}^{t+1})}{\mathbb{Q}(P_i^t)} \sum_{\omega \in P_{i,j}^{t+1}} \frac{\mathbb{Q}(\omega) X(\omega)}{\mathbb{Q}(P_{i,j}^{t+1})} \,. \end{aligned} \tag{4.4}$$

First, by (4.4) and Definition 4.1.4, for any $\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q})$, we can derive that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] &\geq \sum_{\mathbb{Q}(P_{i,j}^{t+1}) \neq 0} \frac{\mathbb{Q}(P_{i,j}^{t+1})}{\mathbb{Q}(P_i^t)} \min_{\omega \in P_i^t} \left\{ \inf_{\mathbb{M} \in \mathcal{Q}} \mathbb{E}_{\mathbb{M}}\big[X|\mathcal{F}_{t+1}\big](\omega) \right\} \\ &= \min_{\omega \in A} \left\{ \inf_{\mathbb{M} \in \mathcal{Q}} \mathbb{E}_{\mathbb{M}}[X|\mathcal{F}_{t+1}](\omega) \right\} . \end{aligned}$$

Hence,

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] = \inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] \geq \min_{\omega \in A} \left\{ \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}](\omega) \right\} .$$

Then, (4.2) holds true for each $P_i^t \in \Upsilon^t$ and therefore for all $A \in \mathcal{F}_t$.

Second, by (4.4) and Definition 4.1.4, for any $\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q})$, we can derive that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[X|P_i^t] \geq & \sum_{\mathbb{Q}(P_{i,j}^{t+1})\neq 0} \frac{\mathbb{Q}(P_{i,j}^{t+1})}{\mathbb{Q}(P_i^t)} \inf_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}[X|P_{i,j}^{t+1}] \\
= & \sum_{\mathbb{Q}(P_{i,j}^{t+1})\neq 0} \frac{\mathbb{Q}(P_{i,j}^{t+1})}{\mathbb{Q}(P_i^t)} \inf_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}[X|P_{i,j}^{t+1}] + \sum_{\mathbb{Q}(P_{i,j}^{t+1})=0} 0 \inf_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}[X|P_{i,j}^{t+1}] \\
= & \mathbb{E}_{\mathbb{Q}}[\inf_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}[X|\mathcal{F}_{t+1}]|P_i^t] \, .
\end{aligned}
$$

Consequently, after taking infimum of the right hand side in previous inequality, we deduct that

$$
\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \geq \inf_{\widetilde{\mathbb{Q}}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\widetilde{\mathbb{Q}}}[\inf_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}[X|\mathcal{F}_{t+1}]|\mathcal{F}_t] \, , \quad \mathbb{Q} \in \mathcal{Q}^{t,i} \, ,
$$

By Lemma B.0.1 and Definition 4.1.4, (4.3) follows. □

The following useful corollary is a direct consequence of Proposition 4.1.2.

Corollary 4.1.1. *If a set of probability measures $\mathcal{Q} \subseteq \mathcal{P}$ is weakly consistent, then $\{\mathcal{Q}_t\}_{t=0}^T$, with $\mathcal{Q}_t = \mathcal{Q}$, $t \in \mathcal{T}$, is a dynamically consistent sequence of sets of probability measures.*

Using Proposition 4.1.1 and Corollary 4.1.1, we also conclude the following result.

Corollary 4.1.2. *If a set of probability measures $\mathcal{Q} \subseteq \mathcal{P}$ is strongly consistent, then $\{\mathcal{Q}_t\}_{t=0}^T$, with $\mathcal{Q}_t = \mathcal{Q}$, $t \in \mathcal{T}$, is a dynamically consistent sequence of sets of probability measures.*

4.1.2 Examples. The rest of the section is dedicated to examples of dynamically consistent sequences of sets of probability measures.

Example 4.1.1. *Singleton set $\mathcal{Q} = \{\mathbb{Q}\}$, with $\mathbb{Q} \in \mathcal{P}^e$, is full-support and clearly strongly consistent. By Corollary 4.1.2 the constant sequence $\{\mathcal{Q}, \mathcal{Q}, \ldots, \mathcal{Q}\}$ is dynamically consistent. For simplicity of writing, we will denote this sequence by $\mathcal{Q}^s$.*

Lemma 4.1.1. *For any $t \in \mathcal{T}$, we have*

$$\sum_{i=1}^{n_t} 1_{P_i^t} \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[D|\mathcal{F}_t] = \sum_{i=1}^{n_t} 1_{P_i^t} \min_{\omega\in P_i^t} D(\omega)\,.$$

Proof. It is enough to show that for each $P_i^t \in \Upsilon^t$,

$$\inf_{\mathbb{Q}\in \mathrm{eff}^{t,i}(\mathcal{P})} \mathbb{E}_{\mathbb{Q}}[D|P_i^t] = \min_{\omega\in P_i^t} D(\omega)\,.$$

Note that for all $\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{P})$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[D|P_i^t] &= \sum_{\omega\in P_i^t} D(\omega)\frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \\ &\geq \min_{\omega\in P_i^t} D(\omega) \sum_{\omega\in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} = \min_{\omega\in P_i^t} D(\omega)\end{aligned}$$

Hence,

$$\inf_{\mathbb{Q}\in \mathrm{eff}^{t,i}(\mathcal{P})} \mathbb{E}_{\mathbb{Q}}[D|P_i^t] \geq \min_{\omega\in P_i^t} D(\omega)\,. \tag{4.5}$$

If we take $\bar{\omega}$ such that $D(\bar{\omega}) = \min_{\omega\in P_i^t} D(\omega)$, and $\mathbb{Q}^n$ such that $\mathbb{Q}^n(\bar{\omega}) = (1-1/n)\mathbb{Q}^n(P_i^t)$ and uniformly distributed in other states, we can prove that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}[D|P_i^t] = \min_{\omega\in P_i^t} D(\omega)\,.$$

Note that $\mathbb{Q}^n \in \mathrm{eff}^{t,i}(\mathcal{P})$, then

$$\inf_{\mathbb{Q}\in \mathrm{eff}^{t,i}(\mathcal{P})} \mathbb{E}_{\mathbb{Q}}[D|P_i^t] \leq \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}[D|P_i^t] = \min_{\omega\in P_i^t} D(\omega)\,.$$

The above inequality together with (4.5) implies

$$\inf_{\mathbb{Q}\in \mathrm{eff}^{t,i}(\mathcal{P})} \mathbb{E}_{\mathbb{Q}}[D|P_i^t] = \min_{\omega\in P_i^t} D(\omega)\,.$$

□

Example 4.1.2. *Lemma 4.1.1 implies that the set $\mathcal{P}$ of all absolutely continuous probability measures with respect to $\mathbb{P}$, is strongly consistent. Hence, the constant sequence $\{\mathcal{P}, \mathcal{P}, \ldots, \mathcal{P}\}$ is dynamically consistent.*

Example 4.1.3. *By similar argument with Lemma 4.1.1, we can prove that for any* $P_i^t \in \Upsilon^t$, $\inf_{\mathbb{Q}\in\mathcal{P}^e} \mathbb{E}_{\mathbb{Q}}[D|P_i^t] = \min_{\omega\in P_i^t} D(\omega)$*. It implies that the set* $\mathcal{P}^e$ *of all equivalent probability measures with respect to* $\mathbb{P}$*, is strongly consistent. Hence, the constant sequence* $\{\mathcal{P}^e, \mathcal{P}^e, \ldots, \mathcal{P}^e\}$ *is dynamically consistent.*

Example 4.1.4. *Let* $a \geq 1$ *be a real number. The following set of probability measures*

$$\mathcal{Q}^{a,u} := \{\mathbb{Q} \in \mathcal{P}^e \mid \mathbb{E}_{\mathbb{P}}[d\mathbb{Q}/d\mathbb{P}|\mathcal{F}_t] \leq a\mathbb{E}_{\mathbb{P}}[d\mathbb{Q}/d\mathbb{P}|\mathcal{F}_{t-1}] \text{ for all } t \in \{1, \ldots, T\}\}$$

is strongly consistent.

Proof. Indeed, by (4.3) we have

$$\inf_{\widetilde{\mathbb{Q}}\in\mathcal{Q}^{a,u}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[X|\mathcal{F}_t] \geq \inf_{\mathbb{Q}\in\mathcal{Q}^{a,u}} \mathbb{E}_{\mathbb{Q}}[\inf_{\mathbb{M}\in\mathcal{Q}^{a,u}} \mathbb{E}_{\mathbb{M}}[X|\mathcal{F}_{t+1}]|\mathcal{F}_t]\,,$$

for every $t \in \{0, \ldots, T-1\}$ and $X \in \mathcal{G}$.

Next we will show that the converse inequality also holds true and hence, by definition, $\mathcal{Q}^{a,u}$ is strongly consistent. Towards this end, assume that $t \in \mathcal{T}$, $X \in \mathcal{G}$, and $a \geq 1$; all arbitrary but fixed in what follows. For convenience, we denote by $P_{i,j}^{t+1}$ the set of partition $(P_1^{t+1}, \ldots, P_{n_{t+1}}^{t+1})$ such that $P_i^t = \cup_{j=1}^{k_i} P_{i,j}^{t+1}$, $i = 1, \ldots, n_t$. Note that $k_1 + k_2 + \cdots + k_{n_t} = n_{t+1}$. Note that since $\mathcal{Q}^{a,u} \subset \mathcal{P}^e$, $\mathcal{Q}^{a,u} = \mathrm{eff}^{t,i}(\mathcal{Q}^{a,u})$ for every $P_i^t \in \Upsilon^t$.

Pick up arbitrarily $n_t + n_{t+1}$ probability measures from $\mathcal{Q}^{a,u}$, and denote them by $(\mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_{n_t}, \mathbb{M}_{1,1}, \mathbb{M}_{1,2}, \ldots, \mathbb{M}_{1,k_1}, \mathbb{M}_{2,1}, \mathbb{M}_{2,2}, \ldots, \mathbb{M}_{2,k_2}, \ldots\ldots, \mathbb{M}_{n_t,1}, \mathbb{M}_{n_t,2}, \ldots, \mathbb{M}_{n_t,k_{n_t}})$. Some of them are allowed to be the same. We will construct a new probability measure based on the above set of probabilities. For any $i \in \{1, 2, \ldots, n_t\}$, $j \in \{1, 2, \ldots, k_i\}$, and $\omega \in P_{i,j}^{t+1}$ we put

$$\mathbb{H}(\omega) := \frac{\mathbb{M}_{i,j}(\omega)}{\mathbb{M}_{i,j}(P_{i,j}^{t+1})} \frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{Q}_i(P_i^t)} \mathbb{P}(P_i^t)\,.$$

Note that $P_{i,j}^{t+1}$, $i \in \{1,2,\dots,n_t\}$, $j \in \{1,2,\dots,k_i\}$, is a partition of Ω, and hence $\mathbb{H}$ is well-defined, and since all probability measures in $\mathcal{Q}$ are of full support, $\mathbb{H}(\omega)$ is finite for all $\omega \in \Omega$. It is also easy to show that $\mathbb{H}(\Omega) = 1$, and thus $\mathbb{H}$ is a probability measure.

Next we will prove that $\mathbb{H} \in \mathcal{Q}^{a,u}$. On any set $P_i^t \in \Upsilon^t$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|P_i^t] &= \sum_{\omega\in P_i^t} \frac{\mathbb{H}(\omega)}{\mathbb{P}(\omega)} \frac{\mathbb{P}(\omega)}{\mathbb{P}(P_i^t)} = \sum_{j=1}^{k_i} \sum_{\omega\in P_{i,j}^{t+1}} \frac{\mathbb{H}(\omega)}{\mathbb{P}(\omega)} \frac{\mathbb{P}(\omega)}{\mathbb{P}(P_i^t)} \\
&= \sum_{j=1}^{k_i} \sum_{\omega\in P_{i,j}^{t+1}} \frac{\mathbb{H}(\omega)}{\mathbb{P}(P_i^t)} = \sum_{j=1}^{k_i} \sum_{\omega\in P_{i,j}^{t+1}} \frac{\mathbb{M}_{i,j}(\omega)}{\mathbb{M}_{i,j}(P_{i,j}^{t+1})} \frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{Q}_i(P_i^t)} \frac{\mathbb{P}(P_i^t)}{\mathbb{P}(P_i^t)} \\
&= \sum_{j=1}^{k_i} \frac{\mathbb{M}_{i,j}(P_{i,j}^{t+1})}{\mathbb{M}_{i,j}(P_{i,j}^{t+1})} \frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{Q}_i(P_i^t)} \frac{\mathbb{P}(P_i^t)}{\mathbb{P}(P_i^t)} = \sum_{j=1}^{k_i} \frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{Q}_i(P_i^t)} = 1\,.
\end{aligned}$$

Thus,

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|\mathcal{F}_t] = \sum_{i=1}^{n_t} 1_{P_i^t} \mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|P_i^t] = \sum_{i=1}^{n_t} 1_{P_i^t} = 1\,.$$

Hence, by tower property, for all $s \le t$,

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|\mathcal{F}_s] = 1\,.$$

Consequently, we get

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|\mathcal{F}_s] \le a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|\mathcal{F}_{s-1}], \quad \text{for all } \ s \le t\,. \tag{4.6}$$

On the other hand, for any $P_{i,j}^{t+1} \in \Upsilon^{t+1}$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|P_{i,j}^{t+1}] &= \sum_{\omega\in P_{i,j}^{t+1}} \frac{\mathbb{H}(\omega)}{\mathbb{P}(\omega)} \frac{\mathbb{P}(\omega)}{\mathbb{P}(P_{i,j}^{t+1})} = \sum_{\omega\in P_{i,j}^{t+1}} \frac{\mathbb{H}(\omega)}{\mathbb{P}(P_{i,j}^{t+1})} \\
&= \sum_{\omega\in P_{i,j}^{t+1}} \frac{\mathbb{M}_{i,j}(\omega)}{\mathbb{M}_{i,j}(P_{i,j}^{t+1})} \frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{Q}_i(P_i^t)} \frac{\mathbb{P}(P_i^t)}{\mathbb{P}(P_{i,j}^{t+1})} \\
&= \frac{\mathbb{M}_{i,j}(P_{i,j}^{t+1})}{\mathbb{M}_{i,j}(P_{i,j}^{t+1})} \frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{Q}_i(P_i^t)} \frac{\mathbb{P}(P_i^t)}{\mathbb{P}(P_{i,j}^{t+1})} = \frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{Q}_i(P_i^t)} \frac{\mathbb{P}(P_i^t)}{\mathbb{P}(P_{i,j}^{t+1})}\,.
\end{aligned}$$

Since $\mathbb{Q}_i \in \mathcal{Q}^{a,u}$, then

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}_i}{d\mathbb{P}}|\mathcal{F}_{t+1}] \leq a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}_i}{d\mathbb{P}}|\mathcal{F}_t]\,,$$

and thus,

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}_i}{d\mathbb{P}}|P_{i,j}^{t+1}] \leq a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}_i}{d\mathbb{P}}|P_{i,j}^{t+1}] = a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}_i}{d\mathbb{P}}|P_i^t]\,.$$

This implies that,

$$\frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{P}(P_{i,j}^{t+1})} \leq a\frac{\mathbb{Q}_i(P_i^t)}{\mathbb{P}(P_i^t)}\,.$$

Hence,

$$\frac{\mathbb{Q}_i(P_{i,j}^{t+1})}{\mathbb{P}(P_{i,j}^{t+1})}\frac{\mathbb{P}(P_i^t)}{\mathbb{Q}_i(P_i^t)} \leq a\,,$$

and therefore,

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|P_{i,j}^{t+1}] \leq a = a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|P_{i,j}^{t+1}]\,.$$

Since the above holds true for any $P_{i,j}^{t+1} \in \Upsilon^{t+1}$, we have that

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|\mathcal{F}_{t+1}] \leq a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|\mathcal{F}_t]\,.$$

By similar arguments as above, inductively, one can show that

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|\mathcal{F}_s] \leq a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{H}}{d\mathbb{P}}|\mathcal{F}_t]\,,$$

for any $s > t$. Combining this with (4.6), we conclude that $\mathbb{H} \in \mathcal{Q}^{a,u}$.

Next let us evaluate $\mathbb{E}_{\mathbb{H}}[D|\mathcal{F}_t]$. Consider a new random variable Y, defined as follows:

$$Y := \sum_{i=1}^{n_t}\sum_{j=1}^{k_i} 1_{P_{i,j}^{t+1}}\mathbb{E}_{\mathbb{M}_{i,j}}[D|\mathcal{F}_{t+1}]\,.$$

Then, for any $m \in \{1, 2, \dots, n_t\}$, we deduce

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_m}[Y|P_m^t] &= \mathbb{E}_{\mathbb{Q}_m}[\sum_{i=1}^{n_t}\sum_{j=1}^{k_i} 1_{P_{i,j}^{t+1}}\mathbb{E}_{\mathbb{M}_{i,j}}[D|\mathcal{F}_{t+1}]|P_m^t] \\ &= \sum_{i=1}^{n_t}\sum_{j=1}^{k_i} \mathbb{E}_{\mathbb{Q}_m}[1_{P_{i,j}^{t+1}}\mathbb{E}_{\mathbb{M}_{i,j}}[D|\mathcal{F}_{t+1}]|P_m^t] \\ &= \sum_{j=1}^{k_m} \mathbb{E}_{\mathbb{Q}_m}[1_{P_{m,j}^{t+1}}\mathbb{E}_{\mathbb{M}_{m,j}}[D|\mathcal{F}_{t+1}]|P_m^t]\,. \end{aligned} \tag{4.7}$$

For convenience, we put $c_{m,j}^{t+1} := \mathbb{E}_{\mathbb{M}_{m,j}}[D|P_{m,j}^{t+1}] = \sum_{\omega \in P_{m,j}^{t+1}} \frac{\mathbb{M}_{m,j}(\omega)}{\mathbb{M}_{m,j}(P_{m,j}^{t+1})} D(\omega)$. Then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_m}[Y|P_m^t] &= \sum_{j=1}^{k_m} \mathbb{E}_{\mathbb{Q}_m}[1_{P_{m,j}^{t+1}} \mathbb{E}_{\mathbb{M}_{m,j}}[D|\mathcal{F}_{t+1}]|P_m^t] \\
&= \sum_{j=1}^{k_m} \mathbb{E}_{\mathbb{Q}_m}[1_{P_{m,j}^{t+1}} c_{m,j}^{t+1}|P_m^t] \\
&= \sum_{j=1}^{k_m} \frac{\mathbb{Q}_m(P_{m,j}^{t+1})}{\mathbb{Q}_m(P_m^t)} c_{m,j}^{t+1} \\
&= \sum_{j=1}^{k_m} \sum_{\omega \in P_{m,j}^{t+1}} \frac{\mathbb{Q}_m(P_{m,j}^{t+1})}{\mathbb{Q}_m(P_m^t)} \frac{\mathbb{M}_{m,j}(\omega)}{\mathbb{M}_{m,j}(P_{m,j}^{t+1})} D(\omega) \,.
\end{aligned}$$

From here, using the fact that $\mathbb{H}(P_i^t) = \mathbb{P}(P_i^t)$, we conclude that

$$\mathbb{E}_{\mathbb{Q}_m}[Y|P_m^t] = \mathbb{E}_{\mathbb{H}}[D|P_m^t] \,.$$

Since $\mathbb{H} \in \mathcal{Q}^{a,u}$, we have that $\mathbb{E}_{\mathbb{H}}[D|\mathcal{F}_t] \geq \inf_{\widetilde{\mathbb{Q}} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[D|\mathcal{F}_t]$. Consequently, the following inequality holds true

$$\mathbb{E}_{\mathbb{Q}_m}[Y|P_m^t] \geq \inf_{\widetilde{\mathbb{Q}} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[D|P_m^t] \,.$$

By (4.7), it follows that

$$\sum_{j=1}^{k_m} \mathbb{E}_{\mathbb{Q}_m}[1_{P_{m,j}^{t+1}} \mathbb{E}_{\mathbb{M}_{m,j}}[D|\mathcal{F}_{t+1}]|P_m^t] \geq \inf_{\widetilde{\mathbb{Q}} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[D|P_m^t] \,.$$

Since the above equality holds true for all $\mathbb{M}_{m,j} \in \mathcal{Q}^{a,u}$, by Lemma B.0.1, we have

$$\sum_{j=1}^{k_m} \mathbb{E}_{\mathbb{Q}_m}\Big[1_{P_{m,j}^{t+1}} \inf_{\mathbb{M}_{m,j} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\mathbb{M}_{m,j}}[D|\mathcal{F}_{t+1}]|P_m^t\Big] \geq 1_{P_m^t} \inf_{\widetilde{\mathbb{Q}} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[D|P_m^t] \,,$$

and since, this is true for all $\mathbb{Q}_m \in \mathcal{Q}^{x,u}$, by Lemma B.0.1, we can conclude that

$$\inf_{\mathbb{Q}_m \in \mathcal{Q}^{a,u}} \mathbb{E}_{\mathbb{Q}_m}\Big[\sum_{j=1}^{k_m} 1_{P_{m,j}^{t+1}} \inf_{\mathbb{M}_{m,j} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\mathbb{M}_{m,j}}[D|\mathcal{F}_{t+1}]|P_m^t\Big] \geq \inf_{\widetilde{\mathbb{Q}} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[D|P_m^t] \,.$$

or equivalently,

$$\inf_{\mathbb{Q} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\mathbb{Q}}\Big[\inf_{\mathbb{M} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\mathbb{M}}[D|\mathcal{F}_{t+1}]|P_m^t\Big] \geq \inf_{\widetilde{\mathbb{Q}} \in \mathcal{Q}^{a,u}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[D|P_m^t] \,.$$

This concludes the proof that $\mathcal{Q}^{a,u}$ is dynamically consistent. $\square$

The following shows some property for $\mathcal{Q}^{a,u}$.

Proposition 4.1.3. *For any* $\mathbb{Q} \in \mathcal{Q}^{a,u}$*, we have,*

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t] \leq a^t\,, \quad t \in \mathcal{T}.$$

Proof. From the definition of $\mathcal{Q}^{a,u}$, take $j = 1$, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_1] &\leq a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_0] \\ &= a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}] \\ &= a\,.\end{aligned}$$

Assume $\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t] \leq a^t$ holds, then

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_{t+1}] &\leq a\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t] \\ &\leq aa^t = a^{t+1}\,.\end{aligned}$$

By induction, we have

$$\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t] \leq a^t$$

for all $t \in \mathcal{T}$. □

Corollary 4.1.3. *Using Proposition 4.1.3, we conclude that for any* $\mathbb{Q} \in \mathcal{Q}^{a,u}$*,* $\mathbb{Q}(A) \leq a^t\mathbb{P}(A)$ *for all* $t \in \mathcal{T}$ *and* $A \in \mathcal{F}_t$*.*

Different probabilities in $\mathcal{Q}^{a,u}$ can be regarded as different opinions about the distribution of cash-flows; the above inequality provides an upper bound of these probabilities in terms of the underlying probability $\mathbb{P}$.

Example 4.1.5. *By similar arguments as in previous examples, one can show that the set of probability measures* $\mathcal{Q}^{a,l}$ *defined as follows*

$$\mathcal{Q}^{a,l} := \{\mathbb{Q} \in \mathcal{P}^e \mid \mathbb{E}_{\mathbb{Q}}[d\mathbb{P}/d\mathbb{Q}\,|\,\mathcal{F}_j] \leq a\mathbb{E}_{\mathbb{Q}}[d\mathbb{P}/d\mathbb{Q}\,|\,\mathcal{F}_{j-1}] \textit{ for all } j = 1, \ldots, T, \}$$

is a strongly consistent set of probability measures.

Analogous to Proposition 4.1.3, we can derive the following result.

Proposition 4.1.4. *For any* $\mathbb{Q} \in \mathcal{Q}^{a,u}$*, we have,*

$$\mathbb{E}_{\mathbb{Q}}[\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{F}_t] \leq a^t\,, \quad t \in \mathcal{T}.$$

Corollary 4.1.4. *Using Proposition 4.1.4, we conclude that for any* $\mathbb{Q} \in \mathcal{Q}^{a,u}$*,* $\mathbb{Q}(A) \geq a^{-t}\mathbb{P}(A)$ *for all* $t \in \mathcal{T}$ *and* $A \in \mathcal{F}_t$*.*

The above inequality provides a lower bound of these probabilities in terms of the underlying probability $\mathbb{P}$.

4.2 Representation Theorems for DCRMs and DCAIs

In this section we will present a representation theorem for dynamic coherent risk measures in terms of dynamically consistent set of probabilities. This result combined with the results from Section 3.4 about duality between DCAIs and DCRMs will lead to a representation theorem for dynamic coherent acceptability indices.

Theorem 4.2.1 (Representation Theorem for DCRM)**.** *A function* $\rho : \mathcal{T}\times\mathcal{D}\times\Omega \to \mathbb{R}$ *is a dynamic coherent risk measure if and only if there exists a dynamically consistent sequence of sets of probabilities* $\mathcal{U} := \{\mathcal{Q}_s\}_{s=0}^T$ *such that,*

$$\rho_t(D) = -\inf_{\mathbb{Q}\in\mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|\mathcal{F}_t\big]\,, \quad \text{for all } t \in \mathcal{T},\ D \in \mathcal{D}. \tag{4.8}$$

Proof. **Sufficiency**. Since $\mathcal{U}$ is dynamically consistent, it is full-support with respect to filtration $\mathbb{F}$. Definitions 4.1.3 and 4.1.4 indicate that ρ in (4.8) is well-defined for all $t \in \mathcal{T}$, $D \in \mathcal{D}$ and $\omega \in \Omega$. We will show that ρ satisfies (A1)-(A7) in Definition 3.3.1.

(A1) and (A2) directly come from (4.8) and Definition 4.1.4.

If $D_s(\omega) \geq D'_s(\omega)$ for some $t \in \mathcal{T}$ and $D, D' \in \mathcal{D}$, and for all $s \geq t$ and $\omega \in \Omega$, then fix $P_i^t \in \Upsilon^t$, we have

$$\begin{aligned}\rho_t(D, \bar{\omega}) &= -\inf_{\mathbb{Q} \in \mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s | P_i^t\big] = -\inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}_t)} \Bigg(\sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \sum_{s=t}^{T} D_s(\omega) \Bigg) \\ &\leq -\inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}_t)} \Bigg(\sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \sum_{s=t}^{T} D'_s(\omega) \Bigg) \\ &= -\inf_{\mathbb{Q} \in \mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D'_s | P_i^t\big] = \rho_t(D', \bar{\omega})\end{aligned}$$

for all $\bar{\omega} \in P_i^t$. In general, we have $\rho_t(D, \omega) \leq \rho_t(D', \omega)$. (A3) holds true.

Now we show that ρ satisfies (A4). For all $\lambda > 0, D \in \mathcal{D}, t \in \mathcal{T}, P_i^t \in \Upsilon^t$ and $\omega \in P_i^t$,

$$\begin{aligned}\rho_t(\lambda D, \omega) &= -\inf_{\mathbb{Q} \in \mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} (\lambda D)_s | P_i^t\big] = -\inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}_t)} \Bigg(\sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \lambda \sum_{s=t}^{T} D_s(\omega) \Bigg) \\ &= -\lambda \inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}_t)} \Bigg(\sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \lambda \sum_{s=t}^{T} D_s(\omega) \Bigg) = \lambda \rho_t(D, \omega)\,.\end{aligned}$$

We show that (A5) is satisfied. For all $t \in \mathcal{T}$, $D, D' \in \mathcal{D}$, $P_i^t \in \Upsilon^t$ and $\omega \in P_i^t$, by Definition 4.1.4 and (4.8),

$$\begin{aligned}\rho_t(D + D', \omega) &= -\inf_{\mathbb{Q} \in \mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} (D + D')_s | P_i^t\big] \\ &= -\inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}_t)} \Bigg(\sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \sum_{s=t}^{T} (D + D')_s(\omega) \Bigg) \\ &= -\inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}_t)} \Bigg(\sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \Big(\sum_{s=t}^{T} D_s(\omega) + \sum_{s=t}^{T} D'_s(\omega) \Big) \Bigg)\,.\end{aligned}$$

Then, by Lemma B.0.1,

$$\begin{aligned}&\rho_t(D + D', \omega) \\ \leq& -\inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}_t)} \Bigg(\sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \sum_{s=t}^{T} D_s(\omega) \Bigg) - \inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}_t)} \Bigg(\sum_{\omega \in P_i^t} \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \sum_{s=t}^{T} D'_s(\omega) \Bigg) \\ =& \rho_t(D, \omega) + \rho_t(D', \omega)\,.\end{aligned}$$

We show that ρ satisfies (A6). For all $t \in \mathcal{T}$, $D \in \mathcal{D}$, $P_i^t \in \Upsilon^t$, $\bar{\omega} \in P_i^t$, $\mathcal{F}_t$-measurable random variable m and $s \geq t$, by Definition 4.1.4

$$\begin{aligned}\rho_t(D+m1_{\{s\}},\bar{\omega}) &= -\inf_{\mathbb{Q}\in\mathcal{Q}_t}\mathbb{E}_{\mathbb{Q}}\big[\sum_{l=t}^{T}(D+m1_{\{s\}})_l|P_i^t\big] = -\inf_{\mathbb{Q}\in\mathcal{Q}_t}\mathbb{E}_{\mathbb{Q}}\big[\sum_{l=t}^{T}D_l+m|P_i^t\big]\\ &= -\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q}_t)}\left(\sum_{\omega\in P_i^t}\frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)}\big(\sum_{l=t}^{T}D(\omega)+m(\omega)\big)\right).\end{aligned}$$

Note that m is constant on P_i^t, then

$$\begin{aligned}\rho_t(D+m1_{\{s\}},\bar{\omega}) &= -\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q}_t)}\left(m(\bar{\omega})+\sum_{\omega\in P_i^t}\frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)}\sum_{l=t}^{T}D_l(\omega)\right)\\ &= -\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q}_t)}\left(\sum_{\omega\in P_i^t}\frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)}\sum_{l=t}^{T}D_l(\omega)\right)-m(\bar{\omega}) = \rho_t(D,\bar{\omega})-m(\bar{\omega})\,.\end{aligned}$$

We will show that (A7) – dynamic consistency, is satisfied. Since $\mathcal{U} = \{\mathcal{Q}_t\}_{t=0}^T$ is dynamically consistent, we have,

$$\begin{aligned}1_A\rho_t(D) &= -1_A\inf_{\mathbb{Q}\in\mathcal{Q}_t}\mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T}D_s|\mathcal{F}_t\big] \geq 1_A\min_{\omega\in A}\left\{-\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}}\mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t+1}^{T}D_s|\mathcal{F}_{t+1}\big](\omega)-D_t\right\}\\ &= 1_A\min_{\omega\in A}\left\{\rho_{t+1}(D,\omega)-D_t\right\},\end{aligned}$$

for every $t \in \mathcal{T}$, $D \in \mathcal{D}$ and $\mathcal{Q}_t \in \mathcal{U}$.

Similarly, one can show that $1_A\rho_t(D) \leq 1_A\max_{\omega\in A}\{\rho_{t+1}(D,\omega)-D_t\}$, for every $t \in \mathcal{T}$, $D \in \mathcal{D}$, $\mathcal{Q}_t \in \mathcal{U}$. Thus (A7) is satisfied.

Necessity. The set $\mathcal{U}$ will be constructed explicitly. Fix a time $t \in \mathcal{T}$. Recall that $\{P_1^t, \ldots, P_{n_t}^t\}$ denotes the partition of Ω that corresponds to $\mathcal{F}_t$. Also, we will denote by $\{P_{i,1}^{t,s}, \ldots, P_{i,m_s}^{t,s}\}$ the partition of P_i^t generated by $\mathcal{F}_s$, for some future time $s \geq t$. Thus $P_i^t = \cup_{j=1}^{m_s}P_{i,j}^{t,s}$. Assume that P_i^t is fixed for some $i \in \{1, \ldots, n_t\}$, and define the following probability space $(\Omega_i^t, 2^{\Omega_i^t}, \mathbb{P}^{\mathrm{uni}})$ with,

$$\Omega_i^t := \left\{(s, P_{i,j}^{t,s}) : s \in \{t, t+1, \ldots, T\} \text{ and } j \in \{1, 2, \ldots, m_s\}\right\},$$

and $\mathbb{P}^{\text{uni}}(\omega) = 1/\text{card}(\Omega_i^t)$ for each $\omega \in \Omega_i^t$.

Let us denote by $\mathcal{X}(\Omega_i^t)$ the set of all random variables on Ω_i^t. There exists a one-to-one correspondence between $\mathcal{X}(\Omega_i^t)$ and the set $\mathcal{D}_i^t := \{D1_{\{t,t+1,\dots,T\}}1_{P_i^t} :$ for all $D \in \mathcal{D}\}$. The map can be defined as follows: for any $X \in \mathcal{X}(\Omega_i^t)$, put

$$D_s^X(\omega) := \begin{cases} X((s, P_{i,j}^{t,s})), & \text{if } s \geq t \text{ and } \omega \in P_{i,j}^{t,s} \\ 0, & \text{otherwise}, \end{cases} \tag{4.9}$$

and vice versa, for any $D \in \mathcal{D}_i^t$, define

$$X^D((s, P_{i,j}^{t,s})) := D_s(\omega), \tag{4.10}$$

for $s \geq t$, $j \in \{1, 2, \dots, m_s\}$, and $\omega \in P_{i,j}^{t,s}$.

Consider the following function $\phi : \mathcal{X}(\Omega_i^t) \to \mathbb{R}$ with,

$$\phi(X) := \frac{1}{T - t + 1} \rho_t(D^X, \omega), \quad \omega \in P_i^t. \tag{4.11}$$

We claim that ϕ is a static coherent risk measure, i.e. satisfies the properties (R1)-(R4) of Definition 2.1.2. Indeed, for any $X, Y \in \mathcal{X}(\Omega_i^t)$, such that $X \leq Y$, we have, $D_s^X(\omega) \leq D_s^Y(\omega)$, for all $s \geq t$ and $\omega \in \Omega$. Then, by (A3), the monotonicity of ρ, we get $\rho_t(D^X, \omega) \geq \rho_t(D^Y, \omega)$, for $\omega \in \Omega$. Therefore, by (4.11), $\phi(X) \geq \phi(Y)$, i.e. ϕ satisfies (R1).

Note that for all $X \in \mathcal{X}(\Omega_i^t)$ and $\lambda \geq 0$, by (4.9), we have,

$$D_s^{\lambda X}(\omega) = \lambda X((s, P_{i,j}^{t,s})) = \lambda D_s^X(\omega),$$

for all $s \geq t$ and $\omega \in P_{i,j}^{t,s}$. From here, by (4.11) and using homogeneity of ρ, the homogeneity (R2) of ϕ follows.

Next we will show that ϕ satisfies (R3). For all $X \in \mathcal{X}(\Omega_i^t)$ and $k \in \mathbb{R}$, by (4.9), we have,

$$D_s^{X+k}(\omega) = X((s, P_{i,j}^{t,s})) + k = D_s^X(\omega) + k,$$

for all $s \geq t$ and $\omega \in P_{i,j}^{t,s}$. Therefore, by (4.11) and (A6), translation invariance of ρ, we deduce

$$\begin{aligned}\phi(X+k) &= \frac{1}{T-t+1}\rho_t(D^{X+k},\omega) = \frac{1}{T-t+1}\rho_t(D^X + k1_{\{t,\ldots,T\}},\omega)\\ &= \frac{1}{T-t+1}(\rho_t(D^X,\omega) - (T-t+1)k) = \frac{1}{T-t+1}\rho_t(D^X,\omega) - k\\ &= \phi(X^D) - k\,,\end{aligned}$$

for all $X \in \mathcal{X}(\Omega_i^t)$.

To show that ϕ satisfies (R4), consider an $X \in \mathcal{X}(\Omega_i^t)$. By (4.9)

$$D_s^{X+Y}(\omega) = X((s, P_{i,j}^{t,s})) + Y((s, P_{i,j}^{t,s})) = D_s^X(\omega) + D_s^Y(\omega)\,,$$

for all $s \geq t$ and $\omega \in P_{i,j}^{t,s}$, and therefore, by (4.11) and (A5), subadditivity of ρ, we obtain

$$\begin{aligned}\phi(X+Y) &= \frac{1}{T-t+1}\rho_t(D^X + D^Y,\omega)\\ &\leq \frac{1}{T-t+1}\rho_t(D^X,\omega) + \frac{1}{T-t+1}\rho_t(D^Y,\omega)\\ &= \phi(X) + \phi(Y)\,.\end{aligned}$$

From all the above, we conclude that ϕ is a static coherent risk measure. By Theorem 2.2.1, representation of static coherent risk measures, there exists $\mathcal{M}_i^t$, a set of absolutely continuous probability measures with respect to $\mathbb{P}^{\mathrm{uni}}$ on Ω_i^t, such that

$$\phi(X) = -\inf_{\mathbb{M}\in\mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}}[X]\,.$$

By (4.11), we have,

$$\frac{1}{T-t+1}\rho_t(D^X,\omega) = -\inf_{\mathbb{M}\in\mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}}[X]\,, \quad \omega \in P_i^t. \tag{4.12}$$

Since there is one-to-one map between $\mathcal{X}(\Omega_i^t)$ and $\mathcal{D}_i^t$, for any $D \in \mathcal{D}_i^t$, we also can write

$$\frac{1}{T-t+1}\rho_t(D,\omega) = -\inf_{\mathbb{M}\in\mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}}[X^D]\,. \tag{4.13}$$

Fix a time $t^0 \in \{t, t+1, \dots, T\}$, and denote by $\widetilde{D}$ the process $1_{\{t^0\}}$. By (A6)-translation invariance and (A2)-independence of the past of ρ, it follows that $\rho_t(\widetilde{D}, \omega) = -1$, $\omega \in P_i^t$. Hence, by (4.13),

$$\inf_{\mathbb{M} \in \mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}}[X^{\widetilde{D}}] = \frac{1}{T-t+1}\,. \tag{4.14}$$

Note that $\mathbb{E}_{\mathbb{M}}[X^{\widetilde{D}}] = \mathbb{M}(\{t^0\} \times P_i^t)$. Thus, (4.14) implies

$$\inf_{\mathbb{M} \in \mathcal{M}_i^t} \mathbb{M}(\{t^0\} \times P_i^t) = \frac{1}{T-t+1}\,.$$

Similarly, one can show that $\mathbb{E}_{\mathbb{M}}[X^{-\widetilde{D}}] = -\mathbb{M}(\{t^0\} \times P_i^t)$. Thus we derive that

$$\inf_{\mathbb{M} \in \mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}}[X^{-\widetilde{D}}] = \inf_{\mathbb{M} \in \mathcal{M}_i^t} (-\mathbb{M}(\{t^0\} \times P_i^t)) = -\sup_{\mathbb{M} \in \mathcal{M}_i^t} \mathbb{M}(\{t^0\} \times P_i^t)\,,$$

and consequently

$$\sup_{\mathbb{M} \in \mathcal{M}_i^t} \mathbb{M}(\{t^0\} \times P_i^t) = \frac{1}{T-t+1}\,.$$

This yields that

$$\mathbb{M}(\{t^0\} \times P_i^t) = \frac{1}{T-t+1}\,, \quad t^0 \in \{t, t+1, \dots, T\}. \tag{4.15}$$

For any $s \in \{t, t+1, \dots, T\}$, define $\mathbb{M}^s : \Omega_i^t \to \mathbb{R}$ as follows

$$\mathbb{M}^s((r, P_{i,j}^{t,r})) := \begin{cases} (T-t+1)\mathbb{M}((r, P_{i,j}^{t,r})), & \text{when } r = s \text{ and } j \in \{1, 2, \dots, m_r\} \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\mathbb{M}^s$ is a probability measure on Ω_i^t for every $s \in \{t, t+1, \dots, T\}$.

For all $D \in \mathcal{D}$, we can derive,

$$\begin{aligned} \sum_{s=t}^{T} \mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_s}] &= \sum_{s=t}^{T} \Big(\sum_{r=t}^{T} \sum_{j=1}^{m_r} \mathbb{M}^s((r, P_{i,j}^{t,r}))(D_s 1_s)_r(\omega) \Big), \text{ for some } \omega \in P_{i,j}^{t,r} \\ &= \sum_{s=t}^{T} \Big(\sum_{j=1}^{m_r} \mathbb{M}^s((s, P_{i,j}^{t,s})) D_s(\omega) \Big), \text{ for some } \omega \in P_{i,j}^{t,r} \\ &= \sum_{s=t}^{T} \Big(\sum_{j=1}^{m_r} (T-t+1)\mathbb{M}((s, P_{i,j}^{t,s})) D_s(\omega) \Big), \text{ for some } \omega \in P_{i,j}^{t,r} \\ &= (T-t+1)\mathbb{E}_{\mathbb{M}}[X^D]\,. \end{aligned}$$

Hence, by (4.13), we have

$$\rho_t(D,\omega) = -(T-t+1)\inf_{\mathbb{M}\in\mathcal{M}_i^t}\mathbb{E}_{\mathbb{M}}[X^D] = -\inf_{\mathbb{M}\in\mathcal{M}_i^t}\sum_{s=t}^{T}\mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_s}], \quad \omega\in P_i^t. \quad (4.16)$$

Since ρ satisfies (A6) and (A7), we deduce that

$$\rho_s(D_s 1_{\{s\}} - D_s 1_{\{T\}},\omega) = 0\,, \quad s\geq t,\ D\in\mathcal{D},\ \omega\in P_i^t.$$

Thus, (4.13) and (4.16) imply,

$$\begin{aligned}
-\inf_{\mathbb{M}\in\mathcal{M}_i^t}\left(\mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_{\{s\}}}] - \mathbb{E}_{\mathbb{M}^T}[X^{D_s 1_{\{T\}}}]\right) &= -\inf_{\mathbb{M}\in\mathcal{M}_i^t}\left[\sum_{r=t}^{T}\mathbb{E}_{\mathbb{M}^r}[X^{(D_s 1_{\{s\}} - D_s 1_{\{T\}})_r 1_r}]\right] \\
&= \rho_t(D_s 1_{\{s\}} - D_s 1_{\{T\}},\omega) = 0\,.
\end{aligned}$$

Since the above equality holds true for all $D\in\mathcal{D}$, it also holds true for $-D$. Hence, we have

$$\inf_{\mathbb{M}\in\mathcal{M}_i^t}(\mathbb{E}_{\mathbb{M}^s}[X^{-D_s 1_{\{s\}}}] - \mathbb{E}_{\mathbb{M}^T}[X^{-D_s 1_{\{T\}}}]) = 0\,. \quad (4.17)$$

On the other hand, by (4.10), one gets

$$\inf_{\mathbb{M}\in\mathcal{M}_i^t}(\mathbb{E}_{\mathbb{M}^s}[X^{-D_s 1_{\{s\}}}] - \mathbb{E}_{\mathbb{M}^T}[X^{-D_s 1_{\{T\}}}]) = -\sup_{\mathbb{M}\in\mathcal{M}_i^t}(\mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_{\{s\}}}] - \mathbb{E}_{\mathbb{M}^T}[X^{D_s 1_{\{T\}}}])\,.$$

Thus,

$$\sup_{\mathbb{M}\in\mathcal{M}_i^t}(\mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_{\{s\}}}] - \mathbb{E}_{\mathbb{M}^T}[X^{D_s 1_{\{T\}}}]) = 0 \quad (4.18)$$

By (4.17) and (4.18) we conclude that

$$\sup_{\mathbb{M}\in\mathcal{M}_i^t}(\mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_{\{s\}}}] - \mathbb{E}_{\mathbb{M}^T}[X^{D_s 1_{\{T\}}}]) = 0 = \inf_{\mathbb{M}\in\mathcal{M}_i^t}(\mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_{\{s\}}}] - \mathbb{E}_{\mathbb{M}^T}[X^{D_s 1_{\{T\}}}])\,,$$

and hence

$$\mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_{\{s\}}}] = \mathbb{E}_{\mathbb{M}^T}[X^{D_s 1_{\{T\}}}]\,. \quad (4.19)$$

for all $s\geq t$, and $\mathbb{M}\in\mathcal{M}_i^t$. Therefore, we can rewrite (4.16) as follows,

$$\begin{aligned}
\rho_t(D,\omega) &= -\inf_{\mathbb{M}\in\mathcal{M}_i^t}\Big[\sum_{s=t}^{T}\mathbb{E}_{\mathbb{M}^s}[X^{D_s 1_{\{s\}}}]\Big] \\
&= -\inf_{\mathbb{M}\in\mathcal{M}_i^t}\Big[\mathbb{E}_{\mathbb{M}^T}[\sum_{s=t}^{T}X^{D_s 1_{\{T\}}}]\Big] \\
&= -\inf_{\mathbb{M}\in\mathcal{M}_i^t}\mathbb{E}_{\mathbb{M}^T}\Big[X^{(\sum_{s=t}^{T}D_s)1_{\{T\}}}\Big]\,. \quad (4.20)
\end{aligned}$$

for all $D \in \mathcal{D}$, and $\omega \in P_i^t$.

To summarize, for every P_i^t, $i = 1, \dots, n_t$, we constructed a set of probability measures $\mathcal{M}_i^t$ on Ω_i^t. Having these sets, we define $\mathcal{Q}_t$ as follows:

$$\mathcal{Q}_t := \Big\{ \mathbb{Q} \in \mathcal{P} \, : \, \text{there exists } \{\mathbb{M}_i\}_{i=1}^{n_t} \text{ such that, } \forall \;\; i \in \{1, \dots, n_t\}, j \in \{1, \dots, m_T^i\}, \\ \mathbb{M}_i \in \mathcal{M}_i^t \text{ and } \mathbb{Q}(\omega) = \frac{1}{n_t} \frac{1}{\mathcal{N}(P_{i,j}^{t,T})} \mathbb{M}_i^T((T, P_{i,j}^{t,T})) \text{ for all } \omega \in P_{i,j}^{t,T} \Big\},$$

where $\mathcal{N}(P)$ stands for cardinality of the set $P \subset \Omega$.

By direct evaluations, one can show that $\mathcal{Q}_t$, $t \in \mathcal{T}$, is a set of probability measure on Ω. In addition, one can show that for all $t \in \mathcal{T}$, $P_i^t \in \Upsilon^t$ and $\mathbb{Q} \in \mathcal{Q}_t$, $\mathbb{Q}_t(P_i^t) = \frac{1}{n_t}$. Hence, by Definition 4.1.3, the sequence of sets of probability measures $\{\mathcal{Q}_t\}_{t=0}^T$ is full-support with respect to filtration $\mathbb{F}$.

Next we will show that (4.8) is fulfilled. Note that,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \big[\sum_{s=t}^T D_s | P_i^t \big] &= \sum_{\omega \in P_i^t} \Big[\sum_{s=t}^T D_s(\omega) \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)} \Big] \\ &= \sum_{j=1}^{m_T^i} \sum_{\omega \in P_{i,j}^{t,T}} \Big[\sum_{s=t}^T D_s(\omega) \frac{1}{\mathcal{N}(P_{i,j}^{t,T})} \mathbb{M}_i^T((T, P_{i,j}^{t,T})) \Big] \\ &= \sum_{j=1}^{m_T^i} \big[\sum_{s=t}^T D_s(\omega) \mathbb{M}_i^T((T, P_{i,j}^{t,T})) \big] \\ &= \mathbb{E}_{\mathbb{M}_i^T} [X^{\sum_{s=t}^T D_s 1_{\{T\}}}] \,. \end{aligned}$$

If $\inf_{\mathbb{Q} \in \mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}} \big[\sum_{s=t}^T D_s | P_i^t \big] > \inf_{\mathbb{M}_i \in \mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}_i^T} [X^{\sum_{s=t}^T D_s 1_{\{T\}}}]$, then there exists $\widetilde{\mathbb{M}}_i \in \mathcal{M}_i^t$ such that

$$\mathbb{E}_{\widetilde{\mathbb{M}}_i^T} [X^{\sum_{s=t}^T D_s 1_{\{T\}}}] < \inf_{\mathbb{Q} \in \mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}} \big[\sum_{s=t}^T D_s | \mathcal{F}_t \big](\omega) \,. \tag{4.21}$$

However, for $\widetilde{\mathbb{Q}}$ constructed by $\widetilde{\mathbb{M}}_i$, as previously proved,

$$\mathbb{E}_{\widetilde{\mathbb{M}}_i^T} [X^{\sum_{s=t}^T D_s 1_{\{T\}}}] = \mathbb{E}_{\widetilde{\mathbb{Q}}} \big[\sum_{s=t}^T D_s | P_i^t \big] \geq \inf_{\mathbb{Q} \in \mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}} \big[\sum_{s=t}^T D_s | P_i^t \big] \,,$$

that contradicts (4.21). On the other hand, if

$$\inf_{\mathbb{Q}\in\mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|P_i^t\big] < \inf_{\mathbb{M}_i\in\mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}_i^T}\big[X^{\sum_{s=t}^T D_s 1_{\{T\}}}\big]\,,$$

then there exists $\widetilde{\mathbb{Q}} \in \mathcal{Q}_t$ such that

$$\inf_{\mathbb{M}_i\in\mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}_i^T}\big[X^{\sum_{s=t}^T D_s 1_{\{T\}}}\big] > \mathbb{E}_{\widetilde{\mathbb{Q}}}\big[\sum_{s=t}^{T} D_s|P_i^t\big]\,. \tag{4.22}$$

As previously proved, there exists $\widetilde{\mathbb{M}}_i \in \mathcal{M}_i^t$, such that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\big[\sum_{s=t}^{T} D_s|P_i^t\big] = \mathbb{E}_{\widetilde{\mathbb{M}}_i^T}\big[X^{\sum_{s=t}^T D_s 1_{\{T\}}}\big] \geq \inf_{\mathbb{M}_i\in\mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}_i^T}\big[X^{\sum_{s=t}^T D_s 1_{\{T\}}}\big]\,,$$

which contradicts (4.22). Thus, we conclude that

$$\inf_{\mathbb{Q}\in\mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|P_i^t\big] = \inf_{\mathbb{M}_i\in\mathcal{M}_i^t} \mathbb{E}_{\mathbb{M}_i^T}\big[X^{\sum_{s=t}^T D_s 1_{\{T\}}}\big]\,,$$

and by (4.20),

$$\rho_t(D) = -\inf_{\mathbb{Q}\in\mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|\mathcal{F}_t\big]\,.$$

To complete the proof we need to show that $\{\mathcal{Q}_s\}_{s=0}^T$ is a dynamically consistent sequence of sets of probability measures. Recall that by (A7), dynamic consistency of ρ,

$$1_A(\min_{\omega\in A} \rho_{t+1}(D,\omega) - D_t) \leq 1_A \rho_t(D) \leq 1_A(\max_{\omega\in A} \rho_{t+1}(D,\omega) - D_t)\,, \tag{4.23}$$

for all $D \in \mathcal{D}$ and $A \in \mathcal{F}_t$. Using this, we get

$$1_A(\min_{\omega\in A}\big\{-\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t+1}^{T} D_s|\mathcal{F}_{t+1}\big](\omega)\big\} - D_t) \leq 1_A(-\inf_{\mathbb{Q}\in\mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|\mathcal{F}_t\big])\,,$$

for any $D \in \mathcal{D}$ and $A \in \mathcal{F}_t$. Consequently, we obtain

$$1_A \max_{\omega\in A}\big\{\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|\mathcal{F}_{t+1}\big](\omega)\big\} \geq 1_A \inf_{\mathbb{Q}\in\mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|\mathcal{F}_t\big]\,,\ D\in\mathcal{D},\ A\in\mathcal{F}_t. \tag{4.24}$$

Similarly, by (4.23)

$$1_A(\max_{\omega\in A}\big\{-\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}}\mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t+1}^{T}D_s|\mathcal{F}_{t+1}\big](\omega)\big\}-D_t)\geq 1_A(-\inf_{\mathbb{Q}\in\mathcal{Q}_t}\mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T}D_s|\mathcal{F}_t\big])$$

and hence

$$1_A\min_{\omega\in A}\big\{\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}}\mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T}D_s|\mathcal{F}_{t+1}\big](\omega)\big\}\leq 1_A\inf_{\mathbb{Q}\in\mathcal{Q}_t}\mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T}D_s|\mathcal{F}_t\big]\,,\ D\in\mathcal{D},\ A\in\mathcal{F}_t. \tag{4.25}$$

Combining (4.24) and (4.25) dynamic consistency of $\{\mathcal{Q}_t\}_{t=0}^T$ follows. □

Recall that in Chapter 2 we discussed that any set of probability measures and its closed convex hull generate the same risk measure. We will extend it to dynamic framework.

Proposition 4.2.1. *Let $\mathcal{Q}$ be a subset of $\mathcal{P}$ with $\mathrm{eff}^{t,i}(\mathcal{Q})\neq\emptyset$ for some $P_i^t\in\Upsilon^t$. Denote the closed convex hull of $\mathcal{Q}$ by $\bar{\mathcal{Q}}^C$. We have*

$$\inf_{\mathbb{Q}\in\mathcal{Q}}\mathbb{E}_{\mathbb{Q}}[X|P_i^t]=\inf_{\mathbb{Q}\in\bar{\mathcal{Q}}^C}\mathbb{E}_{\mathbb{Q}}[X|P_i^t]\,,$$

for all $X\in\mathcal{G}$.

Proof. By Definition 4.1.4, it is enough to show that for all $X\in\mathcal{G}$,

$$\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})}\mathbb{E}_{\mathbb{Q}}[X|P_i^t]=\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\bar{\mathcal{Q}}^C)}\mathbb{E}_{\mathbb{Q}}[X|P_i^t]\,.$$

First, note that $\mathrm{eff}^{t,i}(\mathcal{Q})\subset\mathrm{eff}^{t,i}(\bar{\mathcal{Q}}^C)$, thus

$$\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})}\mathbb{E}_{\mathbb{Q}}[X|P_i^t]\geq\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\bar{\mathcal{Q}}^C)}\mathbb{E}_{\mathbb{Q}}[X|P_i^t]\,. \tag{4.26}$$

We will verify the other way in two steps. Denote by $\mathcal{Q}^C$ as the convex hull of $\mathcal{Q}$. For any $\mathbb{Q}_0\in\mathrm{eff}^{t,i}(\mathcal{Q}^C)$, by the Definition 2.2.2 of convex hull, there exists $\{\mathbb{Q}_1,\mathbb{Q}_2,\cdots,\mathbb{Q}_k\}\in\mathcal{Q}$ such that $\mathbb{Q}_0=\sum_{l=1}^k\lambda_l\mathbb{Q}_l$ with $\lambda_i>0$ and $\sum_{l=1}^k\lambda_l=1$.

Since $\mathbb{Q}_0(P_i^t) > 0$, then there exists at least one $\mathbb{Q}_j$ where $j = 1, \cdots, k$ such that $\mathbb{Q}_j(P_i^t) > 0$.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_0}[X|P_i^t] &= \sum_{\omega \in P_i^t} \frac{\mathbb{Q}_0(\omega)X(\omega)}{\mathbb{Q}_0(P_i^t)} \\
&= \sum_{\omega \in P_i^t} \frac{\sum_{l=1}^k \lambda_l \mathbb{Q}_l(\omega)X(\omega)}{\sum_{l=1}^k \lambda_l \mathbb{Q}_l(P_i^t)} \\
&= \sum_{\omega \in P_i^t} \frac{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \mathbb{Q}_l(\omega)X(\omega)}{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \mathbb{Q}_l(P_i^t)} \\
&= \frac{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \big(\sum_{\omega \in P_i^t} \mathbb{Q}_l(\omega)X(\omega)\big)}{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \mathbb{Q}_l(P_i^t)} \qquad (4.27)
\end{aligned}$$

Define a constant $a := \inf\limits_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}[X|P_i^t]$. By Definition 4.1.4, for all $\mathbb{Q}_j$ with $\mathbb{Q}_j(P_i^t) > 0$, we have

$$\mathbb{E}_{\mathbb{Q}_j}[X|P_i^t] = \frac{\sum_{\omega \in P_i^t} \mathbb{Q}_j(\omega)X(\omega)}{\mathbb{Q}_j(P_i^t)} \geq a\,.$$

Thus, $\sum_{\omega \in P_i^t} \mathbb{Q}_j(\omega)X(\omega) \geq a\mathbb{Q}_j(P_i^t)$. By (4.27), we can derive

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_0}[X|P_i^t] &= \frac{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \big(\sum_{\omega \in P_i^t} \mathbb{Q}_l(\omega)X(\omega)\big)}{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \mathbb{Q}_l(P_i^t)} \\
&\geq \frac{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \big(a\mathbb{Q}_l(P_i^t)\big)}{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \mathbb{Q}_l(P_i^t)} \\
&= a\frac{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \mathbb{Q}_l(P_i^t)}{\sum\limits_{\mathbb{Q}_j(P_i^t)>0} \lambda_l \mathbb{Q}_l(P_i^t)} = a\,.
\end{aligned}$$

We verified that for all $\mathbb{Q}_0 \in \mathrm{eff}^{t,i}(\mathcal{Q}^C)$,

$$\mathbb{E}_{\mathbb{Q}_0}[X|P_i^t] \geq \inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}[X|P_i^t]\,.$$

Then, by Lemma B.0.1, we can conclude

$$\inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q}^C)} \mathbb{E}_{\mathbb{Q}_0}[X|P_i^t] \geq \inf_{\mathbb{Q} \in \mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}[X|P_i^t]\,. \qquad (4.28)$$

Next step, note that $\bar{\mathcal{Q}}^C$ is the closure of $\mathcal{Q}^C$, then for any $\mathbb{Q}_0 \in \mathrm{eff}^{t,i}(\bar{\mathcal{Q}}^C)$, since $\mathbb{Q}_0(P_i^t) > 0$, there exists a sequence $(\mathbb{Q}_1, \mathbb{Q}_2, \cdots)$ with each $\mathbb{Q}_n \in \mathrm{eff}^{t,i}(\mathcal{Q}^C)$ such that

$$\lim_{l\to\infty} \mathbb{Q}_l = \mathbb{Q}_0 \,.$$

Since we are in finite probability space, the above limit is state-wise. Hence, by the linearity of expectation and finiteness of space,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_0}[X|P_i^t](\bar{\omega}) &= \sum_{\omega\in P_i^t} \frac{\mathbb{Q}_0(\omega)X(\omega)}{\mathbb{Q}_0(P_i^t)} \\
&= \lim_{l\to\infty} \sum_{\omega\in P_i^t} \frac{\mathbb{Q}_l(\omega)X(\omega)}{\mathbb{Q}_l(P_i^t)} \\
&= \lim_{l\to\infty} \mathbb{E}_{\mathbb{Q}_l}[X|P_i^t] \,.
\end{aligned}$$

Thus, it implies

$$\mathbb{E}_{\mathbb{Q}_0}[X|P_i^t] \geq \inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q}^C)} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] \,.$$

We verified that for all $\mathbb{Q}_0 \in \mathrm{eff}^{t,i}(\bar{\mathcal{Q}}^C)$,

$$\mathbb{E}_{\mathbb{Q}_0}[X|P_i^t] \geq \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] \,.$$

Then, by Lemma B.0.1, we can conclude

$$\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\bar{\mathcal{Q}}^C)} \mathbb{E}_{\mathbb{Q}_0}[X|P_i^t] \geq \inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q}^C)} \mathbb{E}_{\mathbb{Q}}[X|P_i^t] \,.$$

Together with (4.28), we get

$$\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\bar{\mathcal{Q}}^C)} \mathbb{E}_{\mathbb{Q}}[X](\bar{\omega}) \geq \inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}[X](\bar{\omega}) \,.$$

Together with (4.26), we conclude that, for any $X \in \mathcal{G}$,

$$\inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\bar{\mathcal{Q}}^C)} \mathbb{E}_{\mathbb{Q}}[X](\bar{\omega}) = \inf_{\mathbb{Q}\in\mathrm{eff}^{t,i}(\mathcal{Q})} \mathbb{E}_{\mathbb{Q}}[X](\bar{\omega}) \,.$$

□

Corollary 4.2.1. *Using Proposition 4.2.1, we can conclude that*

(1) If an individual set of probability measures is strongly consistent, its closed convex hull is also strongly consistent.

(2) If an individual set of probability measures is weakly consistent, its closed convex hull is also weakly consistent.

(3) If a sequence of sets of probability measures is dynamically consistent, after taking closed convex hull for each individual set, the new sequence is also dynamically consistent.

Together with the Representation Theorem 4.2.1 for DCRMs, we can conclude the following corollary.

Corollary 4.2.2. *A dynamically consistent sequence of sets of probability measures and its closed convex hull sequence generate the same dynamic coherent risk measure.*

Having derived a representation theorem for dynamic coherent risk measures in terms of dynamically consistent sequence of sets of probability measures, and having derived the duality between DCRMs and DCAIs, we can present another important result: representation theorem for DCAIs in terms of dynamically consistent sequence of sets of probability measures.

We shall mention that Proposition 4.2.1 and Hyperplane Separation Theorem B.0.2 are indispensable technical results to prove the representation theorem for DCAIs.

Definition 4.2.1. *Let* $\mathbb{Q}$ *be an absolutely continuous probability measure with respect to reference* $\mathbb{P}$ *with* $\mathbb{Q}(P_i^t) > 0$ *for some* $P_i^t \in \Upsilon^t$*. The conditional probability measure* $\mathrm{cond}^{t,i}(\mathbb{Q})$ *is defined as follows:*

$$\mathrm{cond}^{t,i}(\mathbb{Q})(\omega) := \frac{\mathbb{Q}(\omega)}{\mathbb{Q}(P_i^t)},$$

for all $\omega \in P_i^t$*.*

It is not hard to observe that $\text{cond}^{t,i}(\mathbb{Q})$ indeed is a probability on P_i^t.

Definition 4.2.2. *Let $\mathcal{Q}$ be a subset of $\mathcal{P}$. The set of conditional probabilities* $\text{cond}^{t,i}(\mathcal{Q})$ *is defined as follows:*

$$\text{cond}^{t,i}(\mathcal{Q}) := \{\text{cond}^{t,i}(\mathbb{Q}),\ \forall\ \mathbb{Q} \in \mathcal{Q} \textit{ with } \mathbb{Q}(P_i^t) > 0\}\,,$$

for all $t \in \mathcal{T}$ and $P_i^t \in \Upsilon^t$.

Lemma 4.2.1. *If $\mathcal{Q} \subset \mathcal{P}$ is full-support, closed and convex, then* $\text{cond}^{t,i}(\mathcal{Q})$ *is closed and convex for all $t \in \mathcal{T}$ and $P_i^t \in \Upsilon^t$.*

Proof. Since $\mathcal{Q}$ is full-support, then every $\text{cond}^{t,i}(\mathcal{Q})$ is nonempty set. For any $\text{cond}^{t,i}(\mathbb{Q}) \in \text{cond}^{t,i}(\mathcal{Q})$, by Definition 4.2.2, we have $\mathbb{Q}(P_i^t) > 0$. Since $\mathcal{Q}$ is closed, there is a sequence $(\mathbb{Q}_1, \mathbb{Q}_2, \cdots)$ with each $\mathbb{Q}_n \in \mathcal{Q}$ such that

$$\lim_{l\to\infty} \mathbb{Q}_l = \mathbb{Q}_0\,.$$

Since we are in finite probability space, the above limit is state-wise. There exists a subsequence of $(\mathbb{Q}_1, \mathbb{Q}_2, \cdots)$ which can be assumed to be the sequence itself, such that $\mathbb{Q}_j(P_i^t) > 0$ for all $j = 1, 2, \cdots$. Then,

$$\lim_{l\to\infty} \text{cond}^{t,i}(\mathbb{Q}_l) = \text{cond}^{t,i}(\mathbb{Q}_0)\,.$$

So, $\text{cond}^{t,i}(\mathcal{Q})$ is closed.

Since $\mathcal{Q}$ is convex, there exists $\mathbb{Q}_1, \mathbb{Q}_2, \cdots, \mathbb{Q}_k$ such that $\mathbb{Q}_0 = \sum_{l=1}^k \lambda_l \mathbb{Q}_l$ with $\lambda_i \geq 0$ and $\sum_{l=1}^k \lambda_l = 1$. Since $\mathbb{Q}_0(P_i^t) > 0$, then there exists at least one $\mathbb{Q}_j$ where $j = 1, \cdots, k$ such that $\mathbb{Q}_j(P_i^t) > 0$. We pick up all $\mathbb{Q}_{j_1}, \mathbb{Q}_{j_2}, \cdots, \mathbb{Q}_{j_u}$ such that all of them are not zero when measuring P_i^t. Then, for all $\omega \in P_i^t$, we have

$$\begin{aligned}
\text{cond}^{t,i}(\mathbb{Q}_0)(\omega) &= \frac{\mathbb{Q}_0(\omega)}{\mathbb{Q}_0(P_i^t)} \\
&= \frac{\sum_{l=1}^{u} \lambda_{j_l} \mathbb{Q}_{j_l}(\omega)}{\mathbb{Q}_0(P_i^t)} \\
&= \frac{\sum_{l=1}^{u} \lambda_{j_l} \mathbb{Q}_{j_l}(\omega)}{\mathbb{Q}_0(P_i^t)} \\
&= \sum_{l=1}^{u} \frac{\lambda_{j_l} \mathbb{Q}_{j_l}(P_i^t)}{\mathbb{Q}_0(P_i^t)} \frac{\mathbb{Q}_{j_l}(\omega)}{\mathbb{Q}_{j_l}(P_i^t)} \\
&= \sum_{l=1}^{u} \frac{\lambda_{j_l} \mathbb{Q}_{j_l}(P_i^t)}{\mathbb{Q}_0(P_i^t)} \text{cond}^{t,i}(\mathbb{Q}_{j_l})(\omega)
\end{aligned}$$

Note that $\frac{\lambda_{j_l} \mathbb{Q}_{j_l}(P_i^t)}{\mathbb{Q}_0(P_i^t)} \geq 0$ and $\sum_{l=1}^{u} \frac{\lambda_{j_l} \mathbb{Q}_{j_l}(P_i^t)}{\mathbb{Q}_0(P_i^t)} = \frac{\sum_{l=1}^{u} \lambda_{j_l} \mathbb{Q}_{j_l}(P_i^t)}{\mathbb{Q}_0(P_i^t)} = \frac{\mathbb{Q}_0(P_i^t)}{\mathbb{Q}_0(P_i^t)} = 1$. Hence, $\text{cond}^{t,i}(\mathcal{Q})$ is convex. □

Definition 4.2.3. *A family of sequences of sets of probability measures* $(\mathcal{U}^x := (\mathcal{Q}_t^x)_{t=0}^T)_{x \in (0,+\infty)}$ *is called increasing if* $\text{cond}^{t,i}(\mathcal{Q}_t^x) \supseteq \text{cond}^{t,i}(\mathcal{Q}_t^y)$, *for all* $x \geq y > 0$, $t \in \mathcal{T}$ *and* P_i^t.

Before we present the representation theorem for DCAIs, we shall discuss a technical result as follows. It will be used for representation theorem of ADCAIs in the following chapter as well. It is worth to mention that Hyperplane Separation Theorem B.0.2 is used to verify the lemma.

Lemma 4.2.2. *Given a finite probability space* $(\Omega, \mathbb{P})$ *with full support. Let* N *denotes the number of states in* Ω, *and* $\mathcal{P}$ *denotes the set of all probability measures absolutely continuous with respect to* $\mathbb{P}$. *For any two closed and convex subsets* $U_1 \subseteq \mathcal{P}$ *and* $U_2 \subseteq \mathcal{P}$, *if*

$$\max_{\mathbb{P} \in U_1} \mathbb{E}^{\mathbb{P}}[X] \geq \max_{\mathbb{P} \in U_2} \mathbb{E}^{\mathbb{P}}[X] \,,$$

for all random variables X. *Then, we have* $U_2 \subseteq U_1$.

Proof. First, we can think of U_1 and U_2 are two closed and convex subsets of the N-dimensional space $\mathbb{R}^N$.

Assume there exists a point (probability measure) $\mathbb{Q}_0$ in $\mathbb{R}^N$ such that $\mathbb{Q}_0 \in U_2$ but $\mathbb{Q}_0 \ni U_1$. Singleton $\{\mathbb{Q}_0\}$ is a closed and convex set. Since $\{\mathbb{Q}_0\}$ and U_1 are disjoint, by the Separating Hyperplane Theorem B.0.2, there exists a point p such that

$$\inf_{x\in\{\mathbb{Q}_0\}} p\cdot x > \sup_{y\in U_1} p\cdot y\,. \tag{4.29}$$

We can define a random variable $X_0(\omega) := p(\omega)$ for all $\omega \in \Omega$ with ω is the corresponding dimension in $\mathbb{R}^N$. Then, we can rewrite (4.29) as follows:

$$\inf_{\mathbb{Q}\in\{\mathbb{Q}_0\}} \mathbb{E}^{\mathbb{Q}}[X_0] > \sup_{\mathbb{Q}\in U_1} \mathbb{E}^{\mathbb{Q}}[X_0]\,.$$

Since U_1 is closed and $\{\mathbb{Q}_0\}$ is singleton, we have

$$\mathbb{E}^{\mathbb{Q}_0}[X_0] > \max_{\mathbb{Q}\in U_1} \mathbb{E}^{\mathbb{Q}}[X_0]\,.$$

Note that $\mathbb{Q}_0 \in U_2$. Hence,

$$\max_{\mathbb{P}\in U_2} \mathbb{E}^{\mathbb{P}}[X_0] \ge \mathbb{E}^{\mathbb{Q}_0}[X_0] > \max_{\mathbb{P}\in U_1} \mathbb{E}^{\mathbb{P}}[X_0]\,.$$

which contradicts the assumption $\max_{\mathbb{P}\in U_1} \mathbb{E}^{\mathbb{P}}[X] \ge \max_{\mathbb{P}\in U_2} \mathbb{E}^{\mathbb{P}}[X]$. Finally, we have $U_2 \subseteq U_1$. □

Theorem 4.2.2. *α is a normalized and right-continuous DCAI if and only if there exists an increasing family of dynamically consistent sequences of sets of probability measures $(\mathcal{U}^x := (\mathcal{Q}_t^x)_{t=0}^T)_{x\in(0,+\infty)}$ such that*

$$\alpha_t(D) = \sup\{x \in (0,+\infty) : \inf_{\mathbb{Q}\in\mathcal{Q}_t^x} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|\mathcal{F}_t\big] \ge 0\}\,, \quad t\in\mathcal{T},\ D\in\mathcal{D}. \tag{4.30}$$

Proof. **Sufficiency**. Given an increasing family of dynamically consistent sequences of sets of probability measures $(\mathcal{U}^x := (\mathcal{Q}_t^x)_{t=0}^T)_{x\in(0,+\infty)}$ such that, for all $t \in \mathcal{T}$ and

$D \in \mathcal{D}$,

$$\alpha_t(D) = \sup\{x \in (0, +\infty) : \inf_{\mathbb{Q} \in \mathcal{Q}_t^x} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s | \mathcal{F}_t\big] \geq 0\}\,.$$

Let us define ρ^x such that

$$\rho_t^x(D) := -\inf_{\mathbb{Q} \in \mathcal{Q}_t^x} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s | \mathcal{F}_t\big]\,.$$

By the Representation Theorem 4.2.1 of DCRMs, each ρ^x is a DCRM. By Definition 4.1.4 and Definition 4.2.2, for $\omega \in P_i^t$,

$$\begin{aligned}\rho_t^x(D, \omega) &= -\inf_{\mathbb{Q} \in \mathcal{Q}_t^x} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s | \mathcal{F}_t\big](\omega) \\ &= -\inf_{\mathbb{Q} \in \mathrm{cond}^{t,i}(\mathcal{Q}_t^x)} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s\big]\,.\end{aligned}$$

Since $\mathcal{U}^x$ is increasing, for $y_1 > y_2 > 0$, we have $\mathrm{cond}^{t,i}(\mathcal{Q}_t^{y_1}) \supset \mathrm{cond}^{t,i}(\mathcal{Q}_t^{y_2})$. Then, for all $\omega \in P_i^t$,

$$\rho_t^{y_1}(D, \omega) = -\inf_{\mathbb{Q} \in \mathrm{cond}^{t,i}(\mathcal{Q}_t^{y_1})} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s\big] \geq= -\inf_{\mathbb{Q} \in \mathrm{cond}^{t,i}(\mathcal{Q}_t^{y_2})} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s\big] = \rho^{y_2}(D, \omega)$$

Note that the above inequality holds true for all P_i^t. We know that ρ^x is increasing with respect x. Therefore, we can rewrite α as,

$$\alpha_t(D) = \sup\{x \in (0, +\infty) : \rho_t^x(D) \leq 0\}\,.$$

By the Theorem 3.4.2, we know that α is a right-continuous and normalized dynamic coherent acceptability index.

Necessity. Since α is a normalized and right-continuous dynamic coherent acceptability index, by the Theorem 3.4.3, we have, there exists an increasing and left-continuous family of DCRM $(\rho^x)_{x \in (0, +\infty)}$ such that

$$\alpha_t(D) = \sup\{x \in (0, +\infty) : \rho_t^x(D) \leq 0\}\,.$$

By the Representation Theorem 4.2.1, for each ρ^x, there exists a dynamically consistent sequence of sets of probability measures $\mathcal{U}^x := \{\mathcal{Q}_s^x\}_{s=0}^T$ such that

$$\rho_t^x(D) = -\inf_{\mathbb{Q} \in \mathcal{Q}_t^x} \mathbb{E}_{\mathbb{Q}}[\sum_{s=t}^{T} D_s | \mathcal{F}_t]\,,$$

which implies,

$$\alpha_t(D) = \sup\{x \in (0, +\infty) : \inf_{\mathbb{Q}\in\mathcal{Q}_t^x} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|\mathcal{F}_t\big] \geq 0\}\,.$$

Denote $\bar{\mathcal{U}}^x := \{\bar{\mathcal{Q}}_s^x\}_{s=0}^T$ as the closed convex hull of $\mathcal{U}^x$ with $\bar{\mathcal{Q}}_s^x$ is the closed convex hull of $\mathcal{Q}_s^x$ for every $s = 0, 1, \cdots, T$. Proposition 4.2.1 shows that

$$\rho_t^x(D) = -\inf_{\mathbb{Q}\in\mathcal{Q}_t^x} \mathbb{E}_{\mathbb{Q}}[\sum_{s=t}^{T} D_s|\mathcal{F}_t] = -\inf_{\mathbb{Q}\in\bar{\mathcal{Q}}_t^x} \mathbb{E}_{\mathbb{Q}}[\sum_{s=t}^{T} D_s|\mathcal{F}_t]\,.$$

Note that the above equality holds true for all $D \in \mathcal{D}$ and $t \in \mathcal{T}$. By Definition 4.1.4 and Definition 4.2.2,

$$\rho_t^x(D) = -\inf_{\mathbb{Q}\in\text{cond}^{t,i}(\bar{\mathcal{Q}}_t^x)} \mathbb{E}_{\mathbb{Q}}[\sum_{s=t}^{T} D_s]\,.$$

Since $\bar{\mathcal{Q}}_t^x$ is full-support, closed and convex, Lemma 4.2.1 implies that $\text{cond}^{t,i}(\bar{\mathcal{Q}}_t^x)$ is also closed and convex. Then, Lemma 4.2.2 showes that $\text{cond}^{t,i}(\bar{\mathcal{Q}}_t^x) \supset \text{cond}^{t,i}(\bar{\mathcal{Q}}_t^y)$ if $x > y$ for all $t \in \mathcal{T}$.

The above argument holds true for all $P_i^t \in \Upsilon^t$. We know that $(\mathcal{U}^x := (\mathcal{Q}_t^x)_{t=0}^T)_{x\in(0,+\infty)}$ is increasing with respect to x. □

Theorem 4.2.2, besides being a fundamental theoretical result, can serve as basis for construction of DCAIs by means of constructing increasing sequences of dynamic sets of probability measures. Using this idea, we present here two abstract, non-trivial, examples of DCAIs.

Example 4.2.1. Dynamic upper-limit ratio.

Assume that $h : (0, +\infty) \to [0, +\infty)$ *is an increasing function. Define* $\acute{\mathcal{Q}}^x$ *as follows,*

$$\acute{\mathcal{Q}}^x := \{\mathbb{Q} \in \mathcal{P}^e | \mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_j] \leq (1 + h(x))\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_{j-1}] \text{ for all } j = 1, \ldots, T, \}\,,$$

and let $\mathcal{U}^x := \{\acute{\mathcal{Q}}^x\}_{t=0}^T$*. Note that* $\acute{\mathcal{Q}}^x = \mathcal{Q}^{1+h(x),u}$*,* $x \geq 0$*, where* $\mathcal{Q}^{a,u}, a \geq 1$*, is defined in Example 4.1.4, and thus* $\acute{\mathcal{Q}}^x$ *is dynamically consistent for any* $x > 0$*. Also* *observe that monotonicity of h implies monotonicity of $\acute{\mathcal{Q}}^x$ with respect to x. Hence, by Theorem 4.2.2,*

$$\alpha_t(D) = \sup\{x \in (0,+\infty) : \inf_{\mathbb{Q}\in\acute{\mathcal{Q}}^x} \mathbb{E}_{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s|\mathcal{F}_t\big] \geq 0\}\,.$$

is a normalized and right-continuous dynamic coherent acceptability index. We call it dynamic upper-limit ratio.

Example 4.2.2. Dynamic lower-limit ratio.

Similarly, using Example 4.1.5, we consider $\grave{\mathcal{Q}}^x := \mathcal{Q}^{1+h(x),l}$, for some increasing, non-negative function h. Then, $\mathcal{U}^x := \{\grave{\mathcal{Q}}^x\}_{t=0}^T$ is dynamically consistent, and by Theorem 4.2.2, the function α defined by (4.30) with $\mathcal{Q}_t^x = \grave{\mathcal{Q}}^x$, $x > 0$, is a normalized and right-continuous dynamic coherent acceptability index. We call it dynamic lower-limit ratio.

Proposition 4.2.2. *Static AI is a particular case of the DCAI and corresponds to $T = 1$. Same is true for the representation theorem for static AI in terms of family of sets of probability measures.*

Proof. (i) Assume Fatou Property holds, we are going to prove that right-continuity holds. In fact, if right-continuity doesn't hold, by the monotonicity of CAI, there exists $x \in [0,+\infty)$ such that

$$\lim_{c\to 0^+} \alpha_t(D + c1_{\{t\}}1_\Omega, \omega) > x > \alpha_t(D,\omega)\,.$$

However, by Fatou Property, since $D + c1_{\{t\}}1_\Omega \to D$, we should have $\alpha_t(D,\omega) \geq x$, which contradicts the above inequality. Therefore, right-continuity holds.

(ii) Assume right-continuity holds, we are going to prove that Fatou Property holds. We will define $X'_n(\omega) := |X_n(\omega) - X(\omega)| + X(\omega)$. Hence, we have $X'_n \geq X_n$.

By the monotonicity, we have

$$\alpha(X'_n) \geq \alpha(X_n) \geq x\,.$$

Denote $c_n = \max_{\omega \in \Omega}\{X'_n(\omega) - X(\omega)\}$, we know $c_n \geq 0$. Because of finite space, we can prove that $\lim_{n\to\infty} c_n = 0$. By the monotonicity, we have

$$\alpha(X + c_n) \geq \alpha(X'_n) \geq x\,.$$

By the right-continuity, we have

$$\alpha(X) = \lim_{n\to\infty} \alpha(X + c_n) \geq x\,.$$

Therefore, Fatou Property holds. □

CHAPTER 5

ALTERNATIVE DYNAMIC COHERENT ACCEPTABILITY INDICES

Many researchers have contributed to the theory of DCRMs. The major difference among them is the dynamic consistency. In this chapter, we use DCRMs defined in Appendix A. It provides us an alternative platform to study DCAIs. In fact, this is the first result we got by studying DCAIs. We differentiate it from DCAIs established in previous chapters, by naming it *alternative dynamic coherent acceptability indices* (ADCAIs).

We assume the same mathematical setup and notations in Section 3.1. In particular, we stress that $\Upsilon^t := \{P_1^t, P_2^t, \ldots, P_{n_t}^t\}$ denotes the unique partition of Ω at time t that generates $\mathcal{F}_t$.

Definition 5.0.4. *A basic dynamic acceptability index is a function $\alpha : \mathcal{T} \times \mathcal{D} \times \Omega \to [0, +\infty]$ that satisfies the following set of properties:*

(O1) Adaptiveness. *For any $t \in \mathcal{T}$ and $D \in \mathcal{D}$, $\alpha_t(D)$ is $\mathcal{F}_t$-measurable;*

(O2) Independence of the past. *For any $t \in \mathcal{T}$ and $D, D' \in \mathcal{D}$, if there exists $A \in \mathcal{F}_t$ such that $1_A D_s = 1_A D'_s$ for all $s \geq t$, then $1_A \alpha_t(D) = 1_A \alpha_t(D')$;*

(O3) Strict monotonicity.

(O3.1) For any $t \in \mathcal{T}$ and $D, D' \in \mathcal{D}$, if $D_s(\omega) \geq D'_s(\omega)$ for all $s \geq t$ and $\omega \in \Omega$, then $\alpha_t(D, \omega) \geq \alpha_t(D', \omega)$ for all $\omega \in \Omega$;

(O3.2) If $\alpha_t(D, \omega) \in (0, +\infty)$ for some $t \in \mathcal{T}$, $D \in \mathcal{D}$ and $\omega \in \Omega$, then $\alpha_t(D + c1_{\{t\}}, \omega) > \alpha_t(D, \omega)$ for every strictly positive constant $c \in \mathbb{R}$;

(O4) Arbitrage consistency. *$\alpha_t(1_{\{s\}}) = +\infty$ for all $t \in \mathcal{T}$ and $s \geq t$;*

(O5) Relevancy. *For all $t \in \mathcal{T}$, $P_i^t \in \Upsilon^t$, $\omega, \bar{\omega} \in P_i^t$ and $s \geq t$, we have $\alpha_t(-1_{\{s\}} 1_{\{\bar{\omega}\}}, \omega) = 0$;*

(O6) Scale invariance. $\alpha_t(\lambda D, \omega) = \alpha_t(D, \omega)$ *for all* $\lambda > 0$, $D \in \mathcal{D}$, $t \in \mathcal{T}$, *and* $\omega \in \Omega$;

(O7) Translation invariance. $\alpha_t(D + m1_{\{t\}}, \omega) = \alpha_t(D + m1_{\{s\}}, \omega)$ *for every* $t \in \mathcal{T}$, $D \in \mathcal{D}$, $\omega \in \Omega$, $s \geq t$ *and every* $\mathcal{F}_t$*-measurable random variable* m;

(O8) Right continuity. $\lim_{c \to 0^+} \alpha_t(D + c1_{\{t\}}, \omega) = \alpha_t(D, \omega)$ *for all* $t \in \mathcal{T}$, $D \in \mathcal{D}$, *and* $\omega \in \Omega$.

(O1) and (O2) are the same as (D1) and (D2), which are the natural properties in dynamic framework. (O3) is stronger than (D3) since it has a strict inequality when positive cash is added into a portfolio, as in (O3.2). Arbitrage consistency (O4) indicates that 'an arbitrage' is $+\infty$ index level and hence the basic dynamic acceptability index is unbounded above. Relevancy (O5) implies that a possible loss portfolio has 0 index level. Combined with (O3), a negative portfolio has 0 index level as well. (O6) and (O7) are the same as (D4) and (D6). For technical reasons, we use (O8) to describe the right-continuity of ADCAIs.

Definition 5.0.5. *For any basic dynamic acceptability index* α, *the* x*-level set of positions with respect to* $P_i^t \in \Upsilon^t$, *is defined by*

$$\mathcal{D}_x^{t,i} := \{D \in \mathcal{D} \,|\, \alpha_t(D, \omega) \geq x, \text{ for all } \omega \in P_i^t\}, \tag{5.1}$$

where $x \in [0, +\infty]$.

We observe that all strictly positive positions are belonging to $\mathcal{D}_x^{t,i}$ for any $x \in [0, +\infty]$. By the Definition 5.0.4, $\mathcal{D}_0^{t,i} = \mathcal{D}$, for all $P_i^t \in \Upsilon^t$. Note that (O1) – adaptiveness indicates that $\alpha_t(D)$ is a constant on each P_i^t.

Definition 5.0.6. *For any basic dynamic acceptability index* α, $D \in \mathcal{D}$ *and* $x \in (0, +\infty]$, *the* x*-level minimum cash payment with respect to* $P_i^t \in \Upsilon^t$, *is defined by*

$$\mathrm{mc}^{D,x,t,i} := \inf\{c \in \mathbb{R} : D + c1_{\{t\}} \in \mathcal{D}_x^{t,i}\}. \tag{5.2}$$

For any $D \in \mathcal{D}$, since D is a bounded process, there exists a positive number $k^D \in \mathbb{R}$ such that $|D| < k^D$. Therefore,

$$\{c \in \mathbb{R} : D + c1_{\{t\}} \in \mathcal{D}_x^{t,i}\} \subseteq (-(T+1)k^D, (T+1)k^D)\,,$$

and then $-(T+1)k^D \leq \mathrm{mc}^{D,x,t,i} \leq (T+1)k^D$, which means $\mathrm{mc}^{D,x,t}$ is finite and well-defined.

Definition 5.0.7. *For any basic dynamic acceptability index α, $D \in \mathcal{D}$ and $x \in (0, +\infty]$, the x-level minimum cash payment at time t, is defined as follows:*

$$\mathrm{MC}^{D,x,t}(\omega) := \mathrm{mc}^{D,x,t,i}\,, \ \text{if} \ \ \omega \in P_i^t, \tag{5.3}$$

for all $\omega \in \Omega$.

We can observe that $\mathrm{MC}^{D,x,t}$ is constant on each P_i^t and therefore is a $\mathcal{F}_t$-measurable random variable.

Lemma 5.0.3. *Given a basic dynamic acceptability index α, $D \in \mathcal{D}, x \in (0, +\infty)$ and $P_i^t \in \Upsilon^t$, $\mathrm{mc}^{D,x,t,i}$ is the x-level minimum cash payment with respect to $P_i^t \in \Upsilon^t$ if and only if for all $\omega \in P_i^t$ and $\eta > 0$,*

$$\alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t, \omega) \geq x \quad \text{and} \quad \alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t - \eta 1_t, \omega) < x\,.$$

Proof. **Necessity.** If $\mathrm{mc}^{D,x,t,i}$ is the x-level minimum cash payment with respect to $P_i^t \in \Upsilon^t$, by (5.1) and (5.2), for all $\omega \in P_i^t$ and $\eta > 0$,

$$\alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t - \eta 1_t, \omega) < x\,.$$

If there exists a $\bar{\omega} \in P_i^t$, such that $\alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t, \bar{\omega}) < x$. Then, there exists a positive $\bar{\epsilon} > 0$ such that $\alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t, \bar{\omega}) = x - \bar{\epsilon}$. By (O8) – right continuity of α, there exists $\bar{\gamma} > 0$ such that

$$\alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t + \bar{\gamma}1_t, \bar{\omega}) - \alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t, \bar{\omega}) \leq \frac{\bar{\epsilon}}{2}\,.$$

Then

$$\alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t + \bar{\gamma}1_t, \bar{\omega}) \leq \alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t, \bar{\omega}) + \frac{\bar{\epsilon}}{2}$$
$$= x - \bar{\epsilon} + \frac{\bar{\epsilon}}{2} = x - \frac{\bar{\epsilon}}{2} < x\,.$$

Since $\alpha_t(D)$ is a constant on each P_i^t, by (O3) – strict monotonicity and the above inequality,

$$\begin{aligned}\mathrm{mc}^{D,x,t,i} &= \inf\{c \in \mathbb{R} : D + c1_{\{t\}} \in \mathcal{D}_x^{t,i}\}\\ &= \inf\{c \in \mathbb{R} : \alpha_t(D + c1_t, \bar{\omega}) \geq x, \}\\ &\geq \mathrm{mc}^{D,x,t,i} + \bar{\gamma}\,.\end{aligned}$$

It is a contradiction since $\bar{\gamma} > 0$. Therefore, $\alpha_t(D + mc^{D,x,\xi_t}1_t, \omega) \geq x$ for all $\omega \in P_i^t$.

Sufficiency. Since for all $\omega \in P_i^t$ and $\eta > 0$, $\alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t - \eta 1_t, \omega) < x$, we have $\mathrm{mc}^{D,x,t,i} <= \inf\{c \in \mathbb{R} : D + c1_{\{t\}} \in \mathcal{D}_x^{t,i}\}$.

On the other hand, since for all $\omega \in P_i^t$ and $\eta > 0$, $\alpha_t(D + \mathrm{mc}^{D,x,t,i}1_t, \omega) \geq x$, it implies that $\mathrm{mc}^{D,x,t,i} >= \inf\{c \in \mathbb{R} : D + c1_{\{t\}} \in \mathcal{D}_x^{t,i}\}$. Finally,

$$\mathrm{mc}^{D,x,t,i} = \inf\{c \in \mathbb{R} : D + c1_{\{t\}} \in \mathcal{D}_x^{t,i}\}\,.$$

□

Definition 5.0.8. *A basic dynamic acceptability index is called ADCAI if it satisfies the following two properties:*

(O9) Quasi-concavity. *If $\alpha_t(D, \omega) \geq x$ and $\alpha_t(D', \omega) \geq x$ for some $t \in \mathcal{T}$, $\omega \in \Omega$, $D, D' \in \mathcal{D}$, and $x \in (0, +\infty]$, then $\alpha_t(\lambda D + (1-\lambda)D', \omega) \geq x$ for all $\lambda \in [0, 1]$;*

(O10) Dynamic consistency. *Given two positions $D, D' \in \mathcal{D}$ satisfying $D_t(\omega) = D'_t(\omega)$ for all $\omega \in \Omega$, if there exists a constant $x \in (0, +\infty)$ and $t \in \mathcal{T}$ such that $\mathrm{MC}^{D,x,t+1} = \mathrm{MC}^{D',x,t+1}$, then $\mathrm{MC}^{D,x,t} = \mathrm{MC}^{D',x,t}$.*

Definition 5.0.9. *A family of sets of probability measures* $(U^x)_{x\in(0,+\infty)}$ *is called increasing if* $\mathrm{cond}^{t,i}(U^x) \supseteq \mathrm{cond}^{t,i}(U^y)$*, for all* $x \geq y > 0$*,* $t \in \mathcal{T}$ *and* $P_i^t \in \Upsilon^t$*.*

Now, we shall introduce the representation theorem for ADCAIs in terms of a sequence of dynamic consistent sets of probability measures[3].

Theorem 5.0.3. *A function* α *is an ADCAI if and only if there exists an increasing sequence of closed and convex dynamic consistent sets of probability measures* $(U_x)_{x\in(0,+\infty)}$ *such that* $\rho_t^x(D)$ *defined as*

$$\rho_t^x(D) := \max_{\mathbb{P}\in U_x} \mathbb{E}^{\mathbb{P}}\big[-\sum_{s=t}^{T} D_s|\mathcal{F}_t\big]\,, \tag{5.4}$$

is continuous with respect to x*, and*

$$\alpha_t(D) = \sup\{x \in (0,+\infty) : \rho_t^x(D) \leq 0\}\,. \tag{5.5}$$

Proof. **Sufficiency.** We shall show that α defined in (5.5) satisfies the properties (O1)-(O10).

First, note that since $(U_x)_{x\in(0,+\infty)}$ is increasing, ρ^x defined in (5.4) is increasing with respect to x as well.

(O1) – adaptiveness, (O2) – independence of the past, and (O3.1) are similar to the corresponding proof in Theorem 3.4.2.

We show that α satisfies (O3.2). If $\alpha_t(D,\omega) \in (0,+\infty)$ for some $t \in \mathcal{T}, D \in \mathcal{D}$ and $\omega \in \Omega$, then, for all constant $c > 0$, (O3.1) implies,

$$\alpha_t(D + c1_{\{t\}},\omega) \geq \alpha_t(D,\omega)\,.$$

Denote by $x_0^\omega := \alpha_t(D,\omega)$, we have $x_0^\omega \in (0,+\infty)$. If $\alpha_t(D+c1_{\{t\}},\omega) = \alpha_t(D,\omega) = x_0^\omega$, since ρ^x is an increasing and continuous function on $(0,+\infty)$ with respect to x, by

[3]We refer to Definition A.0.7.

(5.5) and Lemma B.0.2,

$$\rho_t^{x_0^\omega}(D + c1_{\{t\}}, \omega) = \rho_t^{x_0^\omega}(D, \omega) = 0 \,.$$

By (H6) – translation invariance of ρ, we know that the above equation can not hold true. Therefore, $\alpha_t(D + c1_{\{t\}}, \omega) > \alpha_t(D, \omega)$ for all constant $c > 0$ and (O3.2) holds.

For all $t \in \mathcal{T}$ and $s \geq t$,

$$\begin{aligned}\alpha_t(1_{\{s\}}) &= \sup\{x \in (0, +\infty) : \rho_t^x(1_{\{s\}}) \leq 0\} \\ &= \sup\{x \in (0, +\infty) : \max_{\mathbb{P} \in U_x} \mathbb{E}^{\mathbb{P}}\big[-1 | \mathcal{F}_t\big] \leq 0\} \\ &= \sup\{x \in (0, +\infty) : -1 \leq 0\} = \infty \,.\end{aligned}$$

Hence, (O4) – arbitrage consistency holds true.

Next, we show that α satisfies (O5). For all $t \in \mathcal{T}$, $P_i^t \in \Upsilon^t$, $\omega, \bar{\omega} \in P_i^t$ and $s \geq t$, by (H8), we have $\rho_t^x(-1_{\{s\}}1_{\{\bar{\omega}\}})(\omega) > 0$ for all $x \in (0, +\infty)$. By (5.5), we know $\alpha_t(-1_{\{s\}}1_{\{\bar{\omega}\}}, \omega) = 0$.

(O6) – scale invariance, and (O7) – translation invariance are similar to the corresponding proof in Theorem 3.4.2.

We show that α satisfies (O8) – right continuity. By (O3) – strict monotonicity, (O8) holds when $\alpha_t(D, \omega) = +\infty$. When $\alpha_t(D, \omega) \in [0, +\infty)$, if $\lim_{c \to 0^+} \alpha_t(D + c1_{\{t\}}, \omega) \neq \alpha_t(D, \omega)$, by (O3.1), there exists a positive sequence $(c_n)_{n=1}^{\infty}$ with $\lim_{n \to \infty} c_n = 0^+$ and $\epsilon > 0$, such that

$$\alpha_t(D + c_n 1_{\{t\}}, \omega) > \alpha_t(D, \omega) + \epsilon, \tag{5.6}$$

for all $n \in \mathbb{N}$. Denote $a := \alpha_t(D, \omega)$. (5.5) gives

$$a = \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \leq 0\} \,,$$

which implies

$$\rho_t^{a + \frac{\epsilon}{4}}(D, \omega) > 0 \,. \tag{5.7}$$

Put $a_n := \alpha_t(D + c_n 1_{\{t\}}, \omega)$, we have

$$a_n = \sup\{x \in (0, +\infty) : \rho_t^x(D + c_n 1_{\{t\}}, \omega) \leq 0\}\,.$$

By (H4) – translation invariance of ρ_t^x,

$$a_n = \sup\{x \in (0, +\infty) : \rho_t^x(D, \omega) \leq c_n\}\,.$$

Then, for all $n \in \mathbb{N}$,

$$\rho_t^{a_n - \frac{\epsilon}{4}}(D, \omega) \leq c_n\,.$$

Since $\lim_{n\to\infty} c_n = 0^+$, by (5.7), there exists $N \in \mathbb{N}$, such that $0 < c_N < \rho_t^{a+\frac{\epsilon}{4}}(D, \omega)$. Then,

$$\rho_t^{a+\frac{\epsilon}{4}}(D, \omega) > c_N \geq \rho_t^{a_N - \frac{\epsilon}{4}}(D, \omega)\,.$$

Since $\rho_t^x(D, \omega)$ is a continuous and increasing function with respect to x,

$$a + \frac{\epsilon}{4} \geq a_N - \frac{\epsilon}{4}\,,$$

which implies,

$$a + \epsilon > a + \frac{\epsilon}{2} \geq a_N\,.$$

Hence,

$$\alpha_t(D, \omega) + \epsilon > \alpha_t(D + c_N 1_{\{t\}}, \omega)\,,$$

which contradicts (5.6). Therefore,

$$\lim_{c\to 0^+} \alpha_t(D + c1_{\{t\}}, \omega) = \alpha_t(D, \omega)\,.$$

(O9) – quasi-concavity is similar to the corresponding proof in Theorem 3.4.2.

Last, we show that α satisfies (O10). Given two positions $D, D' \in \mathcal{D}$ satisfying $D_t(\omega) = D'_t(\omega)$ for all $\omega \in \Omega$, if there exists a constant $x \in (0, +\infty)$ and $t \in \mathcal{T}$ such that $\mathrm{MC}^{D,x,t+1} = \mathrm{MC}^{D',x,t+1}$, by Lemma 5.0.3, for each $P_i^{t+1} \in \Upsilon^{t+1}$ and $\eta > 0$,

$$\alpha_{t+1}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}}, \omega) \geq x, \quad \alpha_{t+1}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}} - \eta 1_{\{t+1\}}, \omega) < x\,,$$
$$\alpha_{t+1}(D' + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}}, \omega) \geq x, \quad \alpha_{t+1}(D' + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}} - \eta 1_{\{t+1\}}, \omega) < x\,,$$

By (5.5), for $\omega \in P_i^{t+1}$,

$$\sup\{y \in (0, +\infty) : \rho_{t+1}^{y}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}}, \omega) \leq 0\} \geq x\,. \tag{5.8}$$

If there exists $\bar{\omega} \in P_i^{t+1}$ such that $\rho_{t+1}^{x}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}}, \bar{\omega}) > 0$, by the continuity of ρ^x with respect to $x \in (0, +\infty)$, we know there exists $\epsilon > 0$ such that $\rho_{t+1}^{x-\epsilon}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}}, \bar{\omega}) > 0$, which implies

$$\sup\{y \in (0, +\infty) : \rho_{t+1}^{y}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}}, \bar{\omega}) \leq 0\} \leq x - \epsilon < x\,.$$

The above inequality contradicts (5.8). Hence, for all $\omega \in P_i^{t+1}$,

$$\rho_{t+1}^{x}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}}, \omega) \leq 0\,.$$

By (H4) – translation invariance of ρ^x, it follows that

$$\rho_{t+1}^{x}(D, \omega) \leq \mathrm{mc}^{D,x,t,i+1}\,. \tag{5.9}$$

Since $\alpha_{t+1}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}} - \eta 1_{\{t+1\}}, \omega) < x$ for all $\eta > 0$, by (5.5),

$$\sup\{y \in (0, +\infty) : \rho_{t+1}^{y}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}} - \eta 1_{\{t+1\}}, \omega) \leq 0\} < x\,,$$

which implies

$$\rho_{t+1}^{x}(D + \mathrm{mc}^{D,x,t+1,i} 1_{\{t+1\}} - \eta 1_{\{t+1\}}, \omega) > 0\,,$$

for all $\eta > 0$. Hence, by (H4) – translation invariance of ρ^x, it follows that

$$\rho_{t+1}^{x}(D, \omega) > \mathrm{mc}^{D,x,t,i}(\omega) - \eta\,.$$

Take $\eta \to 0^+$, we have $\rho^x_{t+1}(D,\omega) \geq \mathrm{mc}^{D,x,t+1,i}$. Together with (5.9), $\rho^x_{t+1}(D,\omega) = \mathrm{mc}^{D,x,t+1,i}$. Same with position D', we have $\rho^x_{t+1}(D',\omega) = \mathrm{mc}^{D,x,t+1,i}$. The argument holds true for all $P_i^{t+1} \in \Upsilon^{t+1}$. Hence, for all $\omega \in \Omega$,

$$\rho^x_{t+1}(D,\omega) = \rho^x_{t+1}(D',\omega)\,.$$

Since $D_t(\omega) = D'_t(\omega)$ for all $\omega \in \Omega$, by (H7) – dynamic consistency of ρ^x,

$$\rho^x_t(D,\omega) = \rho^x_t(D',\omega)\,,$$

for all $\omega \in \Omega$.

Denote $m(\omega) := \rho^x_t(D,\omega) = \rho^x_t(D',\omega)$, then $\rho^x_t(D,\omega) - m(\omega) = 0$. By (H4) – translation invariance of ρ^x, $\rho^x_t(D + m1_{\{t\}},\omega) = 0$. Then,

$$\sup\{y \in (0,+\infty) : \rho^y_t(D + m1_{\{t\}},\omega) \leq 0\} \geq x\,.$$

By (5.5),

$$\alpha_t(D + m1_{\{t\}},\omega) \geq x\,.$$

Also, for all $\eta > 0$,

$$\rho^x_t(D,\omega) - m(\omega) + \eta = \eta\,.$$

By (H4) – translation invariance of ρ^x,

$$\rho^x_t(D + m1_{\{t\}} - \eta 1_{\{t\}},\omega) = \rho^x_t(D,\omega) - m(\omega) + \eta = \eta > 0\,.$$

By the continuity of ρ^x with respect to $x \in (0,+\infty)$, there exists $\epsilon > 0$ such that

$$\rho^{x-\epsilon}_t(D + m1_{\{t\}} - \eta 1_{\{t\}},\omega) > 0\,,$$

which implies

$$\sup\{y \in (0,+\infty) : \rho^y_t(D + m1_{\{t\}} - \eta 1_{\{t\}},\omega) \leq 0\} \leq x - \epsilon < x\,.$$

By (5.5),

$$\alpha_t(D + m1_{\{t\}} - \eta 1_{\{t\}},\omega) < x\,.$$

Then, Lemma 5.0.3 indicates that $\mathrm{MC}^{D,x,t} = m$. Same with D', we have $\mathrm{MC}^{D',x,t} = m = \mathrm{MC}^{D,x,t}$. (O10) holds true for α.

Finally, we conclude that α satisfies the properties $(O1) - (O10)$. Therefore, α is an ADCAI.

Necessity. Given an ADCAI α, we define $u_t^x(D) := \mathrm{MC}^{D,x,t}$. By Definition 5.0.5 and Definition 5.0.7, for all $\omega \in \Omega$,

$$u_t^x(D,\omega) = \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}\,. \tag{5.10}$$

We first show that for every $x \in (0, +\infty)$, the function u^x is a dynamic coherent risk measure given by Definition A.0.8. We only need to show that u^x satisfies all properties (H1)-(H8) in Definition A.0.8.

(H1) – adaptiveness, (H2) – independence of the past, (H3) – monotonicity, (H4) – translation invariance, (H5) – homogeneity, and (H6) – subadditivity are similar to the corresponding proof in Theorem 3.4.1.

Now, we show that u^x satisfies (H7). Fix $t \in \{0, \ldots, T-1\}$ and $D, D' \in \mathcal{D}$ with $D_t = D'_t$ and $u_{t+1}^x(D) = u_{t+1}^x(D')$. By definition of u^x, we can derive that

$$\mathrm{MC}^{D,x,t+1} = \mathrm{MC}^{D',x,t+1}\,.$$

By (O10) – dynamic consistency of α,

$$\mathrm{MC}^{D,x,t} = \mathrm{MC}^{D',x,t}\,,$$

which implies

$$u_t^x(D,\omega) = u_t^x(D',\omega)\,.$$

Next, we show that u^x satisfies (H8). For all $t \in \mathcal{T}$, $P_i^t \in \Upsilon^t$, $\omega, \bar{\omega} \in P_i^t$, and $s \geq t$, by (O5) – relevancy of α,

$$\alpha_t(-1_{\{s\}}1_{\{\bar{\omega}\}}, \omega) = 0\,, \tag{5.11}$$

Assume $u_t^x(-1_{\{s\}}1_{\{\bar{\omega}\}}, \omega) \leq 0$, by (5.10), it follows

$$\inf\{c \in \mathbb{R} : \alpha_t(-1_{\{s\}}1_{\{\bar{\omega}\}} + c1_{\{t\}}, \omega) \geq x\} \leq 0\,.$$

Then, for any $\bar{c} > 0$, by (O3) – strict monotonicity of α,

$$\alpha_t(-1_{\{s\}}1_{\{\bar{\omega}\}} + \bar{c}1_{\{t\}}, \omega) \geq x\,,$$

Let $\bar{c} \to 0^+$, by (O8) – right continuity for α, we conclude

$$\alpha_t(-1_{\{s\}}1_{\{\bar{\omega}\}}, \omega) = \lim_{\bar{c}\to 0^+} \alpha_t(-1_{\{s\}}1_{\{\bar{\omega}\}} + \bar{c}1_{\{t\}}, \omega) \geq x > 0\,,$$

which contradicts (5.11). Therefore,

$$u_t^x(-1_{\{s\}}1_{\{\bar{\omega}\}}, \omega) > 0\,.$$

From all above, we conclude that u^x is a dynamic coherent risk measure for all $x \in (0, +\infty)$. By Representation Theorem A.0.1 of dynamic coherent risk measure, there exists a closed, convex and dynamic consistent set of probability measures $U_x \in \mathcal{P}^e$ such that

$$u_t^x(D, \omega) = \max_{\mathbb{P}\in U_x} \mathbb{E}^{\mathbb{P}}\big[-\sum_{s=t}^{T} D_s|\mathcal{F}_t\big](\omega)\,. \tag{5.12}$$

By Definition 5.0.5, $\mathcal{D}_x^{t,i} \subseteq \mathcal{D}_y^{t,i}$ for all $x \geq y > 0$. Then, for any $D \in \mathcal{D}$,

$$\{c \in \mathbb{R} : D - c1_{\{t\}} \in \mathcal{D}_x^{t,i}\} \subseteq \{c \in \mathbb{R} : D - c1_{\{t\}} \in \mathcal{D}_y^{t,i}\}.$$

Thus,

$$\begin{aligned} u_t^x(D, \omega) &= \inf\{c \in \mathbb{R} : D + c1_{\{t\}} \in \mathcal{D}_x^{t,i}\} \\ &\geq \inf\{c \in \mathbb{R} : D + c1_{\{t\}} \in \mathcal{D}_y^{t,i}\} \\ &= u_t^y(D, \omega)\,. \end{aligned}$$

By (5.12), for any $D \in \mathcal{D}$,

$$\max_{\mathbb{P}\in U_x} \mathbb{E}^{\mathbb{P}}\big[-\sum_{s=t}^{T} D_s|\mathcal{F}_t\big] \geq \max_{\mathbb{P}\in U_y} \mathbb{E}^{\mathbb{P}}\big[-\sum_{s=t}^{T} D_s|\mathcal{F}_t\big]\,,$$

Then, Lemma 4.2.1 and Lemma 4.2.2 indicates that $\mathrm{cond}^{t,i}(U^x) \supseteq \mathrm{cond}^{t,i}(U^y)$, for all $x \geq y > 0$ and $P_i^t \in \Upsilon^t$. Then, we know U^x is increasing with respect to $x \in (0, \infty)$.

Now, we show that for any $t \in \mathcal{T}, x \in (0, +\infty), \omega \in \Omega$ and $D \in \mathcal{D}$, $u_t^x(D, \omega)$ is continuous with respect to x. For convenience, we denote $f^{\omega,t,D}(x) := u_t^x(D, \omega)$. By (5.10),

$$f^{\omega,t,D}(x) = \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\}\,. \tag{5.13}$$

We observe that $f^{\omega,t,D}(x)$ is increasing with respect to x. Denote

$$c_0^\omega := \sup\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) = 0\}\,,$$
$$c_\infty^\omega := \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) = +\infty\}\,,$$

By (O3.2) and (O8), we know $\alpha_t(D + c1_{\{t\}}, \omega)$ is right continuous and strictly increasing with respect to c on $(c_\infty^\omega, c_0^\omega)$. Moreover, (O4) and (O5) imply $c_\infty^\omega, c_0^\omega$ are finite.

If $f^{\omega,t,D}$ is not continuous at a point $x_0 \in (0, \infty)$, then there are two distinguished cases.

First case, $\lim_{x \to x_0^+} f^{\omega,t,D}(x) \neq f^{\omega,t,D}(x_0)$. Since $f^{\omega,t,D}(x)$ is increasing, there exist a $\eta > 0$ and a sequence $\{\epsilon_n\}_{n=1}^\infty$, such that $\epsilon_n \to 0^+$, and $f^{\omega,t,D}(x_0 + \epsilon_n) > f^{\omega,t,D}(x_0) + \eta$ for all $n \in \mathbb{N}$. By (5.13), it follows

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0 + \epsilon_n\} > \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} + \eta\,,$$

for all $n \in \mathbb{N}$. Then,

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0 + \epsilon_n\} - \frac{\eta}{2} > \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} + \frac{\eta}{2}\,.$$

Hence, there exist c_1, c_2 such that

$$\begin{aligned}\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0 + \epsilon_n\} - \frac{\eta}{2} > c_1 > c_2 \\ > \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} + \frac{\eta}{2}\end{aligned}$$

for all $n \in \mathbb{N}$. From the left hand inequality, by (O3) – strict monotonicity of α, we conclude

$$\alpha_t(D + c_i 1_{\{t\}}, \omega) < x_0 + \epsilon_n, \quad i = 1, 2\,; n \in \mathbb{N}\,.$$

Passing to the limit in the last inequality with $n \to \infty$, we have, for $i = 1, 2$,

$$\alpha_t(D + c_i 1_{\{t\}}, \omega) \leq x_0\,. \tag{5.14}$$

On the other hand, since

$$c_1 > c_2 > \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} + \frac{\eta}{2}\,,$$

we have $\alpha_t(D + c_i 1_{\{t\}}, \omega) \geq x_0$, for $i = 1, 2$. Together with (5.14), it follows that

$$\alpha_t(D + c_1 1_{\{t\}}, \omega) = \alpha_t(D + c_2 1_{\{t\}}, \omega) = x_0, \quad c_1 < c_2\,,$$

which contradicts (O3.2) – strict monotonicity for α.

Second case, $\lim_{x \to x_0^-} f^{\omega,t,D}(x) \neq f^{\omega,t,D}(x_0)$. Since $f^{\omega,t,D}(x)$ is increasing, there exists $\eta > 0$ and a sequence $\{\epsilon_n\}_{n=1}^{\infty}$, such that $\epsilon_n \to 0^+$, and $f^{\omega,t,D}(x_0 + \epsilon_n) < f^{\omega,t,D}(x_0) - \eta$ for all $n \in \mathbb{N}$. By (5.13), it follows that

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0 + \epsilon_n\} < \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} - \eta\,,$$

for all $n \in \mathbb{N}$. Then,

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0 - \epsilon_n\} + \frac{\eta}{2} < \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} - \frac{\eta}{2}\,,$$

Hence, there exists c_3 such that

$$\inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0 - \epsilon_n\} + \frac{\eta}{2} < c_3 < \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} - \frac{\eta}{2}\,,$$

for all $n \in \mathbb{N}$. From the left hand inequality, we conclude that

$$\alpha_t(D + c_3 1_{\{t\}}, \omega) \geq x_0 - \epsilon_n, \quad n \in \mathbb{N}\,.$$

Passing to the limit in the last inequality with $n \to \infty$,

$$\alpha_t(D + c_3 1_{\{t\}}, \omega) \geq x_0 \,. \tag{5.15}$$

On the other hand, since

$$c_3 < \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\} - \frac{\eta}{2}\,,$$

we have $\alpha_t(D + c_3 1_{\{t\}}, \omega) < x_0$, which contradicts (5.15).

From the above two cases, we know

$$\lim_{x \to x_0} \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x\} = \inf\{c \in \mathbb{R} : \alpha_t(D + c1_{\{t\}}, \omega) \geq x_0\}\,,$$

which implies

$$\lim_{x \to x_0} u_t^x(D, \omega) = u_t^{x_0}(D, \omega)\,.$$

Finally, by similar argument as Theorem 3.4.3, we can derive that

$$\alpha_t(D, \omega) = \sup\{x \in (0, +\infty) : u_t^x(D, \omega) \leq 0\}\,.$$

□

CHAPTER 6

EXAMPLES AND APPLICATIONS

In Chapter 1 two classical static acceptability indices were introduced: Gain Loss Ratio and Risk Adjusted Return on Capital. In this chapter, we will generalize these static indices to dynamic versions, and we will examine if these dynamic versions are DCAIs or ADCAIs.

We will present an application of dynamic acceptability indices in the context of optimal portfolio selection problem. The version of optimal portfolio selection problem that we address, amounts to dynamic selection of portfolio of financial assets to maximize expected value of terminal utility of the portfolio. Investors may choose their optimal portfolios relative to various utility functions. We propose to use dGLR defined in the next section to discriminate between these optimal portfolios, in the sense of deciding which utility function is most preferable to be used in the problem of optimal portfolio.

6.1 Examples and Counterexamples

Gain Loss Ratio (GLR), which was presented in Definition 1.0.2, is a typical return-to-risk type of performance measure, very popular among practitioners. A natural generalization of GLR to dynamic framework is defined as follows.

Definition 6.1.1. Dynamic Gain Loss Ratio.
For all $t \in \mathcal{T}$ and $D \in \mathcal{D}$,

$$\mathrm{dGLR}_t(D) := \begin{cases} \frac{\mathbb{E}[\sum_{s=t}^T D_s|\mathcal{F}_t]}{\mathbb{E}[(\sum_{s=t}^T D_s)^- \,|\, \mathcal{F}_t]}, & \text{if } \ \mathbb{E}[\sum_{s=t}^T D_s \,|\, \mathcal{F}_t] > 0\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{6.1}$$

where $(\sum_{s=t}^T D_s)^- := \max\{-\sum_{s=t}^T D_s, 0\}$. By convention, $\mathrm{dGLR}_t(0) = +\infty$.

Remark 6.1.1. *Note that taking $T = 1$, and letting $X = D_0 + D_1$, $dGLR_0$ becomes* *the static GLR given in Definition 1.0.2.*

Proposition 6.1.1. *dGLR is a normalized and right-continuous dynamic coherent acceptability index.*

Proof. Since $\mathrm{dGLR}_t(1_{\{t\}}) = +\infty$ and $\mathrm{dGLR}_t(-1_{\{t\}}) = 0$, we have that dGLR is normalized.

Next, we show that dGLR is right-continuous. Fixed any $t \in \mathcal{T}$ and $D \in \mathcal{D}$. If $\mathbb{E}[\sum_{s=t}^{T} D_s \,|\, \mathcal{F}_t] > 0$, then for any $c > 0$, $\mathbb{E}[\sum_{s=t}^{T}(D + c1_{\{t\}})_s \,|\, \mathcal{F}_t] = \mathbb{E}[\sum_{s=t}^{T} D_s \,|\, \mathcal{F}_t] + c > 0$. Hence,

$$\begin{aligned}\lim_{c\to 0^+} \mathrm{dGLR}_t(D + c1_{\{t\}}) &= \lim_{c\to 0^+} \frac{\mathbb{E}(\sum_{s=t}^{T} D_s|\mathcal{F}_t) + c}{\mathbb{E}(\{\sum_{s=t}^{T} D_s + c\}^-|\mathcal{F}_t)} \\ &= \frac{\mathbb{E}(\sum_{s=t}^{T} D_s|\mathcal{F}_t)}{\mathbb{E}(\{\sum_{s=t}^{T} D_s\}^-|\mathcal{F}_t)} = \mathrm{dGLR}_t(D)\,.\end{aligned}$$

If $\mathbb{E}[\sum_{s=t}^{T} D_s \,|\, \mathcal{F}_t] = 0$, $\mathrm{dGLR}_t(D) = 0$. For any $c > 0$, $\mathbb{E}[\sum_{s=t}^{T}(D + c1_{\{t\}})_s \,|\, \mathcal{F}_t] > 0$. Hence,

$$\begin{aligned}\lim_{c\to 0^+} \mathrm{dGLR}_t(D + c1_{\{t\}}) &= \lim_{c\to 0^+} \frac{\mathbb{E}(\sum_{s=t}^{T} D_s|\mathcal{F}_t) + c}{\mathbb{E}(\{\sum_{s=t}^{T} D_s + c\}^-|\mathcal{F}_t)} \\ &= \lim_{c\to 0^+} \frac{c}{\mathbb{E}(\{\sum_{s=t}^{T} D_s + c\}^-|\mathcal{F}_t)} = 0 = \mathrm{dGLR}_t(D)\,.\end{aligned}$$

If $\mathbb{E}[\sum_{s=t}^{T} D_s \,|\, \mathcal{F}_t] < 0$, $\mathrm{dGLR}_t(D) = 0$. For some small enough $c > 0$, we have that $\mathbb{E}[\sum_{s=t}^{T}(D + c1_{\{t\}})_s \,|\, \mathcal{F}_t] < 0$. Hence, $\lim_{c\to 0^+} \mathrm{dGLR}_t(D + c1_{\{t\}}) = 0 = \mathrm{dGLR}_t(D)$.

Now, we show that dGLR satisfies (D1)-(D7). Adaptiveness (D1), and independence of the past (D2) of dGLR follow directly from the definition of dGLR. Monotonicity (D3), scale invariance (D4), and quasi-concavity (D5) are verified as in static case with expectation replaced by conditional expectation (for details see [14]).

For any $\mathcal{F}_t$-measurable random variable m, we have

$$\sum_{l=t}^{T}(D + m1_{\{s\}})_l = \sum_{l=t}^{T} D_l + m = \sum_{l=t}^{T}(D + m1_{\{t\}})_l\,.$$

Then,

$$\mathbb{E}(\sum_{l=t}^{T}(D+m1_{\{s\}})_l|\mathcal{F}_t) = \mathbb{E}(\sum_{l=t}^{T}(D+m1_{\{t\}})_l|\mathcal{F}_t)\,,$$

and

$$\mathbb{E}((\sum_{l=t}^{T}(D+m1_{\{s\}})_l)^-|\mathcal{F}_t) = \mathbb{E}((\sum_{l=t}^{T}(D+m1_{\{t\}})_l)^-|\mathcal{F}_t)\,,$$

for all $t \in \mathcal{T},\ D \in \mathcal{D}$. This proves translation invariance.

Finally we will show that dGLR satisfies version (D7-II) of dynamic consistency. For any $t \in \mathcal{T}$ and $D \in \mathcal{D}$, if $D_t(\omega) = 0$ for all $\omega \in \Omega$, all we need to prove is that for each $P_i^t \in \Upsilon^t$,

$$1_{P_i^t} \min_{\omega \in P_i^t} \mathrm{dGLR}_{t+1}(D,\omega) \leq 1_{P_i^t}\mathrm{dGLR}_t(D) \leq 1_{P_i^t} \max_{\omega \in P_i^t} \mathrm{dGLR}_{t+1}(D,\omega).$$

Denote by $m^{t,i} := \max_{\omega \in P_i^t} \mathrm{dGLR}_{t+1}(D,\omega)$. If $m^{t,i} = +\infty$, the above right inequality is obviously satisfied. If $m^{t,i} = 0$, then $\mathrm{dGLR}_{t+1}(D,\omega) = 0$ for all $\omega \in P_i^t$. By Definition 6.1, we can observe that $\mathrm{dGLR}_t(D,\omega) = 0$ for all $\omega \in P_i^t$ as well.

If $m^{t,i} \in (0,\infty)$, we have $\mathrm{dGLR}_{t+1}(D,\omega) \leq m^{t,i}$ for all $\omega \in P_i^t$. By Definition 6.1 of dGLR, for all $\omega \in P_i^t$,

$$\mathbb{E}(\sum_{s=t+1}^{T} D_s|\mathcal{F}_{t+1})(\omega) \leq m^{t,i} \cdot \mathbb{E}(\{\sum_{s=t+1}^{T} D_s\}^-|\mathcal{F}_{t+1})(\omega)\,,$$

and since $D_t = 0$, we have

$$\mathbb{E}(\sum_{s=t}^{T} D_s|\mathcal{F}_t) = \mathbb{E}(\sum_{s=t+1}^{T} D_s|\mathcal{F}_t) = \mathbb{E}(\mathbb{E}(\sum_{s=t+1}^{T} D_s|\mathcal{F}_{t+1})|\mathcal{F}_t)\,.$$

Hence, for all $\omega \in P_i^t$,

$$\begin{aligned}\mathbb{E}(\sum_{s=t}^{T} D_s|\mathcal{F}_t)(\omega) \leq \mathbb{E}\big(m^{t,i}\mathbb{E}(\{\sum_{s=t+1}^{T} D_s\}^-|\mathcal{F}_{t+1})\big|\mathcal{F}_t\big)(\omega) &= m^{t,i}\mathbb{E}(\{\sum_{s=t+1}^{T} D_s\}^-|\mathcal{F}_t)(\omega) \\ &= m^{t,i}\mathbb{E}(\{\sum_{s=t}^{T} D_s\}^-|\mathcal{F}_t)(\omega)\,,\end{aligned}$$

which implies that $\mathrm{dGLR}_t(D,\omega) \leq m^{t,i}$. Then,

$$1_{P_i^t}\mathrm{dGLR}_t(D) \leq 1_{P_i^t} \max_{\omega \in P_i^t} \mathrm{dGLR}_{t+1}(D,\omega)\,.$$

By similar argument, we can show that

$$1_{P_i^t} \min_{\omega \in P_i^t} \mathrm{dGLR}_{t+1}(D,\omega) \le 1_{P_i^t} \mathrm{dGLR}_t(D)\,.$$

Hence, (D7-II) holds true.

Using Corollary 3.2.3, we conclude that dGLR is a DCAI. □

We shall demonstrate now, by means of an example, that dGLR does not satisfy condition (O10) – dynamic consistency of ADCAIs, implying that it is not an ADCAI. Let us consider a two-period model with four states: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Assume $\mathbb{P}$ is the reference probability measure with $\mathbb{P}(\omega_1) = 0.2$, $\mathbb{P}(\omega_2) = 0.3$, $\mathbb{P}(\omega_3) = 0.1$ and $\mathbb{P}(\omega_4) = 0.4$. Denote by $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ the filtration. $\mathcal{F}_0 = \{\emptyset, \Omega\}$; $\mathcal{F}_2$ is generated by the partition $\Big\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\Big\}$; $\mathcal{F}_2$ is generated by the partition $\Big\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\Big\}$. Two dividend processes D and D' are shown in Table 6.1 and Table 6.2.

Table 6.1: Dividend Process D

ω	$D_0(\omega)$	$D_1(\omega)$	$D_2(\omega)$
ω_1	0.5	1	11
ω_2	0.5	1	-5
ω_3	0.5	-3	-8
ω_4	0.5	-3	3

Table 6.2: Dividend Process D'

ω	$D'_0(\omega)$	$D'_1(\omega)$	$D'_2(\omega)$
ω_1	0.5	2.2	2.6
ω_2	0.5	2.2	-3
ω_3	0.5	-4	25
ω_4	0.5	-4	-3

By direct calculation, we have[4] $\mathrm{MC}^{D,0.5,1}(\omega_i) = \mathrm{MC}^{D',0.5,1}(\omega_i) = -0.923$ for $i = 1,2$ and $\mathrm{MC}^{D,0.5,1}(\omega_i) = \mathrm{MC}^{D',0.5,1}(\omega_i) = 3.001$ for $i = 3,4$. We also have $\mathrm{MC}^{D,0.5,0}(\omega_i) = 0.375$, but $\mathrm{MC}^{D,0.5,0}(\omega_i) = 0.649 \neq \mathrm{MC}^{D,0.5,0}(\omega_i)$ for $i = 1,2,3,4$. Note that $D_0 = D'_0 = 0.5$. Hence, dGLR does not satisfy (O10) – dynamic consistency of ADCAI, and it is not an ADCAI.

[4]Recall Definition 5.0.7 for MC

Taking into account the form of the dynamic acceptability index as in (4.30), and the form of the static one as in (2.5), the natural question arises: is it possible in general to 'dynamize' a static coherent acceptability index by taking the appropriate conditional expectation of the cumulative future cash-flow? For example, to dynamize GLR, we considered the static GLR, and we replaced in it the expectation with conditional expectation, and the terminal value with the future cumulative cash-flow. However, this procedure may not lead to desirable results in general, as shown below.

According to the above idea the natural extension of static Risk Adjusted Return on Capital (RAROC) to a dynamic setup should have the following form:

Definition 6.1.2. Dynamic Risk Adjusted Return on Capital
For all $t \in \{0, 1, \dots, T\}$ *and* $D \in \mathcal{D}$*,*

$$\mathrm{dRAROC}_t(D) = \begin{cases} \frac{\mathbb{E}(\sum_{s=t}^T D_s|\mathcal{F}_t)}{-\inf\limits_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\sum_{s=t}^T D_s|\mathcal{F}_t]}, & \text{when } \ \mathbb{E}(\sum_{s=t}^T D_s|\mathcal{F}_t) > 0 \\ 0, & \text{otherwise} \end{cases}$$

with convention that $\mathrm{dRAROC}_t(D) = +\infty$ *if* $\inf\limits_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\sum_{s=t}^T D_s|\mathcal{F}_t] \geq 0$*.*

As it is seen from Figure 6.1, which represents a numerical example, dRAROC does not satisfy property (D7), dynamic consistency. In this example, we consider $\mathcal{Q} = \mathcal{P}^e$. Assume that the states are labeled from top to bottom $\omega_1, \omega_2, \dots, \omega_8$. Note that, $D_1(\omega_1) = 0.2 > 0$, i.e. positive cashflow at time $t = 1$ and state ω_1, but $\mathrm{dRAROC}_1(\omega_1) = 0.31 < 0.33 = \mathrm{dRAROC}_2(\omega_1)$, as well as $\mathrm{dRAROC}_1(\omega_1) = 0.31 < 0.32 = \mathrm{dRAROC}_2(\omega_2)$. Thus dRAROC does not satisfy (D7) and hence it is not a DCAI.

For comparison reasons, we also present in Figure 6.1 the values of dGLR, which is a DCAI.

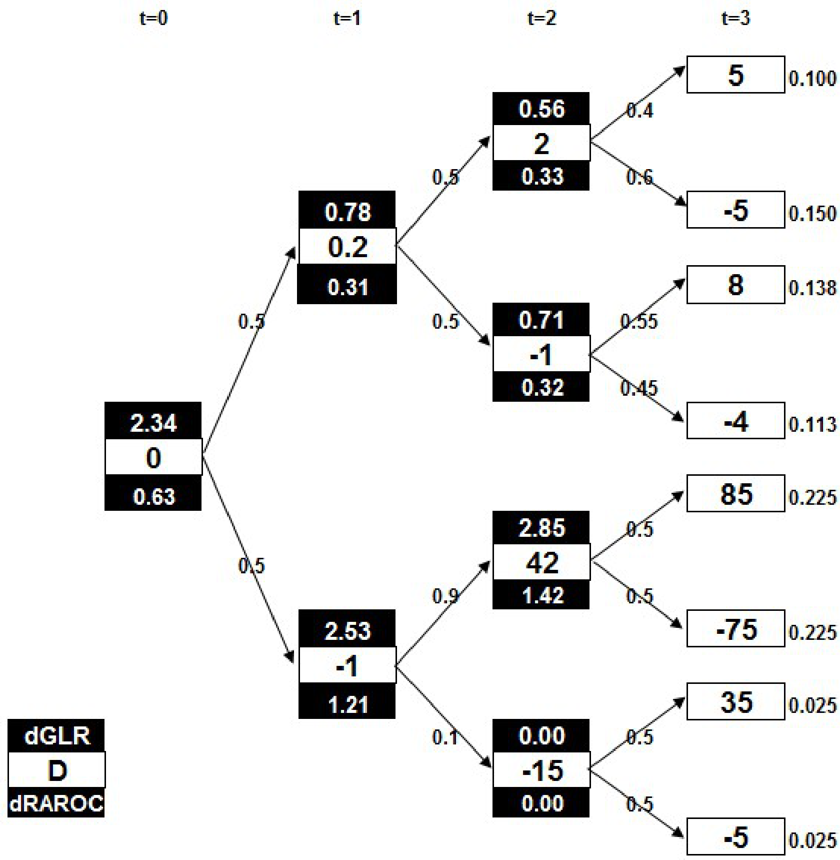


Figure 6.1: dRAROC vs dGLR

6.2 Optimal Portfolio Selection and DCAIs

Optimal portfolio selection problem amounts to determining a strategy for selection of a mix of financial securities that optimizes a given optimization criterion. Typically, optimization criteria are given in terms of utility functions, and the objective is to maximize the expected utility of terminal value of the portfolio.

Different utility functions usually give rise to different optimal portfolios. Dynamic acceptability indices can be used as a tool to select the 'best' optimal portfolio given a set of utility functions. In this section, we will apply our dGLR to discriminate between optimal portfolio strategies corresponding to various utility functions.

The definition of a utility function that we shall use here is the following:

Definition 6.2.1. *(cf. [19]) A utility function is a function* $u : \mathbb{R} \to \mathbb{R}$ *such that it is twice differentiable, concave, and strictly increasing.*

From now on we fix an investment horizon T and a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t,\ t = 0, 1, \ldots, T)$ is the relevant filtration.

We consider a financial market consisting of one risky asset, whose price S_1 is an $\mathcal{F}$-adapted process, and the bank account with a constant price process $S_0(t) = 1$ (this corresponds to assuming zero interest rate). A trading strategy (or a portfolio) $H = (H_0, H_1)$ is a vector of $\mathbb{F}$-predictable stochastic processes[5] $H_n = \{H_n(t); t = 1, 2, \ldots, T\}, n = 0, 1$.

Assuming zero interest rate, $H_0(t)$ should be understood as the amount of money invested in the bank account from time $t-1$ to time t, whereas $H_1(t)$ is the number of units of risky asset S_1 that the investor holds from time $t-1$ to time t. Note also that $H_n(t) < 0$ corresponds to borrowing money from the bank; $H_1(t) < 0$ corresponds to selling short risky asset.

Definition 6.2.2. *(cf. [41]) Given a trading strategy* H*, the value process* $V = \{V_t; t \in \mathcal{T}\}$ *is a stochastic process defined as*

$$V_t := \begin{cases} H_0(1) + H_1(1)S_1(0), & t = 0 \\ H_0(t) + H_1(t)S_1(t), & t \geq 1\,. \end{cases} \tag{6.2}$$

Definition 6.2.3. *(cf. [41]) For a trading strategy* H *with the value process* V*, the dividend process* $D = \{D_t; t \in \mathcal{T}\}$ *is a stochastic process defined as* $D_t = V_t - V_{t-1}$ *for* $t = 1, 2, \cdots, T$ *with* $D_0 = 0$.

[5] A stochastic process H_n is said to be predictable with respect to the filtration $\mathbb{F}$ if each random variable $H_n(t)$ is measurable with respect to $\mathcal{F}_{t-1}$ for all $t = 1, 2, \cdots, T$.

Definition 6.2.4. *(cf. [41]) A trading strategy H is said to be self-financing if*

$$V_t = H_0(t+1) + H_1(t+1)S_1(t), \quad t = 1, 2, \cdots, T-1\,. \tag{6.3}$$

Intuitively, self-financing means that if no money is withdrawn from or added to the portfolio between $t = 0$ and $t = T$, then any change in the portfolio's value must be due to a gain or loss in the investments.

Traditionally, expected utility has been used as a performance measure for financial portfolios in the following sense.

Definition 6.2.5. *(cf. [41]) Given a self-financing trading strategy H, the expected utility of terminal wealth V_T is defined as,*

$$\mathbb{E}u(V_T) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) u(V_T(\omega))\,,$$

where $\mathbb{P}$ is the reference probability measure and u is the given utility function.

Denote by $\mathbb{H}$ the set of all self-financing trading strategies. Given an initial wealth v, the optimal portfolio problem is therefore to choose the optimal self-financing trading strategy H by solving the following optimization problem:

$$\begin{cases} \text{maximize} & \mathbb{E}u(V_T) \\ \text{subject to} & V_0 = v \\ & H \in \mathbb{H} \end{cases} \tag{*}$$

In each of the following examples, we shall consider a pair of utility functions and we shall solve the optimization problem (*) to derive two optimal portfolios. Then, we will apply dGLR to discriminate between the two.

We will denote by (H_0^*, H_1^*) optimal strategy corresponding to each utility function. The values of H_1^* are shown in every example. H_0^* can be calculated through H_1^* and initial wealth V_0 using (6.2) and (6.3) under the assumption of self-financing.

6.2.1 Two-Period Examples.

Example 6.2.1.

Table 6.3: Security Price Process

ω	$S_0(\omega)$	$S_1(\omega)$	$S_2(\omega)$	$\mathbb{P}(\omega)$
ω_1	11	13	20	0.25
ω_2	11	13	12	0.25
ω_3	11	8	12	0.25
ω_4	11	8	6	0.25

In this example, we consider a two-period model with four states: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The security price process and reference probability measure $\mathbb{P}$ are given in Table 6.3. Assume zero interest rate and the initial wealth $V_0 = 5$. We consider two classical utility functions: exponential utility function $u(v) = 1 - \exp\{-v\}$ (cf. [16]) and quadratic utility function $u(v) = v - \frac{1}{2}v^2$ (cf. [16]).

Solving (*) for the exponential utility by the general optimization method, we find the optimal investment strategy being given as $H_1^*(1)(\omega_i) = 0.137$, for $i = 1, 2, 3, 4$; $H_1^*(2)(\omega_i) = -0.116$, for $i = 1, 2$; $H_1^*(2)(\omega_i) = 0.116$, for $i = 3, 4$.

Solving (*) for the quadratic utility by the general optimization method, we find the optimal investment strategy being given as $H_1^*(1)(\omega_i) = -0.400$, for $i = 1, 2, 3, 4$; $H_1^*(2)(\omega_i) = 0.160$, for $i = 1, 2$; $H_1^*(2)(\omega_i) = -0.480$, for $i = 3, 4$.

Table 6.4 and Table 6.5 display the optimal value processes. We would like to determine which optimization criterion, the one corresponding to exponential utility or the one corresponding to quadratic utility is more preferred. In other words, we would like to be able to determine, which of the utility criteria ultimately leads to a better trade-off between the return and the risk of the optimal portfolio. However, it does not seem to be possible to make such determination by merely looking at the

Table 6.4: Optimal Portfolio Value Process for Exponential Utility

ω	$V_0(\omega)$	$V_1(\omega)$	$V_2(\omega)$
ω_1	5.000	5.275	4.466
ω_2	5.000	5.275	5.390
ω_3	5.000	4.588	5.050
ω_4	5.000	4.588	4.357

Table 6.5: Optimal Portfolio Value Process for Quadratic Utility

ω	$V'_0(\omega)$	$V'_1(\omega)$	$V'_2(\omega)$
ω_1	5.000	4.200	5.320
ω_2	5.000	4.200	4.040
ω_3	5.000	6.200	4.280
ω_4	5.000	6.200	7.160

Table 6.6: dGLR Process for Optimal Portfolio V

ω	$\text{dGLR}_0(\omega)$	$\text{dGLR}_1(\omega)$
ω_1	0.000	0.000
ω_2	0.000	0.000
ω_3	0.000	0.000
ω_4	0.000	0.000

Table 6.7: dGLR Process for Optimal Portfolio V'

ω	$\text{dGLR}_0(\omega)$	$\text{dGLR}_1(\omega)$
ω_1	0.476	0.000
ω_2	0.476	0.000
ω_3	0.476	2.000
ω_4	0.476	2.000

optimal values of the wealth process displayed in tables 6.4 and 6.5. Thus, we need to use another tool for this purpose. The tool we propose to use is the dGLR.

Recall that dGLR is a dynamic coherent acceptability index. Table 6.6 and Table 6.7 display values of the dGLR processes corresponding to the two optimal portfolios V and V'. By examining these two tables we observe that at each state, the dGLR for portfolio V' is greater than or equal to the dGLR for portfolio V. Hence, dGLR indicates that for this market conditioning, the optimal portfolio corresponding to the quadratic utility provides a better trade-off between risk and return than the one corresponding to the exponential utility.

Example 6.2.2.

In this example, we consider the same model as in Example 6.2.1, but with

Table 6.8: Security Price Process

ω	$S_0(\omega)$	$S_1(\omega)$	$S_2(\omega)$	$\mathbb{P}(\omega)$
ω_1	11	13	20	0.1
ω_2	11	13	12	0.2
ω_3	11	8	12	0.3
ω_4	11	8	6	0.4

different reference probability measure as shown in Table 6.8. Assume zero interest rate and the initial wealth $V_0 = 5$. We consider the same pair of utility functions: exponential utility function $u(v) = 1 - \exp\{-v\}$, and quadratic utility function $u(v) = v - \frac{1}{2}v^2$.

For the exponential utility, the optimal investment strategy is given as $H_1^*(1)(\omega_i) = -0.270$, for $i = 1, 2, 3, 4$; $H_1^*(2)(\omega_i) = 0.157$, for $i = 1, 2$; $H_1^*(2)(\omega_i) = 0.068$, for $i = 3, 4$, and for the quadratic utility, the optimal investment strategy is given as $H_1^*(1)(\omega_i) = -0.215$, for $i = 1, 2, 3, 4$; $H_1^*(2)(\omega_i) = 0.140$, for $i = 1, 2$; $H_1^*(2)(\omega_i) = 0.022$, for $i = 3, 4$.

Table 6.9: Optimal Portfolio Value Process for Exponential Utility

ω	$V_0(\omega)$	$V_1(\omega)$	$V_2(\omega)$
ω_1	5.000	4.460	5.559
ω_2	5.000	4.460	4.303
ω_3	5.000	5.810	6.082
ω_4	5.000	5.810	5.674

Table 6.10: Optimal Portfolio Value Process for Quadratic Utility

ω	$V_0'(\omega)$	$V_1'(\omega)$	$V_2'(\omega)$
ω_1	5.000	4.570	5.551
ω_2	5.000	4.570	4.429
ω_3	5.000	5.646	5.734
ω_4	5.000	5.646	5.602

Table 6.9 and Table 6.10 display the optimal value processes. Similarly to the previous example, it is not possible to determine which portfolio performs better (as a combined measure of both return and risk) by just looking across all numbers. Using

Table 6.11: dGLR Process for Optimal Portfolio V

ω	$\mathrm{dGLR}_0(\omega)$	$\mathrm{dGLR}_1(\omega)$
ω_1	3.664	0.000
ω_2	3.664	0.000
ω_3	3.664	Inf
ω_4	3.664	Inf

Table 6.12: dGLR Process for Optimal Portfolio V'

ω	$\mathrm{dGLR}_0(\omega)$	$\mathrm{dGLR}_1(\omega)$
ω_1	3.520	0.000
ω_2	3.520	0.000
ω_3	3.520	Inf
ω_4	3.520	Inf

dGLR (displayed in Table 6.11 and Table 6.12), we note that dGLR for portfolio V is greater than or equal to the dGLR for portfolio V'. Hence, for this specific example, dGLR indicates that the optimal portfolio corresponding to the exponential utility provides a better trade-off between risk and return than the one corresponding to the quadratic utility.

Conclusion 6.2.1. *By using dGLR, Example 6.2.1 indicates that quadratic utility function gives an optimal portfolio with better performance, whereas Example 6.2.2 shows that exponential utility function gives an optimal portfolio with better performance. We conclude that utility function can not be the unique factor in selecting the optimal portfolio.*

6.2.2 Four-Period Binomial Model. The binomial model is an important model for the price evolution of a risky asset.

At each period there are two possibilities: the asset price either goes up by the factor u ($u > 1$) or it goes down by the factor d ($0 < d < 1$). The probability of an up move during a period is equal to the parameter p, and the moves over time are independent of each other.

Example 6.2.3. *Consider a four-period binomial model with* $d = 0.95$, $u = 1.2$, $p = 0.5$ *and* $S_1(0) = 10$. *We assume zero interest rate and the initial wealth* $V_0 = 20$.

Two classical utility functions are examined in this example: exponential utility function $u(v) = 1 - \exp\{-v\}$ and logarithmic utility function $u(v) = \log(v)$ (cf. [16]).

As shown in [41], the optimization problem (*) can be solved by the backward induction method. For utility function $u(v) = 1 - \exp\{-v\}$, we find the optimal portfolio to be $H_1^*(t)(\omega_i) = \frac{13.86}{S_1(t-1)(\omega_i)}$ for all $t = 1, 2, 3, 4$ and $i = 1, 2, \cdots, 16$. For utility function $u(v) = \log(v)$, we find the optimal portfolio to be $H_1^*(t)(\omega_i) = 18.75$ for all $t = 1, 2, 3, 4$ and $i = 1, 2, \cdots, 16$.

Similarly to the previous example, by merely looking at the optimal value processes displayed in Figure 6.2, it is hard to determine which portfolio has better performance. Using dGLR, we find that at each node displayed in Figure 6.3, the dGLR corresponding to the logarithmic utility is greater than or equal to the dGLR corresponding to the exponential utility. Hence, dGLR indicates that the optimal portfolio selected by logarithmic utility provides a better trade-off between risk and return than the one corresponding to exponential utility.

Example 6.2.4. *We consider another four-period binomial model, but with a different set of parameters* $d = 0.92$, $u = 1.02$, $p = 0.65$ *and* $S_1(0) = 10$. *We still assume zero interest rate and the initial wealth* $V_0 = 20$.

The same pair of utility functions are examined in this example: exponential utility function $u(v) = 1 - \exp\{-v\}$ and logarithmic utility function $u(v) = \log(v)$.

For utility function, we find the optimal portfolio to be $H_1^*(t)(\omega_i) = \frac{-7.67}{S_1(t-1)(\omega_i)}$ for all $t = 1, 2, 3, 4$ and $i = 1, 2, \cdots, 16$. For utility function, we find the optimal portfolio to be $H_1^*(t)(\omega_i) = -9.375$ for all $t = 1, 2, 3, 4$ and $i = 1, 2, \cdots, 16$.

By merely looking across all the numbers the Figure 6.4 (optimal value pro-

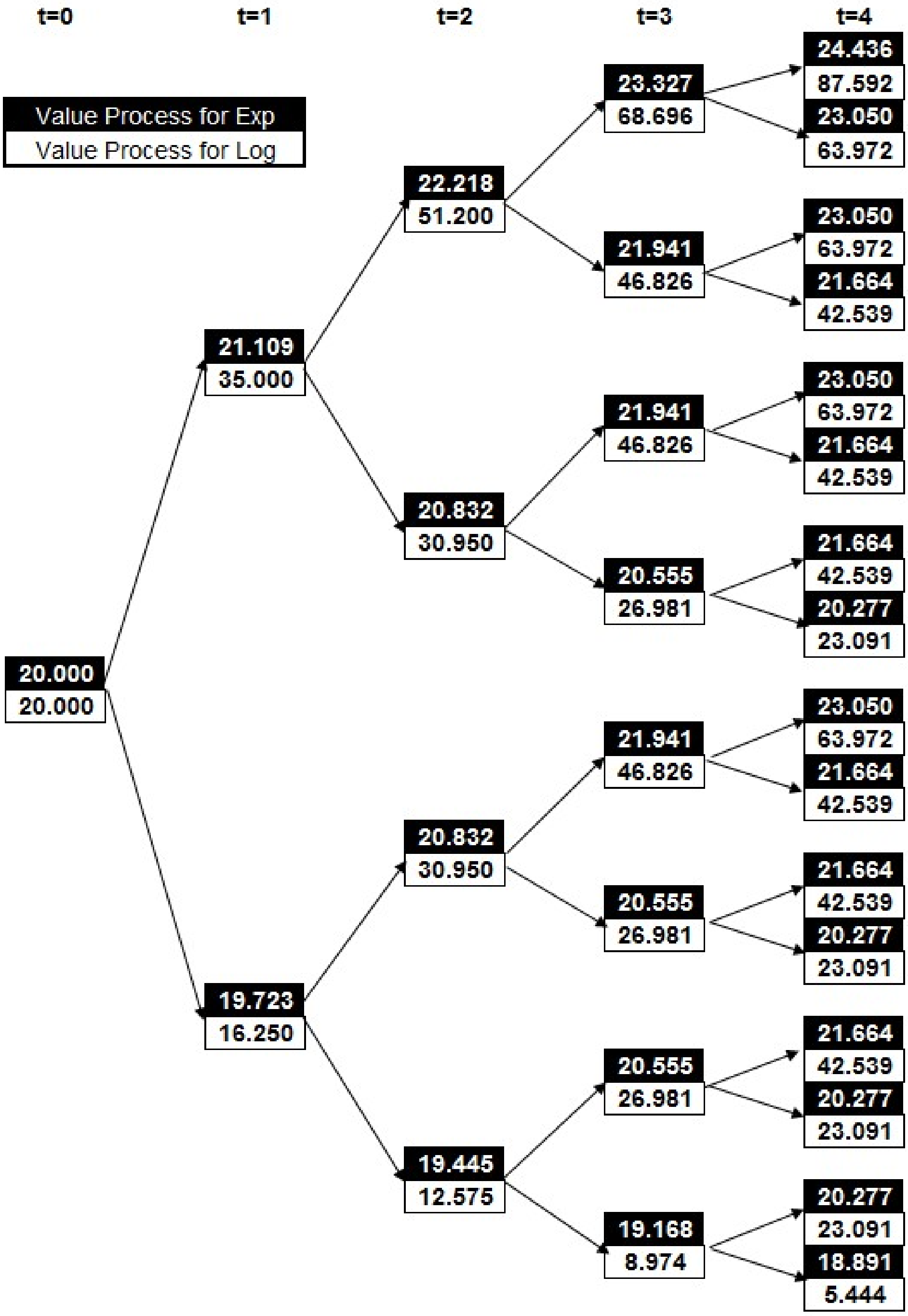


Figure 6.2: Value Processes for Optimal Portfolios

cesses), it is not possible to determine which utility function leads to a better optimal portfolio. However, by examining the dGLR processes shown in Figure 6.5, we note

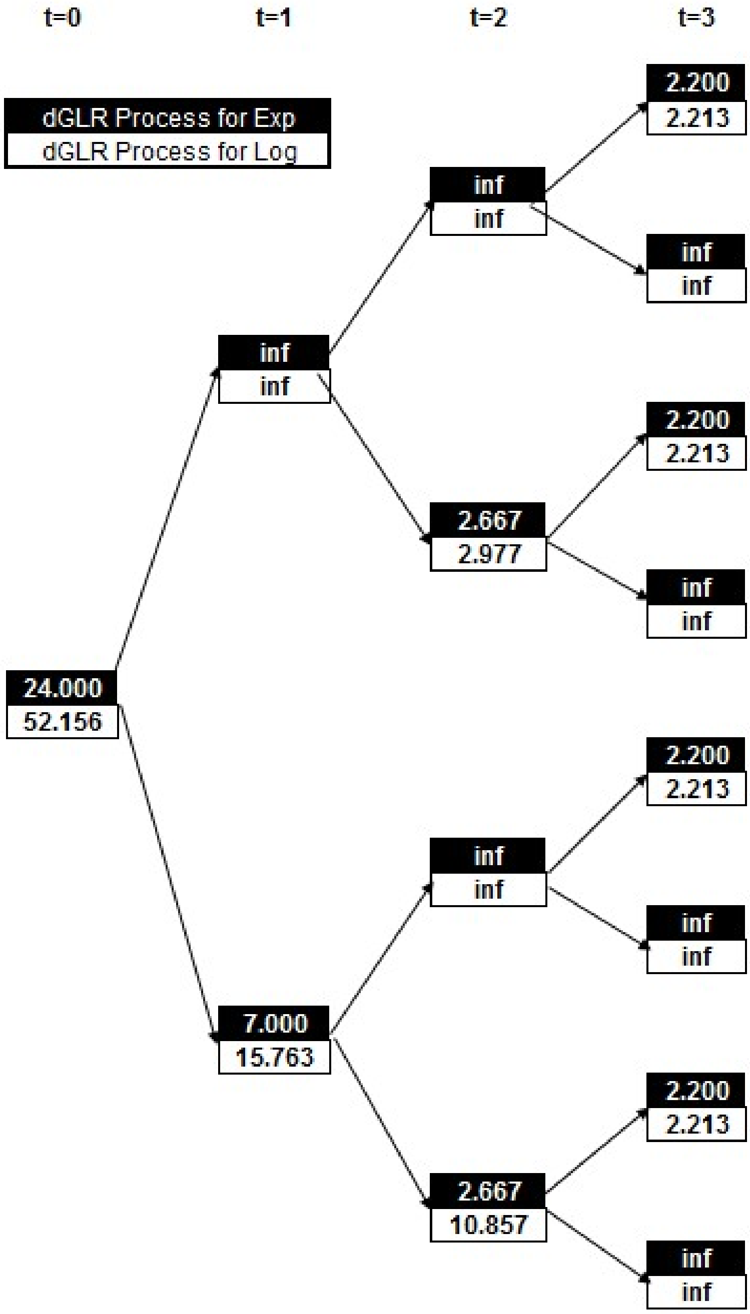


Figure 6.3: dGLR Processes for Optimal Portfolios

that at most nodes, the dGLR corresponding to the exponential utility is greater than or equal to the dGLR corresponding to the logarithmic utility. Hence, for this specific example, we conclude that the optimal portfolio corresponding to the exponential

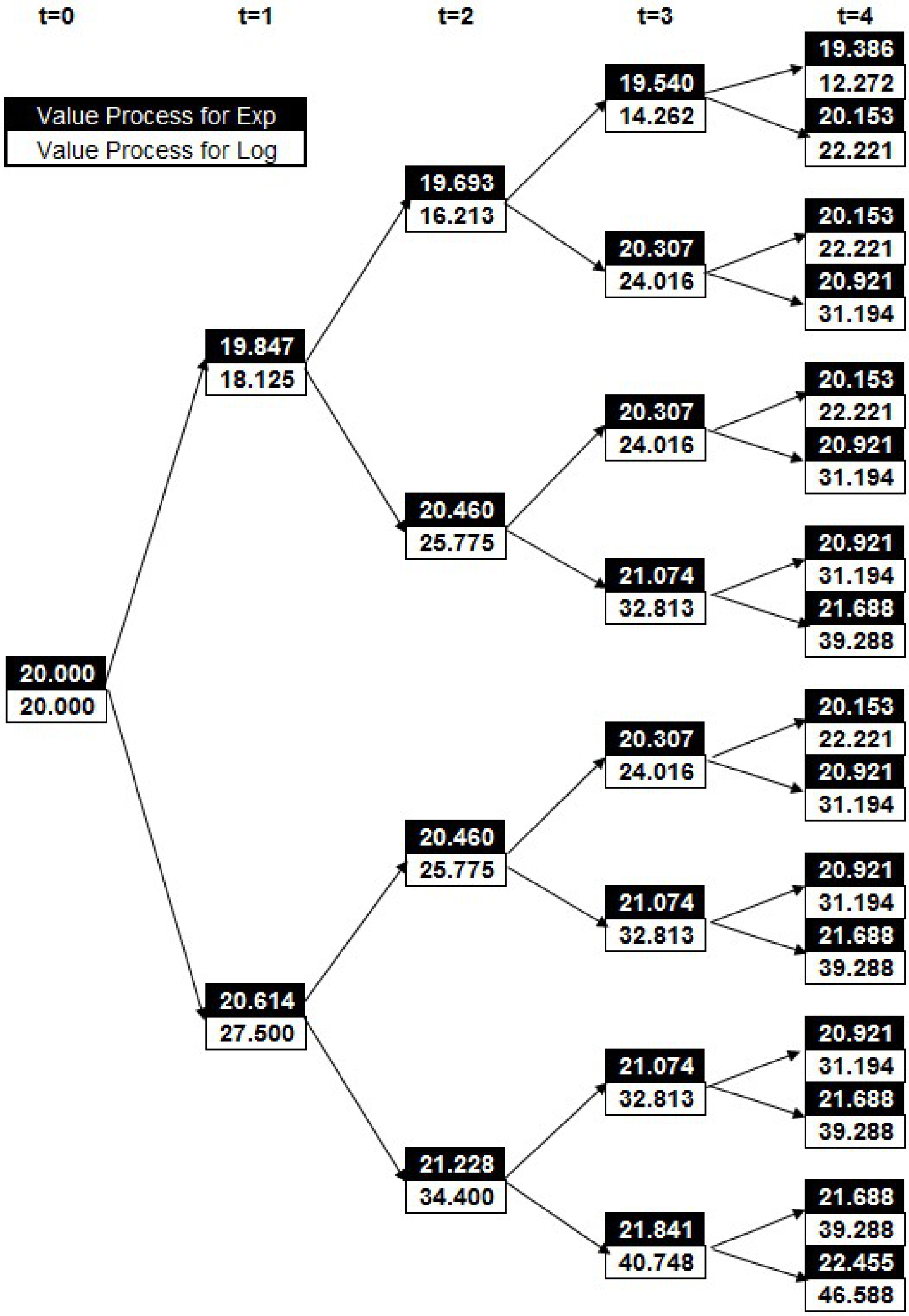


Figure 6.4: Value Processes for Optimal Portfolios

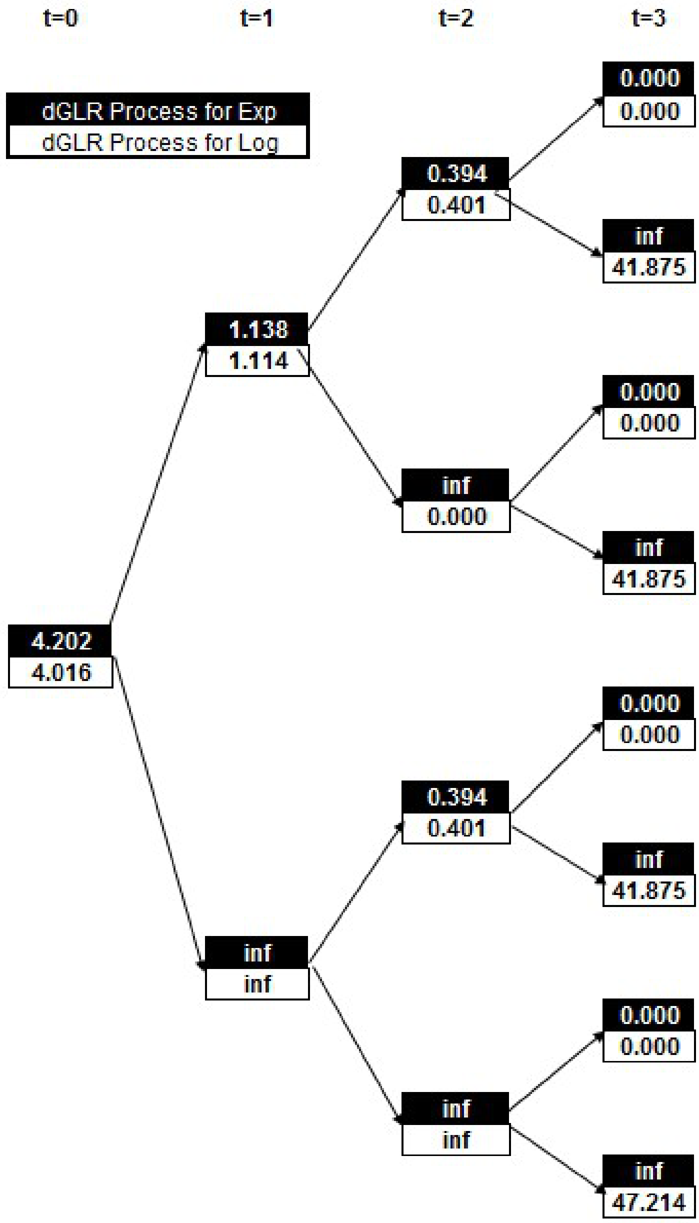


Figure 6.5: dGLR Processes for Optimal Portfolios

utility provides a better trade-off between risk and return.

Examples 6.2.3 and 6.2.4 together support Conclusion 6.2.1 that utility function can not be the unique factor in selecting the optimal portfolio.

6.2.3 Discrimination Between Optimal Portfolios Corresponding to Different Risk Aversion Coefficients. Now, we shall apply dGLR to discriminate between the optimal portfolios corresponding to different risk aversion coefficients for a certain class of utility functions. The risk aversion is a measure of investor's general preference for certainty over uncertainty, defined as follows:

Definition 6.2.6. *For a given utility function $u(v)$, the coefficient of relative risk aversion is defined as,*

$$R(v) = -\frac{vu''(v)}{u'(v)} .$$

We focus on discussion in case of isoelastic utility function $u(v) = \frac{v^{1-r}}{1-r}$ (cf. [40]). The relative risk aversion for such utility function is equal to r.

Example 6.2.5. *Assume a four-period single stock model with sixteen states given in Figure 6.6. We assume that at each node the probability for going up and going down is* 0.5*. Suppose that the investor starts with an initial wealth* $V_0 = 20$*.*

We find the optimal portfolios that correspond to the relative risk aversion coefficients $r = 3$ and $r = 9$, respectively.

By the backward induction method (see for instance [41] p.153), we can derive the holdings of assets of the optimal portfolio at each state and time instant:

$$H_1(t) = \frac{S_t\left[\left(\frac{S_t - S^d_{t+1}}{S^u_{t+1} - S_t}\frac{1-p}{p}\right)^{\frac{1}{r}} - 1\right]}{S^d_{t+1} - S_t - \left(S^u_{t+1} - S_t\right)^{\frac{r-1}{r}}\left(S_t - S^d_{t+1}\right)^{\frac{1}{r}}\left(\frac{1-p}{p}\right)^{\frac{1}{r}}} , \tag{6.4}$$

where S_t is the stock price at time t, p is the up probability and S^u_{t+1} (up price) and S^d_{t+1} (down price) are two successive prices.

The values of optimal portfolios are summarized in Figure 6.7. By merely looking at the table, it does not seem to be possible to determine which utility criteria ultimately leads to a better trade-off between the return and the risk of the optimal portfolio.

We will apply dGLR to analyze the acceptability of this two portfolios. The computed values of dGLR processes that correspond to these portfolios are displayed in Figure 6.8. By examining the table we observe that at all states, the dGLR for portfolio corresponding to $r = 9$ is greater than the dGLR for portfolio corresponding to $r = 3$. Hence, dGLR indicates that for this particular market, the optimal portfolio corresponding to relative risk aversion coefficient $r = 9$ provides a better trade-off between risk and return.

However, as next example will show, a higher risk aversion coefficient does not necessarily imply a better performance, with performance understood in the sense of dynamic acceptability indices such as dGLR.

Example 6.2.6. *We consider a different stock price evolution as in Figure 6.9, and a different probabilities of going to an up state to* 0.65 *and respectively* 0.35 *of going to a down state.*

Using (6.4), we find the optimal portfolios that correspond to the relative risk aversion coefficients $r = 3$ and $r = 9$. The results are displaced in Figure 6.10.

Similarly as above, by visual inspection of the values of the optimal portfolios, it is not clear what portfolio is more. However, computing dGLR (see Figure 6.11), we notice that the dGLR for portfolio corresponding to $r = 3$ is greater than or equal to the dGLR of the portfolio corresponding to $r = 9$. Hence, for this specific example, dGLR indicates that the optimal portfolio corresponding to $r = 3$ provides a better trade-off between risk and return.

Conclusion 6.2.2. *Examples 6.2.5 and 6.2.6 imply that risk aversion coefficient can not be the unique factor in selecting the optimal portfolio (from point of view of ranking portfolios by a dynamic acceptability index).*

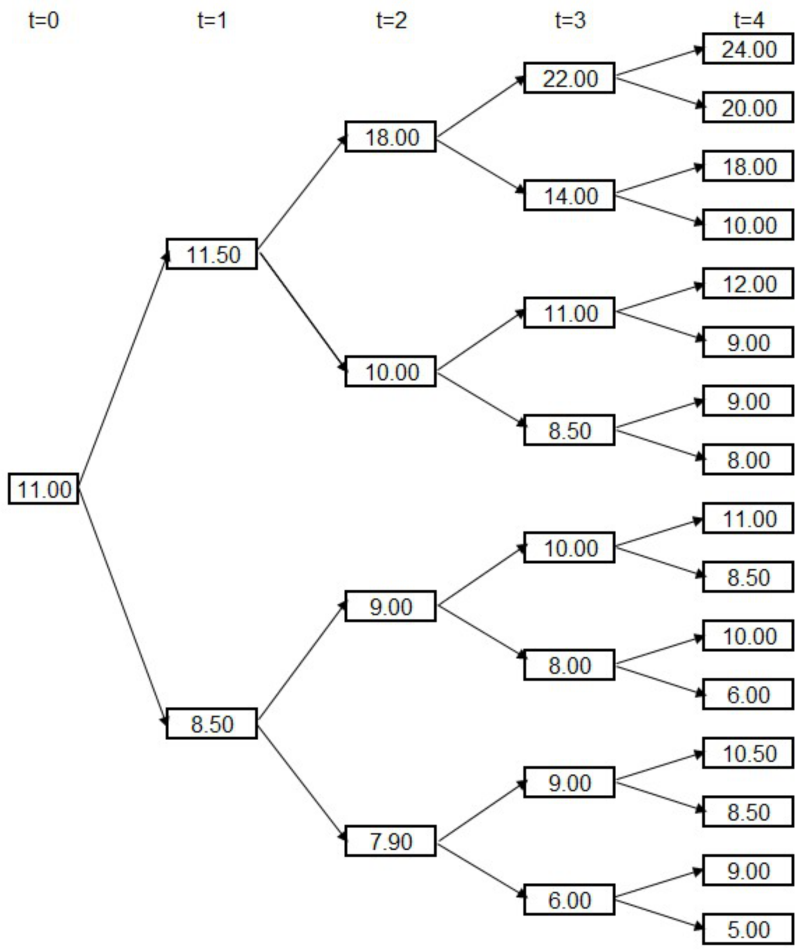


Figure 6.6: Security Price Process

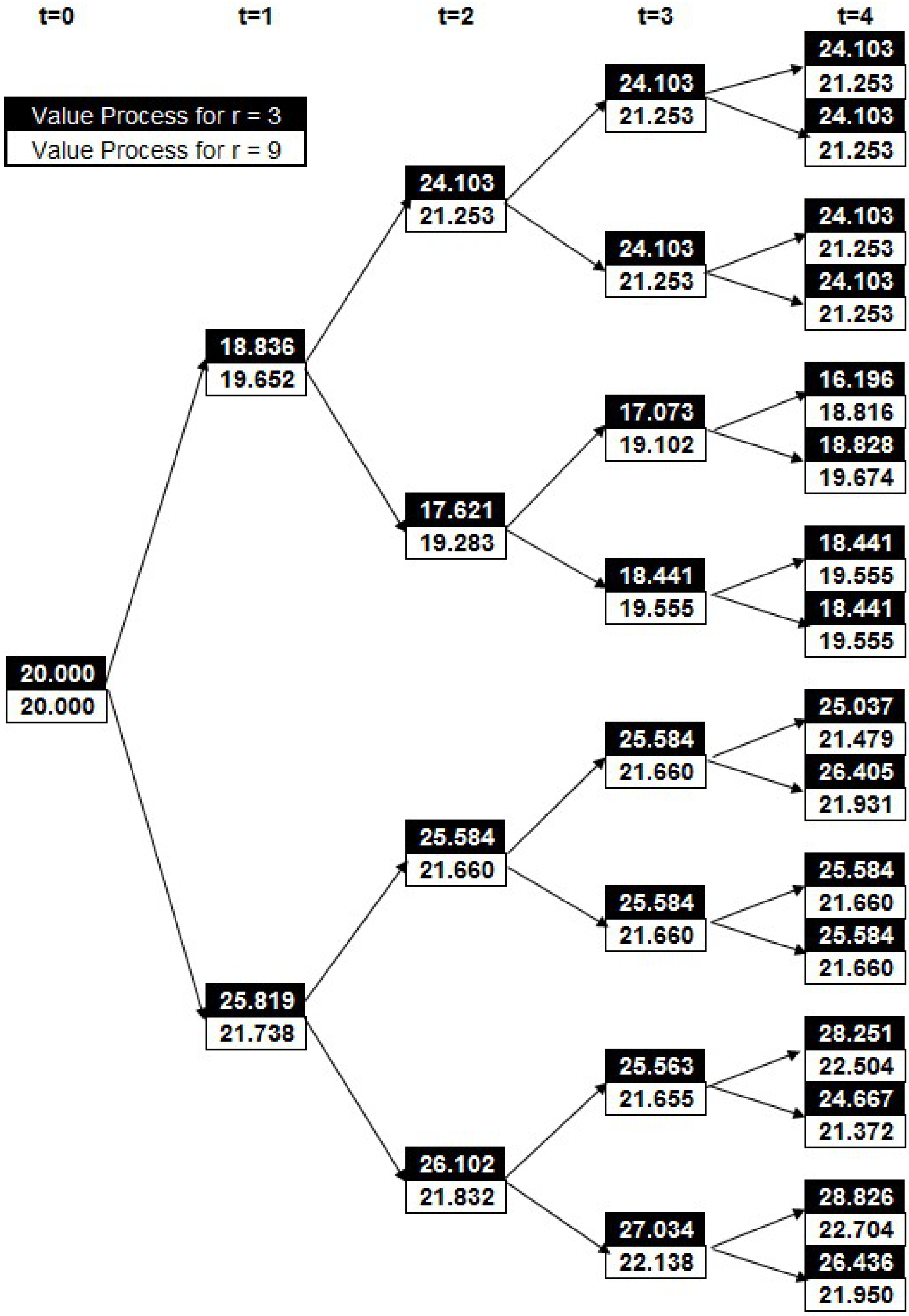


Figure 6.7: Value Processes for Optimal Portfolios

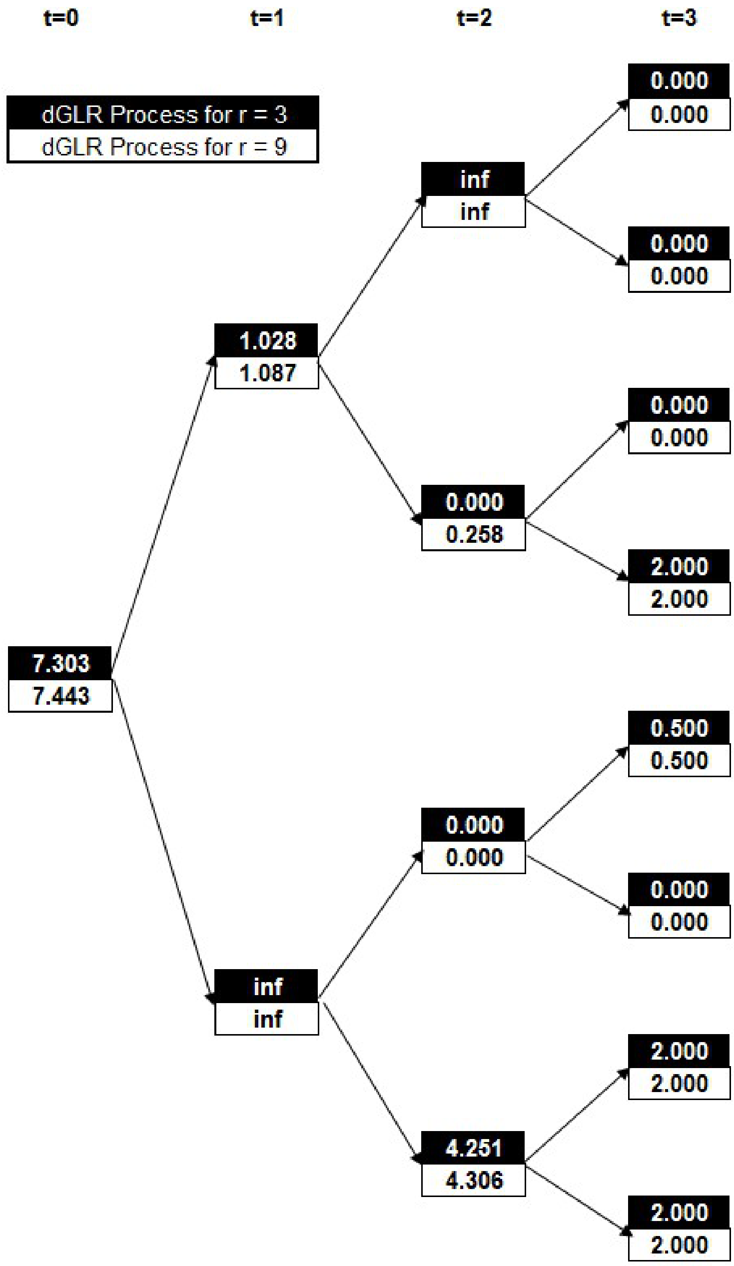


Figure 6.8: dGLR Processes for Optimal Portfolios

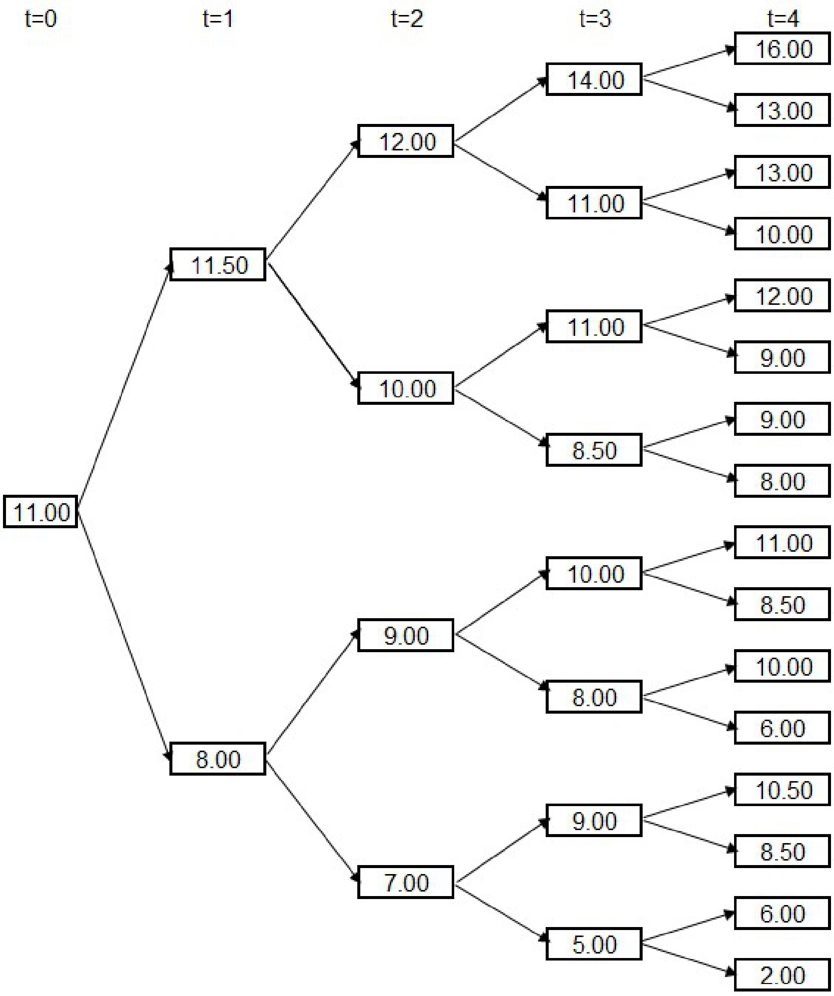


Figure 6.9: Security Price Process

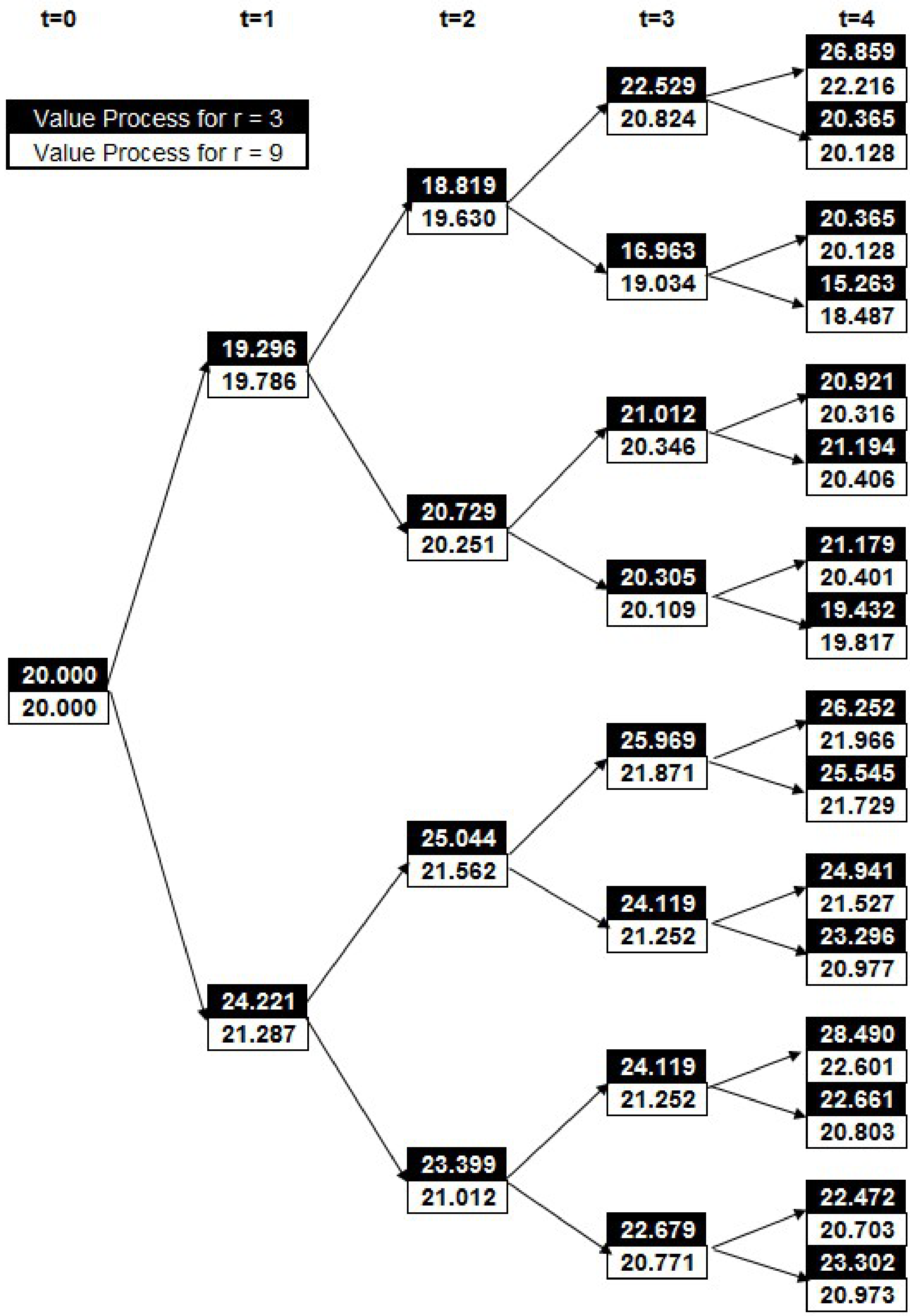


Figure 6.10: Value Processes for Optimal Portfolios

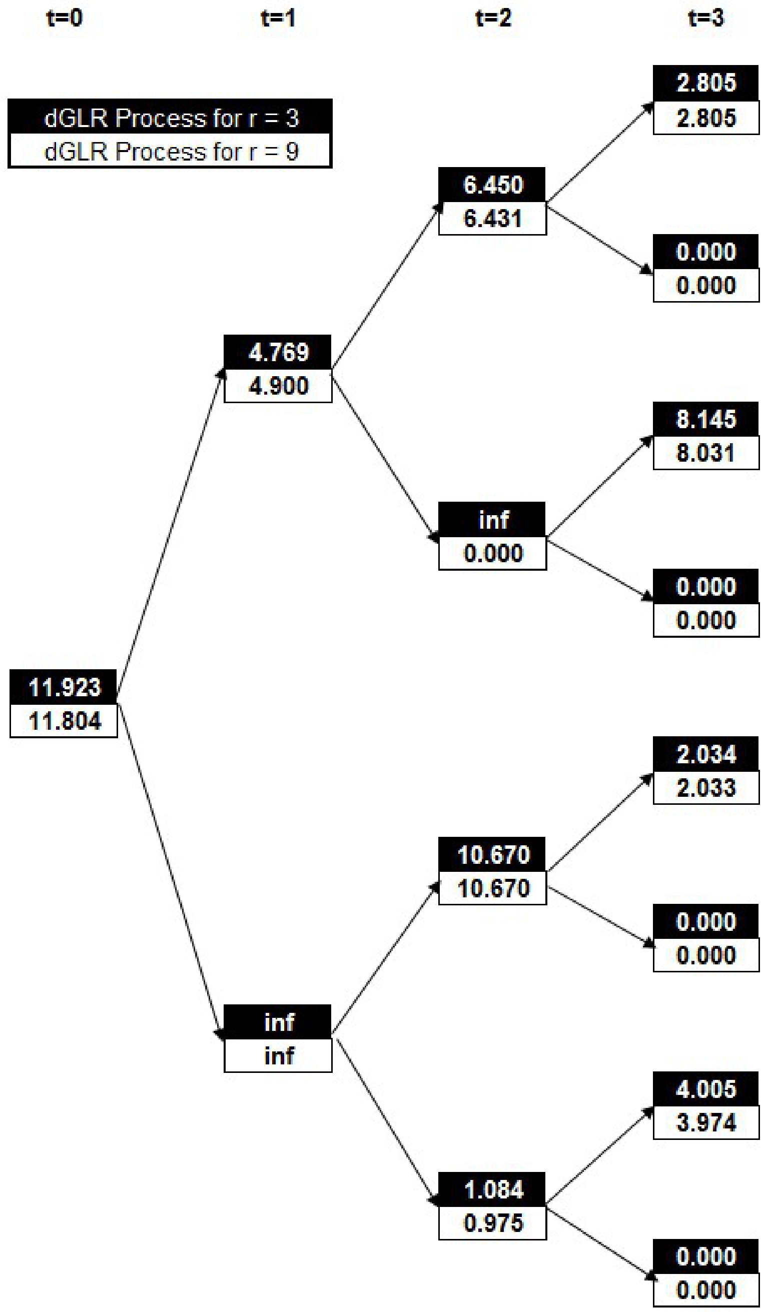


Figure 6.11: dGLR Processes for Optimal Portfolios

CHAPTER 7

FUTURE WORK

In this work, we studied dynamic coherent acceptability indices assuming a finite probability space and a finite discrete time space. An open research problem is to extend the theory developed here to the case of infinite probability space and/or continuous time.

Most of our results can be extended in a direct way to the case of general probability space. This requires techniques of general probability theory to be used. In particular, one will not be able to work with partitions any more, and the general theory of filtrations will need to be used instead.

The extension to continuous time space is much more delicate, as this will need to employ techniques from general theory of stochastic processes.

One of the main research objectives is to construct examples for DCAIs. In this thesis, we extended GLR to the dynamic setup in a natural way and showed that the dynamic GLR (dGLR) is indeed a DCAI. Besides GLR, some other coherent acceptability indices such as AIT, AIW, AIMIN, AIMAX, AIMINMAX, AIMAXMIN were introduced in [14]. Hence, a natural research problem is to look for their natural counterparties in dynamic framework, and then examine if these dynamic counterparties are DCAIs. Inspired by the representation theorem for DCAIs, another way to construct the examples is to find new examples for dynamically consistent sequences of sets of probability measures.

As has been recently demonstrated by Cherny and Madan [15] acceptability indices may play an important role in the area of so called *conic finance*, which studies, among others, the questions of formation of bid- and ask- prices in illiquid markets. Cherny and Madan use static acceptability indices in their approach to conic finance, which may lead to pricing, which is inconsistent in time. Therefore, an important research topic will be to study use of dynamic acceptability indices in conic finance, with a view at developing the theory of time-consistent pricing and hedging.

APPENDIX A

DYNAMIC COHERENT RISK MEASURES ACCORDING TO RIEDEL

In this appendix, we give an overview of DCRMs introduced by F. Riedel in [42]. The main difference between Riedel's and our theory on DCRMs is the dynamic consistency property. Corollary 3.3.1 has shown that our dynamic consistency (A7) is weaker than Riedel's (A7-I) (or (H7) in this appendix). Therefore, set of examples for our theory is richer. More importantly, the weaker dynamic consistency leads to an *if and only if* duality between dynamic acceptability indices and dynamic risk measures.

We assume the same mathematical setup and notations as in Section 3.1. To avoid technical problems, we also assume zero interest rate, whereas in [42] interest rate is not necessary to be zero.

Definition A.0.7. *A set of probability measures $\mathcal{Q} \subset \mathcal{P}^e$ is dynamic consistent if it is of full-support*[6] *and for all $t \in \mathcal{T}$, $X \in \mathcal{G}$,*

$$\min_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] = \min_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[\min_{\mathbb{M}\in\mathcal{Q}} \mathbb{E}_{\mathbb{M}}[X|\mathcal{F}_{t+1}]\big|\mathcal{F}_t\big]\,.$$

We shall mention that the above definition is shown as a lemma in [42], which is equivalent to the original definition of dynamic consistent set of probability measures in [42].

Definition A.0.8. *Dynamic coherent risk measure is a function $\rho : \mathcal{T} \times \mathcal{D} \times \Omega \to \mathbb{R}$ that satisfies the following properties:*

(H1) Adaptiveness. *For all $t \in \mathcal{T}$ and $D \in \mathcal{D}$, $\rho_t(D)$ is $\mathcal{F}_t$-measurable;*

(H2) Independence of the past. *For all $D, D' \in \mathcal{D}$ and $t \in \mathcal{T}$: if $D_s(\omega) = D'_s(\omega)$ for all $s \geq t$ and all $\omega \in \Omega$, then $\rho_t(D,\omega) = \rho_t(D',\omega)$;*

(H3) Monotonicity. *Given $D, D' \in \mathcal{D}$, if $D_t(\omega) \geq D'_t(\omega)$ for all $t \in \mathcal{T}$, $\omega \in \Omega$, then $\rho_t(D,\omega) \leq \rho_t(D',\omega)$ for all $\omega \in \Omega$;*

---

[6]Recall the Definition 4.1.2

(H4) Translation invariance. $\rho_t(D + m1_{\{s\}}) = \rho_t(D) - m$ *for every* $t \in \mathcal{T}$, $D \in \mathcal{D}$, $\mathcal{F}_t$*-measurable random variable* m*, and all* $s \geq t$*;*

(H5) Homogeneity. $\rho_t(\lambda D, \omega) = \lambda \rho_t(D, \omega)$ *for all* $\lambda > 0$, $D \in \mathcal{D}$, $t \in \mathcal{T}$*, and* $\omega \in \Omega$*;*

(H6) Subadditivity. $\rho_t(D + D', \omega) \leq \rho_t(D, \omega) + \rho_t(D', \omega)$ *for all* $t \in \mathcal{T}$, $D, D' \in \mathcal{D}$*, and* $\omega \in \Omega$*;*

(H7) Dynamic consistency. *For all times* $t = 0, \ldots, T-1$ *and positions* $D, D' \in \mathcal{D}$ *with* $D_t = D'_t$ *the following holds true:* $\rho_{t+1}(D, \omega) = \rho_{t+1}(D', \omega)$ *for all* $\omega \in \Omega$ *implies* $\rho_t(D, \omega) = \rho_t(D', \omega)$ *for all* $\omega \in \Omega$*;*

(H8) Relevancy. *For all* $t \in \mathcal{T}$, $P_i^t \in \Upsilon^t$, $\omega, \bar{\omega} \in P_i^t$*, and* $s \geq t$*,*

$$\rho_t(-1_{\{s\}} 1_{\{\bar{\omega}\}})(\omega) > 0\,.$$

In [42] the author established the following representation theorem for DCRMs in terms of a closed, convex and dynamic consistent set of probability measures.

Theorem A.0.1. ρ *is a dynamic coherent risk measure if and only if there exists a closed, convex, and dynamic consistent set of probability measures* $\mathcal{Q} \in \mathcal{P}^e$ *such that*

$$\rho_t(D) = \max_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[-\sum_{s=t}^{T} D_s \big| \mathcal{F}_t \big]\,.$$

APPENDIX B

TECHNICAL RESULTS

We present here some important technical results which are necessary throughout this thesis.

Lemma B.0.1. *Let f be a real-valued function defined on a linear vector space $\mathcal{X}$, and let us assuming that $\mathcal{B}$ is a subset of $\mathcal{X}$.*

(1) If $f(x) \geq c$ for all $x \in \mathcal{B}$, where $c \in \mathbb{R}$, then $\inf_{x\in\mathcal{B}} f(x) \geq c$. If $f(x) \leq \tilde{c}$ for all $x \in \mathcal{B}$, where $\tilde{c} \in \mathbb{R}$, then $\sup_{x\in\mathcal{B}} f(x) \leq \tilde{c}$.

(2)

$$\sup_{x\in\mathcal{B}} \big(f(x) + g(x)\big) \leq \sup_{x\in\mathcal{B}} f(x) + \sup_{x\in\mathcal{B}} g(x)\,,$$
$$\inf_{x\in\mathcal{B}} \big(f(x) + g(x)\big) \geq \inf_{x\in\mathcal{B}} f(x) + \inf_{x\in\mathcal{B}} g(x)\,.$$

Proof. (1) is a direct result from the definitions of infimum and supremum.

(2) is the direct result of (1) by noting that for all $x \in \mathcal{B}$,

$$f(x) + g(x) \leq \sup_{x\in\mathcal{B}} f(x) + \sup_{x\in\mathcal{B}} g(x)\,,$$
$$f(x) + g(x) \geq \inf_{x\in\mathcal{B}} f(x) + \inf_{x\in\mathcal{B}} g(x)\,.$$

□

Lemma B.0.2. *If f is a real valued function, increasing and continuous on an open interval (a, b), such that*

$$\sup\{x \in (a, b) : f(x) \leq 0\} = x_0 \in (a, b)\,,$$

then $f(x_0) = 0$.

Proof. If $f(x_0) < 0$, since $x_0 \in (a, b)$, $f(x)$ is continuous at x_0, and there exists $\eta > 0$ such that $x_0 + \eta \in (a, b)$ and $f(x_0 + \eta) - f(x_0) < -\frac{1}{2} f(x_0)$. This implies that

$f(x_0 + \eta) < \frac{1}{2} f(x_0) < 0$. Then,

$$\sup\{x \in (a, b) : f(x) \leq 0\} \geq x_0 + \eta > x_0 \, .$$

This contradicts the fact that $\sup\{x \in (a, b) : f(x) \leq 0\} = x_0$. Hence, $f(x_0) \geq 0$.

If $f(x_0) > 0$, there exists $\epsilon > 0$ such that $f(x_0 - \epsilon) - f(x_0) > -\frac{1}{2} f(x_0)$, which implies $f(x_0 - \epsilon) > \frac{1}{2} f(x_0) > 0$. Then, since $f(x)$ is increasing,

$$\sup\{x \in (a, b) : f(x) \leq 0\} \leq x_0 - \epsilon < x_0 \, ,$$

which contradicts the fact that $\sup\{x \in (a, b) : f(x) \leq 0\} = x_0$. Hence, $f(x_0) = 0$. □

Finally, we present the *Separation Hyperplane Theorem*, which we use to prove the representation theorems for dynamic coherent risk measure and dynamic coherent acceptability index. For our purpose, we present a simplified version of this theorem, and for more details, see for instance Chapter 3 in [45].

Definition B.0.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of n-dimensional space $\mathbb{R}^n$. The sets $\mathcal{A}$ and $\mathcal{B}$ are called strongly separated if there exists a $p \in \mathbb{R}^n$ such that*

$$\inf_{x \in \mathcal{A}} p \cdot x > \sup_{y \in \mathcal{B}} p \cdot y \, .$$

Theorem B.0.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint nonempty subsets of $\mathbb{R}^n$. If both $\mathcal{A}$ and $\mathcal{B}$ are closed and convex, then there exists a nonzero $p \in \mathbb{R}^n$ that strongly separates $\mathcal{A}$ and $\mathcal{B}$.*

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