TOPICS IN COUNTERPARTY RISK AND DYNAMIC CONIC FINANCE

BY

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ABSTRACT

This thesis consists of three essays about modeling counterparty risk and pricing derivative securities.

In the first essay, we analyze the counterparty risk embedded in CDS contracts, in presence of a bilateral margin agreement. We focus on the pricing of collateralized counterparty risk, and we derive the bilateral Credit Valuation Adjustment (CVA), unilateral Credit Valuation Adjustment (UCVA), and Debt Valuation Adjustment (DVA). We propose a model for the collateral by incorporating all related factors such as the thresholds, haircuts and margin period of risk. We derive the dynamics of the bilateral CVA in a general form with related jump martingales. Counterparty risky and the counterparty risk-free spread dynamics are derived and the dynamics of the Spread Value Adjustment (SVA) is found as a consequence. We finally employ a Markovian copula model for default intensities and illustrate our findings with numerical results.

In the second essay we address the issue of computation of the bilateral CVA under rating triggers in presence of ratings-linked margin agreements. We consider collateralized OTC contracts, that are subject to rating triggers, between two parties – an investor and a counterparty. Moreover, we model the margin process as a function of the credit ratings of the counterparty and the investor. We employ a Markovian approach for modeling of the rating transitions and of the default probabilities of the counterparties. In this framework, we derive the representation for bilateral CVA. We also introduce a new component in the decomposition of the counterparty risky price: namely the rating valuation adjustment (RVA) that accounts for the rating triggers. We consider several dynamic collateralization schemes where the margin thresholds are linked to the credit ratings of the counterparties. We account for the rehypothecation risk in the presence of independent amounts. Our results are

illustrated in terms of a CDS contract and an IRS contract.

In the third essay, we study the problem of pricing in incomplete markets with risk measures and acceptability indices. We propose a model for finding the dynamic ask and bid prices of derivative securities using Dynamic Coherent Acceptability Indices (DCAI) in the presence of transaction costs. In this framework, we define and prove a representation theorem for dynamic bid ask prices. We show that our prices can be computed using dynamic Gain-Loss Ratio (dGLR), which is a DCAI. To illustrate our results, we provide several numerical examples, by pricing barrier options with dGLR.

CHAPTER 1

INTRODUCTION

Pricing, hedging, and risk measurement are the fundamental keystones of the modern mathematical finance. This thesis is motivated by three problems arising from these foundational concepts. In the first part, we study the problem of quantifying the counterparty risk in CDS contracts. The second part addresses the problem of counterparty risk assessment in the presence of rating triggers on the underlying contracts. In the final part, we consider the problem of pricing and hedging derivatives in markets with transaction costs using acceptability indices.

Counterparty risk modeling has gained paramount importance since the financial crisis in 2008. As it is noted in Benjamin [Ben10], just shy of one-third of the losses in the crisis were actually due to realized default events, whereas about two-third were due to mark-to-market losses associated with the counterparty credit risk arising from the OTC derivatives transactions. Evidently, this highlights the significance of the role of the counterparty credit risk in the financial world. As a result, accurate and efficient measurement, mitigation and hedging of the counterparty credit risk have been the main focus of attention between the market participants. Quantification of the counterparty risk requires the computation of potential future exposures, expected exposures and relevant price adjustments, by considering the default risks of the counterparties as well as their own default probabilities. One of the main objectives of the counterparty risk measurement is to calculate the Credit Valuation Adjustment (CVA). CVA is defined as the difference between the values of a portfolio of OTC contracts, with and without considering the counterparty risk. Essentially, CVA specifies the price of the counterparty risk, and indicates the expected loss or gain incurred in the case of a default of a counterparty or in the case of an own default. In order to set limits against these losses originating from the presence of the counterparty risk, financial institutions utilize various techniques that allow them to mitigate and hedge their counterparty exposure.

Mitigation of counterparty risk involves incorporation of protective factors such as collateralization, netting, and break clauses. Collateralization is a procedure between two parties in a financial contract, where the borrower pledges an asset to the lender as a reassurance against his default. In case of swap contracts, collateralization is carried out bilaterally, thereby reducing the exposure for both counterparties. Netting is the practice of mutually settling the financial transactions between two counterparties to a net amount by canceling out the transactions having positive value with the ones having negative value. Furthermore, netting is an effective tool for mitigating the counterparty risk by reducing the overall exposure as well as the operational risks. On the other hand, break clauses are used to decrease the counterparty exposure by imposing optional or mandatory termination of the underlying contract whenever a predefined termination event occurs. Such events are often defined by incorporating credit rating triggers into the underlying contracts, so that the termination event occurs if one of the counterparties' credit rating decreases a threshold level. These provisions provide protection against the losses associated with the default events which occur after a termination event. Finally, hedging of counterparty risk is performed in virtue of trading securities such as CCDS (Contingent CDS), CDS, and IRS contracts. Although neither CCDS nor CDS contracts completely hedge or eliminate the counterparty exposure, they offer an efficient way to transfer the default risk. Reducing and controlling the counterparty exposure (or in general CVA) is one of the main objectives of all major financial institutions. We present our contributions to these efforts in the first two chapters of this thesis.

The importance of risk measurement and management is beyond the scope of counterparty credit risk, which specifically focuses on the risks arising from the probability of the defaults of the counterparties. It is also crucial for financial institutions to measure and manage the risks that are associated with the fluctuations in the security prices, interest rates, and the foreign exchange rates. Incorporating these market risk factors warrant the development of advanced risk measures. From Markowitz [Mar52] to Artzner et al. [ADEH97, ADEH99], the amount of academic literature related to market risk quantification, and in particular risk measures, is immense. In recent years, the theory of risk measures, in combination with the no-arbitrage pricing theory, have been used to price derivatives. The contribution of the last chapter of this thesis is to devise an analogous methodology for pricing derivative securities.

Since the pioneering work of Pliska and Harrison [HP81], the theory of arbitrage has developed significantly, with various generalizations and extensions to continuous-time and the presence of transaction costs. Indeed, the arbitrage pricing theory is the backbone of the entire field of pricing and hedging derivative securities. Broadly speaking, two important results form the foundation of the arbitrage pricing theory: the First Fundamental Theorem of Asset Pricing (FFTAP) and the Second Fundamental Theorem of Asset Pricing (SFTAP). The FFTAP asserts a necessary and sufficient condition for a financial market to not to exhibit arbitrage opportunities, in terms of the existence of a risk-neutral probability measure. On the other hand, the SFTAP provides a necessary and sufficient condition for a risk-neutral probability measure to be unique whenever there are no arbitrage opportunities, which eventually leads to the uniqueness of the price. Other approaches beyond the arbitrage theory have also been considered and studied in the recent literature, from which the no-good-deal pricing is particularly of our interest. Specifically, it is the relation between the no-good-deal pricing approach and the theory of risk measures, as well as the performance measures (i.e. acceptability indices), that makes the no-good-deal pricing approach specially important. As we stated earlier, our contributions to this field are presented in the last chapter of this thesis.

This thesis is organized as follows:

In Chapter 2, we address the problem of counterparty risk modeling in CDS contracts, in presence of a bilateral margin agreement. In Section 2.2, we first define the dividend processes regarding the counterparty risky and the counterparty risk-free CDS contract in case of a bilateral margin agreement. We also define and characterize the CVA, UCVA, and the DVA terms as well as the credit exposures such as PFE, EPE and ENE. We then prove the dynamics of the CVA in Section 2.3. Moreover, we find the fair spread adjustment term and its dynamics in Section 2.3.2. In Section 2.4, we simulate the collateralized exposures, and the CVA using our Markovian copula model of default dependence. The results of this chapter are based on Bielecki, Cialenco and Iyigunler [BCI11]. Parts of these results have also been presented in the AMS 2011 Spring Central Section Meeting, Iowa City, IA, March 18-20, 2011 and in the Stochastic Analysis in Finance and Insurance Workshop, University of Michigan, Ann Arbor, MI, May 17-20, 2011.

We consider the issue of computation of the bilateral credit valuation adjustment (CVA) under rating triggers in presence of ratings-linked margin agreements in Chapter 3. In section 3.2, we present a general framework for the valuation of collateralized credit valuation adjustment in the presence of rating triggers. Moreover, we study dynamic collateralization in Section 3.2.3 and rehypothecation in Section 3.2.4. We employ the Markovian copulae approach for modeling the joint rating transitions of the counterparty and the investor in Section 3.3. Finally, we present numerical results in case of a CDS contract and an IRS contract in Section 3.4. This chapter is based on Bielecki, Cialenco and Iyigunler [BCI12]. Part of this chapter has been presented in the 7th World Congress of the Bachelier Finance Society, Sydney, Australia, June 19-22, 2012.

In Chapter 4, the problem of pricing derivatives using dynamic coherent acceptability indices is examined. We define the no-arbitrage condition and the no-good-deal condition in our set-up, and then present the Fundamental Theorem of Good-Deal Pricing, in Section 4.2. Next, in Section 4.3.1, we introduce the definitions of the good-deal ask and bid prices, and proceed by proving a representation theorem for them. Finally, in Section 4.4, we derive an increasing family of dynamically consistent sets of probability measures corresponding to the dynamic Gain-Loss Ratio. We show that it satisfies some desirable properties, and then use it to compute the good-deal ask and bid prices of some Barrier options. Results of this chapter are based on Bielecki, Cialenco, Iyigunler and Rodriguez [BCIR12]. Parts of the results of this chapter have also been presented in the Workshop on the Mathematics of Financial Risk Management, Penn State University, University Park, PA, May 10-11, 2012.

CHAPTER 2

COUNTERPARTY RISK AND THE IMPACT OF COLLATERALIZATION IN CDS CONTRACTS

2.1 Introduction

Not very long after the collapse of prestigious institutions like Long-Term Capital Management, Enron and Global Crossing, the financial industry has again witnessed dramatic downfalls of financial institutions such as Lehman Brothers, Bear Stearns and Wachovia in 2008. These recent collapses have stressed out the importance of measuring, managing and mitigating counterparty risk appropriately.

Counterparty risk is defined as the risk that a party in an over-the-counter (OTC) contract will default and will not be able to honor its contractual obligations. Since the exchange-traded derivative contracts are subject to clearing by the exchange, counterparty risk arises from OTC derivatives only. The main challenge in counterparty risk modeling is that the exposures of OTC derivatives are stochastic and involve dependencies and systemic risk factors such as wrong way risks; the additional level of complexity is introduced by risk mitigation techniques such as collateralization, netting and additional termination events. Therefore, one needs to model potential future exposures and price the counterparty risk appropriately according to margin agreements that underlie the collateralization procedures.

Brigo and Capponi [BC09] focuses on a Gaussian copula model and study bilateral counterparty risk using a CIR++ intensity model. Brigo, Capponi, Pallavicini and Papatheodorou [BCPP11] extended this methodology to the collateralized contracts with an application to interest rate swaps under bilateral margin agreements. Hull and White [HW01] propose a static copula model and study unilateral counterparty risk on credit default swaps. Bielecki, Crepey, Jeanblanc and Zargari [BCJZ11] study unilateral counterparty risk with the absence of any margin agreements. Assefa, Bielecki, Crepey and Jeanblanc [ABCJ11] consider the portfolio of credit default swaps using the Markovian copula model and consider only fully collateralized contracts. Furthermore, Bielecki and Crepey [BC11] proposed a methodology for dynamically hedging the unilateral counterparty exposure using the same setup based on the min-variance hedging principles. The problem of hedging the counterparty risk is also studied by Kjaer [Kja11] using single-name credit default swaps and vanilla options on the underlying contracts. Jarrow and Yu [JY01] deal with the counterparty risk by using a dependence structure based on the default intensities of the counterparties. This approach, that also addresses the contagion risk issue, is also considered in Leung and Kwok [LK05]. Note that all these works mentioned above employ the reduced form modeling technology. However, structural models have also been used to model counterparty risk. Good examples of this approach are papers by Lipton and Sepp [LS09] and Blanchet-Scalliet and Patras [BSP11]. Moreover, Stein and Lee [SL11] study and illustrate credit valuation adjustment computations in the fixed income markets.

Alternatively, Albanese, Brigo and Oertel [ABO12] suggest several securitization frameworks for structuring the credit valuation adjustment between counterparties where they also consider an additional party as the margin lender. Recently, Crepey [Cre12a, Cre12b] proposed a general theoretical framework considering the borrowing and lending costs as a price adjustment, which is called the funding valuation adjustment. In addition, Pallavicini, Perini and Brigo [PDB12] studied the problem of incorporating the asymmetric collateral and funding rates, and developed a similar framework in a discrete time setup. The problem of incorporating the funding costs in counterparty risk modeling is also studied in various setups and applications by Morini and Prampolini [MP11], Fries [Fri11], Castagna [Cas11], Piterbarg [Pit10], Burgard and Kjaer [BK11a, BK11b, BK12], and by Fujii, Shimada and Takahashi [FST10].

Various issues regarding the simulation of credit valuation adjustments under margin agreements are studied by Pykhtin in [Pyk09]. The problem of fast computation of credit valuation adjustment sensitivities is considered by Capriotti, Lee and Peacock [CL11] and by Capriotti and Giles [CG12] using the method of algorithmic differentiation. On the other hand, Albanese, Bellaj, Gimonet and Pietronero [ABGP11] proposed a computational framework for the efficient simulation and valuation of counterparty risk.

The manuscripts by Cesari et al. [CAC+10] and Gregory [Gre09] provide thorough treatments of the methods and the applications used in practice regarding the counterparty risk.

In this chapter, we analyze the counterparty risk in a Credit Default Swap (CDS) contract in presence of a bilateral margin agreement. There are three risky names associated with the contract: the reference entity, protection seller (the counterparty) and the protection buyer (the investor). Contrary to the common approach which starts with defining the Potential Future Exposure (PFE) and derives the Credit Valuation Adjustment (CVA) as the price of the counterparty risk, we find the CVA as the difference between the market values of a counterparty risk-free and a counterparty risky CDS contract and deduct the relevant credit exposures accordingly. We consider the problem of bilateral counterparty risk assessment; that is, we consider the situation where the two counterparties of the CDS contract, i.e. the investor and the counterparty, are subject to default risk in a counterparty risky CDS contract.

We focus on collateralized contracts, where a bilateral margin agreement is in force as a vital risk mitigation tool, and it requires the counterparty and the investor to post collateral in case their exposure exceeds specific threshold values. We propose a model for the collateral by incorporating all related factors, such as thresholds, margin period of risk and minimum transfer amount. We derive the dynamics of the bilateral CVA, which are essential for dynamic hedging of counterparty risk. We also compute the decomposition of the fair spread for the CDS, and we analyze the so called Spread Value Adjustment (SVA). Essentially, SVA represents the adjustment to be made to the fair spread to incorporate counterparty risk into the CDS contract. More importantly, our results regarding the CVA and SVA representations are model-free. Therefore, our results can be used under any particular model for relevant quantities such as the default times, interest rates etc. Using the bilateral CVA formula, we derive relevant formulas for assessment of credit exposures, such as PFE, Expected Positive Exposure (EPE) and Expected Negative Exposure (ENE).

In our model, the dependence between defaults and the wrong way risk is represented by a Markovian copula framework that accounts for simultaneous defaults among the three names represented in a CDS contract.

2.2 Pricing Counterparty Risk: CVA, UCVA and DVA

We consider a standard CDS contract. A CDS contract is a swap contract between a *protection buyer* and a *protection seller* referring to an underlying credit name, called the reference name. The mechanics of a vanilla CDS contract can be summarized as follows: the protection buyer periodically pays a fee, which is called the *spread*, to the protection seller in exchange for a one-time payment made by the protection seller to the protection buyer if a pre-specified credit event (such as default) regarding the reference name occurs. We refer to the protection buyer as the investor, and to the protection seller as the counterparty. We label by 1 the counterparty, by 2 the investor, and by 3 the reference name. Traditionally, when pricing CDS contracts, only the reference name's default risk is considered. However, in reality both the counterparty and the investor may default before the maturity of the CDS contract, which is the source of the counterparty risk. The main goal of this chapter is to incorporate the default risks of the counterparty and the investor in the context of a CDS contract.

In what follows, we denote by τ_1, τ_2, and τ_3 the respective default times. These times are modeled as non-negative random variables on a underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. We let T and κ denote the maturity and the spread of our CDS contract, respectively. We assume the recovery at default covenant; that is, we assume that recoveries are paid at times of default.

We introduce right-continuous processes H_t^i by setting $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$ and we denote by $\mathbb{H}^i$ the associated filtrations so that $\mathcal{H}_t^i = \sigma\left(H_u^i : u \le t\right)$ for $i = 1, 2, 3$.

We assume that we are given a market filtration $\mathbb{F}$, and we define the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{H}^3$, that is $\mathcal{G}_t = \sigma\left(\mathcal{F}_t \cup \mathcal{H}_t^1 \cup \mathcal{H}_t^2 \cup \mathcal{H}_t^3\right)$ for any $t \in \mathbb{R}_+$. For each $t \in \mathbb{R}_+$ total information available at time t is captured by the σ-field $\mathcal{G}_t$. In particular, processes H^i are $\mathbb{G}$-adapted and the random times τ_i are $\mathbb{G}$-stopping times for $i = 1, 2, 3$.

Next, we define the first default time as the minimum of τ_1, τ_2 and τ_3 as $\tau = \tau_1 \wedge \tau_2 \wedge \tau_3$, and the corresponding indicator process defined as $H_t := \mathbb{1}_{\{\tau \le t\}}$. In addition, we define the first default time of the two counterparties: $\hat{\tau} := \tau_1 \wedge \tau_2$, and the corresponding indicator process $\hat{H}_t := \mathbb{1}_{\{\hat{\tau} \le t\}}$.

We also denote by B the savings account process, that is

$$B_t := e^{\int_0^t r_s ds},$$

where the $\mathbb{F}$-progressively measurable process r models the short-term interest rate. We also postulate that $\mathbb{Q}$ represents a martingale measure associated with the choice of the savings account B as a discount factor (or numeraire).

2.2.1 Dividend Processes and Marking-to-Market. We start by introducing the *counterparty-risk-free* dividend process D, which describes all cash flows associated with a counterparty-risk-free CDS contract;[1] that is, D does not account for the counterparty risk. Note that all cash flows and the prices are considered from the perspective of the investor.

Definition 2.2.1. *The cumulative dividend process D of a counterparty risk-free CDS contract maturing at time T is given as,*

$$D_t = \int_{]0,t]} \delta^1_u dH^3_u - \kappa \int_{]0,t]} \left(1 - H^3_u\right) du, \tag{2.1}$$

for every $t \in [0,T]$, where $\delta^1 : [0,T] \to \mathbb{R}$ is an $\mathbb{F}$-predictable processes.

Process δ^1 represents the loss given default (LGD); that is $\delta^1 = 1 - R^3_t$, where R^3 is the fraction of the nominal that is recovered in case of the default of the reference name. We assume unit nominal, for simplicity.

The ex-dividend price process of the counterparty risk-free CDS contract, say S, describes the current market value, or the Mark-to-Market (MtM) value of this contract.

Definition 2.2.2. *The ex-dividend price process S of a counterparty risk-free CDS contract maturing at time T is given by,*

$$S_t = B_t \mathbb{E}\left[\int_{]t,T]} B_u^{-1} dD_u \,\middle|\, \mathcal{G}_t \right], \qquad t \in [0,T]. \tag{2.2}$$

Remark 2.2.1. *Accordingly, we define the cumulative (dividend) price process, say $\widehat{S}$, of a counterparty risk-free CDS contract as*

$$\widehat{S}_t = S_t + B_t \int_{]0,t]} B_u^{-1} dD_u\,, \qquad t \in [0,T].$$

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[1]We shall refer to such contract as the clean contract.

Now, we are in position to define the dividend process D^C of a counterparty risky CDS contract, that is the CDS contract that accounts for the counterparty risk associated with the two counterparties of the contract.

Definition 2.2.3. *The dividend process D^C of a T-maturity counterparty-risky CDS contract is given as*

$$\begin{aligned} D_t^C = &\int_{]0,t]} C_u dH_u + \int_{]0,t]} \widetilde{\delta}_u^1 (1 - H_{u-})\, dH_u^3 + \int_{]0,t]} \widetilde{\delta}_u^2 (1 - H_{u-})\, dH_u^1 \\ &+ \int_{]0,t]} \widetilde{\delta}_u^3 (1 - H_{u-})\, dH_u^2 + \int_{]0,t]} \widetilde{\delta}_u^4 (1 - H_{u-})\, d[H^1, H^2]_u \\ &+ \int_{]0,t]} \widetilde{\delta}_u^5 (1 - H_{u-})\, d[\hat{H}, H^3]_u - \kappa \int_{]0,t]} (1 - H_u)\, du\,, \quad t \in [0, T]\,, \end{aligned} \tag{2.3}$$

where $\widetilde{\delta}^i : [0, T] \to \mathbb{R}$, $i = 1, 2, \ldots, 5$ are $\mathbb{F}$-predictable processes representing the close-out cash flows and $C : [0, T] \to \mathbb{R}$ is an $\mathbb{F}$-predictable process representing the collateral amount kept in the margin account.

A margin account is a contractual tool that supplements the CDS contract so as to reduce potential losses that may be incurred by one of the counterparties in case of the default of the other counterparty, while the CDS contract is still alive. For the detailed description of the mechanics of the collateral formation in the margin account we refer to Section 2.2.1.1 (see also [BC11]).

In case of any credit event associated with the collateralized CDS contract, the first cash flow that takes place is the "transfer" of the collateral amount; for example, in case when the underlying entity defaults at time $t = \tau = \tau_3$, (before any of the counterparties defaults) the collateral in the margin account is acquired by one of the counterparties (depending on the sign of C_τ). Thus, consistent with the convention of so called close-out cash flows (cf. [BC11]) we define the $\widetilde{\delta}^i$s as follows:

- We set $\widetilde{\delta}_t^1 = \delta_t^1 - C_t$. This is because after the collateral transfer the counterparty pays the remaining recovery amount $\delta_t^1 - C_t$.

- At time $t = \tau = \tau_1$, when the counterparty defaults, if the uncollateralized mark-to-market (MtM) of the CDS contract, $S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_t^1 - C_t$[2], is negative, then the investor closes out the position by paying the defaulting counterparty the uncollateralized MtM. If the uncollateralized MtM is positive, the investor closes out the position and receives a fraction R_1 of the uncollateralized MtM from the counterparty. Therefore, in this case, the close-out payment is defined as

$$\widetilde{\delta}_t^2 = R_1 \left(S_t + \mathbb{1}_{\{t=\tau_1\}}\delta_t^1 - C_t\right)^+ - \left(S_t + \mathbb{1}_{\{t=\tau_1\}}\delta_t^1 - C_t\right)^- .$$

- In the case of investor default, that is at time $t = \tau = \tau_2$, if the uncollateralized MtM is positive, that is if $S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_t^1 - C_t > 0$, the counterparty closes out the position by paying the uncollateralized MtM. If the uncollateralized MtM is negative, the counterparty receives a fraction R_2 of the uncollateralized MtM. Hence, the close-out payment is defined as

$$\widetilde{\delta}_t^3 = \left(S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_t^1 - C_t\right)^+ - R_2 \left(S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_t^1 - C_t\right)^- .$$

- If the investor and the counterparty default simultaneously at time $t = \tau = \tau_1 = \tau_2$, and if the uncollateralized MtM is negative, the counterparty receives a fraction R_2 of the uncollateralized MtM; however, if the uncollateralized MtM is positive, the investor receives a fraction R_1 of the uncollateralized MtM. Therefore, we set

$$\widetilde{\delta}_t^4 = -\left(S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_t^1 - C_t\right) .$$

- If $t = \tau = \hat{\tau} = \tau_3$, that is when the investor or the counterparty default simultaneously with the reference entity, the investor receives a fraction R_1 of the remaining recovery amount, $(\delta_t^1 - C_t)^+$, when the counterparty defaults.

[2]The term $\mathbb{1}_{\{t=\tau_3\}}\delta_t^1$ represents the exposure in case when the counterparty and the underlying entity default simultaneously.

Likewise, if the investor defaults, the counterparty receives a portion R_2 of the remaining recovery amount, $(\delta_t^1 - C_t)^-$. The close-out payment in joint defaults including the underlying entity has the form

$$\widetilde{\delta}_t^5 = -\left(\delta_t^1 - C_t\right).$$

Remark 2.2.2. *Notice that if $\tau_1 = \tau_2 = \infty$ we have $H_1 = H_2 = 0$ and $H = H_3$, which leads to*

$$D_t^C = \int_{]0,t]} C_u dH_u^3 + \int_{]0,t]} \widetilde{\delta}_u^1 \left(1 - H_{u-}^3\right) dH_u^3 - \kappa \int_{]0,t]} \left(1 - H_u^3\right) du$$

for all $t \in [0,T]$. Substituting $\widetilde{\delta}_u^1$, we get

$$\begin{aligned} D_t^C &= \int_{]0,t]} C_u dH_u^3 + \int_{]0,t]} (\delta_t^1 - C_u) \left(1 - H_{u-}^3\right) dH_u^3 - \kappa \int_{]0,t]} \left(1 - H_u^3\right) du \\ &= \int_{]0,t]} \delta_t^1 dH_u^3 - \kappa \int_{]0,t]} \left(1 - H_u^3\right) du, \end{aligned}$$

and therefore $D^C = D$.

We are now ready to define the price processes associated with a counterparty risky CDS contract.

Definition 2.2.4. *The ex-dividend price process S^C of a counterparty risky CDS contract maturing at time T is given as,*

$$S_t^C = B_t \mathbb{E}\left[\int_{]t,T]} B_u^{-1} dD_u^C \;\middle|\; \mathcal{G}_t \right], \quad t \in [0,T]. \tag{2.4}$$

The cumulative price process $\widehat{S}^C$ of a counterparty risky CDS contract is given by,

$$\widehat{S}_t^C = S_t^C + B_t \int_{]0,t]} B_u^{-1} dD_u^C, \quad t \in [0,T].$$

2.2.1.1 Bilateral Margin Agreement and Collateral Modeling. Collateralization is one of the most important techniques of mitigation of counterparty risk, and modeling the collateral process (also termed the margin call process) is of great practical importance (cf. [Alg09]). In this section, we propose a model to describe the formation of the required collateral amount at every time $t \in [0, T]$ with regard to bilateral margin agreements.[3] The following contractual parameters are essential in bilateral margin agreements and they are precisely defined in CSA documents.

Margin Period of Risk: The margin period of risk consists of several components. Firms usually monitor their exposure on a periodic basis and receive or make appropriate margin calls considering other collateral parameters. The frequency of this process is called the margin call period and it is typically one day. This period includes a number of phases such as computation, negotiation, verification and settlement of the margin call also with possible disputes during the process. According to the ISDA Master Agreement, in case of a potential default, the defaulting counterparty enters into a short forbearance period to recover from a potential default event where the collateral is pledged by the other firm. This time interval is called the cure period. If the default is uncured, liquidation process of the collateral assets starts (cf. [Int10b], page 26). This period mainly depends on the collateral portfolio selection, precise assessment of asset correlation and concentration risks as well as their liquidity, volatility and credit quality parameters. Therefore, the time interval from the last margin call plus the cure period until all collateral assets are liquidated and the resulting market risk is re-hedged is called the margin period of risk (cf. [Pyk09]); we shall denote it as Δ.

Threshold: The threshold is the unsecured credit exposure that both coun-

[3] A bilateral margin agreement is a contractual agreement governed by a Credit Support Annex (CSA), which is a regulatory part of the ISDA Master Agreement (cf. [Int05], page 34) describing the use of collateral which is either directly transferred between counterparties or held by a third party such as a clearing house (cf. [Int05], page 68).

terparties are willing to tolerate without holding any collateral. Bilateral margin agreements specify thresholds for both counterparties and require them to post collateral whenever the current credit exposure exceeds their thresholds (cf. [Int10b], page 11). These threshold amounts are defined in the related CSA documents and often set to react to the changes in the credit rating of the counterparties (cf. [Int10a], page 13). We will denote the counterparty and the investor's thresholds by Γ_{cpty} and Γ_{inv}, respectively. Since we perform our analysis from the point of view of the investor, we set the counterparty's threshold Γ_{cpty} to be a non-negative constant, and the investor's threshold Γ_{inv} to be a non-positive constant.

Minimum Transfer Amount: Margin calls for amounts smaller than the MTA are not allowed. The purpose of the MTA is to prevent calling small amounts; this is done so to avoid the operational costs involved in small transactions (cf. [Int10b], page 13). We assume the minimum transfer amount to be the same for the investor and the counterparty. We denote the minimum transfer amount by a positive constant θ.

Re-hypothecation Risk and Segregation: Collateral assets can be reused as a funding source on other derivatives transactions. This is known as rehypothecation. An investor (counterparty) can rehypothecate the collateral received from the counterparty (investor) by selling or lending out the assets to a third party, which dramatically increases the credit risk associated with the collateral. Elimination of this rehypothecation risk is essentially done by segregating the collateral to a third party, such as a clearing house. This procedure carries certain funding risks, since the counterparties will not be receiving funding benefit from the collateral posted, so they need to raise funding in connection with their transactions using their own funding rates.

According to the standard industry practice collateral amounts are adjusted at fixed tenor dates, termed *margin call dates*. Let us denote the margin call dates by $0 < t_1 < \cdots < t_n < T$. On each margin call date, if the exposure is above the counterparty's threshold, Γ_{cpty}, and the difference between the current exposure and the collateral amount is greater than the MTA the counterparty posts collateral and updates the margin account; otherwise, no collateral exchange takes place since the transfer amount is less than the MTA. Likewise, the investor delivers collateral on each margin call date, if the exposure is below investor's threshold, Γ_{inv}, and the difference between the current exposure and the collateral amount is greater than MTA (cf. [Int05], pages 52-56). Note that in this model collateral transfers are allowed only if it is greater than the MTA amount.

In accordance with the above discussion the collateral process is modeled as follows:

We set $C_0 = 0$. Then, for $i = 1, 2, \ldots, n$ we postulate that

$$\begin{aligned} C_{t_i+} = {} & \mathbb{1}_{\{S_{t_i} - \Gamma_{cpty} - C_{t_i} > \theta\}} (S_{t_i} - \Gamma_{cpty} - C_{t_i}) \\ & + \mathbb{1}_{\left\{S_{t_i} - \Gamma_{inv} - C_{t_i} < -\theta\right\}} (S_{t_i} - \Gamma_{inv} - C_{t_i}) + C_{t_i} \,, \end{aligned}$$

on the set $\{t_i < \hat{\tau}\}$, and it is constant on interval $(t_i, t_{i+1}]$. Moreover, $C_t = C_{\hat{\tau}}$ on the set $\{\hat{\tau} < t < \hat{\tau} + \Delta\}$.

Observe that the collateral increments at each margin call date $t_i < \hat{\tau}$ can now be represented as,

$$\begin{aligned} \Delta C_{t_i} & := C_{t_i+} - C_{t_i} \\ & = \mathbb{1}_{\{S_{t_i} - \Gamma_{cpty} - C_{t_i} > \theta\}} (S_{t_i} - \Gamma_{cpty} - C_{t_i}) + \mathbb{1}_{\left\{S_{t_i} - \Gamma_{inv} - C_{t_i} < -\theta\right\}} (S_{t_i} - \Gamma_{inv} - C_{t_i}) \,. \end{aligned}$$

One should also note that the collateral construction given in [Pyk09], which reads

$$C_t = \mathbb{1}_{\{S_t > \Gamma_{cpty} + \theta\}} \left(S_t - \Gamma_{cpty}\right) + \mathbb{1}_{\left\{S_t < \Gamma_{buy} - \theta\right\}} \left(S_t - \Gamma_{inv}\right) \,,$$

allows intermediate collateral updates that are smaller than MTA. In our case, we avoid this intricacy by defining the collateral process as a left-continuous, piecewise constant process.

Remark 2.2.3. *The collateral construction described above is cash based. The net cash value of the collateral portfolio is determined using haircuts.*

The haircut (or, valuation percentage) describes the amount that will be charged from a particular collateral asset. Effective value of the collateral asset is determined by subtracting the mark-to-market value of the asset multiplied by an appropriate haircut (cf. [Int05], page 67). Therefore, the haircuts applied to collateral assets should reflect the market risk on those assets. The haircut is defined as a percentage, where 0% haircut implies complete mark-to-market value of the asset to be used as collateral without any discounting. Government securities having high credit rating such as Treasury bonds and Treasury bills are usually subjected to 1% to 10% haircut, while for more risky, volatile or illiquid securities, such as a stock option, the haircut might be as high as 30%. The only asset that is not subjected to any haircut as collateral is cash where usually both parties mutually agree to the use of an overnight index rate (cf. [Int10b], page 27). The term valuation percentage is also used in Credit Support Annex (CSA) documents. The valuation percentage defines the amount that the market value of the asset is multiplied by to yield the effective collateral value of the asset. Hence, we have $VP_t = 1 - h_t$*, where* VP_t *is the valuation percentage and* h_t *is the total haircut applied to the collateral assets at time* t*. We will not go into the details of the formation of the haircut since it is either pre-determined in the CSA documents or related to market risk measures such as VaR of the collateral assets. (cf. [Int05], page 68). The main purpose of the haircut is to mitigate amortization or depreciation in the collateral asset value at the time of a default and in the margin period of risk. Moreover, the haircut should be updated as frequently as possible to* *reflect the changes in the volatility or liquidity of the collateral assets (cf. [Int05], page 63).*

Therefore, the total value of the collateral portfolio at time t is equal to $(1+h_t)C_t$, where h_t is the appropriate haircut applied to the collateral portfolio.

2.2.2 Bilateral Credit Valuation Adjustment. We are interested in the difference between the price processes S and S^C, representing the counterparty risk-free and the counterparty risky CDS contracts described above. As we stated before, this difference is called the CVA, and it indicates the price of the counterparty risk. In this section, we shall define the CVA of a CDS contract that is subject to a bilateral margin agreement. Moreover, we will prove a representation result for the CVA, which is essential for computational purposes.

Definition 2.2.5. *The bilateral Credit Valuation Adjustment process of a CDS contract maturing at time T is defined as*

$$CVA_t = S_t - S_t^C, \tag{2.5}$$

for every $t \in [0,T]$. The cumulative CVA is defined as,

$$\widehat{CVA}_t = \widehat{S}_{t\wedge\tau} - \widehat{S}^C_{t\wedge\tau},$$

for every $t \in [0,T]$.

We now present a representation of the bilateral CVA.

Proposition 2.2.1. *The bilateral CVA process on a CDS contract maturing at time T satisfies*

$$\begin{aligned} CVA_t = {} & B_t \mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_2\le T\}} B_\tau^{-1} (1-R_1) \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}} \delta_\tau^1 - C_\tau \right)^+ \,\middle|\, \mathcal{G}_t \right] \\ & - B_t \mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_2\le T\}} B_\tau^{-1} (1-R_2) \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}} \delta_\tau^1 - C_\tau \right)^- \,\middle|\, \mathcal{G}_t \right], \end{aligned} \tag{2.6}$$

for every $t \in [0,T]$.

Proof. We begin by observing that

$$\begin{aligned}
\int_{]t,T]} B_u^{-1}\widetilde{\delta}_u^1 (1 - H_{u-})\, dH_u^3 &= B_\tau^{-1}\widetilde{\delta}_\tau^1 \mathbb{1}_{\{t<\tau=\tau_3\le T\}}\,,\\
\int_{]t,T]} B_u^{-1}\widetilde{\delta}_u^2 (1 - H_{u-})\, dH_u^1 &= B_\tau^{-1}\widetilde{\delta}_\tau^2 \mathbb{1}_{\{t<\tau=\tau_1\le T\}}\,,\\
\int_{]t,T]} B_u^{-1}\widetilde{\delta}_u^3 (1 - H_{u-})\, dH_u^2 &= B_\tau^{-1}\widetilde{\delta}_\tau^3 \mathbb{1}_{\{t<\tau=\tau_2\le T\}}\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_{]t,T]} B_u^{-1} dD_u^C = {} & B_\tau^{-1}\widetilde{\delta}_\tau^1 \mathbb{1}_{\{t<\tau=\tau_3\le T\}} + B_\tau^{-1}\widetilde{\delta}_\tau^2 \mathbb{1}_{\{t<\tau=\tau_1\le T\}} + B_\tau^{-1}\widetilde{\delta}_\tau^3 \mathbb{1}_{\{t<\tau=\tau_2\le T\}}\\
& + B_\tau^{-1}\widetilde{\delta}_\tau^4 \mathbb{1}_{\{t<\tau=\tau_1=\tau_2\le T\}} + B_\tau^{-1}\widetilde{\delta}_\tau^5 \mathbb{1}_{\{t<\tau=\tau^*=\tau_3\le T\}} + B_\tau^{-1} C_\tau \mathbb{1}_{\{t<\tau\le T\}}\\
& - \kappa \int_{]t,T]} B_u^{-1} \mathbb{1}_{\{\tau>u\}} du\,. \qquad (2.7)
\end{aligned}$$

Using the definitions of the close-out cash-flows $\widetilde{\delta}_\tau^i$, $i = 1, \dots, 5$, we get from (2.7)

$$\begin{aligned}
\int_{]t,T]} B_u^{-1} dD_u^C = {} & B_\tau^{-1}\left(\delta_\tau^1 - C_\tau\right)\mathbb{1}_{\{t<\tau=\tau_3\le T\}} - \kappa \int_{]t,T]} B_u^{-1}\mathbb{1}_{\{\tau>u\}} du + B_\tau^{-1} C_\tau \mathbb{1}_{\{t<\tau\le T\}}\\
& + B_\tau^{-1}\Big(R_1\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^+ - \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^-\Big)\mathbb{1}_{\{t<\tau=\tau_1\le T\}}\\
& + B_\tau^{-1}\Big(\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^+ - R_2\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^-\Big)\mathbb{1}_{\{t<\tau=\tau_2\le T\}}\\
& - B_\tau^{-1}\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)\mathbb{1}_{\{t<\tau=\tau_2=\tau_3\le T\}} - B_\tau^{-1}\left(\delta_\tau^1 - C_\tau\right)\mathbb{1}_{\{t<\tau=\tau^*=\tau_3\le T\}}.
\end{aligned} \qquad (2.8)$$

Since

$$\mathbb{1}_{\{t<\tau\le T\}} = \mathbb{1}_{\{t<\tau=\tau_1\le T\}} + \mathbb{1}_{\{t<\tau=\tau_2\le T\}} + \mathbb{1}_{\{t<\tau=\tau_3\le T\}} - \mathbb{1}_{\{t<\tau=\tau_1=\tau_2\le T\}} - \mathbb{1}_{\{t<\tau^*=\tau_3\le T\}}\,,$$

using the equality

$$R_i\left(S_\tau - C_\tau\right)^+ - \left(S_\tau - C_\tau\right)^- + C_\tau = S_\tau - (1 - R_i)\left(S_\tau - C_\tau\right)^+$$

and observing that $\mathbb{1}_{\{\tau=\tau_3\}} S_\tau = 0$, we can rearrange the terms in (2.8) as follows,

$$\begin{aligned}\int_{]t,T]} B_u^{-1} dD_u^C &= B_\tau^{-1}\delta_\tau^1 \mathbb{1}_{\{t<\tau=\tau_3\leq T\}} - \kappa \int_{]t,T]} B_u^{-1}\mathbb{1}_{\{\tau>u\}} du \\ &\quad + B_\tau^{-1} S_\tau \left(\mathbb{1}_{\{t<\tau=\tau_1\leq T\}} + \mathbb{1}_{\{t<\tau=\tau_2\leq T\}} - \mathbb{1}_{\{t<\tau=\tau_1=\tau_2\leq T\}}\right)\mathbb{1}_{\{\tau\neq\tau_3\}} \\ &\quad - B_\tau^{-1}(1-R_1)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^+ \mathbb{1}_{\{t<\tau=\tau_1\leq T\}} \\ &\quad + B_\tau^{-1}(1-R_2)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^- \mathbb{1}_{\{t<\tau=\tau_2\leq T\}}.\end{aligned} \tag{2.9}$$

Now, combining (2.9) with (2.1) we see that

$$\begin{aligned}S_t^C &= B_t\mathbb{E}\left[\left(\mathbb{1}_{\{t<\tau=\tau_3\leq T\}} + \mathbb{1}_{\{\tau>T\}}\right)\int_{]t,T]} B_u^{-1} dD_u \,\middle|\, \mathcal{G}_t\right) \\ &\quad + B_t\mathbb{E}\Big(\left(\mathbb{1}_{\{t<\tau=\tau_1\leq T\}} + \mathbb{1}_{\{t<\tau=\tau_2\leq T\}}\right. \\ &\qquad \left. -\mathbb{1}_{\{t<\tau=\tau_1=\tau_2\leq T\}}\right)\mathbb{1}_{\{\tau\neq\tau_3\}}\Big)\mathbb{E}\left[\int_{]\tau,T]} B_u^{-1} dD_u \,\middle|\, \mathcal{G}_\tau\right] \,\Bigg|\, \mathcal{G}_t\Bigg] \\ &\quad - B_t\mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_1\leq T\}} B_\tau^{-1}(1-R_1)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^+ \,\middle|\, \mathcal{G}_t\right] \\ &\quad + B_t\mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_2\leq T\}} B_\tau^{-1}(1-R_2)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^- \,\middle|\, \mathcal{G}_t\right].\end{aligned} \tag{2.10}$$

From here, observing that

$$\begin{aligned}&\mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{\tau>T\}} + \mathbb{1}_{\{t<\tau=\tau_3\leq T\}} \\ &\quad + \left(\mathbb{1}_{\{t<\tau=\tau_1\leq T\}} + \mathbb{1}_{\{t<\tau=\tau_2\leq T\}} - \mathbb{1}_{\{t<\tau=\tau_1=\tau_2\leq T\}}\right)\mathbb{1}_{\{\tau\neq\tau_3\}} = 1,\end{aligned}$$

we get

$$\begin{aligned}S_t^C &= B_t\mathbb{E}\left[\int_{]t,T]} B_u^{-1} dD_u \,\middle|\, \mathcal{G}_t\right] \\ &\quad - B_t\mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_1\leq T\}} B_\tau^{-1}(1-R_1)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^+ \,\middle|\, \mathcal{G}_t\right] \\ &\quad + B_t\mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_2\leq T\}} B_\tau^{-1}(1-R_2)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^- \,\middle|\, \mathcal{G}_t\right],\end{aligned}$$

which is

$$S_t^C = S_t - B_t\mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_1\le T\}} B_\tau^{-1} \left(1-R_1\right) \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^+ \;\middle|\; \mathcal{G}_t \right]$$

$$+ B_t\mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_2\le T\}} B_\tau^{-1} \left(1-R_2\right) \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^- \;\middle|\; \mathcal{G}_t \right].$$

This proves the result. □

The result above shows that the CVA is the difference between the expected loss in case the counterparty defaults first and the expected loss in case the investor defaults first. It is more straightforward to compute the CVA using the representation proved above than computing $S - S^C$.

Remark 2.2.4. *Alternatively, the value of the bilateral CVA can be interpreted as the value of an exotic option. Indeed, the value of the CVA is equal to the sum of the values of a long position in a zero-strike call option on the uncollateralized amount and a short position in a zero-strike put option on the uncollateralized amount.*

2.2.2.1 Unilateral CVA and Debt Value Adjustment. The bilateral nature of the counterparty risk is a consequence of the default risk of the counterparty and the default risk of the investor. The values of potential losses associated with these two components are called unilateral CVA (UCVA) and debt value adjustment (DVA), respectively, and defined below. In practice, these two components are computed separately. This is the main reason why we need to consider UCVA and DVA components in this section.

Definition 2.2.6. *The Unilateral Credit Value Adjustment is defined as,*

$$UCVA_t = B_t\mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_1\le T\}} B_\tau^{-1} \left(1-R_1\right) \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^+ \;\middle|\; \mathcal{G}_t \right],$$

for $t \in [0,T]$, and symmetrically the Debt Value Adjustment is defined as,

$$DVA_t = B_t\mathbb{E}\left[\mathbb{1}_{\{t<\tau=\tau_2\le T\}} B_\tau^{-1} \left(1-R_2\right) \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^- \;\middle|\; \mathcal{G}_t \right],$$

for $t \in [0,T]$.

Remark 2.2.5. *DVA accounts for the risk of investor's own default, and it represents the value of any potential outstanding liabilities of the investors that will not be honored at the time of the investor's default:*

In fact, at time of his/her default, the investor only pays to the counterparty the recovery amount, that is $R_2\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta^1_\tau - C_\tau\right)^-$. Therefore, the investor gains the remaining amount, which is equal to $(1-R_2)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta^1_\tau - C_\tau\right)^-$, on his/her outstanding liabilities by defaulting. Risk management of this component is of great importance for financial institutions.

When considering the unilateral counterparty risk DVA is set to zero.

In view of Proposition 2.2.1 and of the above definition we have that

$$\mathrm{CVA}_t = \mathrm{UCVA}_t - \mathrm{DVA}_t,$$

for all $t \in [0,T]$. Note that the bilateral CVA amount may be negative for the investor due to "own default risk". This also indicates that the price S^C of counterparty risky CDS contract may be greater than the price S of counterparty risk-free contract.

Remark 2.2.6. ***(Upfront CDS Conversion)***

After the "CDS Big Bang" (cf. [Mar09]) a process originated to replace standard CDS contracts with so called upfront CDS contracts. An upfront CDS contract is composed of an upfront payment, which is an amount to be exchanged upon the inception of the contract, and a fixed spread. The fixed spread, say $\widehat{\kappa}$, will be 100bps for investment grade CDS contracts, and 500bps for high yield CDS contracts. The recovery rate is also standardized to two possible values: 20% or 40%, depending on the credit worthiness of the reference name. The corresponding cumulative dividend *process of a counterparty-risk-free CDS contract is described in the following definition.*

Definition 2.2.7. *The cumulative dividend process $\widehat{D}$ of a counterparty-risk-free upfront CDS contract, maturing at time T, is given as*

$$\widehat{D}_t = \int_{]0,t]} \delta^1_u dH^3_u - UP - \widehat{\kappa} \int_{]0,t]} \left(1 - H^3_u\right) du\,, \quad t \in [0,T],$$

where UP is the upfront payment, and $\widehat{\kappa}$ is the fixed spread.

Recall that the spread κ_0 of a standard CDS contract is set such that the protection leg PL_0 and fixed leg $\kappa_0 DV01_0$ are equal at initiation (making the price of the contract equal to zero). Similarly, in the case of an upfront CDS contract, with $\widehat{\kappa}$ being fixed, the upfront payment UP is chosen such that the contract has zero value at initiation. It is easy to convert the conventional spread κ_0 into an upfront payment PU and vice versa. Indeed, directly from the Definition 2.2.7, and definitions of PL_0 and $DV01_0$, we have

$$PL_0 - UP - \widehat{\kappa} DV01_0 = PL_0 - \kappa_0 DV01_0 = 0\,,$$

which implies the following representations

$$UP = (\kappa_0 - \widehat{\kappa})\, DV01_0 \qquad \text{and} \qquad \kappa_0 = \frac{UP}{DV01_0} + \widehat{\kappa}\,.$$

In view of the conversion formulae presented above the discussion of CVA, DVA and UCVA done for standard CDS contracts can be adopted to the case of the upfront CDS contracts in a straightforward manner.

2.2.2.2 CVA via Credit Exposures. Credit exposure is defined as the potential loss that may be suffered by either one of the counterparties due to the other party's default. Here, we discuss some measures commonly used to quantify credit exposure, such as *Potential Future Exposure* (PFE), *Expected Positive Exposure* (EPE) and *Expected Negative Exposure* (ENE), and their relation to CVA. These notions are commonly used in practice, since CVA computation can be performed using them.

Potential Future Exposure is the basic measure of credit exposure:

Definition 2.2.8. *The Potential Future Exposure of a CDS contract with a bilateral margin agreement is defined as follows,*

$$\begin{aligned} PFE = \mathbb{1}_{\{\tau=\tau_1\}} (1-R_1) \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta^1_\tau - C_\tau\right)^+ \\ - \mathbb{1}_{\{\tau=\tau_2\}} (1-R_2) \left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta^1_\tau - C_\tau\right)^- \ . \end{aligned}$$

Note that there exist several forms by which the potential future exposure is defined by financial institutions. The PFE definition given above, as a random variable, is in line with the PFE definitions in (cf. [DPR]), as opposed to the rather classical definition of the PFE as the quantile of the exposure distribution (cf. [CAC$^+$10]).

Remark 2.2.7. *Observe that the CVA can be computed using PFE as follows,*

$$CVA_t = B_t \mathbb{E}\left[\mathbb{1}_{\{t<\tau\le T\}} B_\tau^{-1} PFE \;\middle|\; \mathcal{G}_t\right], \qquad t \in [0,T].$$

Expected Positive Exposure is defined as the expected amount the investor will lose if the counterparty default happens at time t, and Expected Negative Exposure is defined as the expected amount the investor will lose if his own default happens at time t. Note that the losses are conditional on default at time t. EPE and ENE are necessary quantities to price and hedge counterparty risk.

Definition 2.2.9. *The Expected Positive Exposure of a CDS contract with a bilateral margin agreement is defined as,*

$$EPE_t = \mathbb{E}\left[(1-R_1)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta^1_\tau - C_\tau\right)^+ \;\middle|\; \tau = \tau_1 = t\right],$$

and the Expected Negative Exposure is defined as,

$$ENE_t = \mathbb{E}\left[(1-R_2)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta^1_\tau - C_\tau\right)^- \;\middle|\; \tau = \tau_2 = t \right]$$

for every $t \in [0,T]$.

Remark 2.2.8. *It is shown (cf. [ABCJ11]) that in case of a deterministic discount factor, the CVA process can be represented in terms of EPE and ENE as follows*

$$\begin{aligned} CVA_t = {} & B_t \int_t^T B_s^{-1} EPE_s G^{-1}(t)\mathbb{Q}\left(\tau = \tau_2 \in ds\right) \\ & - B_t \int_t^T B_s^{-1} ENE_s G^{-1}(t)\mathbb{Q}\left(\tau = \tau_3 \in ds\right) \end{aligned}$$

for every $t \in [0,T]$.

2.3 Dynamics of CVA

In this section, we derive the dynamics of the CVA. The dynamics of the CVA are important for deriving formulae for dynamic hedging of counterparty risk. This problem is left for future work.

We begin with defining some auxiliary stopping times, that will be useful later on:

$$\tau^{\{1\}} := \begin{cases} \tau_3 & \text{if } \tau_3 \neq \tau_1,\ \tau_3 \neq \tau_2 \\ \infty & \text{otherwise} \end{cases}, \qquad \tau^{\{2\}} := \begin{cases} \tau_1 & \text{if } \tau_1 \neq \tau_3, \tau_1 \neq \tau_2 \\ \infty & \text{otherwise} \end{cases},$$

$$\tau^{\{3\}} := \begin{cases} \tau_2 & \text{if } \tau_2 \neq \tau_3, \tau_2 \neq \tau_1 \\ \infty & \text{otherwise} \end{cases}, \qquad \tau^{\{4\}} := \begin{cases} \tau_1 & \text{if } \tau_1 = \tau_2,\ \tau_1 \neq \tau_3 \\ \infty & \text{otherwise} \end{cases},$$

$$\tau^{\{5\}} := \begin{cases} \tau_3 & \text{if } \tau_3 = \tau_1,\ \tau_3 \neq \tau_2 \\ \infty & \text{otherwise} \end{cases}, \qquad \tau^{\{6\}} := \begin{cases} \tau_3 & \text{if } \tau_3 = \tau_2,\ \tau_3 \neq \tau_1 \\ \infty & \text{otherwise} \end{cases},$$

$$\tau^{\{7\}} := \begin{cases} \tau_1 & \text{if } \tau_1 = \tau_2 = \tau_3 \\ \infty & \text{otherwise} \end{cases}.$$

Accordingly, we define the default indicator processes:

$$H_t^{\{1\}} := \mathbb{1}_{\{\tau_3 \le t, \tau_3 \ne \tau_1, \tau_3 \ne \tau_2\}} = \mathbb{1}_{\left\{\tau^{\{1\}} \le t\right\}},$$

$$H_t^{\{2\}} := \mathbb{1}_{\{\tau_1 \le t, \tau_1 \ne \tau_3, \tau_1 \ne \tau_2\}} = \mathbb{1}_{\left\{\tau^{\{2\}} \le t\right\}},$$

$$H_t^{\{3\}} := \mathbb{1}_{\{\tau_2 \le t, \tau_2 \ne \tau_3, \tau_2 \ne \tau_1\}} = \mathbb{1}_{\left\{\tau^{\{3\}} \le t\right\}},$$

$$H_t^{\{4\}} := \mathbb{1}_{\{\tau_1 = \tau_2 \le t, \tau_3 \ne \tau_1\}} = \mathbb{1}_{\left\{\tau^{\{4\}} \le t\right\}},$$

$$H_t^{\{5\}} := \mathbb{1}_{\{\tau_3 = \tau_1 \le t, \tau_3 \ne \tau_2\}} = \mathbb{1}_{\left\{\tau^{\{5\}} \le t\right\}},$$

$$H_t^{\{6\}} := \mathbb{1}_{\{\tau_3 = \tau_2 \le t, \tau_3 \ne \tau_1\}} = \mathbb{1}_{\left\{\tau^{\{6\}} \le t\right\}},$$

$$H_t^{\{7\}} := \mathbb{1}_{\{\tau_3 = \tau_1 = \tau_2 \le t\}} = \mathbb{1}_{\left\{\tau^{\{7\}} \le t\right\}}.$$

Remark 2.3.1. *Note that one can represent processes $H_t^{\{i\}}, i = 1, \dots, 7$, as follows*

$$H_t^{\{1\}} = H_t^3 - [H^3, H^1]_t - [H^3, H^2]_t + [[H^3, H^1], H^2]_t,$$

$$H_t^{\{2\}} = H_t^1 - [H^3, H^1]_t - [H^1, H^2]_t + [[H^3, H^1], H^2]_t,$$

$$H_t^{\{3\}} = H_t^2 - [H^3, H^2]_t - [H^1, H^2]_t + [[H^3, H^1], H^2]_t,$$

$$H_t^{\{4\}} = [H^1, H^2]_t - [[H^3, H^1], H^2]_t,$$

$$H_t^{\{5\}} = [H^3, H^1]_t - [[H^3, H^1], H^2]_t,$$

$$H_t^{\{6\}} = [H^3, H^2]_t - [[H^3, H^1], H^2]_t,$$

$$H_t^{\{7\}} = [[H^3, H^1], H^2]_t,$$

where $[X, Y]$ denotes the quadratic covariation of processes X and Y. In particular, these processes are $\mathbb{G}$-adapted processes.

Let $G(t) = \mathbb{Q}\left(\tau > t \mid \mathcal{F}_t\right)$ be the survival probability process of τ with respect to filtration $\mathbb{F}$. It is a $\mathbb{F}$ supermartingale and it admits unique Doob-Meyer decomposition $G = \mu - \nu$ where μ is the martingale part and ν is a predictable increasing process. We assume that G is a continuous process and v is absolutely continuous with respect to the Lebesgue measure, so that $d\nu_t = v_t dt$ for some $\mathbb{F}$-progressively measurable, non-negative process v. We denote by l the $\mathbb{F}$-progressively measurable process defined as $l_t = G(t)^{-1}v_t$. Finally, we assume that all $\mathbb{F}$ martingales are continuous.

We assume that hazard process of each stopping time $\tau^{\{i\}}$ admits an $(\mathbb{F}, \mathbb{G})$-intensity process q^i for every $i = 1, \dots, 7$, so that the process $M^{\{i\}}$, given by the formula,

$$M_t^{\{i\}} = H_t^{\{i\}} - \int_0^t \left(1 - H_u^{\{i\}}\right) q_u^i du$$

is a $\mathbb{G}$-martingale for every $t \in [0, T]$ and $i = 1, \dots, 7$.

We now have the following technical result,

Lemma 2.3.1. *The processes*

$$M_t^i := M_{t\wedge\tau}^{\{i\}} = H_{t\wedge\tau}^{\{i\}} - \int_0^{t\wedge\tau} l_u^i du, \qquad t \geq 0,\ i = 1, 2, \dots, 7,$$

and

$$M_t := H_{t\wedge\tau} - \int_0^{t\wedge\tau} l_u du, \quad t \geq 0,$$

where

$$l_t^i = \mathbb{1}_{\{\tau \geq t\}} q_t^i \text{ and } l_t = \sum_{i=1}^{7} l_t^i, \qquad t \geq 0,\ i = 1, 2, \dots, 7,$$

are $\mathbb{G}$-martingales.

Proof. Fix $i = 1, \dots, 7$. Process M^i follows a $\mathbb{G}$-martingale, since it is $\mathbb{G}$-martingale $M^{\{i\}}$ stopped at the $\mathbb{G}$ stopping time τ. Moreover, we have that $M_t = \sum_{i=1}^{7} M_t^i$, so that process M is also a $\mathbb{G}$-martingale. $\square$

We shall now proceed with deriving some useful representations for the processes S^C and S.

Lemma 2.3.2. *The ex-dividend price process, S^C, of a counterparty risky CDS contract, given in* (2.4)*, can be represented as follows,*

$$S_t^C = B_t \mathbb{E}\left[B_\tau^{-1} \sum_{i=1}^{7} \mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}} \bar{\delta}_\tau^i - \kappa \int_{]t,T]} B_u^{-1} \mathbb{1}_{\{\tau>u\}} du \,\middle|\, \mathcal{G}_t \right] \tag{2.11}$$

where

$$\bar{\delta}_t^1 = \delta_t^1,$$

$$\bar{\delta}_t^2 = S_t - (1-R_2)(S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_\tau^1 - C_t)^+$$

$$\bar{\delta}_t^3 = S_t + (1-R_3)(S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_\tau^1 - C_t)^-,$$

$$\bar{\delta}_t^4 = S_t - (1-R_2)(S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_\tau^1 - C_t)^+ + (1-R_3)(S_t + \mathbb{1}_{\{t=\tau_3\}}\delta_\tau^1 - C_t)^-$$

$$\bar{\delta}_t^5 = \delta_t^1 - (1-R_2)(\delta_t^1 - C_t)^+,$$

$$\bar{\delta}_t^6 = \delta_t^1 + (1-R_3)(\delta_t^1 - C_t)^-$$

$$\bar{\delta}_t^7 = \delta_t^1 - (1-R_2)(\delta_t^1 - C_t)^+ + (1-R_3)(\delta_t^1 - C_t)^-.$$

Proof. Let us rewrite (2.10) using (2.9) in the following form,

$$\begin{aligned} S_t^C = B_t \mathbb{E}\Bigg[& B_\tau^{-1}\delta_\tau^1 \sum_{i=1,5,6,7} \mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}} + B_\tau^{-1} S_\tau \sum_{i=2,3,4} \mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}} \\ & - B_\tau^{-1}(1-R_1)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^+ \sum_{i=2,4,5,7} \mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}} \\ & + B_\tau^{-1}(1-R_2)\left(S_\tau + \mathbb{1}_{\{\tau=\tau_3\}}\delta_\tau^1 - C_\tau\right)^- \sum_{i=3,4,6,7} \mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}} \\ & -\kappa \int_{]t,T]} B_u^{-1}\mathbb{1}_{\{\tau>u\}} du \,\Bigg|\, \mathcal{G}_t \Bigg], \end{aligned}$$

which, after rearranging terms, leads to

$$\begin{aligned}
S_t^C = B_t\mathbb{E}\Big[& B_\tau^{-1}\delta_\tau^1 \mathbb{1}_{\left\{t<\tau=\tau^{\{1\}}\leq T\right\}} + B_\tau^{-1}\left(S_\tau-(1-R_1)\left(S_\tau-C_\tau\right)^+\right)\mathbb{1}_{\left\{t<\tau=\tau^{\{2\}}\leq T\right\}} \\
& + B_\tau^{-1}\left(S_\tau+(1-R_2)\left(S_\tau-C_\tau\right)^-\right)\mathbb{1}_{\left\{t<\tau=\tau^{\{3\}}\leq T\right\}} \\
& + B_\tau^{-1}\left(S_\tau-(1-R_1)\left(S_\tau-C_\tau\right)^+ + (1-R_2)\left(S_\tau-C_\tau\right)^-\right)\mathbb{1}_{\left\{t<\tau=\tau^{\{4\}}\leq T\right\}} \\
& + B_\tau^{-1}\left(\delta_\tau^1-(1-R_1)\left(\delta_\tau^1-C_\tau\right)^+\right)\mathbb{1}_{\left\{t<\tau=\tau^{\{5\}}\leq T\right\}} \\
& + B_\tau^{-1}\left(\delta_\tau^1+(1-R_2)\left(\delta_\tau^1-C_\tau\right)^-\right)\mathbb{1}_{\left\{t<\tau=\tau^{\{6\}}\leq T\right\}} \\
& + B_\tau^{-1}\left(\delta_\tau^1-(1-R_1)\left(\delta_\tau^1-C_\tau\right)^+ + (1-R_2)\left(\delta_\tau^1-C_\tau\right)^-\right)\mathbb{1}_{\left\{t<\tau=\tau^{\{7\}}\leq T\right\}} \\
& -\kappa\int_{]t,T]} B_u^{-1}\mathbb{1}_{\{\tau>u\}}du \;\Big|\; \mathcal{G}_t\Big].
\end{aligned}$$

This proves the result. □

In case when $R_1 = R_2 = 1$ process S is the same as process S^C. Thus, we obtain from the above

Corollary 2.3.1. *The ex-dividend price process S of a counterparty risk-free CDS contract, can be represented* [4] *as follows,*

$$S_t = B_t\mathbb{E}\left[B_\tau^{-1}\sum_{i=1}^{7}\mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}}\hat{\delta}_\tau^i - \kappa\int_{]t,T]} B_u^{-1}\mathbb{1}_{\{\tau>u\}}du \;\middle|\; \mathcal{G}_t\right], \tag{2.12}$$

where $\hat{\delta}_t^1 = \hat{\delta}_t^5 = \hat{\delta}_t^6 = \hat{\delta}_t^7 = \delta_t^1$, and $\hat{\delta}_t^2 = \hat{\delta}_t^3 = \hat{\delta}_t^4 = S_t$. Thus,

$$\begin{aligned}
S_t = B_t\mathbb{E}\Big[& B_\tau^{-1}\mathbb{1}_{\{t<\tau=\tau_3\leq T\}}\delta_\tau^1 + B_\tau^{-1}\sum_{i=2}^{4}\mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}}S_\tau \\
& -\kappa\int_{]t,T]} B_u^{-1}\mathbb{1}_{\{\tau>u\}}du \;\Big|\; \mathcal{G}_t\Big].
\end{aligned} \tag{2.13}$$

---

[4] We note that formula (2.13) provides a representation of S_t, which is convenient for our purposes. The traditional representation of S_t, typically used in the context of counterparty risk free CDS contracts is

$$S_t = B_t\mathbb{E}\left[B_{\tau_3}^{-1}\mathbb{1}_{\{t<\tau_3\leq T\}}\delta_{\tau_3}^1 - \kappa\int_{]t,T]} B_u^{-1}\mathbb{1}_{\{\tau_3>u\}}du \;\middle|\; \mathcal{G}_t\right].$$

The following result is borrowed from [BJR08] (see Lemma 3.1 therein)

Lemma 2.3.3. *The following equality holds ($\mathbb{Q}$-a.s.)*

$$B_t \mathbb{E}\left[\mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}} B_\tau^{-1} \overline{\delta}_\tau^i \;\middle|\; \mathcal{G}_t \right] = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G(t)} \mathbb{E}_{\mathbb{Q}} \left(\int_t^T B_u^{-1} l_u^i \overline{\delta}_u^i G(u)\, du \;\middle|\; \mathcal{F}_t \right], \tag{2.14}$$

for every $t \in [0, T]$.

The pre-default ex-dividend price processes, say $\widetilde{S}$ and $\widetilde{S}^C$, are defined as the (unique) $\mathbb{F}$-adapted processes (cf. [BJR08]) such that

$$S_t^C = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t^C, \quad S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t.$$

In view of the above we thus obtain the following result

Lemma 2.3.4. *We have that, for every $t \in [0, T]$,*

$$\widetilde{S}_t^C = \frac{B_t}{G(t)} \mathbb{E}_{\mathbb{Q}} \left[\int_t^T B_u^{-1} G(u) \left(\sum_{i=1}^{7} l_u^i \overline{\delta}_u^i - \kappa \right) du \;\middle|\; \mathcal{F}_t \right], \tag{2.15}$$

and

$$\widetilde{S}_t = \frac{B_t}{G(t)} \mathbb{E}_{\mathbb{Q}} \left[\int_t^T B_u^{-1} G(u) \left(\sum_{i=1}^{7} l_u^i \hat{\delta}_u^i - \kappa \right) du \;\middle|\; \mathcal{F}_t \right]. \tag{2.16}$$

Proof. From Lemma 2.3.2 we have that

$$S_t^C = B_t \mathbb{E}\left[B_\tau^{-1} \sum_{i=1}^{7} \mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}} \overline{\delta}_t^i \;\middle|\; \mathcal{G}_t \right] - \kappa B_t \mathbb{E}\left[\int_{]t,T]} B_u^{-1} \mathbb{1}_{\{\tau>u\}} du \;\middle|\; \mathcal{G}_t \right].$$

Now, in view of (2.14) we see that

$$B_t \mathbb{E}\left[B_\tau^{-1} \sum_{i=1}^{7} \mathbb{1}_{\left\{t<\tau=\tau^{\{i\}}\leq T\right\}} \overline{\delta}_t^i \middle| \mathcal{G}_t \right] = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G(t)} \mathbb{E}\left[\sum_{i=1}^{7} \int_t^T B_u^{-1} l_u^i \overline{\delta}_u^i G(u)\, du \middle| \mathcal{F}_t \right].$$

Let us now fix $t \geq 0$, and define $Y_s := -\kappa \int_{]t,s]} B_u^{-1} du$ for $s \geq t$. Thus, we get

$$-\kappa B_t \mathbb{E}\left[\int_{]t,T]} B_u^{-1} \mathbb{1}_{\{\tau>u\}} du \;\middle|\; \mathcal{G}_t \right] = B_t \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}} Y_T \;\middle|\; \mathcal{G}_t \right]$$

$$+ B_t \mathbb{E}\left[\mathbb{1}_{\{t<\tau\leq T\}} Y_\tau \;\middle|\; \mathcal{G}_t \right].$$

It is known from [BJR08], that

$$B_t \mathbb{E}\left[\mathbb{1}_{\{t<\tau\leq T\}} Y_\tau \;\middle|\; \mathcal{G}_t \right] = -\mathbb{1}_{\{t<\tau\}} \frac{B_t}{G(t)} \mathbb{E}\left[\int_t^T Y_u dG(u) \;\middle|\; \mathcal{F}_t \right]$$

and

$$B_t \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}} Y_T \;\middle|\; \mathcal{G}_t \right] = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G(t)} \mathbb{E}\left[G(T) Y_T \;\middle|\; \mathcal{F}_t \right].$$

Finally, since Y is of finite variation, (2.15) follows by applying the integration by parts formula

$$G(t) Y_T - \int_t^T Y_s dG(s) = \int_t^T G(s)\, dY_s = -\kappa \int_t^T G(s)\, B_u^{-1} du.$$

Equality (2.16) is obtained as a special case of (2.15), by setting $R_1 = R_2 = 1$. □

We are ready now to derive dynamics of the pre-default price processes, that we shall use in order to derive the dynamics of the CVA process.

Lemma 2.3.5.

(i) The pre-default ex-dividend price of a counterparty risky CDS contract follows the dynamics given as

$$d\widetilde{S}_t^C = \left((r_t + l_t)\, \widetilde{S}_t^C - \left(\sum_{i=1}^{7} l_t^i \overline{\delta}_t^i - \kappa \right) \right) dt + G^{-1}(t) \left(B_t dm_t^C - \widetilde{S}_t^C d\mu \right)$$
$$+ G^{-2}(t) \left(\widetilde{S}_t^C d\langle \mu \rangle_t - B_t d\left\langle \mu, m^C \right\rangle_t \right),$$

for $t \in [0, T]$, where

$$m_t^C = \mathbb{E}\left[\int_0^T B_u^{-1} G(u) \left(\sum_{i=1}^{7} l_u^i \overline{\delta}_u^i - \kappa \right) du \;\middle|\; \mathcal{F}_t \right]$$

(ii) The pre-default ex-dividend price of a counterparty risk-free CDS contract follows the dynamics given as

$$d\widetilde{S}_t = \left((r_t + l_t)\, \widetilde{S}_t - \left(\sum_{i=1}^{7} l_t^i \hat{\delta}_t^i - \kappa \right) \right) dt + G^{-1}(t) \left(B_t dm_t - \widetilde{S}_t d\mu \right)$$
$$+ G^{-2}(t) \left(\widetilde{S}_t d\langle \mu \rangle_t - B_t d\langle \mu, m \rangle_t \right),$$

for $t \in [0,T]$*, where*

$$m_t = \mathbb{E}\left[\int_0^T B_u^{-1} G\left(u\right) \left(\sum_{i=1}^{7} l_u^i \hat{\delta}_u^i - \kappa \right) du \;\middle|\; \mathcal{F}_t \right].$$

Proof. The argument below follows the one in the proof of Proposition 1.2 in [BJR08].

In view of (2.15) we may write $\widetilde{S}_t^C$ as

$$\widetilde{S}_t^C = B_t G^{-1}(t) U_t,$$

where

$$U_t = m_t^C - \int_0^t B_u^{-1} G\left(u\right) \left(\sum_{i=1}^{7} l_u^i \overline{\delta}_u^i - \kappa \right) du.$$

Since $G = \mu - v$, then applying Itô's formula one obtains

$$\begin{aligned} d\left(G^{-1}(t)U_t\right) = G^{-1}(t) dm_t^C - B_t^{-1} \left(\sum_{i=1}^{7} l_t^i \overline{\delta}_t^i - \kappa \right) dt \\ + U_t \left(G^{-3}(t) d\left\langle \mu \right\rangle_t - G^{-2}(t) \left(d\mu_t - dv_t \right) \right) \\ - G^{-2}(t) d\left\langle \mu, m^C \right\rangle_t. \end{aligned}$$

Consequently,

$$\begin{aligned} d\widetilde{S}_t^C &= B_t G^{-1}(t) dm_t^C - \left(\sum_{i=1}^{7} l_t^i \overline{\delta}_t^i - \kappa \right) dt \\ &\quad + B_t U_t \left(G^{-3}(t) d\left\langle \mu \right\rangle_t - G^{-2}(t) \left(d\mu_t - l_t G(t) dt \right) \right) \\ &\quad - B_t G^{-2}(t) d\left\langle \mu, m^C \right\rangle + r_t B_t G^{-1}(t) U_t dt \\ &= \left(\left(r_t + l_t \right) \widetilde{S}_t^C - \left(\sum_{i=1}^{7} l_t^i \overline{\delta}_t^i - \kappa \right) \right) dt + G^{-1}(t) \left(B_t dm_t^C - \widetilde{S}_t d\mu \right) \\ &\quad + G^{-2}(t) \left(\widetilde{S}_t d\left\langle \mu \right\rangle_t - B_t d\left\langle \mu, m^C \right\rangle_t \right), \end{aligned}$$

which verifies the result stated in (i).

Starting from (2.16), and using computations analogous to the ones done in (i), one can derive the result stated in (ii). □

Using the lemma above, we derive the dynamics of the CVA process as follows.

Proposition 2.3.1. *The bilateral CVA process satisfies,*

$$\begin{aligned} dCVA_t &= r_t CVA_t dt - CVA_{t-} dM_t - (1-H_t)\left(\sum_{i=1}^{7} l_t^i \xi_t^i\right) dt \\ &\quad + (1-H_t)\, B_t G^{-1}(t) dn_t - G^{-1}(t) CVA_t d\mu_t + G^{-2}(t) CVA_t d\langle \mu\rangle_t \\ &\quad - (1-H_t)\, G^{-2}(t) B_t \left(d\langle \mu, m\rangle_t - d\langle \mu, m^C\rangle_t\right), \end{aligned}$$

for all $t \in [0,T]$ *where* $CVA_{T\wedge\tau} = 0$ *and*

$$n_t = \mathbb{E}\left[\int_0^T B_u^{-1} G(u) \left(\sum_{i=1}^{7} l_u^i \xi_u^i\right) du \;\middle|\; \mathcal{F}_t\right], \; t \in [0,T],$$

with

$$\begin{aligned} \xi_t^1 &= 0, \\ \xi_t^2 &= (1-R_1)(S_t - C_t)^+, \\ \xi_t^3 &= -(1-R_2)(S_t - C_t)^-, \\ \xi_t^4 &= (1-R_1)(S_t - C_t)^+ - (1-R_2)(S_t - C_t)^-, \\ \xi_t^5 &= (1-R_1)(\delta_t^1 - C_t)^+, \\ \xi_t^6 &= -(1-R_2)(\delta_t^1 - C_t)^-, \\ \xi_t^7 &= (1-R_1)(\delta_t^1 - C_t)^+ - (1-R_2)(\delta_t^1 - C_t)^-. \end{aligned}$$

Proof. Applying the integration by parts formula we get that

$$d\mathrm{CVA}_t = (1-H_t)(d\widetilde{S}_t - d\widetilde{S}_t^C) - (\widetilde{S}_t - \widetilde{S}_t^C) dH_t.$$

This together with Lemma 2.3.5 implies

$$dCVA_t = -\left(S_{t-} - S^C_{t-}\right) dM_t + (1-H_t)\left(r_t\left(S_t - S^C_t\right) - \sum_{i=1}^{7} l^i_t\left(\hat{\delta}^i_t - \overline{\delta}^i_t\right)\right) dt \quad (2.17)$$

$$+ (1-H_t)\, B_t G^{-1}(t)\left(dm_t - dm^C_t\right) - (1-H_t)\, G^{-1}(t)\left(S_t - S^C_t\right) d\mu_t$$

$$+ (1-H_t)\, G^{-2}(t)\left(S_t - S^C_t\right) d\left\langle \mu \right\rangle_t$$

$$- (1-H_t)\, G^{-2}(t) B_t\left(d\left\langle \mu, m\right\rangle_t - d\left\langle \mu, m^C\right\rangle_t\right),$$

which proves the result. □

2.3.1 Dynamics of CVA when the immersion property holds. Here we adapt the results derived above to the case when the immersion property holds between filtrations $\mathbb{F}$ and $\mathbb{G}$, that is the case when every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale under $\mathbb{Q}$. In this case, the continuous martingale μ in the Doob-Meyer decomposition of G vanishes, so that the survival process G is a non-increasing process represented as $G = -v$. Frequently, the immersion property is referred to as Hypothesis $(\mathcal{H})$. For an excellent discussion of the immersion property we refer to [JLC09].

Assumption 2.3.1. *Hypothesis $(\mathcal{H})$ holds between the filtrations $\mathbb{F}$ and $\mathbb{G}$ under $\mathbb{Q}$.*

In view of the results (and the notation) from Proposition 2.3.1 we obtain the following result.

Corollary 2.3.2. *2.3.1 Assume that Assumption 2.3.1 is satisfied. Then,*

$$dCVA_t = r_t CVA_t dt - CVA_{t-} dM_t - (1-H_t)\left(\sum_{i=1}^{7} l^i_t \xi^i_t\right) dt$$

$$+ (1-H_t)\, B_t G^{-1}(t) dn_t,$$

for all $t \in [0,T]$ where $CVA_{T\wedge\tau} = 0$.

Remark 2.3.2. *If we assume that the filtration $\mathbb{F}$ is generated by a Brownian motion, then, in view of the Brownian martingale representation theorem, there exists an $\mathbb{F}$-predictable process ζ such that $dn_t = \zeta_t dW_t$, and*

$$dCVA_t = r_t CVA_t dt - CVA_{t-} dM_t - (1 - H_t)\left(\sum_{i=1}^{7} l_t^i \xi_t^i\right) dt$$

$$+ (1 - H_t)\, B_t G^{-1}(t) \zeta_t dW_t,$$

for all $t \in [0, T]$ where $CVA_{T\wedge\tau} = 0$.

We have the following important result regarding the cumulative CVA dynamics.

Lemma 2.3.6. *Dynamics of the $\widehat{CVA}$ are found as follows,*

$$d\widehat{CVA}_t = (1 - H_t)(d\widetilde{S}_t - d\widetilde{S}_t^C) + (Z_t - \widehat{CVA}_{t-}) dH_t,$$

for all $t \in [0, T]$ where $\widehat{CVA}_{T\wedge\tau} = 0$ and

$$Z_t = \sum_{i=1}^{7} \xi_t^i.$$

Proof. We have,

$$\begin{aligned} d\widehat{\mathrm{CVA}}_t &= (1 - H_{t-})(d\widehat{S}_t - d\widehat{S}_t^C) \\ &= (1 - H_t)(dS_t - dS_t^C) + (\Delta\widehat{S}_\tau - \Delta\widehat{S}_\tau^C) dH_t \\ &= (1 - H_t)(dS_t - dS_t^C) + (Z_\tau - \widehat{\mathrm{CVA}}_{\tau-}) dH_t \\ &= (1 - H_t)(dS_t - dS_t^C) + (Z_t - \widehat{\mathrm{CVA}}_t) dH_t \\ &= (1 - H_t)(dS_t - dS_t^C) + (Z_t - \mathrm{CVA}_{t-}) dH_t \end{aligned}$$

where

$$Z_t = \sum_{i=1}^{7} \xi_t^i$$

for all $t \in [0, T]$. $\square$

Corollary 2.3.3. *We have*

$$d\widehat{CVA}_t = r_t \widehat{CVA}_t dt + \sum_{i=1}^{7} \left(\xi_t^i - CVA_{t-}\right) dM_t^i + (1 - H_t)\, B_t G^{-1}(t)\, dn_t,$$

where

$$n_t = \mathbb{E}\left[\int_0^T B_u^{-1} G(u) \left(\sum_{i=1}^{7} l_u^i \xi_u^i \right) du \;\middle|\; \mathcal{F}_t \right]$$

and

$$\xi_t^1 = 0,$$

$$\xi_t^2 = (1 - R_1)(S_t - C_t)^+,$$

$$\xi_t^3 = -(1 - R_2)(S_t - C_t)^-,$$

$$\xi_t^4 = (1 - R_1)(S_t - C_t)^+ - (1 - R_2)(S_t - C_t)^-,$$

$$\xi_t^5 = (1 - R_1)(\delta_t^1 - C_t)^+,$$

$$\xi_t^6 = -(1 - R_2)(\delta_t^1 - C_t)^-,$$

$$\xi_t^7 = (1 - R_1)(\delta_t^1 - C_t)^+ - (1 - R_2)(\delta_t^1 - C_t)^-,$$

for all $t \in [0, T]$.

Proof. Substituting the terms $d\widetilde{S}_t$ and $d\widetilde{S}_t^C$ found in Lemma 2.3.5, we get,

$$\begin{aligned} d\widehat{\text{CVA}}_t &= (1 - H_t) r_t (\widetilde{S}_t - \widetilde{S}_t^C) dt \\ &\quad + (1 - H_t) l_t (\widetilde{S}_t - \widetilde{S}_t^C) dt - (1 - H_t) \sum_{i=1}^{7} l_t^i (\hat{\delta}_t^i - \overline{\delta}_t^i) dt \\ &\quad + (\sum_{i=1}^{7} (\hat{\delta}_t^i - \overline{\delta}_t^i) - \widehat{\text{CVA}}_{t-}) dH_t + (1 - H_t) B_t G^{-1}(t) dn_t, \end{aligned}$$

which is equal to,

$$\begin{aligned} d\widehat{\text{CVA}}_t &= r_t \widehat{\text{CVA}}_t dt - \text{CVA}_{t-} dM_t \\ &\quad + \sum_{i=1}^{7} \xi_t^i dM_t^i + (1 - H_t) B_t G^{-1}(t) dn_t. \end{aligned}$$

□

Remark 2.3.3. *It has been shown in (cf. [BCJZ11], page 10) that in a specific case of a Markovian copula model of unilateral counterparty risk and assuming that the filtration $\mathbb{F}$ is generated by a Brownian motion with $r = 0$, the dynamics of $\widehat{CVA}$ reduce to the form found in Corollary 2.3.3 with $R_2 = 1$ and $\tau_2 = \infty$.*

2.3.2 Fair Spread Value Adjustment. CDS contracts are quoted in terms of their spreads[5], which do not take the counterparty risk into account. Therefore, computing and monitoring the counterparty risk embedded in the CDS spreads is of great importance for financial institutions. In this section, we introduce the Spread Value Adjustment (SVA) as the difference between the counterparty risk-free and the counterparty risky CDS spreads. The SVA provides a more practical way to quantify the counterparty risk, and also it is a very useful indicator for the trading decisions in practice (cf. [Gre09]).

Let us fix $t \in [0, T]$, and let us denote by κ_t the market spread of the counterparty risk-free CDS contract at time t; that is, κ_t is that level of spread which makes the pre-default values of the two legs of a counterparty risk-free CDS contract equal to each other at time t,

$$\widetilde{S}_t(\kappa_t) = 0. \tag{2.18}$$

It is convenient to write the above equation in the form that is common in practice:

$$PL_t - \kappa_t RDV01_t = 0, \tag{2.19}$$

where PL and $RDV01$ are processes representing (pre-default) values of the protec-

[5]As we mentioned in Remark 2.2.6, CDS contracts can be quoted in terms of upfront payments. However, it is always possible to convert the upfront payments to running spreads.

tion leg and the risky annuity [6] , respectively, so that

$$PL_t = \frac{B_t}{G^3(t)} \mathbb{E}\left[\int_{]t,T]} B_u^{-1} G^3(u)\, \delta_u^1 \left(\sum_{i=1,5,6,7} l_u^i \right) du \;\middle|\; \mathcal{F}_t \right], \tag{2.20}$$

and

$$RDV01_t = \frac{B_t}{G^3(t)} \mathbb{E}\left[\int_{]t,T]} B_u^{-1} G^3(u)\, du \;\middle|\; \mathcal{F}_t \right], \tag{2.21}$$

where

$$G^3(t) = \mathbb{Q}\left(\tau_3 > t \mid \mathcal{F}_t \right).$$

Therefore, we get,

$$\kappa_t = \frac{\mathbb{E}\left[\int_{]t,T]} B_u^{-1} G^3(u)\, \delta_u^1 \left(\sum_{i=1,5,6,7} l_u^i \right) du \;\middle|\; \mathcal{F}_t \right]}{\mathbb{E}\left[\int_{]t,T\wedge\tau_3]} B_u^{-1} G^3(u)\, du \;\middle|\; \mathcal{F}_t \right]}. \tag{2.22}$$

We denote by κ_t^C the spread which makes the values of the two pre-first-default legs of a counterparty risky CDS contract equal to each other at every $t \in [0,T]$ as

$$\widetilde{S}_t^C\left(\kappa_t^C\right) = PL_t^C - \kappa_t^C RDV01_t^C = 0. \tag{2.23}$$

Likewise, we use the spread κ_0^C initiated at time $t=0$ in order to compute the fair price of a counterparty risky CDS contract at any time $t \in [0,T]$. Using Lemma 3.1, κ_t^C admits the following representation for every $t \in [0,T]$,

$$\kappa_t^C = \frac{PL_t^C}{RDV01_t^C},$$

where

---

[6]We note that formula (2.20) provides a representation of PL_t, which is convenient for our purposes. The traditional representation of PL_t, typically used in the context of counterparty risk free CDS contracts is

$$PL_t = \frac{B_t}{G^3(t)} \mathbb{E}\left[\int_{]t,T]} B_u^{-1} G^3(u)\, \delta_u^1 \lambda_u^3 du \;\middle|\; \mathcal{F}_t \right],$$

where λ^3 is the $\mathbb{F}$-intensity of τ_3.

$$PL_t^C = \frac{B_t}{G(t)}\mathbb{E}\left[\int_t^T B_u^{-1}G(u)\left(\sum_{i=1}^{7} l_u^i \overline{\delta}_u^i\right)du \,\middle|\, \mathcal{F}_t\right] \tag{2.24}$$

and

$$RDV01_t^C = \frac{B_t}{G(t)}\mathbb{E}\left[\int_{]t,T]} B_u^{-1}G(u)\,du \,\middle|\, \mathcal{F}_t\right]. \tag{2.25}$$

We may now introduce the definition of the spread value adjustment.

Definition 2.3.1. *The SVA of a counterparty risky CDS contract maturing at time T is defined as,*

$$SVA_t = \kappa_t - \kappa_t^C$$

for every $t \in [0,T]$.

We have the following useful representation.

Proposition 2.3.2. *The SVA of a CDS contract maturing at time T can be represented as*

$$SVA_t = \frac{\widetilde{CVA}_t}{B_t G^{-1}(t)\mathbb{E}\left[\int_{]t,T]} B_u^{-1}G(u)\,du \,\middle|\, \mathcal{F}_t\right]}$$

for all $t \in [0,T]$, where the pre-first-default bilateral Credit Valuation Adjustment process $\widetilde{CVA}$ is given as

$$\widetilde{CVA}_t = \widetilde{S}_t - \widetilde{S}_t^C, \tag{2.26}$$

for every $t \in [0,T]$.

Proof. Let us rewrite PL^C as

$$PL_t^C = PL_t^C - \kappa_t RDV01_t^C + \kappa_t RDV01_t^C$$

by a simple modification. Now, using (2.5) and (2.23), we conclude that

$$PL_t^C = \widetilde{S}_t^C(\kappa_t) + \kappa_t RDV01_t^C = \widetilde{S}_t(\kappa_t) - \widetilde{\mathrm{CVA}}_t + \kappa_t RDV01_t^C.$$

Since $\widetilde{S}_t(\kappa_t) = 0$, then κ_t^C has the following form,

$$\kappa_t^C = \frac{-\widetilde{\mathrm{CVA}}_t + \kappa_t RDV01_t^C}{RDV01_t^C},$$

which is

$$\kappa_t^C = -\frac{\widetilde{\mathrm{CVA}}_t}{RDV01_t^C} + \kappa_t.$$

$\square$

Observe that the SVA can be computed using the CVA via the representation above. This is particularly important, since in practice the CVA is a commonly computed quantity; therefore, SVA can be found without any additional effort. Furthermore, this result is model-free, which means that it is valid under any particular model.

2.3.2.1 SVA Dynamics. Applying Itô formula one obtains the dynamics of the fair spread process and of the counterparty risk adjusted spread process as

$$\begin{aligned} d\kappa_t = \frac{1}{\widetilde{RDV01}_t} &\left(B_t^{-1} G^1(t) \left(\kappa_t - \delta_t^1 l_t^1\right) dt + \frac{\kappa_t}{\widetilde{RDV01}_t} d\left\langle \eta^2 \right\rangle_t \right. \\ &\left. - \frac{1}{\widetilde{RDV01}_t} d\left\langle \eta^1, \eta^2 \right\rangle_t \right) + \frac{1}{\widetilde{RDV01}_t} \left(d\eta_t^1 - \kappa_t d\eta_t^2 \right), \qquad t \in [0,T], \end{aligned} \tag{2.27}$$

where

$$\widetilde{RDV01}_t := \mathbb{E}\left[\int_{]t,T]} B_u^{-1} G^3(u)\, du \;\middle|\; \mathcal{F}_t \right],$$

with

$$\eta_t^1 := \mathbb{E}\left[\int_{]0,T]} B_u^{-1} G^3(u)\, \delta_u^1 l_u^1 du \;\middle|\; \mathcal{F}_t \right],$$

and

$$\eta_t^2 = \mathbb{E}\left[\int_{]0,T]} B_u^{-1} G^3(u)\, du \;\middle|\; \mathcal{F}_t \right] = \widetilde{RDV01}_t + \int_{]0,t]} B_u^{-1} G^3(u)\, du.$$

Moreover,

$$d\kappa_t^C = \frac{1}{\widetilde{RDV01}_t^C} \left(B_t^{-1} G(t) \left(\kappa_t^C - \sum_{i=1}^{7} \widetilde{\delta}_t^i l_t^i \right) dt + \frac{\kappa_t^C}{\widetilde{RDV01}_t^C} d\left\langle \zeta^2 \right\rangle_t \right. \tag{2.28}$$

$$\left. - \frac{1}{\widetilde{RDV01}_t^C} d\left\langle \zeta^1, \zeta^2 \right\rangle_t \right) + \frac{1}{\widetilde{RDV01}_t^C} \left(d\zeta_t^1 - \kappa_t^C d\zeta_t^2 \right),$$

where

$$\widetilde{RDV01}_t = \mathbb{E}\left[\int_{]t,T]} B_u^{-1} G^3(u)\, du \;\middle|\; \mathcal{F}_t \right],$$

with

$$\zeta_t^1 = \mathbb{E}\left[\int_{]0,T]} B_u^{-1} G(u) \left(\sum_{i=1}^{7} l_u^i \overline{\delta}_u^i \right) du \;\middle|\; \mathcal{F}_t \right],$$

and

$$\zeta_t^2 = \mathbb{E}\left[\int_{]0,T]} B_u^{-1} G(u)\, du \;\middle|\; \mathcal{F}_t \right] = \widetilde{RDV01}_t^C + \int_{]0,t]} B_u^{-1} G(u)\, du.$$

Combining the above results, we find the dynamics of the SVA process:

$$d\,\mathrm{SVA}_t = d\kappa_t - d\kappa_t^C, \quad t \in [0, T].$$

Dynamics of the SVA are of great importance for observing the behavior of the difference between the fair spread and the counterparty risk adjusted spread. Counterparty risk dynamics can be assessed in a more intuitive manner by computing the SVA dynamics.

2.4 Multivariate Markovian Default Model In this section, we propose an underlying stochastic model following the lines of [BCJZ11]. Towards this end we define a Markovian model of multivariate default times with factor processes $X = (X^1, X^2, X^3)$ which will have the following key features:

- The pair (X, H) is Markov in its natural filtration,
- Each pair (X^i, H^i) is a Markov process,

- At every instant, either each counterparty defaults individually or simultaneously with other counterparties.

Note that the second property grants quick valuation of the CDS and independent calibration of each model marginal (X^i, H^i), whereas the third property will allow us to account for dependence between defaults. We present here some numerical results as an application of the above theory. The default intensities are assumed to be of the affine form

$$l_i\left(t, X_t^i\right) = a_i + X_t^i,$$

where a_i is a constant and X^i is a homogenous CIR process generated by,

$$dX_t^i = \zeta_i\left(\mu_i - X_t^i\right)dt - \sigma_i\sqrt{X_t^i}dW_t^i,$$

for $i = 1, 2, 3$. Each collection of the parameters $(\zeta_i, \mu_i, \sigma_i)$ may take values corresponding to a low, a medium or a high regime which are given as follows.

Table 2.1. CIR parameters for different risk profiles

Credit Risk Level	ζ	μ	σ	X_0
Low	0.9	0.001	0.01	0.001
Medium	0.8	0.02	0.1	0.02
High	0.5	0.05	0.2	0.05

Moreover, following the methodology in [BCJZ11], we specify the marginal default intensity processes as follows

$$q_t^1 = l_t^2 + l_t^4 + l_t^5 + l_t^7, \quad q_t^2 = l_t^3 + l_t^4 + l_t^6 + l_t^7, \quad q_t^3 = l_t^1 + l_t^5 + l_t^6 + l_t^7,$$

where the related survival probabilities are found as

$$\mathbb{Q}\left(\tau_i > t\right) = \mathbb{E}\left[e^{-\int_0^t q_u^i du}\right] \quad \text{and} \quad \mathbb{Q}\left(\tau > t\right) = \mathbb{E}\left[e^{-\int_0^t l_u du}\right].$$

For a detailed discussion including implementation and the calibration of the model, we refer to [BCJZ11] and [ABCJ11].

2.4.1 Results. Our aim here is to assess by means of numerical experiments the impact of collateralization on the counterparty risk exposure. We present numerical results for different collateralization regimes distinguished by different threshold values. The numerical experiments below have been done using the parametrization given in [BCJZ11], the recovery rates are fixed to 40%, the risk-free rate r is taken as 0 and the maturity is set to $T = 5$ years.

Table 2.2 shows the values of CVA_0 and SVA_0 for different threshold regimes. Threshold values in Cases A–F are chosen as a fraction of the notional (cf. [Pyk09]). In practice, institutions pick their collateral threshold levels before initiating the contracts. Therefore, it is very important to see how much the predetermined threshold levels actually reduce the overall exposure, as well as the CVA values.

Computations are done assuming that (see Table 2.1) the underlying entity, the counterparty, and the investor has high risk levels. Simulated fair spread without counterparty risk is found as 153bps. Case A represents the uncollateralized regime where there is no collateral exchanged (this is done by setting the thresholds infinity), whereas other Case F corresponds to the full collateralization where the thresholds are set to 0. In each case, computations are done by setting MTA to zero and assuming there is no margin period. One can observe that decreasing threshold value dramatically decreases the initial CVA and therefore the SVA values.

In Figure 2.2, we present the EPE and ENE curves for each case A to F, and we also plot the mean collateral values. Computations are carried out by running 10^4 Monte Carlo simulations. It is apparent that the behavior of the EPE and ENE values decrease as a result of increased collateralization. Note that there are peaks in

Table 2.2. Collateral thresholds, initial CVA and SVA values

	Γ_{cpty}	Γ_{inv}	CVA_0	SVA_0
Case A	∞	$-\infty$	1.01×10^{-4}	0.2153
Case B	1.5×10^{-3}	0.4×10^{-3}	6.13×10^{-5}	0.1305
Case C	1×10^{-3}	0.2×10^{-3}	4.36×10^{-5}	0.0931
Case D	0.5×10^{-3}	0.1×10^{-3}	2.18×10^{-5}	0.0464
Case E	0.25×10^{-3}	0.05×10^{-3}	1.14×10^{-5}	0.0243
Case F	0	0	0	0

the collateral value in the very beginning and through the maturity. This effect can be explained as follows: Observe from Table 2.2 that the investor has lower threshold than the counterparty in each cases from A to F. As a result, having a lower threshold value, investor will be posting collateral before the counterparty. Therefore, until the counterparty's exposure reaches the threshold, the collateral value remains negative; meaning that there will be margin calls for the investor before the counterparty.

Figure 2.4 plots the mean of sample CVA paths. Starting from CVA_0 we compute the mean sample paths in each case. The behavior of CVA as a credit hybrid option, as indicated in Remark 2.2.4, can be clearly observed in the graphs. CVA values decrease over time as a result of time decay since the expected loss decreases close to the expiration. The effect of collateralization on the CVA values is apparent in the graphs. Observe that increased initial threshold values are of great importance since one can significantly reduce the future CVA values by appropriately setting the collateral thresholds. Moreover, one can also use dynamic thresholds by linking the threshold values to the counterparties' default intensities or credit ratings. In this way, counterparties will have more control on the future values of the CVA of the CDS contract and dynamically manage the CVA since the collateral thresholds will be reacting to the changes in the default intensities or credit ratings.

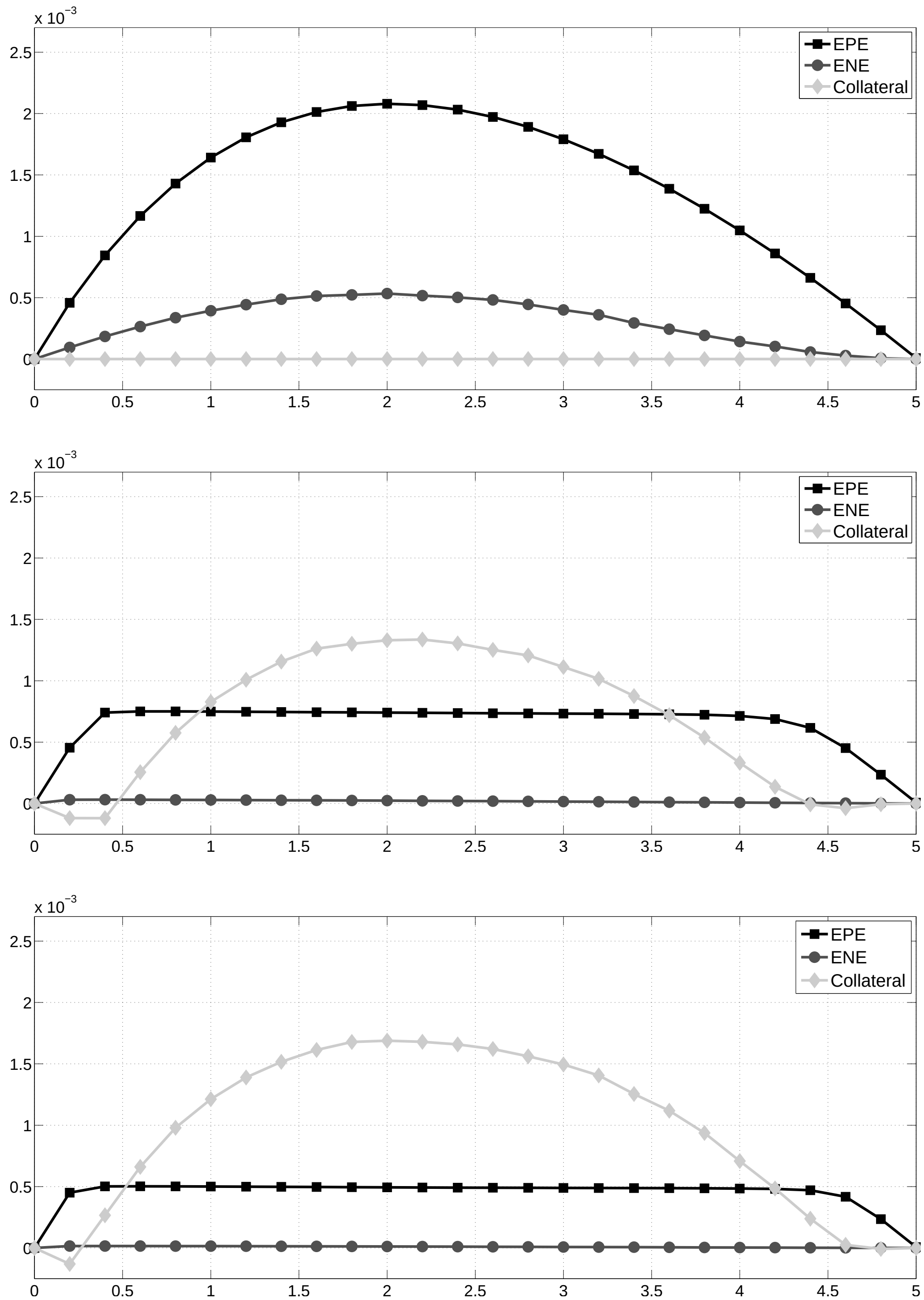

Figure 2.1. EPE, ENE and the Collateral curves for Cases A, B, and C

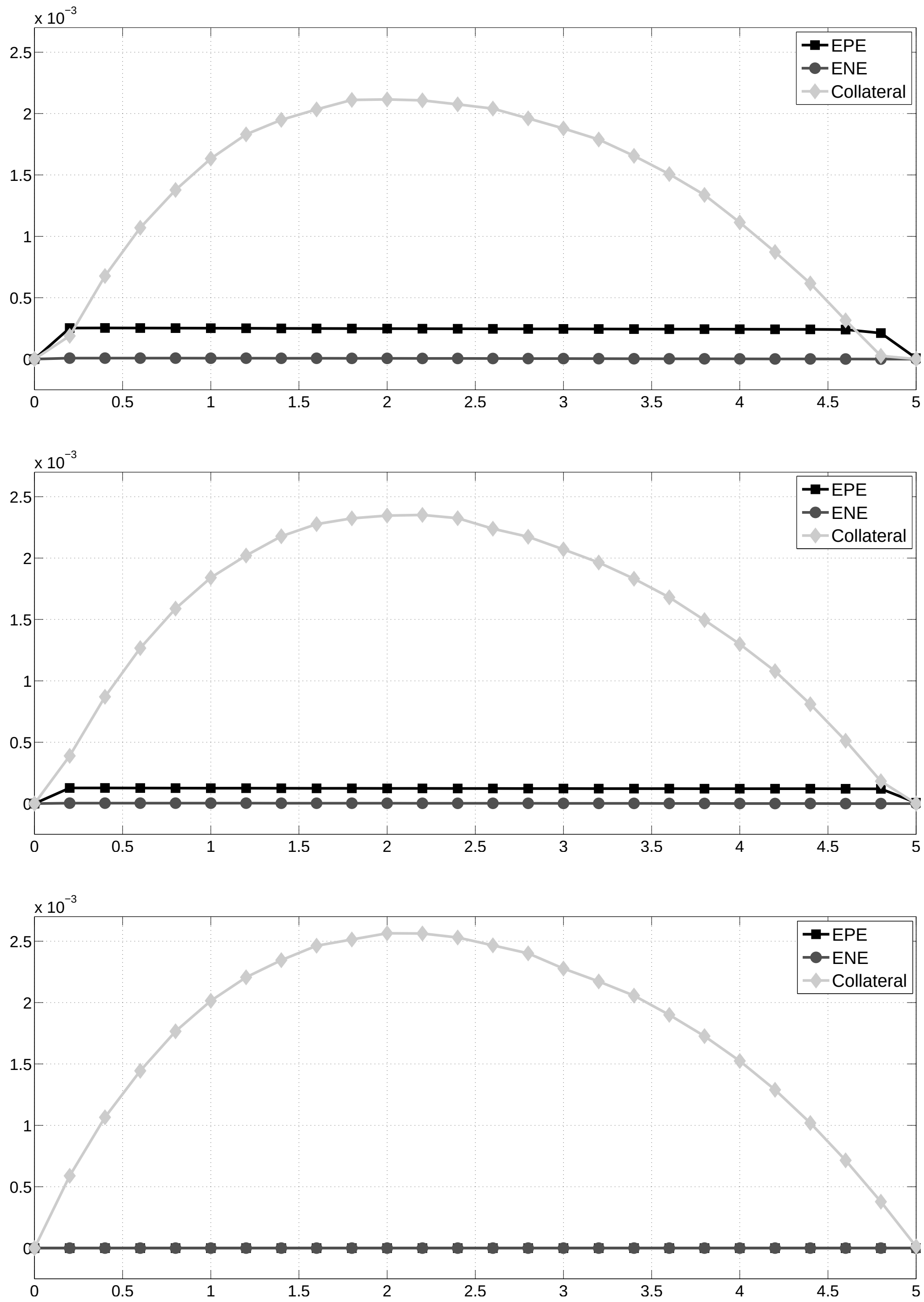

Figure 2.2. EPE, ENE and the Collateral curves for Cases D, E, and F

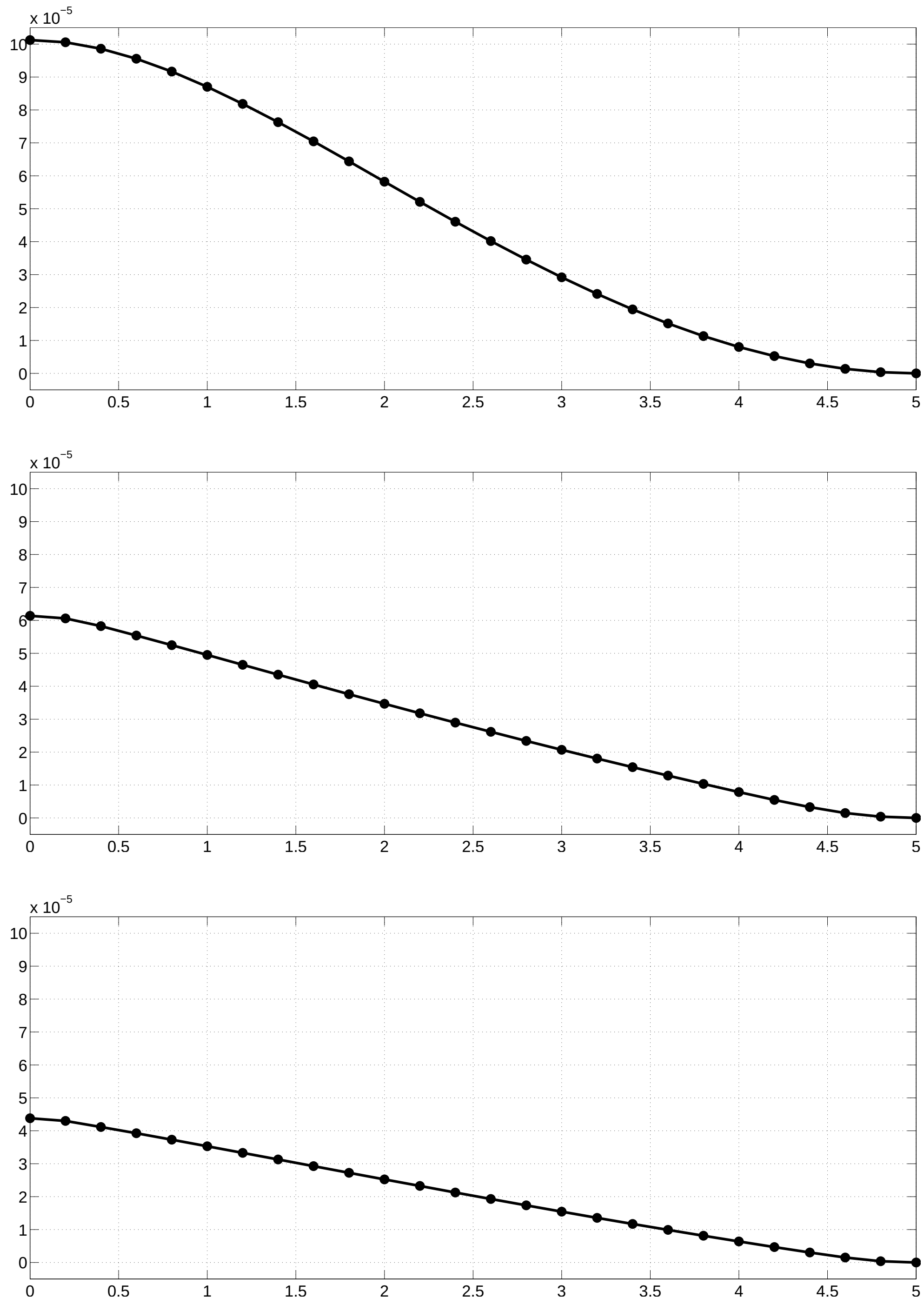

Figure 2.3. Forward CVA curves for Case A, B, and C

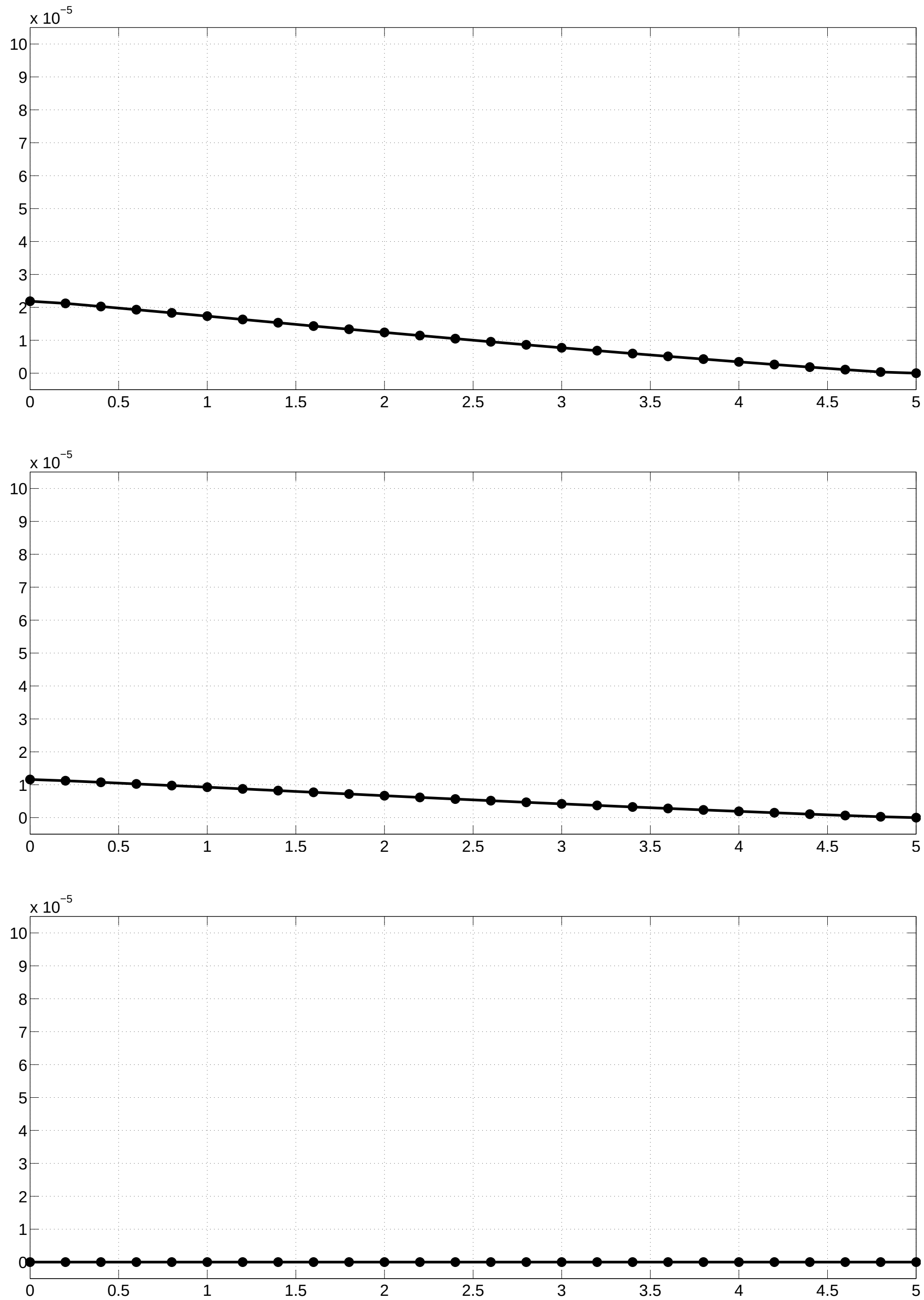

Figure 2.4. Forward CVA curves for Case D, E, and F

CHAPTER 3

COLLATERALIZED CVA VALUATION WITH RATING TRIGGERS AND CREDIT MIGRATIONS

3.1 Introduction

Modeling, managing and mitigating counterparty risk is a crucial task for all financial institutions. One of the most popular mitigation techniques used by the market participants is including additional termination events (ATE) in OTC transactions. As defined in Section 5(b)(vi) of the ISDA Master Agreement (see [Int02]), ATEs allow institutions to terminate and close out the derivatives transactions with a counterparty if a termination event occurs. We consider a particular, and in fact the most common, termination event: rating triggers.

A rating trigger is defined as a threshold credit rating level, which is agreed upon the initiation of the contract. If the credit rating of the counterparty or the investor decreases below the trigger level, before the maturity, the contract is terminated and closed out. Therefore, rating triggers provide additional protection from a counterparty with a deteriorating credit rating, by allowing the investor to terminate the contract prior to a default event. Furthermore, since a significant credit deterioration is usually followed by a default event, adding rating triggers serves as a cushion against such defaults. On the other hand, rating-triggers are also very effective in mitigating counterparty credit risk.

The counterparty credit risk modeling literature has grown significantly since the credit crunch in 2008. We refer to Bielecki, Cialenco, and Iyigunler [BCI11], Assefa, Bielecki, Crepey and Jeanblanc [ABCJ11], Brigo, Capponi, Pallavicini and Papatheodorou [BCPP11] and also Crepey [Cre12a, Cre12b] for recent general results in counterparty risk modeling. Nevertheless, the literature on counterparty risk modeling with rating triggers is very limited. In Yi [Yi11], CVA valuation with rating triggers is studied for optional and mandatory termination events, and a compound Poisson model is introduced for modeling rating transitions and default probabilities. Zhou [Zho10] considers practical problems regarding CVA valuation with additional termination events under a simple model from a practitioner's point of view. Recently, Mercurio [Mer11] studied a similar problem and introduced a valuation model by proposing several generalizing assumptions to simplify the CVA computations considering the unilateral counterparty risk. However, a comprehensive approach which involves the joint modeling of rating transitions in a risk-neutral setting and the dynamic, ratings-dependent collateralization has not been studied in the literature.

In this chapter we consider the problem of collateralized bilateral CVA valuation with rating triggers and credit migrations. We first find the CVA representation in presence of rating triggers. We show that the value of the underlying OTC contract needs to be adjusted also for the rating triggers. This new adjustment term is called the rating valuation adjustment (RVA). We show that RVA represents the expected loss in case of a default event that is preceded by a trigger event. In the bilateral case, we see that RVA is decomposed into two components: URVA and DRVA, representing the rating valuation adjustments for the counterparty's and the investor's rating triggers. Furthermore, we consider dynamic collateralization using the rating transitions. In this framework, the collateral thresholds are defined as the functionals of the current credit ratings of the counterparty and the investor. In practice such rating-dependent margin agreements are standard and they are described in the Credit Support Annex (CSA). Moreover, we consider the rehypothecation risk of the collateral in the presence of independent amounts. These results described above are model-free. Therefore, we employ a Markovian approach for modeling the joint rating transitions and the default probabilities of the counterparty and the investor. The applications of the Markov copulae is previously studied in Bielecki, Crepey, Jeanblanc, and Rutkowski [BCJR06] in the context of basket credit derivatives. Moreover, Bielecki, Vidozzi, and Vidozzi [BVV06, BVV08] applied Markov copulae to the collateralized debt obligations and ratings-triggered corporate step-up bonds. Theoretical aspects of the Markov copulae can be found in Bielecki, Jakubowski, Vidozzi, and Vidozzi [BJVV08] and Bielecki, Jakubowski, and Nieweglowski [BJN11]. We finally illustrate our results with numerical examples. We analyze the impact of early termination clauses and dynamic collateralization on the bilateral and unilateral CVA, as well as the DVA and RVA in case of a CDS and an IRS contract.

3.2 Credit Value Adjustment and Collateralization under Rating Triggers

We consider a generic OTC contract between two names: the investor and the counterparty. In the model we propose in this chapter, the counterparty risk associated with this contract will be sensitive to the current creditworthiness of the two parties. We postulate that the creditworthiness of each party is represented by the same $\mathcal{K} := \{1, 2, \ldots, K\}$ rating categories. We postulate that the ratings are ordered from the best, i.e. 1, to the worst, i.e. K, with the convention that the level K corresponds to a default.

To model the evolution of the credit worthiness we introduce two right continuous processes X^1 and X^2 on $(\Omega, \mathcal{G}, \mathbb{Q})$, with values in $\mathcal{K}$, and we denote by $\mathbb{X}^1$ and $\mathbb{X}^2$ the associated filtrations: $\mathbb{X}^i = (\mathcal{X}_t^i)_{t\geq 0}$ with $\mathcal{X}_t^i = \sigma(X_u^i, u \leq t)$ for $t \in \mathbb{R}_+$, $i = 1, 2$.[7] Processes X^1 and X^2 represent the evolution of the credit ratings of the counterparty and the investor. In what follows we shall make additional specific assumptions about processes X^i, $i = 1, 2$.

We assume that we are given a market filtration $\mathbb{F}$ containing the information about the relevant market variables (i.e. short rate process), and a filtration $\widetilde{\mathbb{F}}$

[7]All filtrations considered in this chapter are assumed to satisfy the usual conditions.

that contains the information regarding the financial contract underlying our OTC contract. Accordingly, we define $\mathbb{G} := \mathbb{F} \vee \widetilde{\mathbb{F}} \vee \mathbb{X}^1 \vee \mathbb{X}^2$.

Recall that the savings account process B is given as,

$$B_t := e^{\int_0^t r_s ds}, \qquad t \in [0, T],$$

where the $\mathbb{F}$-adapted process r models the short-term interest rate. We postulate that $\mathbb{Q}$ represents a pricing measure corresponding to the discount factor $\beta = B^{-1}$.

As mentioned above, both the counterparty and the investor are defaultable, and the respective default times are given as

$$\tau_i := \inf\{t > 0 : X_t^i = K\}, \quad i = 1, 2.$$

We shall also consider the first default time $\tau := \tau_1 \wedge \tau_2$.

We denote by $R_i \in [0, 1]$ a $\mathcal{G}_{\tau_i}$–measurable random variable, which represents the recovery rate of party $i = 1, 2$. The recovery rates represent the fraction of the mark-to-market value of the underlying contract recovered from the defaulting names, which appears in the close-out amounts.

Let D represent the counterparty risk-free cumulative dividend process of our OTC contract over a finite time horizon $[0, T]$, which is the "clean" version of the contract that does not account for the counterparty risk.[8] We assume that D is of finite variation.

In accordance with the classical risk neutral valuation we define the counterparty risk-free ex-dividend price (*clean price* from now on) process of the contract:

Definition 3.2.1. *The ex-dividend price process of a counterparty risk-free contract*

[8]All cash flows are considered from the point of view of the investor.

is defined as, [9]

$$S_t = B_t \mathbb{E}\Big[\int_{]t,T]} B_u^{-1} dD_u \Big| \mathcal{G}_t \Big],$$

for all $t \in [0,T]$.

The process S is also called the clean mark-to-market process. Let us also define the process $S^{\Delta} := S + \Delta D$.

We consider collateralized contracts, therefore we define a $\mathbb{G}$-predictable process C on $[0,T]$ representing cumulative collateral amount in the margin account. Mechanics and the modeling of the collateral process are discussed in Section 3.2.3.

3.2.1 Pricing Bilateral Counterparty Risk. Let us consider the case $K = 2$, therefore allowing only the default and the pre-default states prevail. This case corresponds to the model presented in Chapter 2 in the context of CDS contracts, and also discussed by Bielecki et al. [BCI11, BC11, BCJZ11] and Assefa et al. [ABCJ11].

Recall that, we denote $H_t^1 := \mathbb{1}_{\{\tau_1 \le t\}}$ and $H_t^2 := \mathbb{1}_{\{\tau_2 \le t\}}$ as the default indicator processes of τ_1 and τ_2 respectively. We also define $\tau := \tau_1 \wedge \tau_2$ as the first default time of the counterparty and the investor and let $H := \mathbb{1}_{\{\tau \le t\}}$ be the default indicator process corresponding to τ. We now have $X_t^1 = 1 + H_t^1$ and $X_t^2 = 1 + H_t^2$.

Let $\mathcal{D}^C$ represent the counterparty risky cumulative dividend processes of the contract that is subject to counterparty default risk. Therefore, given D, we define the counterparty risky cumulative dividend process $\mathcal{D}^C$ as follows.

Definition 3.2.2. *Counterparty-risky cumulative dividend process has the following*

[9] Required integrability properties are assumed implicitly.

form

$$\begin{aligned}\mathcal{D}_t &= (1-H_t)D_t + H_t D_{\tau-} + \mathbb{1}_{\{\tau<T\}}(C_\tau \mathbb{1}_{\{\tau\le t\}} \\ &\quad + (R_1(S^\Delta_\tau - C_\tau)^+ - (S^\Delta_\tau - C_\tau)^-)\mathbb{1}_{\{\tau=\tau_1\le t\}} \\ &\quad - (R_2(S^\Delta_\tau - C_\tau)^- - (S^\Delta_\tau - C_\tau)^+)\mathbb{1}_{\{\tau=\tau_2\le t\}} - (S^\Delta_\tau - C_\tau)\mathbb{1}_{\{\tau=\tau_1=\tau_2\le t\}})\,,\end{aligned}$$

for all $t \in [0,T]$.

We proceed with defining the ex-dividend price of a counterparty risky contract,

Definition 3.2.3. *The ex-dividend price process of a counterparty risky contract is given as,*

$$S^C_t = B_t \mathbb{E}\Big[\int_{]t,T]} B_u^{-1} d\mathcal{D}_u \,\Big|\, \mathcal{G}_t \Big]\,,$$

for all $t \in [0,T]$.

Having defined a counterparty risk-free and a counterparty risky contract, we are now interested in the difference between their ex-dividend prices. Recall that, this difference is called the *Credit Valuation Adjustment* (CVA). Since we consider bilateral case, both the investor and the counterparty can default on their contractual obligations. Therefore, we refer to the CVA as the *bilateral* credit valuation adjustment.

Definition 3.2.4. *The bilateral credit valuation adjustment is defined as,*

$$CVA_t = S_t - S^C_t\,,$$

for all $t \in [0, \tau \wedge T]$.

Bilateral counterparty valuation adjustment process has the following representation.

Proposition 3.2.1. *The credit valuation adjustment process can be represented as*

$$\begin{aligned}CVA_t =& B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_1\le T\}}B_\tau^{-1}(1-R_1)(S_\tau^\Delta - C_\tau)^+ \,\Big|\, \mathcal{G}_t\Big] \\ &- B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_2\le T\}}B_\tau^{-1}(1-R_2)(S_\tau^\Delta - C_\tau)^- \,\Big|\, \mathcal{G}_t\Big]\,, \end{aligned} \tag{3.1}$$

for all $t \in [0, \tau \wedge T]$.

A proof of this proposition, where the underlying is assumed to be a CDS contract, is given in Proposition 2.2.1 in Chapter 2, and also in Bielecki et al. [BCI11]. Recall that the bilateral CVA can be decomposed as

$$\mathrm{UCVA}_t = B_t\mathbb{E}[\mathbb{1}_{\{\tau=\tau_1\le T\}}B_\tau^{-1}(1-R_1)(S_\tau^\Delta - C_\tau)^+ \,|\, \mathcal{G}_t\,]\,, \tag{3.2}$$

$$\mathrm{DVA}_t = B_t\mathbb{E}[\mathbb{1}_{\{\tau=\tau_2\le T\}}B_\tau^{-1}(1-R_2)(S_\tau^\Delta - C_\tau)^- \,|\, \mathcal{G}_t\,]\,,$$

for all $t \in [0, \tau \wedge T]$. These components represent the two legs of bilateral CVA, namely the Unilateral Credit Valuation Adjustment (UCVA) and the Debt Valuation Adjustment (DVA), representing the expected losses in case of the counterparty's and the investor's defaults, respectively.

3.2.2 Pricing Bilateral Counterparty Risk with Rating Triggers. We now proceed with introducing the rating trigger times, and the close-out cash flows in the CVA valuation. We also show how the clean price of our OTC contract can be adjusted for the counterparty risk and the rating triggers.

3.2.2.1 Trigger Times. As we stated before, the counterparty risk of the OTC contract we consider is sensitive to the creditworthiness of the investor and the counterparty. Specifically, we consider an OTC contract that is subject to a *rating trigger clause*:

> If the investor's or the counterparty's credit rating deteriorates to or below the *trigger level* (except the default level), the contract is terminated and closed out. Note that there are no mark-to-market losses associated with the trigger events.

The trigger levels are set as K_1 for the counterparty, and K_2 for the investor,[10] where $1 < K_1, K_2 \leq K$. Let τ_i^R represent the first time that the i-th party's credit rating crosses the his rating trigger, that is

$$\tau_i^R := \inf\{t > 0 : X_t^i \geq K_i\}\,, \quad i = 1, 2.$$

The corresponding rating trigger event times[11] are defined as

$$\widehat{\tau}_i^R := \inf\{t > 0 : X_t^i \in \{K_i, K_{i+1}, \ldots, K-1\}\}\,, \quad i = 1, 2,$$

and we set

$$\tau^R := \tau_1^R \wedge \tau_2^R \quad \text{and} \quad \widehat{\tau}^R := \widehat{\tau}_1^R \wedge \widehat{\tau}_2^R\,.$$

Clearly, $\tau_i^R = \widehat{\tau}_i^R \wedge \tau_i$ for $i = 1, 2$.

We denote by $H_t^R := \mathbb{1}_{\{\tau^R \leq t\}}$ and $\widehat{H}_t^R := \mathbb{1}_{\{\widehat{\tau}^R \leq t\}}$ the rating trigger indicator processes including and not including the default event, respectively.

3.2.2.2 Cash Flows, Prices and Adjustments. The close-out portion of the cumulative dividend process of the counterparty risky contract needs to account for the MtM exchange without incurring any losses at a trigger time other than default. On the other hand, if a trigger event occurs simultaneously with a default event, the deal will be settled according to the default event. Consequently, we propose the following definition of the cumulative dividend process of the counterparty risky contract,

Definition 3.2.5. *The counterparty-risky cumulative dividend process of an OTC*

---

[10]It is implicitly assumed that $X_0^i < K_i$ for $i = 1, 2$.

[11]That is, excluding default.

contract subject to rating triggers is defined as

$$
\begin{aligned}
\mathcal{D}_t^R = (1 - H_t^R) D_t + D_{\tau^R -} H_t^R + \mathbb{1}_{\{\tau^R \le T\}} \Big(& C_{\tau^R} H_t^R \\
& + \left(R_1 (S^{\Delta}_{\tau^R} - C_{\tau^R})^+ - (S^{\Delta}_{\tau^R} - C_{\tau^R})^- \right) [H^R, H^1]_t \\
& - \left(R_2 (S^{\Delta}_{\tau^R} - C_{\tau^R})^- - (S^{\Delta}_{\tau^R} - C_{\tau^R})^+ \right) [H^R, H^2]_t \\
& - \left(S^{\Delta}_{\tau^R} - C_{\tau^R} \right) [[H^R, H^1], H^2]_t + \left(S^{\Delta}_{\tau^R} - C_{\tau^R} \right) [H^R, \widehat{H}^R]_t \Big) ,
\end{aligned}
$$

for all $t \in [0, T]$.

Accordingly, the ex-dividend price processes associated with a counterparty risky contract with rating triggers is defined as follows.

Definition 3.2.6. *The ex-dividend price process* S_t^R *of a counterparty risky contract with rating triggers, maturing at time* T*, is defined as*

$$
S_t^R = B_t \mathbb{E}\Big[\int_{]t,T]} B_u^{-1} d\mathcal{D}_u^R \,\Big|\, \mathcal{G}_t \Big] ,
$$

for all $t \in [0, T]$.

We now introduce the credit valuation adjustment term when the underlying contract is subject to rating triggers.

Definition 3.2.7. *The bilateral credit valuation adjustment with rating triggers is defined as,*

$$
CVA_t^R = S_t - S_t^R \,, \tag{3.3}
$$

for $t \in [0, \tau^R \wedge T]$.

The following representation generalizes the results derived in the previous chapter and in Bielecki et al. [BCI11].

Proposition 3.2.2. *The bilateral credit valuation adjustment defined in* (3.3) *can be represented as*

$$
\begin{aligned}
CVA_t^R =& B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R = \tau_1 \le T\}} B_{\tau^R}^{-1} (1 - R_1)(S_{\tau^R}^{\Delta} - C_{\tau^R})^+ \,\Big|\, \mathcal{G}_t \Big] \\
& - B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R = \tau_2 \le T\}} B_{\tau^R}^{-1} (1 - R_2)(S_{\tau^R}^{\Delta} - C_{\tau^R})^- \,\Big|\, \mathcal{G}_t \Big] ,
\end{aligned} \tag{3.4}
$$

for $t \in [0, \tau^R \wedge T]$.

Proof. Using Definition 3.2.5, we get,

$$
\begin{aligned}
dD_t - d\mathcal{D}_t^R = dD_t &- (1 - H_t^R) dD_t - D_{t-} dH_t^R + D_{\tau^R -} dH_t^R - \mathbb{1}_{\{\tau^R \le T\}} C_{\tau^R} dH_t^R \\
&- \mathbb{1}_{\{\tau^R \le T\}} \big(R_1 (S_{\tau^R}^{\Delta} - C_{\tau^R})^+ - (S_{\tau^R}^{\Delta} - C_{\tau^R})^- \big) d[H^R, H^1]_t \\
&+ \mathbb{1}_{\{\tau^R \le T\}} \big(R_2 (S_{\tau^R}^{\Delta} - C_{\tau^R})^- - (S_{\tau^R}^{\Delta} - C_{\tau^R})^+ \big) d[H^R, H^2]_t \\
&+ \mathbb{1}_{\{\tau^R \le T\}} (S_{\tau^R}^{\Delta} - C_{\tau^R}) d[[H^R, H^1], H^2]_t \\
&- \mathbb{1}_{\{\tau^R \le T\}} (S_{\tau^R}^{\Delta} - C_{\tau^R}) d[H^R, \widehat{H}^R]_t \,.
\end{aligned}
$$

Integrating both sides leads to,

$$
\begin{aligned}
\int_{]t,T]} B_u^{-1} (dD_u - d\mathcal{D}_u^R) = & \int_{]t,T]} B_u^{-1} H_u^R dD_u - \int_{]t,T]} B_u^{-1} D_{u-} dH_u^R + \int_{]t,T]} B_u^{-1} D_{\tau^R -} dH_u^R \\
& - \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1} \big(R_1 (S_{\tau^R}^{\Delta} - C_{\tau^R})^+ - (S_{\tau^R}^{\Delta} - C_{\tau^R})^- \big) d[H^R, H^1]_u \\
& + \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1} \big(R_2 (S_{\tau^R}^{\Delta} - C_{\tau^R})^- - (S_{\tau^R}^{\Delta} - C_{\tau^R})^+ \big) d[H^R, H^2]_u \\
& + \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1} (S_{\tau^R}^{\Delta} - C_{\tau^R}) d[[H^R, H^1], H^2]_u \\
& - \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1} (S_{\tau^R}^{\Delta} - C_{\tau^R}) d[H^R, \widehat{H}^R]_u - \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1} C_{\tau^R} dH_u^R \,.
\end{aligned}
$$

Since,

$$
\int_{]t,T]} B_u^{-1} D_{\tau^R -} dH_u^R - \int_{]t,T]} B_u^{-1} D_{u-} dH_u^R = 0 \,,
$$

we obtain,

$$\int_{]t,T]} B_u^{-1}(dD_u - d\mathcal{D}_u^R) = \int_{]t,T]} B_u^{-1} H_u^R dD_u - \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-2} C_{\tau^R} dH_u^R$$
$$- \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1}\big(R_1(S^\Delta_{\tau^R} - C_{\tau^R})^+ - (S^\Delta_{\tau^R} - C_{\tau^R})^-\big) d[H^R, H^1]_u$$
$$+ \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1}\big(R_2(S^\Delta_{\tau^R} - C_{\tau^R})^- - (S^\Delta_{\tau^R} - C_{\tau^R})^+\big) d[H^R, H^2]_u$$
$$+ \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1}(S^\Delta_{\tau^R} - C_{\tau^R}) d[[H^R, H^1], H^2]_u$$
$$- \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1}(S^\Delta_{\tau^R} - C_{\tau^R}) d[H^R, \widehat{H}^R]_u \,.$$

Conditioning on τ^R, we get

$$\begin{aligned}
\mathbb{1}_{\{t \le \tau^R \wedge T\}}\ \mathbb{E}\Big[\int_{]t,T]} B_u^{-1}(dD_u - d\mathcal{D}_u^R) \,\Big|\, \mathcal{G}_{\tau^R}\Big] &= \mathbb{1}_{\{t \le \tau^R \wedge T\}}\ \mathbb{E}\Bigg[\int_{]t,T]} B_u^{-1} H_u^R dD_u \\
&- \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1} C_{\tau^R} dH_u^R \\
&- \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1}\big(R_1(S^\Delta_{\tau^R} - C_{\tau^R})^+ - (S^\Delta_{\tau^R} - C_{\tau^R})^-\big) d[H^R, H^1]_u \\
&+ \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1}\big(R_2(S^\Delta_{\tau^R} - C_{\tau^R})^- - (S^\Delta_{\tau^R} - C_{\tau^R})^+\big) d[H^R, H^2]_u \\
&+ \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1}(S^\Delta_{\tau^R} - C_{\tau^R}) d[[H^R, H^1], H^2]_u \qquad (3.5) \\
&- \int_{]t,T]} \mathbb{1}_{\{\tau^R \le T\}} B_u^{-1}(S^\Delta_{\tau^R} - C_{\tau^R}) d[H^R, \widehat{H}^R]_u \,\Bigg|\, \mathcal{G}_{\tau^R}\Bigg].
\end{aligned}$$

Notice that, since $t \in [0, \tau^R \wedge T]$, we have

$$\begin{aligned}
\int_{]t,T]} B_u^{-1} H_u^R dD_u &= \int_{]t,\tau^R[} B_u^{-1} H_u^R dD_u + \int_{[\tau^R,T]} B_u^{-1} H_u^R dD_u \\
&= \int_{[\tau^R,T]} B_u^{-1} H_u^R dD_u \,. \qquad (3.6)
\end{aligned}$$

Therefore,

$$\mathbb{1}_{\{t \le \tau^R \wedge T\}}\mathbb{E}\Big[\int_{]t,T]} B_u^{-1} H_u^R dD_u \,\Big|\, \mathcal{G}_{\tau^R}\Big] = \mathbb{1}_{\{t \le \tau^R \wedge T\}}\mathbb{E}\Big[\int_{[\tau^R,T]} B_u^{-1} H_u^R dD_u \,\Big|\, \mathcal{G}_{\tau^R}\Big]$$
$$= \mathbb{1}_{\{t \le \tau^R \wedge T\}}\mathbb{1}_{\{\tau^R \le T\}} B_{\tau^R}^{-1}(S_{\tau^R} + \Delta D_{\tau^R}) = \mathbb{1}_{\{t \le \tau^R\}}\mathbb{1}_{\{\tau^R \le T\}} B_{\tau^R}^{-1}(S_{\tau^R} + \Delta D_{\tau^R}) \,. \quad (3.7)$$

Taking conditional expectation given $\mathcal{G}_t$ and using the tower property in (3.12) reads

$$\mathbb{1}_{\{t\le\tau^R\wedge T\}}(S_t - S_t^R) = \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[\int_{]t,T]} B_u^{-1}(dD_u - d\mathcal{D}_u^R)\,\Big|\,\mathcal{G}_t\Big] \tag{3.8}$$

$$= \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[B_{\tau^R}^{-1}\big(\mathbb{1}_{\{\tau^R\le T\}}(S^\Delta_{\tau^R} - C_{\tau^R})$$

$$- (R_1(S^\Delta_{\tau^R} - C_{\tau^R})^+ - (S^\Delta_{\tau^R} - C_{\tau^R})^-)\mathbb{1}_{\{\tau^R=\tau_1\le T\}}$$

$$+ (R_2(S^\Delta_{\tau^R} - C_{\tau^R})^- - (S^\Delta_{\tau^R} - C_{\tau^R})^+)\mathbb{1}_{\{\tau^R=\tau_2\le T\}}$$

$$+ (S^\Delta_{\tau^R} - C_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_1=\tau_2\le T\}} - (S^\Delta_{\tau^R} - C_{\tau^R})\mathbb{1}_{\{\tau^R=\widehat{\tau}^R\le T\}}\big)\,\Big|\,\mathcal{G}_t\Big]\,.$$

Since

$$(S^\Delta_{\tau^R} - C_{\tau^R}) = (S^\Delta_{\tau^R} - C_{\tau^R})^+ - (S^\Delta_{\tau^R} - C_{\tau^R})^-\,,$$

it follows that (3.15) is equivalent to

$$\mathbb{1}_{\{t\le\tau^R\wedge T\}}(S_t - S_t^R) = \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[B_{\tau^R}^{-1}\big(\mathbb{1}_{\{\tau^R\le T\}}(S^\Delta_{\tau^R} - C_{\tau^R})$$

$$- (R_1(S^\Delta_{\tau^R} - C_{\tau^R})^+ + (S^\Delta_{\tau^R} - C_{\tau^R}) - (S^\Delta_{\tau^R} - C_{\tau^R})^+)\mathbb{1}_{\{\tau^R=\tau_1\le T\}}$$

$$+ (R_2(S^\Delta_{\tau^R} - C_{\tau^R})^- - (S^\Delta_{\tau^R} - C_{\tau^R})^- - (S^\Delta_{\tau^R} - C_{\tau^R}))\mathbb{1}_{\{\tau^R=\tau_2\le T\}}$$

$$+ (S^\Delta_{\tau^R} - C_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_1=\tau_2\le T\}} - (S^\Delta_{\tau^R} - C_{\tau^R})\mathbb{1}_{\{\tau^R=\widehat{\tau}^R\le T\}}\big)\,\Big|\,\mathcal{G}_t\Big]\,.$$

After simplifying the terms above, we obtain

$$\mathbb{1}_{\{t\le\tau^R\wedge T\}}(S_t - S_t^R) = \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[B_{\tau^R}^{-1}\big(\mathbb{1}_{\{\tau^R\le T\}}(S^\Delta_{\tau^R} - C_{\tau^R})$$

$$+ (1-R_1)(S^\Delta_{\tau^R} - C_{\tau^R})^+\mathbb{1}_{\{\tau^R=\tau_1\le T\}} - (S^\Delta_{\tau^R} - C_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_1\le T\}}$$

$$- (1-R_2)(S^\Delta_{\tau^R} - C_{\tau^R})^-\mathbb{1}_{\{\tau^R=\tau_2\le T\}} - (S^\Delta_{\tau^R} - C_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_2\le T\}}$$

$$+ (S^\Delta_{\tau^R} - C_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_1=\tau_2\le T\}} - (S^\Delta_{\tau^R} - C_{\tau^R})\mathbb{1}_{\{\tau^R=\widehat{\tau}^R\le T\}}\big)\,\Big|\,\mathcal{G}_t\Big]\,,$$

which is equivalent to

$$\mathbb{1}_{\{t\le\tau^R\wedge T\}}(S_t - S_t^R) = \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[B_{\tau^R}^{-1}\big[\mathbb{1}_{\{\tau^R\le T\}}(S^\Delta_{\tau^R} - C_{\tau^R}) - \mathbb{1}_{\{\tau^R\le T\}}(S^\Delta_{\tau^R} - C_{\tau^R})$$

$$+ (1-R_1)(S^\Delta_{\tau^R} - C_{\tau^R})^+\mathbb{1}_{\{\tau^R=\tau_1\le T\}} - (1-R_2)(S^\Delta_{\tau^R} - C_{\tau^R})^-\mathbb{1}_{\{\tau^R=\tau_2\le T\}}\big]\,\Big|\,\mathcal{G}_t\Big]\,.$$

Finally, we find that

$$S_t - S_t^R = B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\le T\}} B_{\tau^R}^{-1}(1-R_1)(S_{\tau^R}^{\Delta} - C_{\tau^R})^+ \,\Big|\, \mathcal{G}_t\Big]$$
$$- B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\le T\}} B_{\tau^R}^{-1}(1-R_2)(S_{\tau^R}^{\Delta} - C_{\tau^R})^- \,\Big|\, \mathcal{G}_t\Big]\,,$$

on the set $t \in [0, \tau^R \wedge T]$, which proves our claim. □

Note that since there are no losses associated with the trigger events other than defaults, and since CVA (as well as CVA^R) only reflects the expected losses, these cases do not appear directly in (3.4).

Similar to (3.2), we can define

$$\mathrm{UCVA}_t^R := B_t \mathbb{E}[\mathbb{1}_{\{\tau^R=\tau_1\le T\}} B_{\tau^R}^{-1}(1-R_1)(S_{\tau^R}^{\Delta} - C_{\tau^R})^+ \,|\, \mathcal{G}_t\,]\,,$$
$$\mathrm{DVA}_t^R := B_t \mathbb{E}[\mathbb{1}_{\{\tau^R=\tau_2\le T\}} B_{\tau^R}^{-1}(1-R_2)(S_{\tau^R}^{\Delta} - C_{\tau^R})^- \,|\, \mathcal{G}_t\,]\,,$$

for $t \in [0, \tau^R \wedge T]$. Therefore, the credit valuation adjustment representation found in (3.4) can be decomposed as $\mathrm{CVA}^R = \mathrm{UCVA}^R - \mathrm{DVA}^R$.

Remark 3.2.1. *Note that although banks report on DVA (or DVA^R in our case) in their earnings reports, it is not included in determining the capital levels. This is also stated in [Ban11, Paragraph 75] as*

> *Derecognise in the calculation of Common Equity Tier 1, all unrealised gains and losses that have resulted from changes in the fair value of liabilities that are due to changes in the banks own credit risk.*

Therefore, Basel III framework does not allow the banks to account for DVA in their regulatory capital calculations (see also [Ban12] for a detailed discussion). The main reason of this treatment of DVA in Basel III is to not to allow banks to have the value of their liabilities decrease while their credit risk is increasing.

The difference between CVA and CVA^R indicates the change, either as a reduction or as an increase, in the CVA due to rating triggers. Therefore, the counterparties can determine the appropriate rating trigger levels at the initiation of the contracts. This is very important for the financial institutions; since, as we mentioned before, rating triggers are commonly used tools to mitigate the CVA. This leads us to introduce the following concept.

Definition 3.2.8. *The Rating Valuation Adjustment (RVA) process, is defined as*

$$RVA_t = CVA_t - CVA_t^R\,, \tag{3.9}$$

for all $t \in [0, \tau^R \wedge T]$.

The rating valuation adjustment term defined above has the following representation.

Proposition 3.2.3. *The* RVA *process can be represented as*

$$\begin{aligned}\mathrm{RVA}_t &= B_t \mathbb{E}[\mathbb{1}_{\{\tau^R < \tau = \tau_1 \le T\}} B_\tau^{-1}(1-R_1)(S_\tau^\Delta - C_\tau)^+ \,|\, \mathcal{G}_t\,] \\ &\quad - B_t \mathbb{E}[\mathbb{1}_{\{\tau^R < \tau = \tau_2 \le T\}} B_\tau^{-1}(1-R_2)(S_\tau^\Delta - C_\tau)^- \,|\, \mathcal{G}_t\,]\,,\end{aligned}$$

for all $t \in [0, \tau^R \wedge T]$.

Proof. From (3.1) and (3.4) we obtain

$$\begin{aligned}\mathrm{CVA}_t - \mathrm{CVA}_t^R = B_t \mathbb{E}\Big[&\mathbb{1}_{\{\tau = \tau_1 \le T\}} B_\tau^{-1}(1-R_1)(S_\tau^\Delta - C_\tau)^+ \,\Big|\, \mathcal{G}_t\Big] \\ &- B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau = \tau_2 \le T\}} B_\tau^{-1}(1-R_2)(S_\tau^\Delta - C_\tau)^- \,\Big|\, \mathcal{G}_t\Big] \\ &- B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R = \tau_1 \le T\}} B_{\tau^R}^{-1}(1-R_1)(S_{\tau^R}^\Delta - C_{\tau^R})^+ \,\Big|\, \mathcal{G}_t\Big] \\ &+ B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R = \tau_2 \le T\}} B_{\tau^R}^{-1}(1-R_2)(S_{\tau^R}^\Delta - C_{\tau^R})^- \,\Big|\, \mathcal{G}_t\Big]\,,\end{aligned}$$

which can be written as,

$$
\begin{aligned}
\mathrm{CVA}_t - \mathrm{CVA}_t^R = {} & B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_1\le T\}} B_{\tau_1}^{-1}(1-R_1)(S_{\tau_1}^{\Delta} - C_{\tau_1})^+ \,\Big|\, \mathcal{G}_t\Big] \\
& - B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_2\le T\}} B_{\tau_2}^{-1}(1-R_2)(S_{\tau_2}^{\Delta} - C_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big] \\
& - B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\le T\}} B_{\tau_1}^{-1}(1-R_1)(S_{\tau_1}^{\Delta} - C_{\tau_1})^+ \,\Big|\, \mathcal{G}_t\Big] \\
& + B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\le T\}} B_{\tau_2}^{-1}(1-R_2)(S_{\tau_2}^{\Delta} - C_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big] .
\end{aligned}
$$

Therefore, simplifying the terms above yields

$$
\begin{aligned}
\mathrm{CVA}_t - \mathrm{CVA}_t^R = {} & B_t \mathbb{E}\Big[(\mathbb{1}_{\{\tau=\tau_1\le T\}} - \mathbb{1}_{\{\tau^R=\tau_1\le T\}}) B_{\tau_1}^{-1}(1-R_1)(S_{\tau_1}^{\Delta} - C_{\tau_1})^+ \,\Big|\, \mathcal{G}_t\Big] \\
& - B_t \mathbb{E}\Big[(\mathbb{1}_{\{\tau=\tau_2\le T\}} - \mathbb{1}_{\{\tau^R=\tau_2\le T\}}) B_{\tau_2}^{-1}(1-R_2)(S_{\tau_2}^{\Delta} - C_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big] ,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\mathrm{CVA}_t - \mathrm{CVA}_t^R = {} & B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R<\tau=\tau_1\le T\}} B_{\tau_1}^{-1}(1-R_1)(S_{\tau_1}^{\Delta} - C_{\tau_1})^+ \,\Big|\, \mathcal{G}_t\Big] \\
& - B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R<\tau=\tau_2\le T\}} B_{\tau_2}^{-1}(1-R_2)(S_{\tau_2}^{\Delta} - C_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big] ,
\end{aligned}
$$

for $t \in [0, \tau^R \wedge T]$, which proves the result in view of (3.9). □

Remark 3.2.2. *Note that RVA can be positive or negative. If RVA is positive then there is a decrease in the bilateral CVA. If RVA is negative then this indicates an increase in the bilateral CVA due to adding rating triggers. Furthermore, RVA is always non-negative in case of measuring unilateral counterparty risk ($\tau_2 = \infty$).*

Let us define

$$
\begin{aligned}
\mathrm{URVA}_t &:= B_t \mathbb{E}[\mathbb{1}_{\{\tau^R<\tau=\tau_1\le T\}} B_{\tau}^{-1}(1-R_1)(S_{\tau}^{\Delta} - C_{\tau})^+ \,|\, \mathcal{G}_t\,] , \\
\mathrm{DRVA}_t &:= B_t \mathbb{E}[\mathbb{1}_{\{\tau^R<\tau=\tau_2\le T\}} B_{\tau}^{-1}(1-R_2)(S_{\tau}^{\Delta} - C_{\tau})^- \,|\, \mathcal{G}_t\,] .
\end{aligned}
$$

for $t \in [0, \tau^R \wedge T]$. Therefore, RVA has the following decomposition,

$$
\mathrm{RVA}_t = \mathrm{URVA}_t - \mathrm{DRVA}_t ,
$$

for $t \in [0, \tau^R \wedge T]$. Here URVA represents the expected loss if the counterparty defaults first which is preceded by a rating trigger. Similarly, DRVA is the expected loss in case the investor defaults first after a rating trigger. Therefore, including rating triggers provision in an OTC contract provides protection from losses due to default events which happen after a credit downgrade. Accordingly, the value of the contract is adjusted for this protection, as shown in the following result.

Corollary 3.2.1. *We have the following decomposition for the counterparty risky price process*

$$\begin{aligned} S_t^R &= S_t - CVA_t^R \\ &= S_t - CVA_t + RVA_t \\ &= S_t - UCVA_t + DVA_t + RVA_t \\ &= S_t - UCVA_t + DVA_t + URVA_t - DRVA_t \,, \end{aligned}$$

for $t \in [0, \tau^R \wedge T]$.

The above result is particularly important because it provides an explicit view of the adjustments we consider in case the underlying contracts have rating triggers. In practice, each term in the above decomposition is computed separately. Moreover, each term in the decomposition is treated differently, in the sense that different desks and departments are responsible for applying the above adjustments. We will see that it is possible to have further refinements of the above decomposition, by considering additional risks associated with the counterparty risk.

3.2.3 Dynamic Collateralization. As we mentioned in Section 2.2.1.1 in Chapter 2, in bilateral margin agreements, counterparties are required to post collateral as soon as the clean price of the contract exceeds thresholds, which are defined in CSA (see [Int94]). In particular, these thresholds are defined in terms of the credit ratings of the counterparties. Specifically, the collateral threshold of a counterparty decreases as a result of a credit rating downgrade and increases as a result of a credit rating upgrade. Consequently, a counterparty with higher credit rating will have higher threshold than a counterparty with a lower credit rating.

It is important to note that there is an adverse relation between the margin requirements and the credit ratings. A credit downgrade along with higher borrowing rates and exposures forces the companies to post increasing amounts of collateral to their counterparties, which can be fatal. For example, the ratings linked collateral thresholds, coupled with rehypothecation, have been considered to be one of the key drivers of AIG's collapse in 2008. Before 2007, as a 'AAA' rated company, AIG had not been required to post any collateral for most of its derivatives transactions. However, after several downgrades AIG had posted more than $40 billion in collateral as of November 2008 (see [Int09] for details).

Thus, one of the key issues in modeling of the collateral process[12] is the issue of modeling of the thresholds. In what follows, we shall model the collateral threshold for the counterparty at time t, say Γ^1_t, as $\Gamma^1_t = \gamma^1(t, X^1_t, S_t)$, where $\gamma^1 : [0,T] \times \mathcal{K} \times \mathbb{R} \to \mathbb{R}_+$ is a measurable function. Likewise, we shall model the collateral threshold for the investor at time t, say Γ^2_t, as $\Gamma^2_t = \gamma^2(t, X^2_t, S_t)$, where $\gamma^2 : [0,T] \times \mathcal{K} \times \mathbb{R} \to \mathbb{R}_-$ is a measurable function.

For a proper modeling of the collateral we need to consider the so called *independent amounts* (i.e. initial margins) posted by the counterparty and the investor by the constants $\beta_1 \in \mathbb{R}_+$ and $\beta_2 \in \mathbb{R}_-$, respectively. We also need to consider the so called *minimum transfer amount* (MTA), which is a positive constant denoted by θ and the *margin period of risk*, which is again a positive constant denoted by Δ.[13]

[12]Since in this chapter we only consider symmetric cash flows (form the point of view of both the parties), we only need to model a single collateral process.

[13]We refer to Section 2.2.1.1, and to Bielecki et al. [BCI11] for a detailed dis-

Let us denote the margin call dates by $0 < t_1 < \ldots < t_n < T$ as in Section 2.2.1.1. On the margin call date t_i, if the exposure is above the counterparty's current threshold, $\Gamma^1_{t_i}$, and if the difference between the current exposure and the collateral amount is greater than the MTA the counterparty posts collateral and updates the margin account; otherwise, no collateral exchange takes place since the transfer amount is less than the MTA. Likewise, the investor delivers collateral on the margin call date t_i, if the exposure is below investor's threshold, $\Gamma^2_{t_i}$, and the difference between the current exposure and the collateral amount is greater than MTA (cf. [Int05], pages 52–56).

In accordance with the above discussion the collateral process is modeled as follows:

We set $C_0 = 0$. Then, for $i = 1, 2, \ldots, n$, we define

$$\begin{aligned} C_t := {} & \mathbb{1}_{\{S_{t_i}+B_{t_i}(\beta_1-\beta_2)-\Gamma^1_{t_i}-C_{t_i}>\theta\}}(S_{t_i} + B_{t_i}(\beta_1 - \beta_2) - \Gamma^1_{t_i} - C_{t_i}) \\ & + \mathbb{1}_{\{S_{t_i}+B_{t_i}(\beta_2-\beta_1)-\Gamma^2_{t_i}-C_{t_i}<-\theta\}}(S_{t_i} + B_{t_i}(\beta_2 - \beta_1) - \Gamma^2_{t_i} - C_{t_i}) + C_{t_i}, \end{aligned}$$

for $t \in (t_i, t_{i+1}]$, on the set $\{t_i < \tau^R\}$. Moreover, $C_t = C_{\tau^R}$ on the set $\{\tau^R \leq t \leq \tau^R + \Delta\}$, where Δ represents the margin period of risk.

Observe that the collateral increments at each margin call date $t_i < \tau^R$ can now be represented as,

$$\begin{aligned} \Delta C_{t_i} : &= C_{t_i+} - C_{t_i} \\ &= \mathbb{1}_{\{S_{t_i}+B_{t_i}(\beta_1-\beta_2)-\Gamma^1_{t_i}-C_{t_i}>\theta\}}(S_{t_i} + B_{t_i}(\beta_1 - \beta_2) - \Gamma^1_{t_i} - C_{t_i}) \\ &+ \mathbb{1}_{\{S_{t_i}+B_{t_i}(\beta_2-\beta_1)-\Gamma^2_{t_i}-C_{t_i}<-\theta\}}(S_{t_i} + B_{t_i}(\beta_2 - \beta_1) - \Gamma^2_{t_i} - C_{t_i}). \end{aligned}$$

In Section 3.4 we assume, for simplicity, that the margin period of risk, in-

cussion and definitions.

dependent amounts and minimum transfer amount are equal to zero. Thus, the collateral amount at time t (from the point of view of the investor) is given as

$$C_t = \mathbb{1}_{\{S_{t_i}-\Gamma^1_{t_i}>C_{t_i}\}}(S_{t_i} - \Gamma^1_{t_i} - C_{t_i}) + \mathbb{1}_{\{S_{t_i}-\Gamma^2_{t_i}<C_{t_i}\}}(S_{t_i} - \Gamma^2_{t_i} - C_{t_i}) + C_{t_i},$$

for $t \in (t_i, t_{i+1}]$. Furthermore, we consider the following structure for the collateral thresholds

$$\gamma^i(t, x, s) = \rho^i(t, x)s, \qquad i = 1, 2,$$

where $\rho^i : [0, T] \times \mathcal{K} \to [0, 1]$ is a measurable function. The functions ρ^1 and ρ^2 represent the *collateral rates* for the counterparty and the investor at time t, respectively. Essentially, the collateral rates indicate the percentage of exposure at time t.

In practice, the threshold levels are set in CSA documents (available upon request) for different credit rating levels. However, these levels usually do not follow a pattern, and they are not formulated as functions of the credit rating levels. Here, we propose two forms of the collateral threshold levels. We introduce two specifications of collateral rates:[14]

- The *linear case*:
$$\rho^i_l(t, x) := \frac{K - x}{K - 1}$$
for all $i = 1, 2$. In particular, $\rho^i(t, 1) = 1$ and $\rho^i(t, K) = 0$.

- The *exponential case*:
$$\rho^i_e(t, x) := \begin{cases} e^{1-x}, & \text{if } x < K \\ 0, & \text{if } x = K \end{cases}$$
for all $i = 1, 2$.

[14] Recall that the credit ratings of each credit name take values in the set $\mathcal{K} = \{1, 2, \ldots, K\}$, where K represents the default and where 1 represents the highest possible rating.

In the linear case, collateral thresholds change linearly with the credit qualities of the counterparties. Likewise, in the exponential case, the collateral thresholds exponentially change with the credit ratings. Therefore, the collateral rates in the exponential case are always less than the ones in the linear case, which leads to lower collateral thresholds, and as a result more collateral being kept in the margin account. In both cases, the amount of collateral to be posted increases with the decreasing credit ratings. We will use these collateralization schemes for our experiments in the last section.

3.2.4 Rehypothecation Risk. We now consider the case that the collateral receiver (counterparty or the investor) can rehypothecate the collateral. Rehypothecation[15] refers to the usage of the collateral in the margin account for (risky) investment and funding purposes. Naturally, it may not be possible to fully recover the collateral in case of a default, if the counterparties rehypothecate the collateral. Therefore, rehypothecation risk is defined as the risk of not fully recovering collateral as a result of rehypothecation. The vital importance of considering rehypothecation risk is also stated in ISDA's AIG report (see [Int09]) as follows.

> Normally, the lender (i.e., AIG) would invest collateral received in highly liquid and safe short-term securities such as Treasury bills to earn a modest return.AIG, however, invested the collateral it received in subprime mortgage backed securities.As borrowers began returning the securities they had borrowed and demanding repayment of collateral.AIG found it could not sell the mortgage-backed securities in which it had invested the cash collateral and had to search for alternative sources of funds.

Let us define a $\mathcal{G}_{\tau_1}$-measurable random variable R_1^h and a $\mathcal{G}_{\tau_2}$-measurable random variable R_2^h as the recovery rates of the rehypothecated collateral for the investor

[15]Origin of the word *rehypothecate* comes from the Medieval Latin *hypothecare* to pledge, from Late Latin *hypotheca* pledge, from Greek *hypotheke*, from *hypotithenai* to put under, deposit as a pledge

and the counterparty. Following Brigo et al. [BCPP11], we assume that $R_1 \leq R_1^h$ and $R_2 \leq R_2^h$, since in case of a default the collateral has priority among other liabilities.

Let us now define the cumulative dividend process associated with the counterparty risky contract with rehypothecation.

Definition 3.2.9. *Cumulative dividend process of a counterparty risky contract that takes rehypothecation risk into account is represented as,*

$$
\begin{aligned}
\mathcal{D}_t^{R,h} = (1 - H_t^R) D_t + D_{\tau^R -} H_t^R + \mathbb{1}_{\{\tau^R \leq T\}} \Big(& \widetilde{C}_{\tau^R} H_t^R \\
& + \Big(R_1 (S_{\tau^R}^{\Delta} - \widetilde{C}_{\tau^R})^+ - (S_{\tau^R}^{\Delta} - \widetilde{C}_{\tau^R})^- \Big) [H^R, H^1]_t \\
& - \Big(R_2 (S_{\tau^R}^{\Delta} - \widetilde{C}_{\tau^R})^- - (S_{\tau^R}^{\Delta} - \widetilde{C}_{\tau^R})^+ \Big) [H^R, H^2]_t \\
& - \Big(S_{\tau^R}^{\Delta} - \widetilde{C}_{\tau^R} \Big) [[H^R, H^1], H^2]_t + \Big(S_{\tau^R}^{\Delta} - \widetilde{C}_{\tau^R} \Big) [H^R, \widehat{H}^R]_t \Big),
\end{aligned}
$$

for all $t \in [0, T]$, where

$$
\begin{aligned}
\widetilde{C}_{\tau^R} = C_{\tau^R} \Big[& \mathbb{1}_{\tau^R = \tau_1 \neq \tau_2} (R_1^h \mathbb{1}_{C_\tau^R > 0} + \mathbb{1}_{C_{\tau^R} \leq 0}) + \mathbb{1}_{\tau^R = \tau_2 \neq \tau_1} (\mathbb{1}_{C_{\tau^R} > 0} + R_2^h \mathbb{1}_{C_{\tau^R} \leq 0}) \\
& + \mathbb{1}_{\tau^R = \tau_1 = \tau_2} (R_1^h \mathbb{1}_{C_{\tau^R} > 0} + R_2^h \mathbb{1}_{C_{\tau^R} \leq 0}) + \mathbb{1}_{\tau^R = \widehat{\tau}^R} \Big].
\end{aligned}
$$

The definition above can be interpreted as follows. If the counterparty defaults first and if he also holds the collateral, then he delivers only a fraction, R_1^h, of the collateral posted to the margin account. Likewise, if the investor defaults first and if he holds the collateral, then he delivers only a fraction, R_2^h, of the collateral posted to the margin account.

We are now ready to define the ex-dividend price processes associated with a counterparty risky contract with rating triggers and rehypothecation risk.

Definition 3.2.10. *The ex-dividend price process $S^{R,h}$ of a counterparty risky contract maturing at time T, with rating triggers and rehypothecation risk is defined as,*

$$
S_t^{R,h} = B_t \mathbb{E} \Big[\int_{]t,T]} B_u^{-1} d\mathcal{D}_u^{R,h} \,\Big|\, \mathcal{G}_t \Big],
$$

for all $t \in [0, T]$.

Next, we give the definition of credit valuation adjustment of a contract with rating triggers in presence of rehypothecation risk.

Definition 3.2.11. *The credit valuation adjustment with rating triggers taking the rehypothecation risk into account is defined as,*

$$CVA_t^{R,h} = S_t - S_t^{R,h}, \tag{3.10}$$

for all $t \in [0, \tau^R \wedge T]$.

This form of the counterparty-risky cumulative dividend process leads to the following representation for the bilateral CVA.

Proposition 3.2.4. *The bilateral Credit Valuation Adjustment process with rehypothecation risk defined in* (3.10) *can be represented as*

$$\begin{aligned} CVA_t^{R,h} = {} & B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R = \tau_1 \le T\}} B_{\tau^R}^{-1} (1 - R_1)(S_{\tau^R}^{\Delta} - \widetilde{C}_{\tau^R}^1)^+ \Big| \mathcal{G}_t \Big] \\ & - B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R = \tau_2 \le T\}} B_{\tau^R}^{-1} (1 - R_2)(S_{\tau^R}^{\Delta} - \widetilde{C}_{\tau^R}^2)^- \Big| \mathcal{G}_t \Big], \end{aligned} \tag{3.11}$$

for all $t \in [0, \tau^R \wedge T]$, *where*

$$\widetilde{C}_{\tau^R}^1 = C_{\tau^R} \Big[\mathbb{1}_{\tau^R = \tau_1 \neq \tau_2} (R_1^h \mathbb{1}_{C_{\tau}^R > 0} + \mathbb{1}_{C_{\tau^R} \le 0}) + \mathbb{1}_{\tau^R = \tau_1 = \tau_2} (R_1^h \mathbb{1}_{C_{\tau^R} > 0} + R_2^h \mathbb{1}_{C_{\tau^R} \le 0}) \Big| \mathcal{G}_t \Big],$$

and

$$\widetilde{C}_{\tau^R}^2 = C_{\tau^R} \Big[\mathbb{1}_{\tau^R = \tau_2 \neq \tau_1} (\mathbb{1}_{C_{\tau^R} > 0} + R_2^h \mathbb{1}_{C_{\tau^R} \le 0}) + \mathbb{1}_{\tau^R = \tau_1 = \tau_2} (R_1^h \mathbb{1}_{C_{\tau^R} > 0} + R_2^h \mathbb{1}_{C_{\tau^R} \le 0}) \Big| \mathcal{G}_t \Big].$$

Proof. Using Definition 3.2.9, we have

$$
\begin{aligned}
dD_t - d\mathcal{D}^{R,h}_t = {} & dD_t - (1-H^R_t)dD_t - D_{t-}dH^R_t + D_{\tau^R-}dH^R_t - \mathbb{1}_{\{\tau^R\le T\}}\widetilde{C}_{\tau^R}dH^R_t \\
& - \mathbb{1}_{\{\tau^R\le T\}}\big(R_1(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+ - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^-\big)d[H^R,H^1]_t \\
& + \mathbb{1}_{\{\tau^R\le T\}}\big(R_2(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^- - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+\big)d[H^R,H^2]_t \\
& + \mathbb{1}_{\{\tau^R\le T\}}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})d[[H^R,H^1],H^2]_t \\
& - \mathbb{1}_{\{\tau^R\le T\}}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})d[H^R,\widetilde{H}^R]_t \,.
\end{aligned}
$$

Integrating both sides leads to,

$$
\begin{aligned}
\int_{]t,T]} B_u^{-1}(dD_u - d\mathcal{D}^{R,h}_u) = {} & \int_{]t,T]} B_u^{-1}H^R_u dD_u - \int_{]t,T]} B_u^{-1}D_{u-}dH^R_u + \int_{]t,T]} B_u^{-1}D_{\tau^R-}dH^R_u \\
& - \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}\big(R_1(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+ - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^-\big)d[H^R,H^1]_u \\
& + \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}\big(R_2(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^- - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+\big)d[H^R,H^2]_u \\
& + \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})d[[H^R,H^1],H^2]_u \\
& - \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})d[H^R,\widetilde{H}^R]_u - \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}\widetilde{C}_{\tau^R}dH^R_u \,.
\end{aligned}
$$

Since,

$$
\int_{]t,T]} B_u^{-1}D_{\tau^R-}dH^R_u - \int_{]t,T]} B_u^{-1}D_{u-}dH^R_u = 0 \,,
$$

we obtain,

$$
\begin{aligned}
\int_{]t,T]} B_u^{-1}(dD_u - d\mathcal{D}^{R,h}_u) = {} & \int_{]t,T]} B_u^{-1}H^R_u dD_u - \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-2}\widetilde{C}_{\tau^R}dH^R_u \\
& - \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}\big(R_1(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+ - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^-\big)d[H^R,H^1]_u \\
& + \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}\big(R_2(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^- - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+\big)d[H^R,H^2]_u \\
& + \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})d[[H^R,H^1],H^2]_u \\
& - \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}}B_u^{-1}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})d[H^R,\widetilde{H}^R]_u \,.
\end{aligned}
$$

Conditioning on τ^R, we get

$$\begin{aligned}
\mathbb{1}_{\{t\le\tau^R\wedge T\}}\ \mathbb{E}\Big[\int_{]t,T]} B_u^{-1}(dD_u - d\mathcal{D}_u^{R,h})\,\Big|\,\mathcal{G}_{\tau^R}\Big] &= \mathbb{1}_{\{t\le\tau^R\wedge T\}}\ \mathbb{E}\Bigg[\int_{]t,T]} B_u^{-1}H_u^R dD_u \\
&- \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}} B_u^{-1}\widetilde{C}_{\tau^R} dH_u^R \\
&- \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}} B_u^{-1}\big(R_1(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+ - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^-\big) d[H^R,H^1]_u \\
&+ \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}} B_u^{-1}\big(R_2(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^- - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+\big) d[H^R,H^2]_u \\
&+ \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}} B_u^{-1}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R}) d[[H^R,H^1]\,,H^2]_u \qquad (3.12)\\
&- \int_{]t,T]} \mathbb{1}_{\{\tau^R\le T\}} B_u^{-1}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R}) d[H^R,\widetilde{H}^R]_u \,\Bigg|\, \mathcal{G}_{\tau^R}\Bigg].
\end{aligned}$$

Notice that, since $t\in[0,\tau^R\wedge T]$, we have

$$\begin{aligned}
\int_{]t,T]} B_u^{-1}H_u^R dD_u &= \int_{]t,\tau^R[} B_u^{-1}H_u^R dD_u + \int_{[\tau^R,T]} B_u^{-1}H_u^R dD_u \\
&= \int_{[\tau^R,T]} B_u^{-1}H_u^R dD_u\,. \qquad (3.13)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{1}_{\{t\le\tau^R\wedge T\}}\mathbb{E}\Big[\int_{]t,T]} B_u^{-1}H_u^R dD_u\,\Big|\,\mathcal{G}_{\tau^R}\Big] = \mathbb{1}_{\{t\le\tau^R\wedge T\}}\mathbb{E}\Big[\int_{[\tau^R,T]} B_u^{-1}H_u^R dD_u\,\Big|\,\mathcal{G}_{\tau^R}\Big] \\
&= \mathbb{1}_{\{t\le\tau^R\wedge T\}}\mathbb{1}_{\{\tau^R\le T\}}B_{\tau^R}^{-1}(S_{\tau^R}+\Delta D_{\tau^R}) = \mathbb{1}_{\{t\le\tau^R\}}\mathbb{1}_{\{\tau^R\le T\}}B_{\tau^R}^{-1}(S_{\tau^R}+\Delta D_{\tau^R})\,. \qquad (3.14)
\end{aligned}$$

Taking conditional expectation given $\mathcal{G}_t$ and using the tower property in (3.12) reads

$$\begin{aligned}
\mathbb{1}_{\{t\le\tau^R\wedge T\}}(S_t - S_t^R) &= \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[\int_{]t,T]} B_u^{-1}(dD_u - d\mathcal{D}_u^{R,h})\,\Big|\,\mathcal{G}_t\Big] \qquad (3.15)\\
&= \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[B_{\tau^R}^{-1}\big(\mathbb{1}_{\{\tau^R\le T\}}(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R}) \\
&- (R_1(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+ - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^-)\mathbb{1}_{\{\tau^R=\tau_1\le T\}} \\
&+ (R_2(S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^- - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})^+)\mathbb{1}_{\{\tau^R=\tau_2\le T\}} \\
&+ (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_1=\tau_2\le T\}} - (S^\Delta_{\tau^R}-\widetilde{C}_{\tau^R})\mathbb{1}_{\{\tau^R=\widetilde{\tau}^R\le T\}}\big)\,\Big|\,\mathcal{G}_t\Big]\,.
\end{aligned}$$

Since

$$(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R}) = (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^+ - (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^- ,$$

it follows that (3.15) is equivalent to

$$\begin{aligned}\mathbb{1}_{\{t\le\tau^R\wedge T\}}(S_t - S^R_t) = \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[B^{-1}_{\tau^R}\big(&\mathbb{1}_{\{\tau^R\le T\}}(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\\ &- (R_1(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^+ + (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R}) - (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^+)\mathbb{1}_{\{\tau^R=\tau_1\le T\}}\\ &+ (R_2(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^- - (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^- - (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R}))\mathbb{1}_{\{\tau^R=\tau_2\le T\}}\\ &+ (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_1=\tau_2\le T\}} - (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\mathbb{1}_{\{\tau^R=\widetilde{\tau}^R\le T\}}\big)\,\Big|\,\mathcal{G}_t\Big] .\end{aligned}$$

After simplifying the terms above, we obtain

$$\begin{aligned}\mathbb{1}_{\{t\le\tau^R\wedge T\}}(S_t - S^R_t) = \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[B^{-1}_{\tau^R}\big(&\mathbb{1}_{\{\tau^R\le T\}}(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\\ &+ (1-R_1)(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^+\mathbb{1}_{\{\tau^R=\tau_1\le T\}} - (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_1\le T\}}\\ &- (1-R_2)(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^-\mathbb{1}_{\{\tau^R=\tau_2\le T\}} - (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_2\le T\}}\\ &+ (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\mathbb{1}_{\{\tau^R=\tau_1=\tau_2\le T\}} - (S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\mathbb{1}_{\{\tau^R=\widetilde{\tau}^R\le T\}}\big)\,\Big|\,\mathcal{G}_t\Big] ,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathbb{1}_{\{t\le\tau^R\wedge T\}}(S_t - S^R_t) = \mathbb{1}_{\{t\le\tau^R\wedge T\}}B_t\mathbb{E}\Big[B^{-1}_{\tau^R}\big[\mathbb{1}_{\{\tau^R\le T\}}(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R}) - \mathbb{1}_{\{\tau^R\le T\}}(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})\\ + (1-R_1)(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^+\mathbb{1}_{\{\tau^R=\tau_1\le T\}} - (1-R_2)(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^-\mathbb{1}_{\{\tau^R=\tau_2\le T\}}\big]\,\Big|\,\mathcal{G}_t\Big] .\end{aligned}$$

Finally, we find that

$$\begin{aligned}S_t - S^R_t = {}&B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\le T\}}B^{-1}_{\tau^R}(1-R_1)(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^+\,\Big|\,\mathcal{G}_t\Big]\\ &- B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\le T\}}B^{-1}_{\tau^R}(1-R_2)(S^{\Delta}_{\tau^R} - \widetilde{C}_{\tau^R})^-\,\Big|\,\mathcal{G}_t\Big] ,\end{aligned}$$

on the set $t \in [0, \tau^R \wedge T]$, which proves our claim. □

Remark 3.2.3. *Observe that if we set $R^h_1 = 1$ and $R^h_2 = 1$, which means no rehypothecation risk, then we have $CVA^{R,h} = CVA^R$.*

Next, we define the rating valuation adjustment in the presence of rehypothecation risk.

Definition 3.2.12. *The Rating Valuation Adjustment process (RVA^h) with rehypothecation risk is defined as*

$$RVA_t^h = CVA_t - CVA_t^{R,h}\,,$$

for $t \in [0, \tau^R \wedge T]$.

We have the following representation for $\mathrm{RVA}^{R,h}$.

Lemma 3.2.1. *$RVA^{R,h}$ can be represented as*

$$\begin{aligned}
RVA_t^h &= RVA_t \\
&+ B_t\mathbb{E}_t\Big[\mathbb{1}_{\{\tau^R=\tau_1\neq\tau_2\leq T\}}B_{\tau_1}^{-1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}((S_{\tau_1}^{\Delta}-C_{\tau_1})^+ - (S_{\tau_1}^{\Delta}-R_1^hC_{\tau_1})^+)] \\
&\quad + \mathbb{1}_{\{\tau^R=\tau_1=\tau_2\leq T\}}B_{\tau_1}^{-1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}((S_{\tau_1}^{\Delta}-C_{\tau_1})^+ - (S_{\tau_1}^{\Delta}-R_1^hC_{\tau_1})^+) \\
&\qquad + \mathbb{1}_{C_{\tau_1}<0}((S_{\tau_1}^{\Delta}-C_{\tau_1})^+ - (S_{\tau_1}^{\Delta}-R_2^hC_{\tau_1})^+)]\Big] \\
&+ B_t\mathbb{E}_t\Big[\mathbb{1}_{\{\tau^R=\tau_2\neq\tau_1\leq T\}}B_{\tau_2}^{-1}(1-R_2)[\mathbb{1}_{C_{\tau_2}<0}((S_{\tau_2}^{\Delta}-R_2^hC_{\tau_2})^- - (S_{\tau_2}^{\Delta}-C_{\tau_2})^-)] \\
&\quad + \mathbb{1}_{\{\tau^R=\tau_2=\tau_1\leq T\}}B_{\tau_2}^{-1}(1-R_2)[\mathbb{1}_{C_{\tau_2}>0}((S_{\tau_2}^{\Delta}-R_1^hC_{\tau_2})^- - (S_{\tau_2}^{\Delta}-C_{\tau_2})^-) \\
&\qquad + \mathbb{1}_{C_{\tau_2}<0}((S_{\tau_2}^{\Delta}-R_2^hC_{\tau_2})^- - (S_{\tau_2}^{\Delta}-C_{\tau_2})^-)]\Big]\,,
\end{aligned}$$

for $t \in [0, \tau^R \wedge T]$.

Proof. Using (3.1) and (3.11) we obtain

$$\begin{aligned}
\mathrm{CVA}_t - \mathrm{CVA}_t^{R,h} &= B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_1\leq T\}}B_\tau^{-1}(1-R_1)(S_\tau^{\Delta}-C_\tau)^+\,\Big|\,\mathcal{G}_t\Big] \\
&- B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_2\leq T\}}B_\tau^{-1}(1-R_2)(S_\tau^{\Delta}-C_\tau)^-\,\Big|\,\mathcal{G}_t\Big] \\
&- B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\leq T\}}B_{\tau^R}^{-1}(1-R_1)(S_{\tau^R}^{\Delta}-\widetilde{C}_{\tau^R}^1)^+\,\Big|\,\mathcal{G}_t\Big] \\
&+ B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\leq T\}}B_{\tau^R}^{-1}(1-R_2)(S_{\tau^R}^{\Delta}-\widetilde{C}_{\tau^R}^2)^-\,\Big|\,\mathcal{G}_t\Big]\,,
\end{aligned}$$

where

$$\widetilde{C}^1_{\tau^R} = \mathbb{1}_{\tau^R=\tau_1\neq\tau_2}(R_1^h C^+_{\tau^R} + C^-_{\tau^R}) + \mathbb{1}_{\tau^R=\tau_1=\tau_2}(R_1^h C^+_{\tau^R} + R_2^h C^-_{\tau^R}),$$

and

$$\widetilde{C}^2_{\tau^R} = \mathbb{1}_{\tau^R=\tau_2\neq\tau_1}(C^+_{\tau^R} + R_2^h C^-_{\tau^R}) + \mathbb{1}_{\tau^R=\tau_1=\tau_2}(R_1^h C^+_{\tau^R} + R_2^h C^-_{\tau^R}).$$

Rearranging the terms above yields

$$\begin{aligned}
\mathrm{CVA}_t - \mathrm{CVA}_t^{R,h} = {} & B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_1\leq T\}}B_{\tau_1}^{-1}(1-R_1)(S^\Delta_{\tau_1} - C_{\tau_1})^+ \,\Big|\, \mathcal{G}_t\Big] \\
& - B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_2\leq T\}}B_{\tau_2}^{-1}(1-R_2)(S^\Delta_{\tau_2} - C_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big] \\
& - B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\leq T\}}B_{\tau_1}^{-1}(1-R_1)(S^\Delta_{\tau_1} - \widetilde{C}^1_{\tau_1})^+ \,\Big|\, \mathcal{G}_t\Big] \\
& + B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\leq T\}}B_{\tau_2}^{-1}(1-R_2)(S^\Delta_{\tau_2} - \widetilde{C}^2_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big].
\end{aligned} \tag{3.16}$$

Plugging in the terms $\widetilde{C}^1_{\tau_1}$ and $\widetilde{C}^2_{\tau_2}$ into (3.16), we get

$$\begin{aligned}
& \mathrm{CVA}_t - \mathrm{CVA}_t^{R,h} = B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_1\leq T\}}B_{\tau_1}^{-1}(1-R_1)(S^\Delta_{\tau_1} - C_{\tau_1})^+ \,\Big|\, \mathcal{G}_t\Big] \\
& - B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_2\leq T\}}B_{\tau_2}^{-1}(1-R_2)(S^\Delta_{\tau_2} - C_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big] \\
& - B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\neq\tau_2\leq T\}}B_{\tau_1}^{-1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}(S^\Delta_{\tau_1} - R_1^h C_{\tau_1})^+ + \mathbb{1}_{C_{\tau_1}<0}(S^\Delta_{\tau_1} - C_{\tau_1})^+)] \\
& \quad + \mathbb{1}_{\{\tau^R=\tau_1=\tau_2\leq T\}}B_{\tau_1}^{-1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}(S^\Delta_{\tau_1} - R_1^h C_{\tau_1})^+ + \mathbb{1}_{C_{\tau_1}<0}(S^\Delta_{\tau_1} - R_2^h C_{\tau_1})^+)] \,\Big|\, \mathcal{G}_t\Big] \\
& + B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\neq\tau_1\leq T\}}B_{\tau_2}^{-1}(1-R_2)[\mathbb{1}_{C_{\tau_2}>0}(S^\Delta_{\tau_2} - C_{\tau_2})^- + \mathbb{1}_{C_{\tau_2}<0}(S^\Delta_{\tau_2} - R_2^h C_{\tau_2})^-] \\
& \quad + \mathbb{1}_{\{\tau^R=\tau_2=\tau_1\leq T\}}B_{\tau_2}^{-1}(1-R_2)[\mathbb{1}_{C_{\tau_2}>0}(S^\Delta_{\tau_2} - R_1^h C_{\tau_2})^-] + \mathbb{1}_{C_{\tau_2}<0}(S^\Delta_{\tau_2} - R_2^h C_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big].
\end{aligned}$$

It follows from (3.9) that

$$\begin{aligned}
\mathrm{CVA}_t - \mathrm{CVA}_t^{R,h} &= \mathrm{RVA}_t + B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\le T\}}B_{\tau^R}^{-1}(1-R_1)(S_{\tau^R}^{\Delta}-C_{\tau^R})^+ \,\Big|\, \mathcal{G}_t\Big] \\
&- B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\le T\}}B_{\tau^R}^{-1}(1-R_2)(S_{\tau^R}^{\Delta}-C_{\tau^R})^- \,\Big|\, \mathcal{G}_t\Big] \\
&- B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\ne\tau_2\le T\}}B_{\tau_1}^{-1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}(S_{\tau_1}^{\Delta}-R_1^hC_{\tau_1})^+ + \mathbb{1}_{C_{\tau_1}<0}(S_{\tau_1}^{\Delta}-C_{\tau_1})^+)] \\
&\quad + \mathbb{1}_{\{\tau^R=\tau_1=\tau_2\le T\}}B_{\tau_1}^{-1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}(S_{\tau_1}^{\Delta}-R_1^hC_{\tau_1})^+ + \mathbb{1}_{C_{\tau_1}<0}(S_{\tau_1}^{\Delta}-R_2^hC_{\tau_1})^+)] \,\Big|\, \mathcal{G}_t\Big] \\
&+ B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\ne\tau_1\le T\}}B_{\tau_2}^{-1}(1-R_2)[\mathbb{1}_{C_{\tau_2}>0}(S_{\tau_2}^{\Delta}-C_{\tau_2})^- + \mathbb{1}_{C_{\tau_2}<0}(S_{\tau_2}^{\Delta}-R_2^hC_{\tau_2})^-] \\
&\quad + \mathbb{1}_{\{\tau^R=\tau_2=\tau_1\le T\}}B_{\tau_2}^{-1}(1-R_2)[\mathbb{1}_{C_{\tau_2}>0}(S_{\tau_2}^{\Delta}-R_1^hC_{\tau_2})^-] + \mathbb{1}_{C_{\tau_2}<0}(S_{\tau_2}^{\Delta}-R_2^hC_{\tau_2})^- \,\Big|\, \mathcal{G}_t\Big].
\end{aligned}$$

Finally, we find

$$\begin{aligned}
\mathrm{CVA}_t - \mathrm{CVA}_t^{R,h} &= \mathrm{RVA}_t \\
&+ B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\ne\tau_2\le T\}}B_{\tau_1}^{-1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}((S_{\tau^R}^{\Delta}-C_{\tau^R})^+ - (S_{\tau_1}^{\Delta}-R_1^hC_{\tau_1})^+)] \\
&\quad + \mathbb{1}_{\{\tau^R=\tau_1=\tau_2\le T\}}B_{\tau_1}^{-1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}((S_{\tau^R}^{\Delta}-C_{\tau^R})^+ - (S_{\tau_1}^{\Delta}-R_1^hC_{\tau_1})^+) \\
&\qquad + \mathbb{1}_{C_{\tau_1}<0}((S_{\tau^R}^{\Delta}-C_{\tau^R})^+ - (S_{\tau_1}^{\Delta}-R_2^hC_{\tau_1})^+)] \,\Big|\, \mathcal{G}_t\Big] \\
&+ B_t\mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\ne\tau_1\le T\}}B_{\tau_2}^{-1}(1-R_2)[\mathbb{1}_{C_{\tau_2}<0}((S_{\tau_2}^{\Delta}-R_2^hC_{\tau_2})^- - (S_{\tau^R}^{\Delta}-C_{\tau^R})^-)] \\
&\quad + \mathbb{1}_{\{\tau^R=\tau_2=\tau_1\le T\}}B_{\tau_2}^{-1}(1-R_2)[\mathbb{1}_{C_{\tau_2}>0}((S_{\tau_2}^{\Delta}-R_1^hC_{\tau_2})^- - (S_{\tau^R}^{\Delta}-C_{\tau^R})^-)] \\
&\qquad + \mathbb{1}_{C_{\tau_2}<0}((S_{\tau_2}^{\Delta}-R_2^hC_{\tau_2})^- - (S_{\tau^R}^{\Delta}-C_{\tau^R})^-) \,\Big|\, \mathcal{G}_t\Big],
\end{aligned}$$

for $t \in [0, \tau^R \wedge T]$. □

Remark 3.2.4. *Note that RVA^h can either be negative or positive. If the difference is positive, then there is a decrease in the bilateral CVA, however if it is negative then there is an increase in the bilateral CVA.*

Let us define

$$\begin{aligned}
\mathrm{URVA}^h_t :&= B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_1\neq\tau_2\leq T\}} B^{-1}_{\tau_1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}((S^\Delta_{\tau_1}-C_{\tau_1})^+ - (S^\Delta_{\tau_1}-R^h_1 C_{\tau_1})^+)] \\
&\quad + \mathbb{1}_{\{\tau^R=\tau_1=\tau_2\leq T\}} B^{-1}_{\tau_1}(1-R_1)[\mathbb{1}_{C_{\tau_1}>0}((S^\Delta_{\tau_1}-C_{\tau_1})^+ - (S^\Delta_{\tau_1}-R^h_1 C_{\tau_1})^+) \\
&\quad + \mathbb{1}_{C_{\tau_1}<0}((S^\Delta_{\tau_1}-C_{\tau_1})^+ - (S^\Delta_{\tau_1}-R^h_2 C_{\tau_1})^+)] \Big| \mathcal{G}_t \Big], \\
\mathrm{DRVA}^h_t :&= B_t \mathbb{E}\Big[\mathbb{1}_{\{\tau^R=\tau_2\neq\tau_1\leq T\}} B^{-1}_{\tau_2}(1-R_2)[\mathbb{1}_{C_{\tau_2}<0}((S^\Delta_{\tau_2}-R^h_2 C_{\tau_2})^- - (S^\Delta_{\tau_2}-C_{\tau_2})^-)] \\
&\quad + \mathbb{1}_{\{\tau^R=\tau_2=\tau_1\leq T\}} B^{-1}_{\tau_2}(1-R_2)[\mathbb{1}_{C_{\tau_2}>0}((S^\Delta_{\tau_2}-R^h_1 C_{\tau_2})^- - (S^\Delta_{\tau_2}-C_{\tau_2})^-)] \\
&\quad + \mathbb{1}_{C_{\tau_2}<0}((S^\Delta_{\tau_2}-R^h_2 C_{\tau_2})^- - (S^\Delta_{\tau_2}-C_{\tau_2})^-) \Big| \mathcal{G}_t \Big]
\end{aligned}$$

for $t \in [0, \tau^R \wedge T]$. Therefore, RVA^h has the following decomposition,

$$\mathrm{RVA}^h_t = \mathrm{RVA}_t + \mathrm{URVA}^h_t + \mathrm{DRVA}^h_t \,,$$

for $t \in [0, \tau^R \wedge T]$.

Here URVA^h represents the expected loss if the counterparty defaults first which is preceded by a rating trigger. Likewise, DRVA^h is the expected loss in case the investor defaults first after a rating trigger. Therefore, including rating triggers provision in an OTC contract provides protection from losses due to default events which happen after a credit downgrade. Accordingly, the value of the contract is adjusted for this protection, as shown in the following result.

Corollary 3.2.2. *We have the following decomposition for the counterparty risky price process*

$$\begin{aligned}
S^{R,h}_t &= S_t - CVA^{R,h}_t \\
&= S_t - CVA_t + RVA^h_t \\
&= S_t - UCVA_t + DVA_t + RVA^h_t \\
&= S_t - UCVA_t + DVA_t + RVA_t + URVA^h_t + DRVA^h_t \,,
\end{aligned}$$

for $t \in [0, \tau^R \wedge T]$.

The above result provides an enhanced form of the decomposition found in Corollary 3.2.1, by taking the rehypothecation risk along with rating triggers into account. As we stated earlier, each term in the above decomposition is computed and treated separately in practice.

3.3 Markovian Approach for Rating-Based Pricing

In this section, we employ Markov copulae for modeling the rating transitions in our framework. Our approach is based on the studies of Bielecki et al. [BCJR06, BVV06, BVV08, BJVV08, BJN11].

3.3.1 Markov Copulae for the Multivariate Markov Chains. Let us first consider two Markov chains X^1 and X^2 on $(\Omega, \mathcal{F}, \mathbb{P})$ with the infinitesimal generators $A^1 := [a^1_{ij}]$ and $A^2 := [a^2_{hk}]$, respectively.

In what follows, we work under the following assumption, which is necessary for the Markovian copulae property.

Assumption 3.3.1. The system of equations,

$$\sum_{k \in \mathcal{K}} a^X_{ih,jk} = a^1_{ij}\,, \quad \forall i,j,h \in \mathcal{K},\ i \neq j, \tag{3.17}$$

$$\sum_{j \in \mathcal{K}} a^X_{ih,jk} = a^2_{hk}\,, \quad \forall i,h,k \in \mathcal{K},\ h \neq k, \tag{3.18}$$

has a positive solution.

The proof of the following proposition can be found in [BVV08].

Proposition 3.3.1. *If Assumption 3.3.1 is satisfied, then* $A^X = [a^X_{ih,jk}]_{i,h,j,k \in \mathcal{K}}$ *(where diagonal elements are defined appropriately) satisfies the conditions for a generator matrix of a bivariate time-homogeneous Markov chain, say* $X = (Y^1, Y^2)$*, whose components are Markov chains with the same laws as* X^1 *and* X^2*.*

Hence, the resulting matrix $A^X = [a^X_{ih,jk}]_{i,h,j,k\in\mathcal{K}}$ satisfies the conditions for a generator matrix of a bivariate time-homogeneous Markov chain, whose marginals are Markov chains with the same distributions as X^1 and X^2. Therefore, the system (3.17)–(3.18) serves as a Markov copula between the Markovian margins Y^1, Y^2 and the bivariate Markov chain X.

Note that the system (3.17)–(3.18) can contain more unknowns than the number of equations, therefore being underdeteremined. Therefore, as it is proposed by Bielecki et al. [BVV08], we impose additional constraints on the variables in the system (3.17)–(3.18). We postulate that

$$a^X_{ih,jk} = \begin{cases} 0\,, & \text{if } i \neq j, h \neq k, j \neq k \\ \\ \alpha \min(a^1_{ij}, a^2_{hk})\,, & \text{if } i \neq j, h \neq k, j = k \end{cases} \tag{3.19}$$

where $\alpha \in [0,1]$. Using the constraints (3.19) the system (3.17)–(3.18) becomes fully decoupled, and we can obtain the generator of the joint process.

We interpret the constraint (3.19) as follows. Y^1 and Y^2 migrate according to their marginal laws. Nevertheless, they can have the same values. The intensity of migrating to the same rating category is measured by the parameter α. If $\alpha = 0$, then the components Y^1 and Y^2 of X migrate independently. However, if $\alpha = 1$, the tendency of Y^1 and Y^2 migrating to the same categories is at maximum.

3.3.2 Markovian Changes of Measure. Since rating transition matrices indicate the historical default probabilities, we need to switch to the risk-neutral probabilities. In practice, the change of measure is done such a way that the resulting risk-neutral probabilities are consistent with the default probabilities inferred from the quoted CDS spreads. We need to apply changes of measure, while preserving Markovian structure of the model X. Therefore, the process X, which is Markovian under the statistical measure, will remain Markovian under the risk-neutral measure as well.

Let Y be a Markov process under $\mathbb{P}$ with generator A and domain $\mathcal{D}(A)$ and define

$$M_t^f := \frac{f(Y_t)}{f(Y_0)} e^{-\int_0^t \frac{Af(Y_s)}{f(Y_s)} ds} .$$

The following definition is borrowed from [PR02].

Definition 3.3.1. *A strictly positive function $f \in \mathcal{D}(A)$ is a good function if M_t^f is a true (genuine) martingale with mean 1 as $\mathbb{E}_{\mathbb{P}}(M_t^f) = 1$.*

Let $f \in \mathcal{D}(A)$ and h be a good function and define

$$A^h f := h^{-1} A(fh) - f A(h) .$$

The proof of the following Theorem can be found in [PR02].

Theorem 3.3.1. *Let $\mathbb{Q}^h$ be the probability measure associated to the density process M_t^h. Then Y is a Markov process under $\mathbb{Q}^h$ with extended generator $(A^h, \mathcal{D}(A))$.*

If Y is a finite state Markov chain, then we have the following result.

Corollary 3.3.1. *Let Y be a finite state Markov chain on $\mathcal{K}$ with cardinality $\mathcal{K}$ and generator $A = a_{ij}$ and let $h = (h_1, \ldots, h_K)$ be a positive vector. Then Y is a Markov process under $\mathbb{Q}^h$ with generator $A^h = [a_{ij} h_j h_i^{-1}]$.*

Using the above corollary, we can change the measure from the statistical measure $\mathbb{P}$ to a risk-neutral measure $\mathbb{Q}$ using a vector $h = (h_{11}, h_{22} \ldots, h_{KK}) \in \mathbb{R}^K$, so that the process X will be a time-homogeneous Markov chain under $\mathbb{Q}$. In this case, the infinitesimal generator under $\mathbb{Q}$ is found as

$$\widetilde{A}^X = [\widetilde{a}_{ih,jk}] ,$$

where

$$\widetilde{a}_{ih,jk} := \begin{cases} a_{ih,jk}\frac{h_{jk}}{h_{ih}} & \text{if } ih \neq jk\,, \\ -\sum_{ih\neq jk} a_{ih,jk}\frac{h_{jk}}{h_{ih}} & \text{if } ih = jk\,. \end{cases}$$

In Bielecki et al [BVV08], it is suggested that the vector h_{ij} can be chosen as

$$h_{ij} = e^{\alpha_1 i + \alpha_2 j}\,, \qquad i, j \in \mathcal{K}\,,$$

where the parameters α_1 and α_2 can be estimated through calibration.

3.4 Applications

In this section, we illustrate our results in the context of a CDS and an IRS contract. We postulate that our CDS and IRS contracts are subject to rating triggers, so that they are terminated in case a trigger event occurs. We compute the adjustments we discussed previously; namely, CVA, DVA, URVA and DRVA of the contracts for different rating trigger levels. Moreover, we compare CVA^R and CVA values and find the impact of adding rating triggers on the adjustments.

For the sake of simplicity, we carry out our analysis with $K = 4$ rating categories: A, B, C and D. The level A represents the highest rating level, whereas D corresponds to the default state. We assume that the counterparty initially has rating B. In what follows, we suppose that the 1-year rating transition matrix is given in Table 3.1.

Moreover, we assume that the current rating of the investor is A. Investor's 1-year rating transition matrix is assumed to be given as in Table 3.2.

We assume that the rating transition matrices given above are already risk-neutral, therefore we set $\alpha_1 = \alpha_2 = 0$. We also assume deterministic recovery rates; $R_1 = R_2 = 0.4$ and $R_1^h = R_2^h = 1$.

3.4.1 CVA of an IRS with Rating Triggers. In this section, we compute the

Table 3.1. Counterparty's rating transition matrix

	A	B	C	D
A	0.9	0.08	0.017	0.003
B	0.05	0.85	0.09	0.01
C	0.01	0.09	0.8	0.1
D	0	0	0	1

Table 3.2. Investor's rating transition matrix

	A	B	C	D
A	0.8	0.1	0.05	0.05
B	0.04	0.9	0.03	0.03
C	0.015	0.1	0.7	0.185
D	0	0	0	1

CVA, DVA, and RVA of a fixed-for-float payer 10-year IRS contract with \$1 notional, in presence of rating triggers as break clauses. We assume that the payments are done every quarter, and the fixed leg pays the swap rate, while the floating leg pays the LIBOR rate. We also assume that the swap is initiated at $T_0 := 0$ and we denote by $T_1 < T_2 < \cdots < T_n$, the payment dates and S by the fixed rate.

As we noted above, the rating transition matrices of the counterparty and the investor are given in Tables 3.1 and 3.2, respectively.

The cumulative dividend process of the IRS contract at time T_i is given by

$$D_{T_i} = \sum_{k=1}^{i} (L(T_k) - S)\delta_k \,,$$

where $L(T_i)$ is time-T_i LIBOR rate and $\delta_k = T_k - T_{k-1}$ for $k = 1, 2, \ldots, n$. We also suppose that the instantaneous interest rate r follows

$$dr_t = (\theta - \alpha\, r_t)dt + \sigma dW_t$$

where we set $r_0 = 0.05$, $\theta = 0.1$, $\alpha = 0.05$ and $\sigma = 0.01$. We find the corresponding swap rate as $S = 0.0496$.

We carry out our analysis for uncollateralized, linearly collateralized and exponentially collateralized cases for $\alpha = 0$ and $\alpha = 1$. In practice, these parameters are estimated using the current market data. Our results are displayed in Tables 3.3–3.12. We observe that the initial URVA values decrease with the decreasing counterparty trigger levels, which we denote by K_1. Similarly, the initial DRVA values also decrease when we decrease the investor's trigger level, which is denoted by K_2. However, the RVA values, which indicate the total bilateral adjustment due to the additional rating triggers, do not follow a certain pattern. For example, in Table 3.3, although we decrease the trigger levels from $K_1 = B$, $K_2 = B$ to $K_1 = C$, $K_2 = C$, the corresponding RVA values do not necessarily decrease, as opposed to URVA and DRVA values. We also observe from in Tables 3.5, 3.6, 3.9, 3.10, 3.13 and 3.14 that adding bilateral rating triggers can actually decrease the initial bilateral CVA values (in absolute terms), compared to the case with no rating triggers, which is $K_1 = D$ or $K_2 = D$. For instance, in Table 3.5, the reduction in CVA^R with no rating triggers is almost four times greater than the absolute value of CVA^R with $K_1 = D$ and $K_2 = B$. In other words, in this case adding rating triggers decreases the absolute value of the bilateral CVA by nearly 80%. However, in some cases such as $K_1 = B$ and $K_2 = D$ or $K_1 = C$ and $K_2 = D$, there is an increase in the bilateral CVA. The changes in the CVA^R values due to rating triggers are also visualized in Figures 3.1–3.6. Also, it can be seen from the Tables 3.4,3.8 and 3.12, where $\alpha = 1$, that the URVA and DRVA values are slightly higher compared to the values in Tables 3.3,3.7 and 3.11, where $\alpha = 0$.

Moreover, we see that the UCVA^R values start increasing as we lower the counterparty trigger levels. We also observe that the DVA^R values also increase with the decreasing trigger levels for the investor. However, the CVA^R values, that is the bilateral CVA, do not change significantly unless we set $K_1 = D$ or $K_2 = D$, which essentially means elimination of rating triggers.

We note that DRVA values are equal to zero whenever $K_2 = D$. This is because by setting the investors trigger to level D, we simply do not have any ratings adjustments for the investor. Likewise, we see that the URVA values are equal to zero where $K_1 = D$. Naturally, the case $K_1 = D$ and $K_2 = D$ corresponds to the CVA computation without any rating triggers.

Table 3.3. CVA and RVA of an IRS, $\alpha = 0$, No collateralization

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	1.212149×10^{-3}	7.466618×10^{-3}	-6.254469×10^{-3}	2.060247×10^{-4}	3.666725×10^{-3}	-3.460700×10^{-3}
B	C	1.132726×10^{-3}	7.306835×10^{-3}	-6.174109×10^{-3}	2.449648×10^{-4}	3.949869×10^{-3}	-3.704904×10^{-3}
C	B	9.032370×10^{-4}	7.477358×10^{-3}	-6.574121×10^{-3}	5.170761×10^{-4}	4.040241×10^{-3}	-3.523165×10^{-3}
C	C	9.110989×10^{-4}	7.316334×10^{-3}	-6.405235×10^{-3}	4.784849×10^{-4}	3.902284×10^{-3}	-3.423799×10^{-3}
B	D	1.207109×10^{-3}	0	1.207109×10^{-3}	2.371698×10^{-4}	1.118618×10^{-2}	-1.094901×10^{-2}
D	B	0	7.777546×10^{-3}	-7.777546×10^{-3}	1.508203×10^{-3}	3.484406×10^{-3}	-1.976203×10^{-3}
C	D	8.362864×10^{-4}	0	8.362864×10^{-4}	5.841795×10^{-4}	1.145506×10^{-2}	-1.087088×10^{-2}
D	C	0	7.719867×10^{-3}	-7.719867×10^{-3}	1.380511×10^{-3}	3.870841×10^{-3}	-2.490330×10^{-3}
D	D	0	0	0	1.400339×10^{-3}	1.140837×10^{-2}	-1.000803×10^{-2}

Table 3.4. CVA and RVA of an IRS, $\alpha = 1$, No collateralization

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	1.561017×10^{-3}	8.904928×10^{-3}	-7.343911×10^{-3}	2.478850×10^{-4}	3.443990×10^{-3}	-3.196105×10^{-3}
B	C	1.394051×10^{-3}	8.258988×10^{-3}	-6.864937×10^{-3}	1.469006×10^{-4}	3.465602×10^{-3}	-3.318701×10^{-3}
C	B	1.010048×10^{-3}	8.916948×10^{-3}	-7.906900×10^{-3}	1.708725×10^{-4}	3.398574×10^{-3}	-3.227701×10^{-3}
C	C	1.211336×10^{-3}	8.262675×10^{-3}	-7.051339×10^{-3}	4.935991×10^{-4}	3.606996×10^{-3}	-3.113397×10^{-3}
B	D	1.489470×10^{-3}	0	1.489470×10^{-3}	3.862817×10^{-5}	1.230561×10^{-2}	-1.226698×10^{-2}
D	B	0	8.971507×10^{-3}	-8.971507×10^{-3}	1.831549×10^{-3}	3.288356×10^{-3}	-1.456807×10^{-3}
C	D	9.994871×10^{-4}	0	9.994871×10^{-4}	6.116215×10^{-5}	1.245009×10^{-2}	-1.238893×10^{-2}
D	C	0	8.450063×10^{-3}	-8.450063×10^{-3}	1.633839×10^{-3}	3.156695×10^{-3}	-1.522856×10^{-3}
D	D	0	0	0	1.681383×10^{-3}	1.237909×10^{-2}	-1.069770×10^{-2}

Table 3.5. Mitigation in CVA of an IRS, $\alpha = 0$, No collateralization

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
65.42%	62.98 %	64.80%	65.79%	-9.40%	80.25%	-8.61%	75.12%

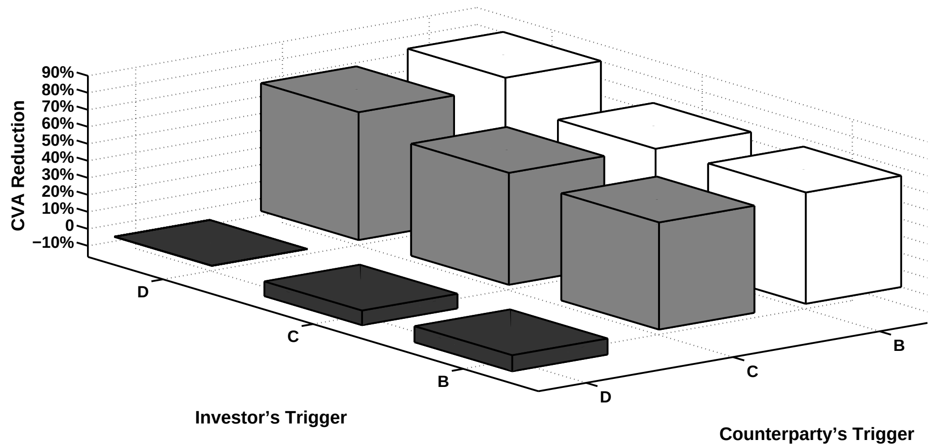

Figure 3.1. Change in CVA of an IRS, $\alpha = 0$, No collateralization

Table 3.6. Mitigation in CVA of an IRS (in %), $\alpha = 1$, No collateralization

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
70.12%	68.98%	69.83%	70.90%	-14.67%	86.38%	-15.81%	85.76%

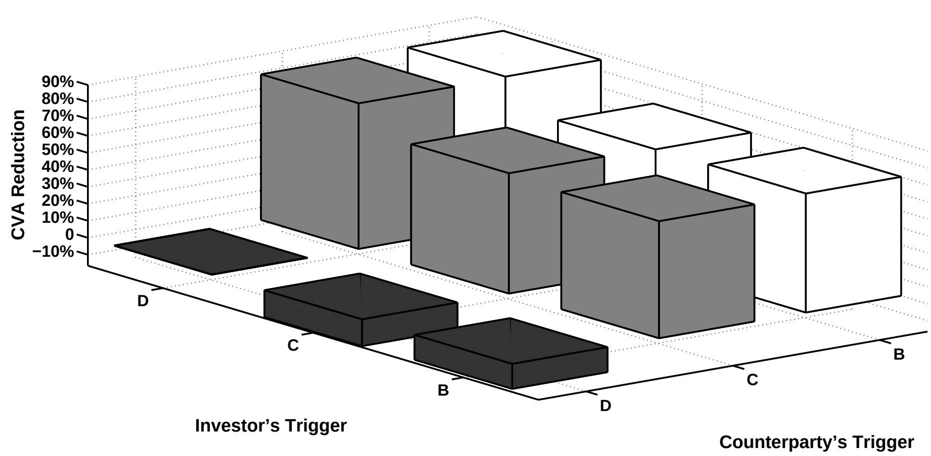

Figure 3.2. Change in CVA of an IRS, $\alpha = 1$, No collateralization

Table 3.7. CVA and RVA of an IRS, $\alpha = 0$, Linear collateral rate: ρ_l^i

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	6.527880×10^{-4}	4.258394×10^{-3}	-3.605606×10^{-3}	1.625832×10^{-4}	1.797607×10^{-3}	-1.635023×10^{-3}
B	C	6.049668×10^{-4}	3.170659×10^{-3}	-2.565692×10^{-3}	1.872098×10^{-4}	3.020700×10^{-3}	-2.833491×10^{-3}
C	B	4.296954×10^{-4}	4.233830×10^{-3}	-3.804135×10^{-3}	4.213768×10^{-4}	1.882251×10^{-3}	-1.460874×10^{-3}
C	C	4.383909×10^{-4}	3.205261×10^{-3}	-2.766871×10^{-3}	3.883170×10^{-4}	3.010062×10^{-3}	-2.621745×10^{-3}
B	D	6.761344×10^{-4}	0	6.761344×10^{-4}	1.860410×10^{-4}	6.047507×10^{-3}	-5.861466×10^{-3}
D	B	0	4.377871×10^{-3}	-4.377871×10^{-3}	9.108784×10^{-4}	1.689992×10^{-3}	-7.791136×10^{-4}
C	D	4.303853×10^{-4}	0	4.303853×10^{-4}	4.616551×10^{-4}	6.083496×10^{-3}	-5.621841×10^{-3}
D	C	0	3.260573×10^{-3}	-2.430122×10^{-3}	8.304507×10^{-4}	2.978720×10^{-3}	-2.148269×10^{-3}
D	D	0	0	0	8.366662×10^{-4}	6.118963×10^{-3}	-5.282297×10^{-3}

Table 3.8. CVA and RVA of an IRS, $\alpha = 1$, Linear collateral rate: ρ_l^i

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	9.087316×10^{-4}	4.952554×10^{-3}	-4.043823×10^{-3}	2.186338×10^{-4}	1.639453×10^{-3}	-1.420819×10^{-3}
B	C	7.717694×10^{-4}	3.571846×10^{-3}	-2.800077×10^{-3}	1.469006×10^{-4}	2.657801×10^{-3}	-2.510901×10^{-3}
C	B	5.017888×10^{-4}	4.971249×10^{-3}	-4.469461×10^{-3}	1.708725×10^{-4}	1.611505×10^{-3}	-1.440632×10^{-3}
C	C	6.194279×10^{-4}	3.585330×10^{-3}	-2.965902×10^{-3}	4.134398×10^{-4}	2.745938×10^{-3}	-2.332498×10^{-3}
B	D	8.500339×10^{-4}	0	-8.500339×10^{-4}	3.862817×10^{-5}	6.603631×10^{-3}	-6.565003×10^{-3}
D	B	0	5.023766×10^{-3}	-5.023766×10^{-3}	1.125967×10^{-3}	1.526974×10^{-3}	-4.010070×10^{-4}
C	D	4.881072×10^{-4}	0	4.881072×10^{-4}	6.116215×10^{-5}	6.683425×10^{-3}	-6.622263×10^{-3}
D	C	0	3.653127×10^{-3}	-3.653127×10^{-3}	9.790810×10^{-4}	2.415920×10^{-3}	-1.436839×10^{-3}
D	D	0	0	0	1.064880×10^{-3}	6.754737×10^{-3}	-5.689858×10^{-3}

Table 3.9. Mitigation in CVA of an IRS (in %), $\alpha = 0$, Lin. collateral rate: ρ_l^i

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
69.05%	46.36%	72.34%	50.37%	-10.97%	85.25%	-6.43%	59.33%

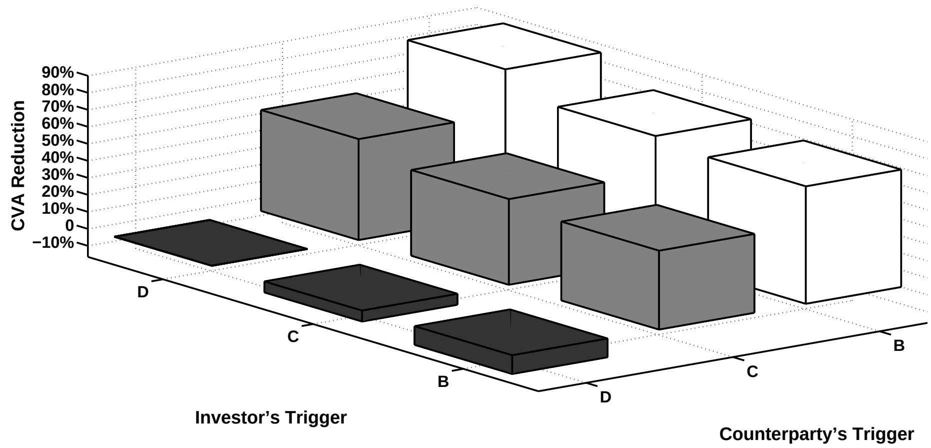

Figure 3.3. Change in CVA of an IRS, $\alpha = 0$, Linear collateral rate: ρ_l^i

Table 3.10. Mitigation in CVA of an IRS (in %), $\alpha = 1$, Lin. collateral rate: ρ_l^i

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
75.03%	55.84%	74.68%	59.01%	-15.38%	92.95%	-16.39%	74.75%

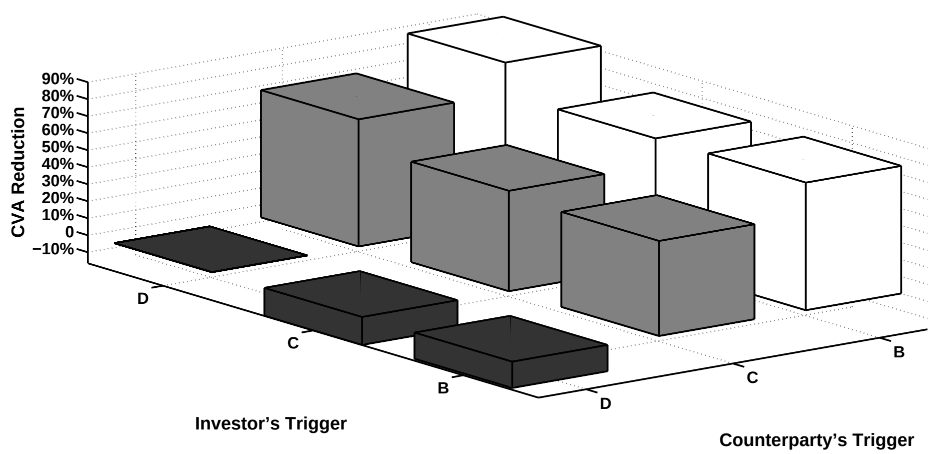

Figure 3.4. Change in CVA of an IRS, $\alpha = 1$, Linear collateral rate: ρ_l^i

Table 3.11. CVA and RVA of an IRS, $\alpha = 0$, Exponential collateral rate: ρ_e^i

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	3.778482×10^{-4}	2.472450×10^{-3}	-2.094602×10^{-3}	1.496812×10^{-4}	1.242484×10^{-3}	-1.092803×10^{-3}
B	C	3.306753×10^{-4}	1.543663×10^{-3}	-1.212988×10^{-3}	1.700568×10^{-4}	2.187830×10^{-3}	-2.017773×10^{-3}
C	B	2.253405×10^{-4}	2.407175×10^{-3}	-2.181834×10^{-3}	3.355956×10^{-4}	1.241334×10^{-3}	-9.057385×10^{-4}
C	C	2.331000×10^{-4}	1.572236×10^{-3}	-1.339136×10^{-3}	3.074939×10^{-4}	2.210308×10^{-3}	-1.902814×10^{-3}
B	D	3.861326×10^{-4}	0	3.861326×10^{-4}	1.708559×10^{-4}	3.696679×10^{-3}	-3.525823×10^{-3}
D	B	0	2.532624×10^{-3}	-2.532624×10^{-3}	5.835989×10^{-4}	1.157057×10^{-3}	-5.734581×10^{-4}
C	D	2.355660×10^{-4}	0	2.355660×10^{-4}	3.518290×10^{-4}	3.651942×10^{-3}	-3.300113×10^{-3}
D	C	0	1.605878×10^{-3}	-1.605878×10^{-3}	5.437258×10^{-4}	2.179058×10^{-3}	-1.635332×10^{-3}
D	D	0	0	0	5.410869×10^{-4}	3.706215×10^{-3}	-3.165128×10^{-3}

Table 3.12. CVA and RVA of an IRS, $\alpha = 1$, Exponential collateral rate: ρ_e^i

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	5.423935×10^{-4}	2.826616×10^{-3}	-2.284222×10^{-3}	2.099463×10^{-4}	1.103511×10^{-3}	-8.935648×10^{-4}
B	C	4.366426×10^{-4}	1.791075×10^{-3}	-1.354432×10^{-3}	1.469006×10^{-4}	1.933720×10^{-3}	-1.786820×10^{-3}
C	B	2.695380×10^{-4}	2.850330×10^{-3}	-2.580792×10^{-3}	1.708725×10^{-4}	1.080751×10^{-3}	-9.098784×10^{-4}
C	C	3.429487×10^{-4}	1.823886×10^{-3}	-1.480937×10^{-3}	3.415880×10^{-4}	1.974119×10^{-3}	-1.632531×10^{-3}
B	D	5.014045×10^{-4}	0	5.014045×10^{-4}	3.862817×10^{-5}	3.958847×10^{-3}	-3.920219×10^{-3}
D	B	0	2.878485×10^{-3}	-2.878485×10^{-3}	7.609933×10^{-4}	1.003849×10^{-3}	-2.428557×10^{-4}
C	D	2.641466×10^{-4}	0	2.641466×10^{-4}	6.116215×10^{-5}	3.991569×10^{-3}	-3.930407×10^{-3}
D	C	0	1.835762×10^{-3}	-1.835762×10^{-3}	6.197874×10^{-4}	1.751918×10^{-3}	-1.132131×10^{-3}
D	D	0	0	0	7.305637×10^{-4}	4.095416×10^{-3}	-3.364853×10^{-3}

Table 3.13. Mitigation in CVA of an IRS (in %), $\alpha = 0$, Exp. collateral rate: ρ_e^i

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
65.47%	36.25%	71.38%	39.88%	-11.40%	81.88%	-4.27%	48.33%

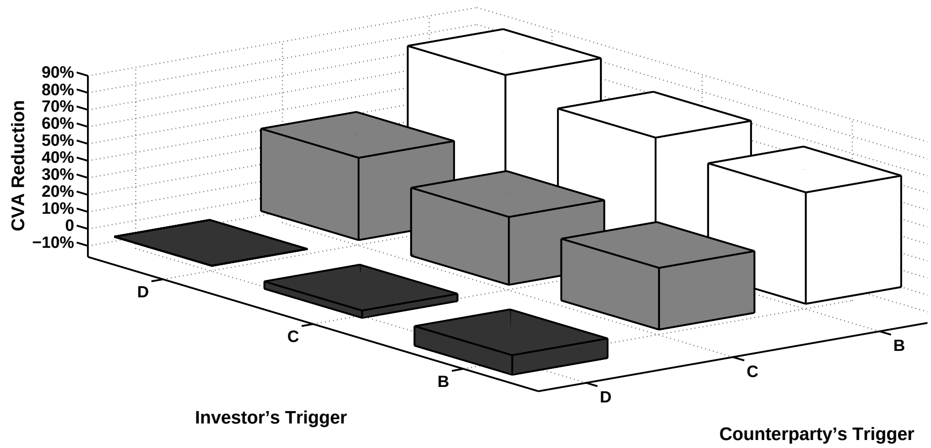

Figure 3.5. Change in CVA of an IRS, $\alpha = 0$, Exponential collateral rate: ρ_e^i

Table 3.14. Mitigation in CVA of an IRS (in %), $\alpha = 1$, Exp. collateral rate: ρ_e^i

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
73.44%	46.90%	72.96%	51.48%	-16.50%	92.78%	-16.81%	66.35%

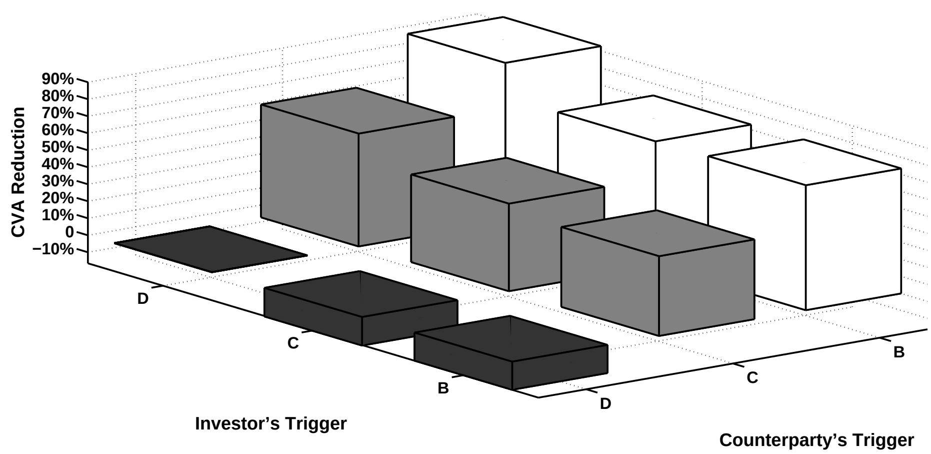

Figure 3.6. Change in CVA of an IRS, $\alpha = 1$, Exponential collateral rate: ρ_e^i

3.4.2 CVA of a CDS with Rating Triggers. In this section, we compute the CVA, DVA, and RVA of a CDS contract in presence of rating triggers as break clauses. Recall that D represents the counterparty risk-free cumulative dividend process of a contract. We assume that the reference entity is free of any trigger events. We denote by τ_3 the default time of the reference entity and R_3 the recovery rate of the reference entity. We assume that the CDS contract has spread κ, expires at T and has nominal value of 1. Consequently, the cumulative dividend process of the CDS contract is given by

$$D_t = (1 - R_3)\mathbb{1}_{\{\tau_3 \leq t\}} - \kappa(t \wedge T \wedge \tau)\,,$$

for all $t \in [0, T]$. We also assume that the underlying entity's 1-year rating transition matrix is given as in Table 3.15.

Table 3.15. Underlying entity's rating transition matrix: P^3

	A	B	C	D
A	0.95	0.03	0.019	0.001
B	0.04	0.85	0.107	0.003
C	0.01	0.19	0.791	0.009
D	0	0	0	1

Similar to the IRS example, we carry out our analysis for uncollateralized, linearly collateralized and exponentially collateralized CDS contracts where $\alpha = 0$ and $\alpha = 1$. We display our results in Tables 3.16–3.25.

The initial URVA values increase with the increasing counterparty trigger levels, and the initial DRVA values increase with the increasing investor trigger levels. However, the absolute values of the RVA numbers can increase or decrease with the changing trigger levels. For example, in Table 3.16, although we decrease the trigger levels from $K_1 = B$, $K_2 = B$ to $K_1 = C$, $K_2 = C$, the corresponding RVA values (in absolute terms) do not necessarily decrease, compared to the URVA and DRVA values.

It can also be observed from in Tables 3.16–3.25 that bilateral rating triggers can actually decrease the initial bilateral CVA values (in absolute values). For instance, the absolute value of CVA^R in Table 3.16 with no rating triggers is almost three times greater than the absolute value of CVA^R with $K_1 = B$ and $K_2 = B$. In addition, the UCVA^R values with no rating triggers are also almost three times greater than the UCVA^R values with $K_1 = B$ and $K_2 = B$. Similarly, the DVA^R values with no rating triggers are almost four times greater than the UCVA^R values with $K_1 = B$ and $K_2 = B$. In other words, as showed in Table 3.16 and Table 3.18, adding rating triggers decreases the UCVA^R value by nearly 60%, DVA^R value by nearly 75%, and the absolute value of the bilateral CVA^R by nearly 80%. Nevertheless, in case $K_1 = B$ and $K_2 = D$ in Table 3.18, there is a slight increase in the bilateral CVA value. Figures 3.7–3.12 illustrate the changes in the bilateral CVA values for each set of rating triggers.

Also, it can be seen from the Tables 3.17,3.21 and 3.25, where $\alpha = 1$, that the URVA and DRVA values are slightly higher compared to the values in Tables 3.16,3.20 and 3.24, where $\alpha = 0$.

Moreover, it can be seen from Tables 3.16–3.25 the UCVA^R values start increasing as we lower the counterparty trigger levels. Likewise, the DVA^R values also increase with the decreasing trigger levels for the investor. However, the CVA^R values do not change significantly unless we set $K_1 = D$ or $K_2 = D$, or eliminate the rating triggers.

The DRVA values are equal to zero whenever $K_2 = D$, and the URVA values are equal to zero where $K_1 = D$, since the rating triggers are set to the default levels.

Table 3.16. CVA and RVA of a CDS, $\alpha = 0$, No collateralization

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	3.492635 $\times 10^{-3}$	3.513821 $\times 10^{-2}$	-3.164558 $\times 10^{-2}$	2.194568 $\times 10^{-3}$	1.723002 $\times 10^{-2}$	-1.503545 $\times 10^{-2}$
B	C	3.421221 $\times 10^{-3}$	3.415462 $\times 10^{-2}$	-3.073340 $\times 10^{-2}$	2.579731 $\times 10^{-3}$	1.6384 $\times 10^{-2}$	-1.380433 $\times 10^{-2}$
C	B	1.859017 $\times 10^{-3}$	3.465935 $\times 10^{-2}$	-3.280033 $\times 10^{-2}$	4.249776 $\times 10^{-3}$	1.5488 $\times 10^{-2}$	-1.123824 $\times 10^{-2}$
C	C	1.883742 $\times 10^{-3}$	3.410477 $\times 10^{-2}$	-3.222103 $\times 10^{-2}$	4.299762 $\times 10^{-3}$	1.8653 $\times 10^{-2}$	-1.435328 $\times 10^{-2}$
B	D	3.204657 $\times 10^{-3}$	0	3.204657 $\times 10^{-3}$	2.139716 $\times 10^{-3}$	5.019431 $\times 10^{-2}$	-4.805459 $\times 10^{-2}$
D	B	0	3.740412 $\times 10^{-2}$	-3.740412 $\times 10^{-2}$	5.552273 $\times 10^{-3}$	1.502219 $\times 10^{-2}$	-9.469917 $\times 10^{-3}$
C	D	1.864508 $\times 10^{-3}$	0	1.864508 $\times 10^{-3}$	3.826232 $\times 10^{-3}$	5.126756 $\times 10^{-2}$	-4.744133 $\times 10^{-2}$
D	C	0	3.429175 $\times 10^{-2}$	-3.429175 $\times 10^{-2}$	5.607620 $\times 10^{-3}$	1.602933 $\times 10^{-2}$	-1.042171 $\times 10^{-2}$
D	D	0	0	0	5.959881 $\times 10^{-3}$	5.381753 $\times 10^{-2}$	-4.785765 $\times 10^{-2}$

Table 3.17. CVA and RVA of a CDS, $\alpha = 1$, No collateralization

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	5.500829×10^{-3}	3.781545×10^{-2}	-3.231462×10^{-2}	2.550964×10^{-3}	1.785338×10^{-2}	-1.530242×10^{-2}
B	C	5.107067×10^{-3}	3.609685×10^{-2}	-3.098979×10^{-2}	2.081542×10^{-3}	1.560634×10^{-2}	-1.352480×10^{-2}
C	B	3.007677×10^{-3}	3.788597×10^{-2}	-3.487829×10^{-2}	1.949816×10^{-3}	1.694930×10^{-2}	-1.499948×10^{-2}
C	C	3.080816×10^{-3}	3.747075×10^{-2}	-3.438993×10^{-2}	4.659889×10^{-3}	1.781806×10^{-2}	-1.315817×10^{-2}
B	D	5.548679×10^{-3}	0	5.548679×10^{-3}	3.183325×10^{-3}	4.701431×10^{-2}	-4.383099×10^{-2}
D	B	0	3.871672×10^{-2}	-3.871672×10^{-2}	3.216778×10^{-3}	1.773985×10^{-2}	-1.452308×10^{-2}
C	D	2.989101×10^{-3}	0	2.989101×10^{-3}	5.395727×10^{-3}	5.024807×10^{-2}	-4.485234×10^{-2}
D	C	0	3.767893×10^{-2}	-3.767893×10^{-2}	7.414729×10^{-3}	1.814965×10^{-2}	-1.073492×10^{-2}
D	D	0	0	0	8.961868×10^{-3}	5.680674×10^{-2}	-4.784488×10^{-2}

Table 3.18. Mitigation in CVA of a CDS (in %), $\alpha = 0$, No collateralization

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
68.58%	71.15%	76.52%	70.01%	-0.41%	80.21%	0.87%	78.22%

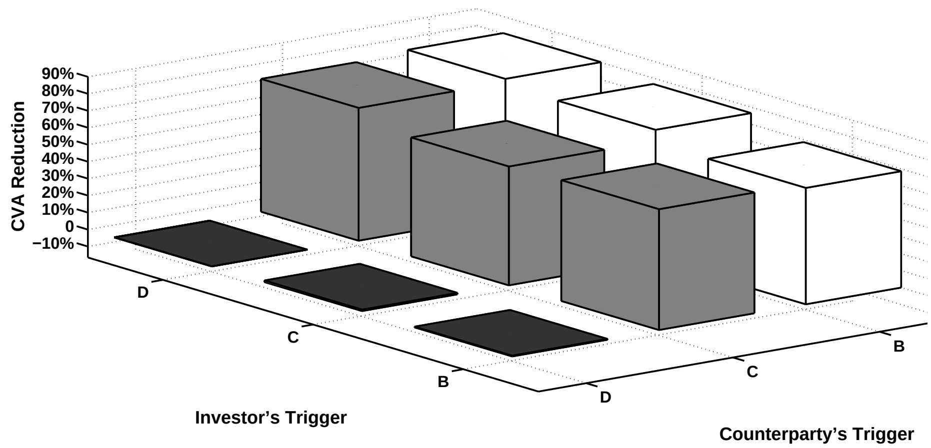

Figure 3.7. Change in CVA of a CDS, $\alpha = 0$, No collateralization

Table 3.19. Mitigation in CVA of a CDS (in %), $\alpha = 1$, No collateralization

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
68.02%	71.73%	68.65%	72.50%	8.39%	69.65%	6.25%	77.56%

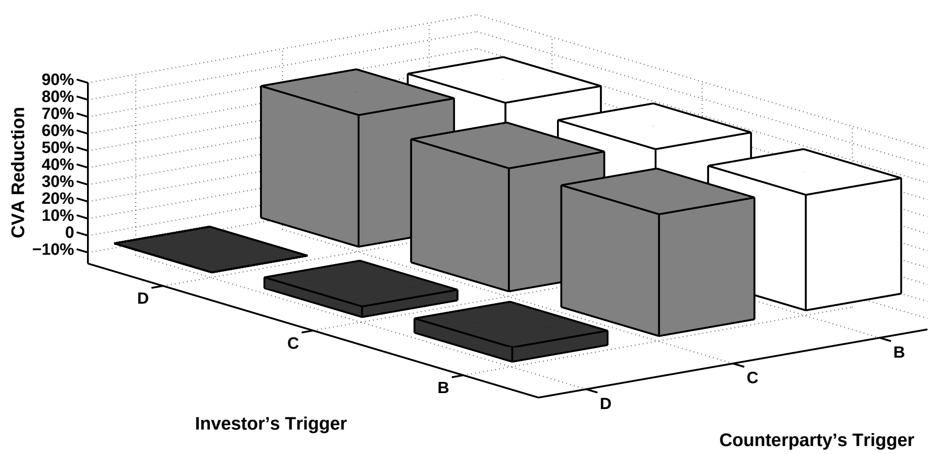

Figure 3.8. Change in CVA of a CDS, $\alpha = 1$, No collateralization

Table 3.20. CVA and RVA of a CDS, $\alpha = 0$, Linear collateral rate: ρ_l^i

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	1.827688×10^{-3}	1.931893×10^{-2}	-1.749124×10^{-2}	1.851721×10^{-3}	8.543911×10^{-3}	-6.692191×10^{-3}
B	C	2.205201×10^{-3}	1.545451×10^{-2}	-1.324931×10^{-2}	2.433040×10^{-3}	1.228745×10^{-2}	-9.854406×10^{-3}
C	B	7.037760×10^{-4}	1.972871×10^{-2}	-1.902494×10^{-2}	3.239934×10^{-3}	8.374923×10^{-3}	-5.134989×10^{-3}
C	C	7.546025×10^{-4}	1.495284×10^{-2}	-1.419824×10^{-2}	3.862542×10^{-3}	1.373431×10^{-2}	-9.871769×10^{-3}
B	D	1.759613×10^{-3}	0	1.759613×10^{-3}	2.011140×10^{-3}	2.860364×10^{-2}	-2.659250×10^{-2}
D	B	0	2.133867×10^{-2}	-2.133867×10^{-2}	3.944832×10^{-3}	8.315853×10^{-3}	-4.371021×10^{-3}
C	D	8.363832×10^{-4}	0	8.363832×10^{-4}	3.202537×10^{-3}	2.813766×10^{-2}	-2.493512×10^{-2}
D	C	0	1.433160×10^{-2}	-1.433160×10^{-2}	3.968530×10^{-3}	1.267252×10^{-2}	-8.703990×10^{-3}
D	D	0	0	0	4.081315×10^{-3}	3.017055×10^{-2}	-2.608923×10^{-2}

Table 3.21. CVA and RVA of a CDS, $\alpha = 1$, Linear collateral rate: ρ_l^i

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	3.206251×10^{-3}	2.086365×10^{-2}	-1.765740×10^{-2}	2.048730×10^{-3}	9.171566×10^{-3}	-7.122835×10^{-3}
B	C	3.206432×10^{-3}	1.529364×10^{-2}	-1.208721×10^{-2}	2.096852×10^{-3}	1.215847×10^{-2}	-1.006162×10^{-2}
C	B	1.272679×10^{-3}	2.294444×10^{-2}	-2.167176×10^{-2}	2.036685×10^{-3}	8.753749×10^{-3}	-6.717064×10^{-3}
C	C	1.103225×10^{-3}	1.592227×10^{-2}	-1.481905×10^{-2}	3.530859×10^{-3}	1.282223×10^{-2}	-9.291374×10^{-3}
B	D	3.206432×10^{-3}	0	3.206432×10^{-3}	2.469920×10^{-3}	2.549394×10^{-2}	-2.302402×10^{-2}
D	B	0	2.294444×10^{-2}	-2.294444×10^{-2}	2.184473×10^{-3}	8.811898×10^{-3}	-6.627426×10^{-3}
C	D	1.381531×10^{-3}	0	1.381531×10^{-3}	4.550913×10^{-3}	2.782208×10^{-2}	-2.327117×10^{-2}
D	C	0	1.639743×10^{-2}	-1.639743×10^{-2}	5.331034×10^{-3}	1.442756×10^{-2}	-9.096524×10^{-3}
D	D	0	0	0	5.864742×10^{-3}	2.985514×10^{-2}	-2.399040×10^{-2}

Table 3.22. Mitigation in CVA of a CDS (in %), $\alpha = 0$, Lin. collateral rate: ρ_l^i

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
74.35%	62.23%	80.32%	62.16%	-1.93%	83.25%	4.42%	66.64%

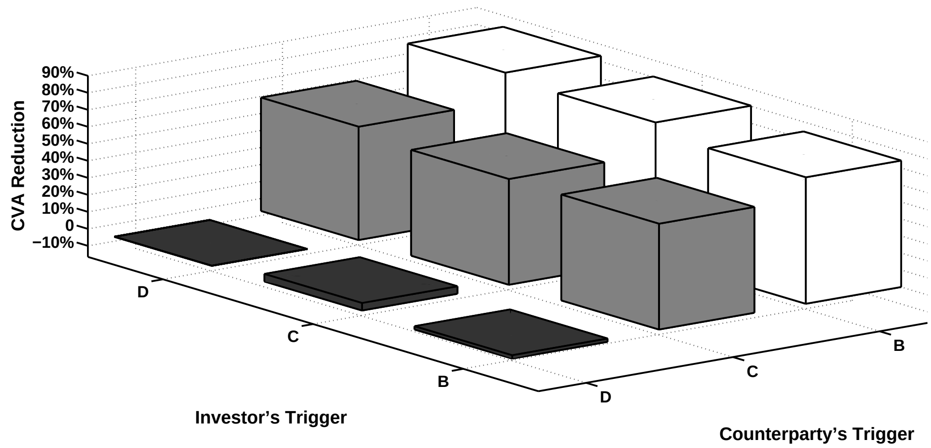

Figure 3.9. Change in CVA of a CDS, $\alpha = 0$, Linear collateral rate: ρ_l^i

Table 3.23. Mitigation in CVA of a CDS (in %), $\alpha = 1$, Lin. collateral rate: ρ_l^i

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
70.31%	58.06%	72%	61.27%	4.03%	72.37%	3%	62.08%

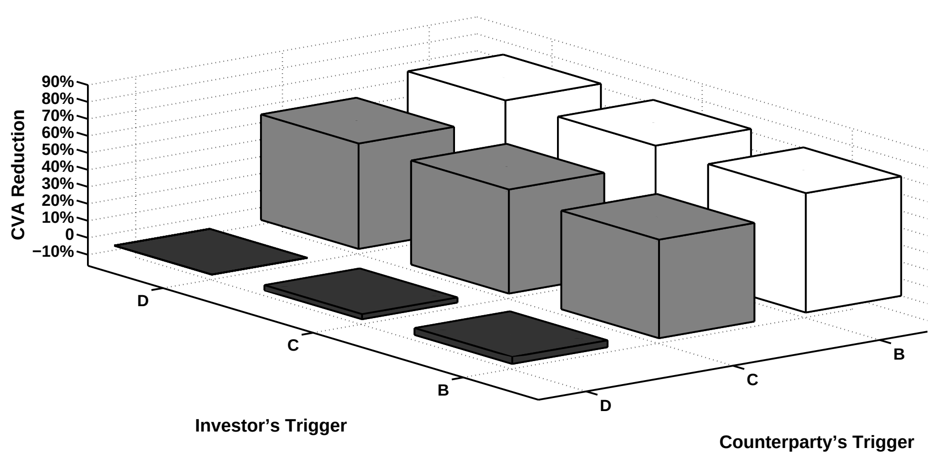

Figure 3.10. Change in CVA of a CDS, $\alpha = 1$, Linear collateral rate: ρ_l^i

Table 3.24. CVA and RVA of a CDS, $\alpha = 0$, Exponential collateral rate: ρ_e^i

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	9.882070×10^{-4}	1.108770×10^{-2}	-1.009949×10^{-2}	1.799380×10^{-3}	6.152931×10^{-3}	-4.353550×10^{-3}
B	C	1.262944×10^{-3}	7.943275×10^{-3}	-6.680331×10^{-3}	2.332527×10^{-3}	9.207949×10^{-3}	-6.875422×10^{-3}
C	B	3.332852×10^{-4}	1.139068×10^{-2}	-1.105739×10^{-2}	2.716527×10^{-3}	6.238456×10^{-3}	-3.521929×10^{-3}
C	C	3.427475×10^{-4}	7.530875×10^{-3}	-7.188127×10^{-3}	3.140173×10^{-3}	1.047286×10^{-2}	-7.332685×10^{-3}
B	D	9.810633×10^{-4}	0	9.810633×10^{-4}	1.936064×10^{-3}	1.803969×10^{-2}	-1.610362×10^{-2}
D	B	0	1.266808×10^{-2}	-1.266808×10^{-2}	2.982484×10^{-3}	6.167720×10^{-3}	-3.185236×10^{-3}
C	D	3.937553×10^{-4}	0	3.937553×10^{-4}	2.753292×10^{-3}	1.764404×10^{-2}	-1.489075×10^{-2}
D	C	0	7.150446×10^{-3}	-7.150446×10^{-3}	3.016230×10^{-3}	9.484559×10^{-3}	-6.468329×10^{-3}
D	D	0	0	0	3.191956×10^{-3}	1.878889×10^{-2}	-1.559694×10^{-2}

Table 3.25. CVA and RVA of a CDS, $\alpha = 1$, Exponential collateral rate: ρ_e^i

K_1	K_2	URVA	DRVA	RVA	UCVA^R	DVA^R	CVA^R
B	B	1.800100×10^{-3}	1.177528×10^{-2}	-9.975181×10^{-3}	2.847972×10^{-3}	6.927269×10^{-3}	-4.079297×10^{-3}
B	C	1.881202×10^{-3}	8.294210×10^{-3}	-6.413008×10^{-3}	2.178360×10^{-3}	8.833110×10^{-3}	-6.654750×10^{-3}
C	B	6.403958×10^{-4}	1.311860×10^{-2}	-1.247821×10^{-2}	2.079241×10^{-3}	6.899252×10^{-3}	-4.820011×10^{-3}
C	C	6.252133×10^{-4}	8.910434×10^{-3}	-8.285220×10^{-3}	3.269873×10^{-3}	1.037452×10^{-2}	-7.104646×10^{-3}
B	D	1.695676×10^{-3}	0	1.695676×10^{-3}	2.625477×10^{-3}	1.523046×10^{-2}	-1.260499×10^{-2}
D	B	0	1.171822×10^{-2}	-1.171822×10^{-2}	2.761786×10^{-3}	7.089100×10^{-3}	-4.327314×10^{-3}
C	D	7.471891×10^{-4}	0	7.471891×10^{-4}	4.134406×10^{-3}	1.643914×10^{-2}	-1.230473×10^{-2}
D	C	0	8.784312×10^{-3}	-8.784312×10^{-3}	3.617629×10^{-3}	1.035936×10^{-2}	-6.741733×10^{-3}
D	D	0	0	0	4.085851×10^{-3}	1.929658×10^{-2}	-1.521073×10^{-2}

Table 3.26. Mitigation in CVA of a CDS (in %), $\alpha = 0$, Exp. collateral rate: ρ_e^i

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
72.09%	55.92%	77.42%	52.98%	-3.25%	79.58%	4.53%	58.53%

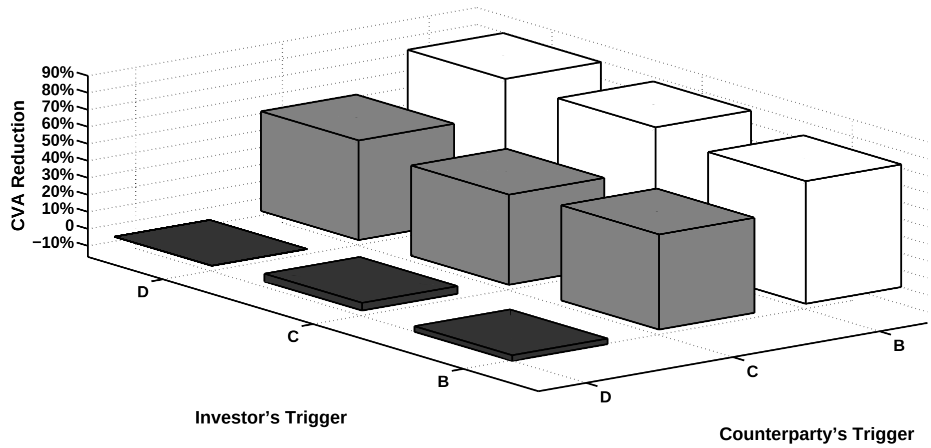

Figure 3.11. Change in CVA of a CDS, $\alpha = 0$, Exponential collateral rate: ρ_e^i

Table 3.27. Mitigation in CVA of a CDS (in %), $\alpha = 1$, Exp. collateral rate: ρ_e^i

(B,B)	(B,C)	(C,B)	(C,C)	(B,D)	(D,B)	(C,D)	(D,C)
73.18%	56.25%	68.31%	53.29%	17.13%	71.55%	19.11%	55.68%

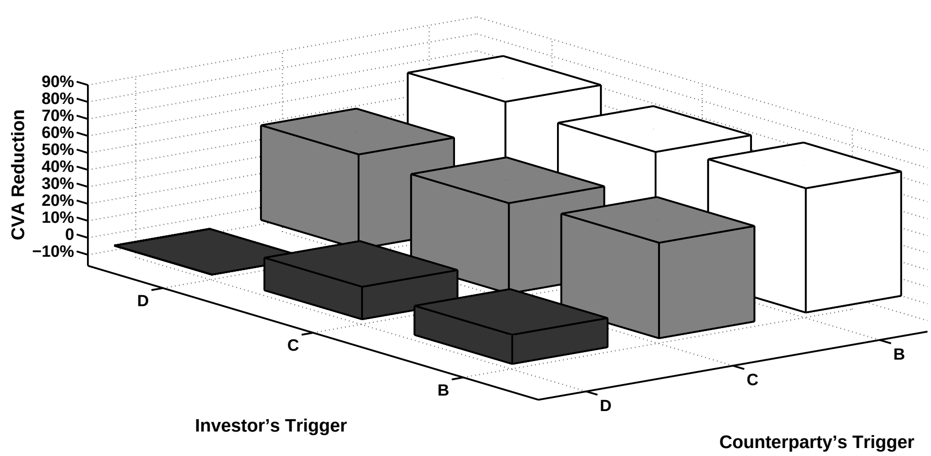

Figure 3.12. Change in CVA of a CDS, $\alpha = 1$, Exponential collateral rate: ρ_e^i

CHAPTER 4

PRICING VIA DYNAMIC COHERENT ACCEPTABILITY INDICES WITH TRANSACTION COSTS

4.1 Introduction

In this chapter, we develop a framework for narrowing the theoretical spread between ask prices and bid prices of derivative securities in markets with transaction costs, using dynamic coherent acceptability indices (DCAIs), developed in Bielecki, Cialenco, and Zhang [BCZ11]. Apart from utilizing the DCAIs, our approach is related to the literature for studying no-good-deal pricing to narrow the no-arbitrage pricing interval.

The literature on models for narrowing the no-arbitrage interval is quite immense. One of the widely studied approaches is indifference pricing, which is based on utility maximization. Specifically, an indifference price is a price at which an agent receives the same expected utility between trading and not trading. A comprehensive collection of articles related to indifference pricing can be found in Carmona [Car09]. However, it is known that the indifference pricing approach has limitations: numerical implementations and explicit calculations for indifference pricing may not be robust, and the resulting bid and ask prices are not necessarily risk-neutral in practice (see for instance Staum [Sta07]). Alternatively, Cochrane and Saá-Requejo [CSR00] introduced the no-good-deal pricing methodology. In this approach, the arbitrage bounds are narrowed by ruling out deals that are too good—cash flows that have high Sharpe ratios. This strengthens the no-arbitrage argument by assuming that any investor is willing to accept a good-deal. In subsequent papers by Bernardo and Ledoit [BL00] and Pinar, Salih, and Camci [PSC10] cash flows are considered good-deals if their corresponding Gain-Loss ratio is high. The no-good-deal pricing approach has been used in other applications and settings by Carr, Geman, and Madan [CGM01], Jaschke and

Kuchler [JK01], Staum [Sta04], Engwerda, Roorda, and Schumacher [RSE05], Bjork and Slinko [BS06], Kloppel and Schweitzer [KS07], Arai and Fukasawa [AF11]. The no-good-deal pricing has also been approached via coherent risk measures in Cherny and Madan [CM06] and Cherny [Che07b]. A comprehensive survey of the theory of pricing and hedging in incomplete markets is provided by Staum [Sta07].

No-good-deal pricing has also been studied by several authors besides Cochrane and Saá-Requejo [CSR00]. In Madan and Pistorious [MPS11], dynamically consistent bid and ask prices for structured products are derived using nonlinear expectations, and in Bion-Nadal [BN09] and Cherny [Che07a] dynamic bid and ask prices are found via dynamic risk measures.

Cherny and Madan [CM10] proposed the conic finance framework for pricing in incomplete, frictionless markets using static acceptability indices, which are introduced in Cherny and Madan [CM09]. The framework is called conic finance because the derivative prices they introduce depend on the direction of trade—the resulting set of cash flows generated by the prices of the derivative is no longer a linear space, it is instead a *convex cone*. Nevertheless, as with any static pricing technique, their prices may lack a dynamic consistency property. This drawback renders the static approach inadequate for pricing exotic derivatives such as path-dependent derivatives. In a recent study, Rosazza-Gianin and Sgarra [RGS12] apply the concepts of dynamic acceptability indices and of BSDEs and g-expectations to determine ask and bid prices of derivatives dynamically in time and to model liquidity risk. Most importantly, their framework is developed without assuming the scale invariance property on the acceptability indices, therefore utilizing a more general class of indices than DCAIs, which are called the quasi-concave acceptability indices.

Our contributions can be summarized as follows. First of all, our framework allows for the (hedging) cash flows to pay dividends, and have transaction costs. In particular, we can apply our no-good-deal pricing approach to the pricing of interest rate swaps and credit default swaps in markets with transaction costs. It is important to stress that our no-good-deal condition is dynamically consistent in time. On the other hand, we construct the good-deal ask and bid prices of a derivative which are dynamically consistent, in the sense that they are defined in terms of dynamic coherent acceptability indices. Furthermore, we prove a representation theorem in terms of risk-neutral measures and dynamically consistent sequences of sets of probability measures This allows us to narrow the no-arbitrage pricing interval. Finally, we propose an application of our framework with the dynamic Gain-Loss ratio, which is a particular dynamic coherent acceptability index.

4.2 Arbitrage and Good-Deals

Let $\mathcal{T} := \{0, 1, \ldots, T\}$, where T is a fixed time horizon. Moreover, let $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ be the underlying filtered probability space. We assume that $\Omega = \{\omega_1, \ldots, \omega_N\}$, and $\mathbb{P}$ is of full support. In what follows, we will denote by $L^0 := L^0(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P})$ the set of all $\mathbb{F}$-adapted processes.

We consider a market consisting of a savings account B and of N traded securities satisfying the following properties:

1. Savings account can be bought and sold via the price process $B := \big((\prod_{s=0}^{t}[1 + r_s])\big)_{t=0}^{T}$, where $(r_t)_{t=0}^{T}$ is the risk-free rate, which is a nonnegative adapted process
2. The securities can be bought by means of the ex-dividend price process $S^{ask} := \big((S_t^{ask,1}, \ldots, S_t^{ask,N})\big)_{t=0}^{T}$, and the corresponding (cumulative) dividend process is denoted by $A^{ask} := \big((A_t^{ask,1}, \ldots, A_t^{ask,N})\big)_{t=1}^{T}$.
3. The securities can be sold in accordance with the ex-dividend price process $S^{bid} := \big((S_t^{bid,1}, \ldots, S_t^{bid,N})\big)_{t=0}^{T}$, and the corresponding (cumulative) dividend

process is denoted by $A^{bid} := \big((A_t^{bid,1}, \ldots, A_t^{bid,N})\big)_{t=1}^{T}$.

The processes S^{ask}, S^{bid}, A^{ask}, A^{bid} are assumed to be adapted. Clearly, all equalities and inequalities involving the above vector-valued processes are understood coordinate-wise. As in Chapter 2 and in Chapter 3, we denote by Δ the backward difference operator: $\Delta Y_t := Y_t - Y_{t-1}$. Without loss of generality, we use the convention that $A_0^{ask} = A_0^{bid} = 0$.

Remark 4.2.1. *Note that for any $t = 1, 2, \ldots, T$ and $j = 1, 2, \ldots, N$, the random variable $\Delta A_t^{ask,j}$ is interpreted as amount of dividend associated with holding a* long *position in security j from time $t-1$ to time t. Likewise, the random variable $\Delta A_t^{bid,j}$ is interpreted as amount of dividend associated with holding a* short *position in security j from time $t-1$ to time t.*

We now make the following standing assumption.

Assumption (A): $S^{ask} \geq S^{bid}$ and $\Delta A^{ask} \leq \Delta A^{bid}$.

Note that if this assumption is violated, then market exhibits arbitrage by simultaneously buying and selling the corresponding security.

4.2.1 Self-Financing Trading Strategies. A *trading strategy* is a predictable process $\phi := \big((\phi_t^0, \phi_t^1, \ldots, \phi_t^N)\big)_{t=1}^{T}$, where ϕ_t^j is interpreted as the number of units of security j held from time $t-1$ to time t. The processes $\phi^1, \ldots, \phi^N$ correspond to the holdings in the N securities, and process ϕ^0 corresponds to the holdings in the savings account B. We take the convention $\phi_0 = (0, \ldots, 0)$.

We define the wealth process associated with a trading strategy as follows.

Definition 4.2.1. *The wealth process $V(\phi)$ associated with a trading strategy ϕ is* *defined as*

$$V_t(\phi) = \begin{cases} \phi_1^0 + \sum_{j=1}^N \mathbb{1}_{\{\phi_1^j \geq 0\}} \phi_1^j S_0^{ask,j} + \sum_{j=1}^N \mathbb{1}_{\{\phi_1^j < 0\}} \phi_1^j S_0^{bid,j}, & \textit{if } \ t = 0, \\ \phi_t^0 B_t + \sum_{j=1}^N \mathbb{1}_{\{\phi_t^j \geq 0\}} \phi_t^j (S_t^{bid,j} + \Delta A_t^{ask,j}) & \\ \qquad + \sum_{j=1}^N \mathbb{1}_{\{\phi_t^j < 0\}} \phi_t^j (S_t^{ask,j} + \Delta A_t^{bid,j}), & \textit{if } \ 1 \leq t \leq T. \end{cases}$$

Remark 4.2.2.

(i) Observe that in Definition 4.2.1, $V_0(\phi)$ is interpreted as the cost of setting up the portfolio associated with ϕ. However, at $t = 1, \ldots, T$, the process $V_t(\phi)$ indicates the sum of the liquidation value of the portfolio associated with trading strategy ϕ before any time t transactions and the dividends associated with the strategy ϕ from $t-1$ to t.

(ii) Furthermore, the wealth process V is not linear, i.e. $V(\phi) + V(\psi) \neq V(\phi + \psi)$, and $V(\alpha\phi) \neq \alpha V(\phi)$ for $\alpha \in \mathbb{R}$ and some trading strategies ϕ and ψ. This is a consequence of the presence of transaction costs and also the main difference from the frictionless setup.

Let us proceed by defining the self-financing condition in our context.

Definition 4.2.2. *A trading strategy ϕ is self-financing if*

$$\begin{aligned} B_t \Delta \phi_{t+1}^0 + \sum_{j=1}^N S_t^{ask,j} \mathbb{1}_{\{\Delta \phi_{t+1}^j \geq 0\}} \Delta \phi_{t+1}^j + \sum_{j=1}^N S_t^{bid,j} \mathbb{1}_{\{\Delta \phi_{t+1}^j < 0\}} \Delta \phi_{t+1}^j \\ = \sum_{j=1}^N \phi_t^j \mathbb{1}_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + \sum_{j=1}^N \phi_t^j \mathbb{1}_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j} \end{aligned} \tag{4.1}$$

for all $t = 1, 2, \ldots, T-1$.

Naturally, the self-financing condition implies that no money can flow in or out of the portfolio.

We define the discounted wealth processes as $V^*(\phi) := B^{-1}V(\phi)$ for all trading strategies ϕ. The next lemma gives a useful characterization of the self-financing condition in terms of the discounted wealth process. We refer to Bielecki et al. [BCR12] for the proof.

Lemma 4.2.1. *A trading strategy ϕ is self-financing if and only if the wealth process $V(\phi)$ satisfies the following equality*

$$\begin{aligned} V_t^*(\phi) = V_0(\phi) &+ \sum_{j=1}^{N} \mathbb{1}_{\{\phi_t^j \geq 0\}} \phi_t^j B_t^{-1} S_t^{bid,j} + \sum_{j=1}^{N} \mathbb{1}_{\{\phi_t^j < 0\}} \phi_t^j B_t^{-1} S_t^{ask,j} \\ &- \sum_{j=1}^{N} \sum_{u=1}^{t} \mathbb{1}_{\{\Delta\phi_u^j \geq 0\}} \Delta\phi_u^j B_{u-1}^{-1} S_{u-1}^{ask,j} - \sum_{j=1}^{N} \sum_{u=1}^{t} \mathbb{1}_{\{\Delta\phi_u^j < 0\}} \Delta\phi_u^j B_{u-1}^{-1} S_{u-1}^{bid,j} \\ &+ \sum_{j=1}^{N} \sum_{u=1}^{t} \mathbb{1}_{\{\phi_u^j \geq 0\}} \phi_u^j B_u^{-1} \Delta A_u^{ask,j} + \sum_{j=1}^{N} \sum_{u=1}^{t} \mathbb{1}_{\{\phi_u^j < 0\}} \phi_u^j B_u^{-1} \Delta A_u^{bid,j} \end{aligned}$$

for $t = 1, 2, \ldots, T$.

Therefore, the wealth process at time t, associated with a self-financing trading strategy ϕ, is equal to the sum of setting up the portfolio associated with ϕ, the liquidation value at time t of the portfolio associated with ϕ, all purchases and sales before time t, and all dividends associated with ϕ up to time t.

Remark 4.2.3. *We recover classic definitions of the wealth process and self-financing condition if there are no transactions costs. In case $S^{ask} = S^{bid}$ and $A^{ask} = A^{bid} = 0$, i.e. if the market is frictionless and there are no dividend-paying securities, see Pliska [Pli97] for the relevant definition. The definition, in case the market is frictionless and there are dividend-paying securities, which is $S^{ask} = S^{bid}$ and $A^{ask} = A^{bid}$, can be found in Kijima [Kij03].*

4.2.2 Arbitrage.

Let us start with defining the following sets of self-financing trading strategies.

$$\mathcal{S}(t) := \begin{cases} \{\phi : \phi \ \text{ is s.f.}, V_0(\phi) = 0\}, & t = 0 \\ \\ \{\phi : \phi \ \text{ is s.f.}, \ \phi_s = \mathbb{1}_{\{s \geq t+1\}}\phi_s \text{ for all } s = 1, 2, \ldots, T\}, & t \in \{1, \ldots, T-1\} \end{cases}$$

Note that in particular $V_t(\phi) = 0$ for any $\phi \in \mathcal{S}(t)$. Moreover, let us define

$$\mathcal{H}^0(t) := \Big\{ \Big(0, \ldots, 0, \Delta V^*_{t+1}(\phi), \ldots, \Delta V^*_T(\phi)\Big) \, : \, \phi \in \mathcal{S}(t) \Big\} \tag{4.2}$$

for $t \in \{0, \ldots, T-1\}$.

In general, the sets $\mathcal{H}^0(t)$ are not convex because of the presence of transaction costs. Therefore, we define the following sets.

$$\mathcal{L}_+(t) := \Big\{ (Z_s)_{s=0}^T : Z_s \in L_+(\Omega, \mathcal{F}_s, \mathbb{P}), \ Z_s = \mathbb{1}_{\{s \geq t+1\}} Z_s, s = 0, \ldots, T \Big\}, \tag{4.3}$$

$$\mathcal{H}(t) := \Big\{ \Big(0, \ldots, 0, \Delta(V^*_{t+1}(\phi) - Z_{t+1}), \ldots, \Delta(V^*_T(\phi) - Z_T)\Big) \, : \, \phi \in \mathcal{S}(t), \ Z \in \mathcal{L}_+(t) \Big\}, \tag{4.4}$$

for $t \in \{0, \ldots, T-1\}$. We call $\mathcal{H}(t)$ as the *set of hedging cash flows initiated at time* t.

Using the fact that the set

$$\{V^*_s(\phi) - X : \phi \text{ is s.f.}, \ X \text{ is } \mathcal{F}_s - \text{measurable, and } X \geq 0\}$$

is a convex cone (see Bielecki et al. [BCR12]), it can be shown that the set $\mathcal{H}(t)$ is also a convex cone.

We continue with defining an arbitrage opportunity in our setup.

Definition 4.2.3. *An* arbitrage opportunity *at time $t \in \{0, \ldots, T-1\}$ for $\mathcal{H}^0(t)$ is a cash flow $H \in \mathcal{H}^0(t)$ such that $\sum_{s=t}^T H_s(\omega) \geq 0$ for all $\omega \in \Omega$, and $\mathbb{E}^{\mathbb{P}}[\sum_{s=t}^T H_s \,|\, \mathcal{F}_t](\omega) > 0$ for some $\omega \in \Omega$.*

The *no-arbitrage condition* holds true at time t for $\mathcal{H}^0(t)$ if there does not exist an arbitrage opportunity at time t for $\mathcal{H}^0(t)$, where $t \in \{0, \ldots, T-1\}$.

Remark 4.2.4. *An arbitrage opportunity is usually defined through a trading strategy rather than a cash flow. However, we work with cash flows in our setup and each hedging cash flow corresponds to a trading strategy.*

Definition 4.2.4. *For any fixed $t \in \{0, \ldots, T-1\}$, we say that a probability measure $\mathbb{Q}$ is* risk-neutral for $\mathcal{H}^0(t)$ *if $\mathbb{Q} \sim \mathbb{P}$, and if $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s \,|\, \mathcal{F}_t](\omega) \leq 0$ for all $\omega \in \Omega$ and all $H \in \mathcal{H}^0(t)$. The set of all risk-neutral measures for $\mathcal{H}^0(t)$ will be denoted by $\mathcal{R}(\mathcal{H}^0(t))$.*

Likewise, we can define the set of risk-neutral probabilities $\mathcal{R}(\mathcal{H}(t))$, and the arbitrage opportunity and no-arbitrage condition for the set $\mathcal{H}(t)$, where $t \in \{0, \ldots, T-1\}$. We see from the following results that we can interchange $\mathcal{H}^0(t)$ by $\mathcal{H}(t)$ in Definition 4.2.3 and Definition 4.2.4.

Lemma 4.2.2.

(i) For all $t \in \{0, \ldots, T-1\}$, we have $\mathbb{Q} \in \mathcal{R}(\mathcal{H}^0(t))$ if and only if $\mathbb{Q} \sim \mathbb{P}$, and if $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s \,|\, \mathcal{F}_t] \leq 0$ for all $H \in \mathcal{H}(t)$.

(ii) For all $t \in \{0, \ldots, T-1\}$, we have that $\mathcal{R}(\mathcal{H}(t)) = \mathcal{R}(\mathcal{H}^0(t))$.

Proof. First, let us fix $t \in \{0, \ldots, T-1\}$.

($\Longrightarrow$) If $\mathbb{Q} \in \mathcal{R}(\mathcal{H}^0(t))$, then $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s^0 \,|\, \mathcal{F}_t] \leq 0$ for all $H^0 \in \mathcal{H}^0(t)$. Hence, $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s^0 - Z_T \,|\, \mathcal{F}_t] \leq 0$ for all $H^0 \in \mathcal{H}^0(t)$ and $Z \in \mathcal{L}_+(t)$. Therefore, $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s \,|\, \mathcal{F}_t] \leq 0$ for all $H \in \mathcal{H}(t)$.

($\Longleftarrow$) Suppose that $\mathbb{Q} \sim \mathbb{P}$, and that $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s \,|\, \mathcal{F}_t] \leq 0$ for all $H \in \mathcal{H}(t)$. Then, $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s^0 - Z_T \,|\, \mathcal{F}_t] \leq 0$ for all $H^0 \in \mathcal{H}^0(t)$ and $Z \in \mathcal{L}_+(t)$. Letting $Z_T = 0$ proves that $\mathbb{Q} \in \mathcal{R}(\mathcal{H}^0(t))$.

$\square$

Lemma 4.2.3. *For each $t \in \{0, \dots, T-1\}$, the no-arbitrage condition holds true at time t for $\mathcal{H}^0(t)$ if and only if for each $H \in \mathcal{H}(t)$ such that $\sum_{s=t}^{T} H_s \geq 0$, we have $\sum_{s=t}^{T} H_s = 0$.*

Proof. Let us fix $t \in \{0, \dots, T-1\}$.

($\Longrightarrow$) Assume that $H \in \mathcal{H}(t)$ is such that $\sum_{s=t}^{T} H_s \geq 0$. Then, by definition of $\mathcal{H}(t)$, there exists $H^0 \in \mathcal{H}^0(t)$ and $Z \in \mathcal{L}_+(t)$ so that $\sum_{s=t}^{T} H_s = \sum_{s=t}^{T} H_s^0 - Z_T$. This give us $\sum_{s=t}^{T} H_s^0 \geq Z_T$. The no-arbitrage condition holds true at time t for $\mathcal{H}^0(t)$, so $\sum_{s=t}^{T} H_s^0 = 0$. Therefore, $Z_T = 0$, which implies $\sum_{s=t}^{T} H_s = 0$.

($\Longleftarrow$) Suppose that $H^0 \in \mathcal{H}^0(t)$ is such that $\sum_{s=t}^{T} H_s^0 \geq 0$. By assumption, for each $H \in \mathcal{H}(t)$ such that $\sum_{s=t}^{T} H_s \geq 0$, we have $\sum_{s=t}^{T} H_s = 0$. From the definition of $\mathcal{H}(t)$, this implies that for each $\hat{H}^0 \in \mathcal{H}^0(t)$, $Z \in \mathcal{L}_+(t)$ such that $\sum_{s=t}^{T} \hat{H}_s - Z_T \geq 0$, we have $\sum_{s=t}^{T} \hat{H}_s^0 - Z = 0$. Taking $Z = 0$ and $\hat{H}^0 := H^0$ gives us $\sum_{s=t}^{T} H_s^0 = 0$.

$\square$

Remark 4.2.5. *Recall that the sets $\mathcal{H}^0(t)$ have a natural financial interpretation, compared to the sets $\mathcal{H}(t)$. Nevertheless, using Lemma 4.2.2 and Lemma 4.2.3, we can make use of either $\mathcal{H}(t)$ or $\mathcal{H}^0(t)$ since $\mathcal{R}(\mathcal{H}(t)) = \mathcal{R}(\mathcal{H}^0(t))$. Therefore, Theorem 4.2.1 and Theorem 4.3.1 can be stated and proved in terms of $\mathcal{H}^0(t)$ as well. It is more convenient to work with the set $\mathcal{H}(t)$, since it is a convex cone. Therefore, we work with $\mathcal{H}(t)$ in the sequel.*

In conclusion, let us present and prove the following proposition which characterizes the no-arbitrage condition for $\mathcal{H}(t)$ via the set of risk neutral measures $\mathcal{R}(\mathcal{H}(t))$.

Proposition 4.2.1. *If $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$, then the no-arbitrage condition holds at time $t \in \{0, \ldots, T-1\}$ for $\mathcal{H}(t)$.*

Proof. Assume that $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))$, and that there exists an arbitrage opportunity H at time $t \in \{0, \ldots, T-1\}$. By the definition of an arbitrage opportunity, $H \in \mathcal{H}(t)$, $\sum_{s=t}^{T} H_s \geq 0$, and $\mathbb{E}^{\mathbb{P}}[\sum_{s=t}^{T} H_s \,|\, \mathcal{F}_t](\omega) > 0$ for some $\omega \in \Omega$. Therefore, we have $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s \,|\, \mathcal{F}_t](\omega) > 0$ for some $\omega \in \Omega$, since $\mathbb{Q} \sim \mathbb{P}$ and $\sum_{s=t}^{T} H_s \geq 0$, However, this contradicts that $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))$. Hence, the no-arbitrage condition holds true at time $t \in \{0, \ldots, T-1\}$ for $\mathcal{H}(t)$. □

We now state definitions related to pricing with the no-arbitrage arguments defined above.

Definition 4.2.5. *Let $D \in L^0$ and $t \in \{0, \ldots, T-1\}$.*

(i) The set of extended cash flows associated with an $\mathcal{F}_t$-measurable random variable S_t and $D \in L^0$ *is defined as*

$$\widetilde{\mathcal{H}}(t, S_t) := \Big\{ \Big(0, \ldots, 0, \xi_t S_t, H_{t+1} - \xi_t D^*_{t+1}, \ldots, H_T - \xi_t D^*_T\Big) \\ : H \in \mathcal{H}(t),\ \xi_t \text{ is an } \mathcal{F}_t\text{-measurable r.v.} \Big\},$$

(ii) The pricing interval associated with a process $D \in L^0$ and a set of probability measures $\mathcal{X}$ *is defined as*

$$\mathcal{I}(t, D; \mathcal{X}) := \Big\{ \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D^*_s \,\Big|\, \mathcal{F}_t \Big] : \mathbb{Q} \in \mathcal{X} \Big\}.$$

A cash flow in $\mathcal{H}(t, S_t)$ is the sum of a position in $\mathcal{H}(t)$ and a position of ξ_t units in the discounted cash flow $(0, \ldots, 0, S_t, -D^*_{t+1}, \ldots, -D^*_T)$.

$\mathcal{I}(t, D; \mathcal{X})$ is called the *no-arbitrage pricing interval* if for each $S_t \in \mathcal{I}(t, D; \mathcal{X})$ the no-arbitrage condition is satisfied for $\widetilde{\mathcal{H}}(t, S_t)$. Similarly, we call $S_t \in \mathcal{I}(t, D)$ a no-arbitrage price if $\mathcal{I}(t, D)$ is a no-arbitrage pricing interval. In other words, $\mathcal{I}(t, D; \mathcal{X})$

is the no-arbitrage pricing interval if for each $S_t \in \mathcal{I}(t, D; \mathcal{X})$ and each $\widetilde{H} \in \widetilde{\mathcal{H}}(t, S_t)$ such that $\sum_{s=t}^{T} \widetilde{H}_s \geq 0$, we have $\sum_{s=t}^{T} \widetilde{H}_s = 0$.

We denote by $\pi_t^U := \sup_{\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\big[\sum_{s=t+1}^{T} D_s^* \,\big|\, \mathcal{F}_t\big]$ *the upper no-arbitrage bound* and $\pi_t^L := \inf_{\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\big[\sum_{s+1}^{T} D_s^* \,\big|\, \mathcal{F}_t\big]$ *the lower no-arbitrage bound*, if $\mathcal{I}(t, D; \mathcal{X})$ is a no-arbitrage pricing interval. Moreover, any $S_t \in \mathcal{I}(t, D; \mathcal{X})$ is called a no-arbitrage price.

The following result provides a necessary condition for $\mathcal{I}(t, D; \mathcal{X})$ to be a no-arbitrage pricing interval.

Lemma 4.2.4. *Let $D \in L^0$ and $t \in \{0, \ldots, T-1\}$. If $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$, then $\mathcal{I}(t, D)$ is the no-arbitrage pricing interval.*

Proof. Fix $D \in L^0$, $t \in \{0, \ldots, T-1\}$ and $S_t \in \mathcal{I}(t, D; \mathcal{R}(\mathcal{H}(t)))$. Let $\widetilde{H} \in \widetilde{\mathcal{H}}(t, S_t)$ be a cash flow such that $\sum_{s=t}^{T} \widetilde{H}_s \geq 0$. By definition of $\widetilde{\mathcal{H}}(t, S_t)$, we have that

$$\xi_t S_t + \sum_{s=t}^{T} (H_s - \xi_t D_s^*) \geq 0 \tag{4.5}$$

for some $H \in \mathcal{H}(t)$ and some $\mathcal{F}_t$-measurable random variable ξ_t.

Next, since $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$ and $S_t \in \mathcal{I}(t, D; \mathcal{R}(\mathcal{H}(t)))$, there exists $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))$ such that $S_t = \mathbb{E}^{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s^* \,\big|\, \mathcal{F}_t\big]$. As a result, $\xi_t \mathbb{E}^{\mathbb{Q}}\big[\sum_{s=t}^{T} D_s^* \,\big|\, \mathcal{F}_t\big] - \xi_t S_t = 0$. In view of (4.5) we deduce that $\mathbb{E}^{\mathbb{Q}}\big[\sum_{s=t}^{T} H_s \,\big|\, \mathcal{F}_t\big] \geq 0$ holds true. Since $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))$, we have that $\mathbb{E}^{\mathbb{Q}}\big[\sum_{s=t}^{T} H_s \,\big|\, \mathcal{F}_t\big] = 0$, which gives us that

$$\xi_t S_t + \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} (H_s - \xi_t D_s^*) \,\Big|\, \mathcal{F}_t\Big] = 0.$$

In virtue of the above result, we conclude that $\xi_t S_t + \sum_{s=t}^{T} (H_s - \xi_t D_s^*) = 0$, which implies that the no-arbitrage condition holds true for $\widetilde{\mathcal{H}}(t, S_t)$. $\square$

4.2.3 Good-Deals. The theory of Dynamic Coherent Acceptability Indices (DCAIs) was developed in Bielecki et al. [BCZ11] (see Appendix A for the definitions and the related results). A DCAI α is associated with a left-continuous, increasing family of DCRMs $(\rho^\gamma)_{\{\gamma\in(0,\infty)\}}$, and consequently with a family of dynamically consistent sequences of sets of probability measures. We fix a family of DCRMs $(\rho^\gamma)_{\{\gamma\in(0,\infty)\}}$, and denote by $\mathcal{Q} = ((\mathcal{Q}_t^\gamma)_{t\in\mathcal{T}})_{\gamma\in(0,\infty)}$ the corresponding family of dynamically consistent sequences of sets of probability measures.

The following definition is a counterpart of Definition 4.2.3.

Definition 4.2.6. *A* good-deal *for $\mathcal{H}(t)$ at time $t \in \{0,\dots,T-1\}$ and level $\gamma > 0$ is a cash flow $H \in \mathcal{H}(t)$ such that $\rho_t^\gamma(H)(\omega) < 0$ for some $\omega \in \Omega$.*

Contrary to the definition of an arbitrage opportunity, a *good-deal* is defined through a family of DCRMs and a level γ. Thus, even though a cash flow stream $H \in \mathcal{H}(t)$ is a good-deal with respect to a family of DCRMs for a fixed acceptability level γ, it may not be a good-deal with respect to another family of DCRMs. Note that if a cash flow is a good-deal for γ_0, then it is also a good-deal for any $\gamma' \leq \gamma_0$, since ρ^γ is monotone increasing in γ. Therefore, a cash flow stream that is a good-deal at level γ_0 for a fixed family of DCRMs, may not be a good-deal at another level $\gamma' > \gamma_0$.

Let us proceed with defining the no-good-deal condition.

Definition 4.2.7. *The* no-good-deal condition *(NGD) holds true for $\mathcal{H}(t)$ at time $t \in \{0,\dots,T-1\}$ and level $\gamma > 0$ if $\rho_t^\gamma(H)(\omega) \geq 0$ for all $H \in \mathcal{H}(t)$ and $\omega \in \Omega$.*

We will make the following technical assumption on $\mathcal{Q}$.

Assumption (B): We assume that, for each $\gamma > 0$ and $t \in \mathcal{T}$, any probability measure $\mathbb{Q} \in \mathcal{Q}_t^\gamma$ is equivalent to $\mathbb{P}$, and the set

$$\mathcal{E}_t^\gamma := \left\{ \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \,:\, \mathbb{Q} \in \mathcal{Q}_t^\gamma \right\}$$

is closed and convex.

Observe that, since Ω is finite and $\mathbb{P}$ is of full support, the set $\mathcal{E}_t^\gamma$ is bounded. Thus, the set $\mathcal{E}_t^\gamma$ is compact for all $\gamma > 0$ and $t \in \mathcal{T}$. We show that a family of densities $\mathcal{E}$ corresponding to the dynamic Gain-Loss Ratio satisfies this assumption, in Section 4.4.

Finally, let us recall from Bielecki et al. [BCIR12] the following result which characterizes the NGD condition at time $t \in \{0, \ldots, T-1\}$ and level $\gamma > 0$ via the $\mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$. Proof of the following theorem can be found in Bielecki et al. [BCIR12].

Theorem 4.2.1. *The NGD condition holds true for $\mathcal{H}(t)$ at time $t \in \{0, \ldots, T-1\}$ and level $\gamma > 0$ if and only if $\mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$.*

Observe that owing to Proposition 4.2.1 and Theorem 4.2.1 that if no-good-deal condition holds true then the no-arbitrage condition also holds true, since $\mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$ implies $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$.

4.3 Dynamic Ask and Bid Prices via DCAI

In this section we define the dynamic ask and bid prices of a derivative contract via DCAIs. Moreover we derive a representation for the prices using risk neutral measures and dynamically consistent sequences of sets of probability measures.

Let us start by defining the *set of extended cash flows*, which is needed to derive the dynamic ask and bid prices. Let $D \in L^0$ be a cash flow associated to a *derivative contract*. For a fixed $t \in \{0, \ldots, T-1\}$, $D \in L^0$, and an $\mathcal{F}_t$-measurable random variable X_t, we define the following sets

$$\widehat{\mathcal{H}}(t) := \Big\{\Big(0,\dots,0,\xi_t X_t^*, H_{t+1} - \xi_t D_{t+1}^*, \dots, H_T - \xi_t D_T^*\Big) \\ : H \in \mathcal{H}(t),\ \xi_t \text{ is } \mathcal{F}_t\text{-measurable},\ \xi_t \geq 0\Big\}, \tag{4.6}$$

$$\overline{\mathcal{H}}(t) := \Big\{\Big(0,\dots,0,-\xi_t X_t^*, H_{t+1} + \xi_t D_{t+1}^*, \dots, H_T + \xi_t D_T^*\Big) \\ : H \in \mathcal{H}(t),\ \xi_t \text{ is } \mathcal{F}_t\text{-measurable},\ \xi_t \geq 0\Big\}, \tag{4.7}$$

where $X_t^* := B_t^{-1} X_t$ and $D^* := B^{-1} D$. We call the pair $(\widehat{\mathcal{H}}(t), \overline{\mathcal{H}}(t))$ as the *set of extended cash flows*.

An element, i.e. a cash flow stream, in $\widehat{\mathcal{H}}(t)$ consists of a position in the underlying market $\mathcal{H}(t)$ and a nonnegative *static* position of ξ_t units in the discounted cash flow $(0,\dots,0,X_t^*,-D_{t+1}^*,\dots,-D_T^*)$. Respectively, a cash flow stream in $\overline{\mathcal{H}}(t)$ consists of a position in the underlying market $\mathcal{H}(t)$ and a nonnegative *static* position of ξ_t units in the discounted cash flow $(0,\dots,0,-X_t^*,D_{t+1}^*,\dots,D_T^*)$.

Observe that $\mathcal{H}(t) \subset \widehat{H}(t) \cap \overline{H}(t)$. Moreover, $H \in \widehat{H}(t)$ and $H \in \overline{H}(t)$ for any $H \in \mathcal{H}(t)$, where $\xi_t = 0$ in (4.6) and (4.7).

Analogously to Definition 4.2.4, a probability measure $\mathbb{Q}$ is *risk-neutral* for $\widehat{\mathcal{H}}(t)$, respectively $\overline{\mathcal{H}}(t)$, if $\mathbb{Q} \sim \mathbb{P}$, and $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t}^{T} H_s \,|\, \mathcal{F}_t] \leq 0$ for all $H \in \widehat{\mathcal{H}}(t)$, respectively for all $H \in \overline{\mathcal{H}}(t)$. Furthermore, the *no-good-deal condition* holds true for $\widehat{\mathcal{H}}(t)$, respectively $\overline{\mathcal{H}}(t)$, at time $t \in \mathcal{T}$ and level $\gamma > 0$, if $\rho_t^\gamma(H) \geq 0$ for all $H \in \widehat{\mathcal{H}}(t)$, respectively $H \in \overline{\mathcal{H}}(t)$. The set of all risk-neutral measures for $\widehat{\mathcal{H}}(t)$ and $\overline{\mathcal{H}}(t)$ is denoted by $\mathcal{R}(\widehat{\mathcal{H}}(t))$ and $\mathcal{R}(\overline{\mathcal{H}}(t))$, respectively.

Let us proceed with the following lemma.

Lemma 4.3.1. *The sets $\widehat{\mathcal{H}}(t)$ and $\overline{\mathcal{H}}(t)$ are convex cones.*

Proof. Let us first show that $\widehat{H}(t)$ is a convex cone. Suppose that $t \in \{0,\dots,T-1\}$,

$\widehat{H}^1, \widehat{H}^2 \in \widehat{\mathcal{H}}(t)$, and $\lambda_1, \lambda_2 \geq 0$. Using the definition of $\widehat{H}(t)$, for a fixed $D \in L^0$ and a fixed $\mathcal{F}_t$-measurable random variable X_t, there exist $H^1, H^2 \in \mathcal{H}(t)$ and nonnegative $\mathcal{F}_t$-measurable random variables ξ_t^1, ξ_t^2 such that

$$\widehat{H}^1 = \left(0, \ldots, 0, \xi_t^1 X_t^*, H_{t+1}^1 - \xi_t^1 D_{t+1}^*, \ldots, H_T^1 - \xi_t^1 D_T^*\right),$$
$$\widehat{H}^2 = \left(0, \ldots, 0, \xi_t^2 X_t^*, H_{t+1}^2 - \xi_t^2 D_{t+1}^*, \ldots, H_T^2 - \xi_t^2 D_T^*\right).$$

Next, we see that

$$\begin{aligned}\lambda_1 \widehat{H}^1 + \lambda_2 \widehat{H}^2 &= \left(0, \ldots, 0, \lambda_1 \xi_t^1 X_t^*, \lambda_1 H_{t+1}^1 - \lambda_1 \xi_t^1 D_{t+1}^*, \ldots, \lambda_1 H_T^1 - \lambda_1 \xi_t^1 D_T^*\right) \\ &\quad + \left(0, \ldots, 0, \lambda_2 \xi_t^2 X_t^*, \lambda_2 H_{t+1}^2 - \lambda_2 \xi_t^2 D_{t+1}^*, \ldots, \lambda_2 H_T^2 - \lambda_2 \xi_t^2 D_T^*\right) \\ &= \Big(0, \ldots, 0, (\lambda_1 \xi_t^1 + \lambda_2 \xi_t^2) X_t^*, (\lambda_1 H_{t+1}^1 + \lambda_2 H_{t+1}^2) - (\lambda_1 \xi_t^1 + \lambda_2 \xi_t^2) D_{t+1}^*, \\ &\quad \ldots, (\lambda_1 H_T^1 + \lambda_2 H_T^2) - (\lambda_1 \xi_t^1 + \lambda_2 \xi_t^2) D_T^*\Big)\end{aligned}$$

Since $\mathcal{H}(t)$ is a convex cone, we have that $\lambda_1 H^1 + \lambda_2 H^2 \in \mathcal{H}(t)$. Moreover, $\lambda_1 \xi_t^1 + \lambda^2 \xi_t^2$ is $\mathcal{F}_t$-measurable and nonnegative. It follows that $\widehat{\mathcal{H}}(t)$ is a convex cone.

Let us proceed by proving that $\overline{H}(t)$ is a convex cone. Let $t \in \{0, \ldots, T-1\}$, $\overline{H}^1, \overline{H}^2 \in \widehat{\mathcal{H}}(t)$, and $\lambda_1, \lambda_2 \geq 0$. By the definition of $\overline{H}(t)$, for a fixed $D \in L^0$ and a fixed $\mathcal{F}_t$-measurable random variable X_t, there exist $H^1, H^2 \in \mathcal{H}(t)$ and nonnegative $\mathcal{F}_t$-measurable random variables ξ_t^1, ξ_t^2 such that

$$\overline{H}^1 = \left(0, \ldots, 0, -\xi_t^1 X_t^*, H_{t+1}^1 + \xi_t^1 D_{t+1}^*, \ldots, H_T^1 + \xi_t^1 D_T^*\right),$$
$$\overline{H}^2 = \left(0, \ldots, 0, -\xi_t^2 X_t^*, H_{t+1}^2 + \xi_t^2 D_{t+1}^*, \ldots, H_T^2 + \xi_t^2 D_T^*\right).$$

Then, we see that

$$\begin{aligned}\lambda_1 \overline{H}^1 + \lambda_2 \overline{H}^2 &= \left(0, \ldots, 0, -\lambda_1 \xi_t^1 X_t^*, \lambda_1 H_{t+1}^1 + \lambda_1 \xi_t^1 D_{t+1}^*, \ldots, \lambda_1 H_T^1 + \lambda_1 \xi_t^1 D_T^*\right) \\ &\quad + \left(0, \ldots, 0, -\lambda_2 \xi_t^2 X_t^*, \lambda_2 H_{t+1}^2 + \lambda_2 \xi_t^2 D_{t+1}^*, \ldots, \lambda_2 H_T^2 + \lambda_2 \xi_t^2 D_T^*\right) \\ &= \Big(0, \ldots, 0, -(\lambda_1 \xi_t^1 + \lambda_2 \xi_t^2) X_t^*, (\lambda_1 H_{t+1}^1 + \lambda_2 H_{t+1}^2) + (\lambda_1 \xi_t^1 + \lambda_2 \xi_t^2) D_{t+1}^*, \\ &\quad \ldots, (\lambda_1 H_T^1 + \lambda_2 H_T^2) + (\lambda_1 \xi_t^1 + \lambda_2 \xi_t^2) D_T^*\Big)\end{aligned}$$

Using the same arguments for $\widehat{H}(t)$, we conclude that $\overline{\mathcal{H}}(t)$ is a convex cone. $\square$

Remark 4.3.1. *Note that, analogously to Proposition 4.2.1, the no-good-deal condition holds true for $\widehat{H}(t)$, respectively $\overline{H}(t)$, at time $t \in \mathcal{T}$ and level $\gamma > 0$ if and only if $\mathcal{R}(\widehat{H}(t)) \cap \mathcal{Q}_t^\gamma = \emptyset$, respectively $\mathcal{R}(\overline{H}(t)) \cap \mathcal{Q}_t^\gamma = \emptyset$. Using the fact that $\widehat{H}(t)$ and $\overline{H}(t)$ are convex cones, we can interchange $\mathcal{H}(t)$ with $\widehat{H}(t)$ or $\overline{H}(t)$ in Proposition 4.2.1.*

For the sake of brevity, let us define the mappings $\delta_t^+, \delta_t : L^0 \to L^0$ as

$$\delta_t^+(D) := \big(0, \quad \ldots, \quad 0, \quad 0, \quad D_{t+1}, \quad \ldots \quad D_T\big), \qquad t \in \{0, \ldots, T-1\},$$

$$\delta_t(D) := \big(0, \quad \ldots, \quad 0, \quad D_t, \quad 0, \quad \ldots, \quad 0\big), \qquad t \in \mathcal{T}.$$

We are now ready to introduce the *the dynamic good-deal ask and bid prices* corresponding to a given DCAI α.

Definition 4.3.1. *The discounted* good-deal ask and bid prices *of a derivative contract $D \in L^0$, at level $\gamma > 0$, at time $t \in \{1, \ldots, T-1\}$ are defined as*

$$\begin{aligned}
\Pi_t^{ask,\gamma}(D)(\omega) : &= \inf\{v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \\
&\qquad \text{such that } \alpha_t(\delta_t(\boldsymbol{1}v) + H - \delta_t^+(D^*))(\omega) \geq \gamma\}, \\
\Pi_t^{bid,\gamma}(D)(\omega) : &= \sup\{v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \\
&\qquad \text{such that } \alpha_t(\delta_t^+(D^*) + H - \delta_t(\boldsymbol{1}v))(\omega) \geq \gamma\},
\end{aligned}$$

for all $\omega \in \Omega$.

Remark 4.3.2. *Clearly, the good-deal prices defined above depend on the choice of DCAI α, level γ, and the set of hedging cash flows $\mathcal{H}(t)$. Furthermore, the good-deal ask (bid) price is non-decreasing (non-increasing) in γ, from the monotonicity property of DCAIs (see property (D3) in Definition A.1). In addition, the good-deal* *ask (bid) price is non-increasing (non-decreasing) in* $\mathcal{H}(t)$ *since*

$$\begin{aligned}\Pi_t^{ask,\gamma}(D)(\omega) &= \inf \bigcup_{H\in\mathcal{H}(t)} \{v \in \mathbb{R} :\ \alpha_t(\delta_t(\boldsymbol{1}v) + H - \delta_t^+(D^*))(\omega) \geq \gamma\},\\ \Pi_t^{bid,\gamma}(D)(\omega) &= \sup \bigcup_{H\in\mathcal{H}(t)} \{v \in \mathbb{R} :\ \alpha_t(\delta_t^+(D^*) + H - \delta_t(\boldsymbol{1}v))(\omega) \geq \gamma\}\end{aligned}$$

for all $\omega \in \Omega$.

Remark 4.3.3. *Note that the choice of the appropriate* γ *level is of great importance when finding the good-deal prices an illiquid derivative. Typically, the* γ *levels are calibrated from the quoted prices using a given* α*, and then used to price an illiquid derivative. Such applications can be found in Cherny and Madan [CM10] and Madan and Schoutens [MS11a, MS11b].*

Remark 4.3.4. *We can interpret the ask price,* $\Pi_t^{ask,\gamma}(D)$*, as the minimum amount of cash* v *such that* v *plus the resulting hedging error acceptable (with respect to the acceptability index* α*) at least at level* γ*. Respectively, we can interpret the bid price,* $\Pi_t^{bid,\gamma}(D)$*, as the maximum amount of cash* v *such that* $-v$ *plus the resulting hedging error is* α*-acceptable at least at level* γ*.*

Remark 4.3.5. *Using Theorem A.3, we see that*

$$\begin{aligned}\alpha_t(\delta_t(\boldsymbol{1}v) + H - \delta_t^+(D^*))(\omega) = \sup \Big\{ &\gamma \in (0,+\infty) :\\ &v + \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\big[\sum_{s=t+1}^{T} H_s - D_s^* \,\big|\, \mathcal{F}_t\big](\omega) \geq 0\Big\}\end{aligned}$$

for all $\omega \in \Omega$*,* $t \in \{1,\ldots,T-1\}$*, and* $D \in L^0$*. Since the cash flows* D^* *and* $H \in \mathcal{H}(t)$ *are discounted, the prices* $\Pi^{ask,\gamma}(D)$ *and* $\Pi^{bid,\gamma}(D)$ *are also discounted.*

The following result gives a justification of our definition of ask and bid prices, in the sense that they are well-defined in Definition 4.3.1.

Proposition 4.3.1. *For any fixed $t \in \{1, \dots, T-1\}$, $D \in L^0$, and $\gamma > 0$, the sets*

$$\{v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) \geq \gamma\},$$

$$\{v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t^+(D^*) + H - \delta_t(\mathbf{1}v))(\omega) \geq \gamma\}$$

are nonempty for all $\omega \in \Omega$.

Proof. Let us fix $t \in \{1, \dots, T-1\}$, $D \in L^0$, and $\gamma > 0$.

Suppose that

$$\alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*)) < \gamma$$

for all $v \in \mathbb{R}$ and $H \in \mathcal{H}(t)$. By Theorem A.3, we have that

$$\alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) = \sup \Big\{ \beta \in (0, +\infty) : \\ v + \inf_{\mathbb{Q} \in \mathcal{Q}_t^\beta} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s - D_s \,\Big|\, \mathcal{F}_t \Big](\omega) \geq 0 \Big\} < \gamma$$

for all $v \in \mathbb{R}$ and $H \in \mathcal{H}(t)$. Since α is normalized, there exists $D' \in L^0$ such that $\alpha_t(D') = +\infty$. Let us define v^* as the scalar

$$v^* := \sup_{\omega \in \Omega} \sup_{H \in \mathcal{H}(t)} \left\{ \sup_{\mathbb{Q} \in \mathcal{Q}_t^\beta} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D'_s \,\Big|\, \mathcal{F}_t \Big](\omega) - \inf_{\mathbb{Q} \in \mathcal{Q}_t^\beta} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s - D_s \,\Big|\, \mathcal{F}_t \Big](\omega) \right\}.$$

Then, we see that

$$v^* + \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s - D_s \,\Big|\, \mathcal{F}_t \Big](\omega) \geq \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D'_s \,\Big|\, \mathcal{F}_t \Big](\omega),$$

for all $\mathbb{Q} \in \mathcal{Q}_t^\gamma$, $\omega \in \Omega$, and $H \in \mathcal{H}(t)$. From the monotonicity property of α, we obtain

$$\alpha_t(\delta_t(\mathbf{1}v^*) + H - \delta_t^+(D^*)) \geq \alpha_t(D') = +\infty,$$

which contradicts $\alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) < \gamma$ for all $v \in \mathbb{R}$. $\square$

Using the following result, we can interpret the ask prices via bid prices and bid prices via ask prices.

Lemma 4.3.2. *For any $D \in L^0$, $\gamma > 0$, and $t \in \{0, \dots, T-1\}$ we have that $\Pi_t^{ask,\gamma}(D) = -\Pi_t^{bid,\gamma}(-D)$.*

Proof. Using the definitions of $\Pi_t^{ask,\gamma}(D)$ and $\Pi_t^{bid,\gamma}(D)$, we deduce that

$$\begin{aligned}
\Pi_t^{ask,\gamma}(D) &= \inf\{v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*)) \geq \gamma\} \\
&= -\sup\{-v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*)) \geq \gamma\} \\
&= -\sup\{v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(-\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*)) \geq \gamma\} \\
&= -\Pi_t^{bid,\gamma}(-D).
\end{aligned}$$

□

4.3.1 Dual Representation of Good-Deal Ask and Bid Prices. In this section we prove a representation theorem for the good-deal ask and bid prices in terms of a family of dynamically consistent sequences of sets of probability measures and risk neutral measures.

We now make the following standing assumption, which is necessary for Theorem 4.3.1.

Assumption (C): The mapping $\gamma \mapsto \rho^\gamma$ is continuous.

In Proposition 4.4.3, we show that the dynamic Gain-Loss Ratio indeed satisfies the following assumption.

The following result states one of the main contributions of this work, which gives a representation of the prices $\Pi_t^{ask,\gamma}$ and $\Pi_t^{bid,\gamma}$ in terms of the sets $\mathcal{R}(\mathcal{H}(t))$ and $\mathcal{Q}_t^\gamma(\mathcal{H}(t))$.

Theorem 4.3.1. *The discounted good-deal ask and bid prices of a derivative contract*

$D \in L^0$, at level $\gamma > 0$, at time $t \in \{1, \ldots, T-1\}$ satisfy

$$\Pi_t^{ask,\gamma}(D) = \sup_{\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t \Big],$$

$$\Pi_t^{bid,\gamma}(D) = \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t \Big].$$

Proof. Let $D \in L^0$, $\gamma > 0$, and $t \in \{1, \ldots, T-1\}$. We first show that the theorem holds true for $\Pi^{ask,\gamma}(D)$.

Step 1.a

We first show that

$$\Pi_t^{ask,\gamma}(D) \leq \sup_{\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t \Big].$$

First notice that, by Definition 4.3.1 and Theorem A.1, we have

$$\begin{aligned}\Pi_t^{ask,\gamma}(D)(\omega) = \inf \Big\{ & v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ such that} \\ & \sup \big\{ \beta \in (0, \infty) : \ \rho_t^\beta(\delta_t(\mathbf{1}v) + H - \delta_t^+(D))(\omega) \leq 0 \big\} \geq \gamma \Big\},\end{aligned}$$

for all $\omega \in \Omega$. By continuity and monotonicity of the map $\gamma \mapsto \rho^\gamma$, we may apply Lemma B.2 to deduce that

$$\sup \big\{ \beta \in (0, \infty) : \ \rho_t^\beta(\delta_t(\mathbf{1}v) + H - \delta_t^+(D))(\omega) \leq 0 \big\} \geq \gamma$$

if and only if $\rho_t^\gamma(\delta_t(\mathbf{1}v) + H - \delta_t^+(D))(\omega) \leq 0$ for all $\omega \in \Omega$. Hence,

$$\begin{aligned}\Pi_t^{ask,\gamma}(D)(\omega) = \inf \Big\{ & v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \\ & \text{such that } \rho_t^\gamma(\delta_t(\mathbf{1}v) + H - \delta_t^+(D))(\omega) \leq 0 \Big\}\end{aligned} \tag{4.8}$$

for all $\omega \in \Omega$.

Now fix an $\mathcal{F}_t$-measurable random variable X_t, and let $\mathcal{P}^t := \{P_1^t, P_2^t, \ldots, P_{n_t}^t\}$ be the unique partition that generates $\mathcal{F}_t$. Fix $P_i^t \neq \emptyset$ and let $\omega_i \in P_i^t$. Then

$\mathbb{1}_{P_i^t}(\omega)X_t(\omega_i) = \mathbb{1}_{P_i^t}(\omega)X_t(\omega)$ for all $\omega \in \Omega$. Using (4.8), we have that $\Pi_t^{ask,\gamma}(D)(\omega_i) > X_t^*(\omega_i)$ if and only if

$$X_t^*(\omega_i) \notin \Big\{v \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ s.t. } \rho_t^\gamma(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega_i) \le 0\Big\}.$$

Now, the above condition holds true if and only if

$$\rho_t^\gamma(\delta_t(\mathbf{1}X_t^*(\omega_i)) + H - \delta_t^+(D^*))(\omega_i) > 0, \qquad H \in \mathcal{H}(t)\,.$$

Since ρ^γ is adapted, the above inequality holds true if and only if

$$\mathbb{1}_{P_i^t}(\omega)\rho_t^\gamma(\delta_t(\mathbf{1}X_t^*(\omega_i)) + H - \delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t.$$

By property (A2) in Definition A.2 of ρ^γ, the above holds true if and only if

$$\mathbb{1}_{P_i^t}(\omega)\rho_t^\gamma(\delta_t(\mathbf{1}\mathbb{1}_{P_i^t}X_t^*(\omega_i)) + \mathbb{1}_{P_i^t}H - \mathbb{1}_{P_i^t}\delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t.$$

Since, $\mathbb{1}_{P_i^t}(\omega)X_t(\omega_i) = \mathbb{1}_{P_i^t}(\omega)X_t(\omega)$ for all $\omega \in \Omega$, the above inequality holds true if and only if

$$\mathbb{1}_{P_i^t}(\omega)\rho_t^\gamma(\delta_t(\mathbf{1}\mathbb{1}_{P_i^t}X_t^*) + \mathbb{1}_{P_i^t}H - \mathbb{1}_{P_i^t}\delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t.$$

Again, using property (A2) in Definition A.2, the last inequality holds true if and only if

$$\mathbb{1}_{P_i^t}(\omega)\rho_t^\gamma(\delta_t(\mathbf{1}X_t^*) + H - \delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t.$$

Since $P_i^t \neq \emptyset$, the above holds true if and only if

$$\rho_t^\gamma(\delta_t(\mathbf{1}X_t^*) + H - \delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t,$$

which ultimately implies

$$\rho_t^\gamma(\delta_t(\mathbf{1}X_t^*) + H - \delta_t^+(D^*))(\omega) \ge 0, \quad H \in \mathcal{H}(t), \omega \in P_i^t.$$

Since $\omega_i \in \Omega$ is arbitrary, the partition P_i^t is also arbitrary. As a result, if $\Pi_t^{ask,\gamma}(D) > X_t^*$, then

$$\rho_t^\gamma(\delta_t(\mathbf{1}X_t^*) + H - \delta_t^+(D^*)) \geq 0, \qquad H \in \mathcal{H}(t)\,.$$

By property (A6) of ρ^γ, we get

$$-X_t + \rho_t^\gamma(H - \delta_t^+(D^*)) \geq 0, \qquad H \in \mathcal{H}(t)\,.$$

In virtue of Theorem A.2, the above is equivalent to

$$-X_t^* - \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s - D_s^* \,\Big|\, \mathcal{F}_t\Big] \geq 0, \qquad H \in \mathcal{H}(t).$$

Hence, for any nonnegative $\mathcal{F}_t$-measurable random variable ξ_t, we have that

$$-\xi_t X_t^* - \xi_t \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s - D_s^* \,\Big|\, \mathcal{F}_t\Big] \geq 0, \qquad H \in \mathcal{H}(t).$$

Similarly, in view of Theorem A.2 and property (A6), and since ξ_t is $\mathcal{F}_t$-measurable, we have that

$$\rho_t^\gamma(\delta_t(\mathbf{1}\xi_t X_t^*) + \xi_t H - \xi_t\delta_t^+(D^*)) \geq 0$$

for any $H \in \mathcal{H}(t)$ and any nonnegative $\mathcal{F}_t$-measurable random variable ξ_t. Since $\mathcal{H}(t)$ is closed under multiplication of nonnegative $\mathcal{F}_t$-measurable random variables, the inequality above is equivalent to

$$\rho_t^\gamma(\xi_t\delta_t(\mathbf{1}X_t^*) + H - \xi_t\delta_t^+(D^*)) \geq 0$$

for any $H \in \mathcal{H}(t)$ and any nonnegative $\mathcal{F}_t$-measurable random variable ξ_t.

Therefore, by the definition of $\widehat{\mathcal{H}}(t)$, we deduce that

$$\rho_t^\gamma(\widehat{H}) \geq 0, \qquad \widehat{H} \in \widehat{\mathcal{H}}(t),$$

and hence NGD holds true for $\widehat{\mathcal{H}}(t)$, at time t and level γ. It follows that $\mathcal{R}(\widehat{\mathcal{H}}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$ (see Remark 4.3.1). Let $\mathbb{Q}^* \in \mathcal{R}(\widehat{\mathcal{H}}(t)) \cap \mathcal{Q}_t^\gamma$.

From the definition of $\mathcal{R}(\widehat{\mathcal{H}}(t))$, we have that

$$\mathbb{E}^{\mathbb{Q}^*}\Big[\sum_{u=t+1}^{T}(H_u - \xi_t D_u^*)\Big|\mathcal{F}_t\Big] + \xi_t X_t^* \leq 0 \tag{4.9}$$

for all $H \in \mathcal{H}(t)$ and all nonnegative $\mathcal{F}_t$-measurable random variables ξ_t. Note that $\mathcal{R}(\mathcal{H}(t)) \supseteq \mathcal{R}(\widehat{\mathcal{H}}(t))$ since $\mathcal{H}(t) \subset \widehat{\mathcal{H}}(t)$. Thus, $\mathbb{Q}^* \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$. Because $0 \in \mathcal{H}(t)$, we may let $H = 0$ in (4.9) to conclude that, if $\Pi_t^{ask,\gamma}(D) > X_t^*$, then there exists $\mathbb{Q}^* \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$ such that

$$\mathbb{E}^{\mathbb{Q}^*}\Big[\sum_{s=t+1}^{T} D_s^*\Big|\mathcal{F}_t\Big] \geq X_t^*.$$

Now, for any $\epsilon > 0$, let us define $X_t^{*,\epsilon} = \Pi_t^{ask,\gamma}(D) - \epsilon$. From the inequality above, we deduce that there exists $\mathbb{Q}^{*,\epsilon} \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$ such that

$$\mathbb{E}^{\mathbb{Q}^{*,\epsilon}}\Big[\sum_{s=t+1}^{T} D_s^*\Big|\mathcal{F}_t\Big] \geq \Pi_t^{ask,\gamma}(D) - \epsilon,$$

which leads to

$$\sup_{\mathbb{Q}\in\mathcal{R}(\mathcal{H}(t))\cap\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^*\Big|\mathcal{F}_t\Big] \geq \Pi_t^{ask,\gamma}(D) - \epsilon.$$

Since ϵ is arbitrary, we have that

$$\Pi_t^{ask,\gamma}(D) \leq \sup_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^*\Big|\mathcal{F}_t\Big]. \tag{4.10}$$

Step 1.b

We proceed by showing that

$$\Pi_t^{ask,\gamma}(D) \geq \sup_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^*\Big|\mathcal{F}_t\Big].$$

Suppose $\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))$. By Theorem A.2,

$$\rho_t^\gamma(H - \delta_t^+(D^*)) = \sup_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* - H_s\Big|\mathcal{F}_t\Big]. \tag{4.11}$$

Moreover, we have that

$$\begin{aligned}\sup_{\mathbb{Q}\in\mathcal{Q}_t^\gamma}\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* - H_s\,\Big|\,\mathcal{F}_t\Big] &\geq \sup_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))}\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* - H_s\,\Big|\,\mathcal{F}_t\Big]\\ &\geq \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* - H_s\,\Big|\,\mathcal{F}_t\Big]\\ &= \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^*\,\Big|\,\mathcal{F}_t\Big] - \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s\,\Big|\,\mathcal{F}_t\Big]\\ &\geq \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^*\,\Big|\,\mathcal{F}_t\Big], \end{aligned} \tag{4.12}$$

for all $H\in\mathcal{H}(t)$ and $\mathbb{Q}\in\mathcal{R}(\mathcal{H}(t))\cap\mathcal{Q}_t^\gamma$. The last inequality follows since $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t+1}^{T} H_s\,|\,\mathcal{F}_t] \leq 0$. Hence, combining (4.11) and (4.12) and we deduce

$$\rho_t^\gamma(H-\delta_t^+(D^*)) \geq \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^*\,\Big|\,\mathcal{F}_t\Big], \quad \mathcal{H}\in\mathcal{H}(t),\ \mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t)). \tag{4.13}$$

Recall that $\Pi^{ask,\gamma}$ is defined as,

$$\Pi_t^{ask,\gamma}(D) = \inf\{v\in\mathbb{R}:\ \text{there exists } H\in\mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t(\mathbf{1}v)+H-\delta_t^+(D^*))\geq\gamma\}.$$

Also, it is true that

$$\begin{aligned}&\inf\{v\in\mathbb{R}:\ \text{there exists } H\in\mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t(\mathbf{1}v)+H-\delta_t^+(D^*))\geq\gamma\}\\ &=\inf\bigcup_{H\in\mathcal{H}(t)}\{v\in\mathbb{R}:\ \alpha_t(\delta_t(\mathbf{1}v)+H-\delta_t^+(D^*))(\omega)\geq\gamma\}\\ &=\inf_{H\in\mathcal{H}(t)}\inf\{v\in\mathbb{R}:\ \alpha_t(\delta_t(\mathbf{1}v)+H-\delta_t^+(D^*))\geq\gamma\}.\end{aligned}$$

Therefore,

$$\Pi_t^{ask,\gamma}(D)(\omega) = \inf_{H\in\mathcal{H}(t)}\inf\{v\in\mathbb{R}:\ \alpha_t(\delta_t(\mathbf{1}v)+H-\delta_t^+(D^*))(\omega)\geq\gamma\}$$

for all $\omega\in\Omega$. In virtue of Theorem A.1,

$$\Pi_t^{ask,\gamma}(D) = \inf_{H\in\mathcal{H}(t)}\rho_t^\gamma(H-\delta_t^+(D^*)).$$

Applying (4.13), we see that

$$\Pi_t^{ask,\gamma}(D) \geq \inf_{H\in\mathcal{H}(t)} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big].$$

for all $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^{\gamma}$. Hence,

$$\Pi_t^{ask,\gamma}(D) \geq \sup_{\mathbb{Q}\in\mathcal{Q}_t^{\gamma}\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big]. \tag{4.14}$$

Step 1.c

In conclusion, having shown that (4.10) and (4.14) holds true, we deduce that

$$\Pi_t^{ask,\gamma}(D) = \sup_{\mathbb{Q}\in\mathcal{Q}_t^{\gamma}\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big].$$

Let us now proceed by showing that claim holds true for $\Pi_t^{bid,\gamma}(D)$.

Step 2.a

We first show that,

$$\Pi_t^{bid,\gamma}(D) \geq \inf_{\mathbb{Q}\in\mathcal{Q}_t^{\gamma}\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big].$$

Using Definition 4.3.1 and Theorem A.1, we get

$$\Pi_t^{bid,\gamma}(D)(\omega) = \sup\Big\{v \in \mathbb{R}:\ \text{there exists } H \in \mathcal{H}(t) \text{ such that}$$
$$\sup\big\{\beta \in (0,\infty):\ \rho_t^{\beta}(-\delta_t(\mathbf{1}v) + H + \delta_t^{+}(D))(\omega) \leq 0\big\} \geq \gamma\Big\}$$

for all $\omega \in \Omega$. By continuity and monotonicity of the map $\gamma \mapsto \rho^{\gamma}$, we may apply Lemma B.2 to deduce that

$$\sup\big\{\beta \in (0,\infty):\ \rho_t^{\beta}(-\delta_t(\mathbf{1}v) + H + \delta_t^{+}(D))(\omega) \leq 0\big\} \geq \gamma$$

if and only if $\rho_t^{\gamma}(-\delta_t(\mathbf{1}v) + H + \delta_t^{+}(D))(\omega) \leq 0$ for all $\omega \in \Omega$. Hence,

$$\Pi_t^{bid,\gamma}(D)(\omega) = \sup\Big\{v \in \mathbb{R} : \text{there exists } H \in \mathcal{H}(t) \tag{4.15}$$
$$\text{such that } \rho_t^{\gamma}(-\delta_t(\mathbf{1}v) + H + \delta_t^{+}(D))(\omega) \leq 0\Big\}$$

for all $\omega \in \Omega$.

Now fix an $\mathcal{F}_t$-measurable random variable X_t, and let $\mathcal{P}^t := \{P_1^t, P_2^t, \dots, P_{n_t}^t\}$ be the unique partition that generates $\mathcal{F}_t$. Fix $P_i^t \neq \emptyset$ and let $\omega_i \in P_i^t$. Then $\mathbb{1}_{P_i^t}(\omega)X_t(\omega_i) = \mathbb{1}_{P_i^t}(\omega)X_t(\omega)$ for all $\omega \in \Omega$. By (4.15), we have that $\Pi_t^{ask,\gamma}(D)(\omega_i) > X_t^*(\omega_i)$ if and only if

$$X_t^*(\omega_i) \notin \Big\{v \in \mathbb{R} :\ \text{there exists } H \in \mathcal{H}(t) \text{ s.t. } \rho_t^\gamma(-\delta_t(\mathbf{1}v) + H + \delta_t^+(D^*))(\omega_i) \leq 0\Big\}.$$

Now, the above condition holds true if and only if

$$\rho_t^\gamma(-\delta_t(\mathbf{1}X_t^*(\omega_i)) + H + \delta_t^+(D^*))(\omega_i) > 0, \qquad H \in \mathcal{H}(t)\,.$$

Since ρ^γ is adapted, the above inequality holds true if and only if

$$\mathbb{1}_{P_i^t}(\omega)\rho_t^\gamma(-\delta_t(\mathbf{1}X_t^*(\omega_i)) + H + \delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t.$$

By property (A2) in Definition A.2 of ρ^γ, the above holds true if and only if

$$\mathbb{1}_{P_i^t}(\omega)\rho_t^\gamma(-\delta_t(\mathbf{1}\mathbb{1}_{P_i^t}X_t^*(\omega_i)) + \mathbb{1}_{P_i^t}H + \mathbb{1}_{P_i^t}\delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t.$$

Since, $\mathbb{1}_{P_i^t}(\omega)X_t(\omega_i) = \mathbb{1}_{P_i^t}(\omega)X_t(\omega)$ for all $\omega \in \Omega$, the above inequality holds true if and only if

$$\mathbb{1}_{P_i^t}(\omega)\rho_t^\gamma(-\delta_t(\mathbf{1}\mathbb{1}_{P_i^t}X_t^*) + \mathbb{1}_{P_i^t}H + \mathbb{1}_{P_i^t}\delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t.$$

By property (A2) in Definition A.2, the last inequality holds true if and only if

$$\mathbb{1}_{P_i^t}(\omega)\rho_t^\gamma(-\delta_t(\mathbf{1}X_t^*) + H + \delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t.$$

Since $P_i^t \neq \emptyset$, the above holds true if and only if

$$\rho_t^\gamma(-\delta_t(\mathbf{1}X_t^*) + H + \delta_t^+(D^*))(\omega) > 0, \qquad H \in \mathcal{H}(t), \omega \in P_i^t,$$

which implies

$$\rho_t^\gamma(-\delta_t(\mathbf{1}X_t^*) + H + \delta_t^+(D^*))(\omega) \geq 0, \quad H \in \mathcal{H}(t), \omega \in P_i^t.$$

Since $\omega_i \in \Omega$ is arbitrary, the partition P_i^t is also arbitrary. It follows that if $\Pi_t^{bid,\gamma}(D) < X_t^*$, then

$$\rho_t^\gamma(-\delta_t(\mathbf{1}X_t^*) + H + \delta_t^+(D^*)) \geq 0, \qquad H \in \mathcal{H}(t)\,.$$

By property (A6) of ρ^γ, we have that

$$X_t + \rho_t^\gamma(H + \delta_t^+(D^*)) \geq 0, \qquad H \in \mathcal{H}(t)\,.$$

Due to Theorem A.2, the above is equivalent to

$$X_t^* - \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s + D_s^* \,\Big|\, \mathcal{F}_t\Big] \geq 0.$$

Hence, for any nonnegative $\mathcal{F}_t$-measurable random variable ξ_t, we have that

$$\xi_t X_t^* - \xi_t \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s + D_s^* \,\Big|\, \mathcal{F}_t\Big] \geq 0.$$

Again, by Theorem A.2 and property (A6), and since ξ_t is $\mathcal{F}_t$-measurable, we have that

$$\rho_t^\gamma(-\delta_t(\mathbf{1}\xi_t X_t^*) + \xi_t H + \xi_t\delta_t^+(D^*)) \geq 0$$

for any $H \in \mathcal{H}(t)$ and any nonnegative $\mathcal{F}_t$-measurable random variable ξ_t. Since $\mathcal{H}(t)$ is closed under multiplication of nonnegative $\mathcal{F}_t$-measurable random variables, the inequality above is equivalent to

$$\rho_t^\gamma(-\xi_t\delta_t(\mathbf{1}X_t^*) + H + \xi_t\delta_t^+(D^*)) \geq 0$$

for any $H \in \mathcal{H}(t)$ and any nonnegative $\mathcal{F}_t$-measurable random variable ξ_t.

Therefore, by the definition of $\overline{\mathcal{H}}(t)$, we have that

$$\rho_t^\gamma(\overline{H}) \geq 0, \qquad \overline{H} \in \overline{\mathcal{H}}(t),$$

and hence NGD holds true for $\overline{\mathcal{H}}(t)$, at time t and level γ. Hence, $\mathcal{R}(\overline{\mathcal{H}}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$ (see Remark 4.3.1). Let $\mathbb{Q}^* \in \mathcal{R}(\overline{\mathcal{H}}(t)) \cap \mathcal{Q}_t^\gamma$. From the definition of $\mathcal{R}(\overline{\mathcal{H}}(t))$, we have that

$$\mathbb{E}^{\mathbb{Q}^*}\Big[\sum_{u=t+1}^{T}(H_u + \xi_t D_u^*) \,\Big|\, \mathcal{F}_t\Big] - \xi_t X_t^* \leq 0 \tag{4.16}$$

for all $H \in \mathcal{H}(t)$ and all nonnegative $\mathcal{F}_t$-measurable random variables ξ_t. Note that $\mathcal{R}(\mathcal{H}(t)) \supseteq \mathcal{R}(\overline{\mathcal{H}}(t))$ since $\mathcal{H}(t) \subset \overline{\mathcal{H}}(t)$. Thus, $\mathbb{Q}^* \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$. Because $0 \in \mathcal{H}(t)$, we may let $H = 0$ in (4.9) to conclude that, if $\Pi_t^{bid,\gamma}(D) < X_t^*$, then there exists $\mathbb{Q}^* \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$ such that

$$\mathbb{E}^{\mathbb{Q}^*}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big] \leq X_t^*.$$

Next, for an arbitrary $\epsilon > 0$, let us define $X_t^{*,\epsilon} = \Pi_t^{bid,\gamma}(D) + \epsilon$. From the inequality above, we have that there exists $\mathbb{Q}^{*,\epsilon} \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$ such that

$$\mathbb{E}^{\mathbb{Q}^{*,\epsilon}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big] \geq \Pi_t^{bid,\gamma}(D) + \epsilon,$$

which leads to

$$\inf_{\mathbb{Q}\in\mathcal{R}(\mathcal{H}(t))\cap\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big] \leq \Pi_t^{bid,\gamma}(D) + \epsilon.$$

Therefore, since ϵ is arbitrary, we have

$$\Pi_t^{bid,\gamma}(D) \geq \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big]. \tag{4.17}$$

Step 2.b

We proceed by showing that

$$\Pi_t^{bid,\gamma}(D) \leq \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big].$$

By Theorem A.2,

$$\rho_t^\gamma(H + \delta_t^+(D^*)) = -\inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* + H_s \,\Big|\, \mathcal{F}_t\Big], \quad H \in \mathcal{H}(t).$$

Since $\mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t)) \subseteq \mathcal{Q}_t^\gamma$,

$$-\inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* + H_s \,\Big|\, \mathcal{F}_t\Big] \geq -\inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* + H_s \,\Big|\, \mathcal{F}_t\Big], \quad H \in \mathcal{H}(t).$$

Since $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$, we deduce that

$$
\begin{aligned}
-\inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* + H_s \,\Big|\, \mathcal{F}_t\Big] &\geq -\mathbb{E}^{\mathbb{Q}^*}\Big[\sum_{s=t+1}^{T} D_s^* + H_s \,\Big|\, \mathcal{F}_t\Big] \\
&= -\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big] - \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} H_s \,\Big|\, \mathcal{F}_t\Big]. \\
&\geq -\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big],
\end{aligned}
$$

since $\mathbb{E}^{\mathbb{Q}}[\sum_{s=t+1}^{T} H_s \,|\, \mathcal{F}_t] \leq 0$. Therefore,

$$
\rho_t^\gamma(H + \delta_t^+(D^*)) \geq -\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big], \tag{4.18}
$$

for all $\mathcal{H} \in \mathcal{H}(t)$ and $\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))$.

Recall that $\Pi^{bid,\gamma}$ is defined as,

$$
\Pi_t^{bid,\gamma}(D) = \sup\{v \in \mathbb{R} :\ \text{there exists } H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(-\delta_t(\mathbf{1}v) + H + \delta_t^+(D^*)) \geq \gamma\}.
$$

Moreover, we have

$$
\begin{aligned}
&\sup\{v \in \mathbb{R} :\ \text{there exists } H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(-\delta_t(\mathbf{1}v) + H + \delta_t^+(D^*)) \geq \gamma\} \\
&= \sup \bigcup_{H\in\mathcal{H}(t)} \{v \in \mathbb{R} :\ \alpha_t(\delta_t^+(D^*) + H - \delta_t(\mathbf{1}v))(\omega) \geq \gamma\} \\
&= \sup_{H\in\mathcal{H}(t)} \sup\{v \in \mathbb{R} :\ \alpha_t(-\delta_t(\mathbf{1}v) + H + \delta_t^+(D^*)) \geq \gamma\}.
\end{aligned}
$$

Hence, we find

$$
\begin{aligned}
\Pi_t^{bid,\gamma}(D) &= \sup_{H\in\mathcal{H}(t)} \sup\{v \in \mathbb{R} :\ \alpha_t(-\delta_t(\mathbf{1}v) + H + \delta_t^+(D^*)) \geq \gamma\} \\
&= \sup_{H\in\mathcal{H}(t)} \{-\inf\{v \in \mathbb{R} :\ \alpha_t(\delta_t(\mathbf{1}v) + H + \delta_t^+(D^*)) \geq \gamma\}\}.
\end{aligned}
$$

Now, using Theorem A.1 we get

$$
\Pi_t^{bid,\gamma}(D) = \sup_{H\in\mathcal{H}(t)} \{-\rho_t^\gamma(H + \delta_t^+(D^*))\}.
$$

Applying (4.18), we obtain

$$\Pi_t^{bid,\gamma}(D) = \sup_{H\in\mathcal{H}(t)} \{-\rho_t^\gamma(H+\delta_t^+(D^*))\} \le \sup_{H\in\mathcal{H}(t)} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big]$$
$$= \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big].$$

for all $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$. Consequently,

$$\Pi_t^{bid,\gamma}(D) \le \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big]. \tag{4.19}$$

Step 2.c

Finally, using (4.17) and (4.19), we conclude that

$$\Pi_t^{bid,\gamma}(D) = \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma\cap\mathcal{R}(\mathcal{H}(t))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big].$$

Hence the proof is complete.

□

We proceed with the following important remarks on Theorem 4.3.1.

Remark 4.3.6. *Note that if the NGD does not hold true for $\mathcal{H}(t)$, at time $t \in \{1,\dots,T-1\}$, at level γ, then*

$$\Pi_t^{ask,\gamma}(D)(\omega) = -\infty,$$

$$\Pi_t^{bid,\gamma}(D)(\omega) = \infty,$$

for all $\omega \in \Omega$ and $D \in L^0$.

Remark 4.3.7. *If the set of hedging cash flows $\mathcal{H}(t)$ satisfies the no-arbitrage condition, and $\mathcal{H}(T-1)$ is complete (for any $D \in L^0$, there exists $H \in \mathcal{H}(T-1)$ so that $H_T = D_T$), then it follows that $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$, for $t = 1, 2, \dots, T-2$,* *and* $\mathcal{R}(\mathcal{H}(T-1)) = \{\mathbb{Q}^*\}$. *Since* $\mathcal{R}(\mathcal{H}(0)) \subseteq \cdots \subseteq \mathcal{R}(\mathcal{H}(T-1))$, *we have that* $\mathcal{R}(\mathcal{H}(t)) = \{\mathbb{Q}^*\} \neq \emptyset$ *for* $t = 0, 1, \ldots, T-2$. *By Theorems 4.2.1 and 4.3.1, if NGDB holds then the good-deal ask and bid prices of a derivative contract* $D \in L^0$, *at time* $t \in \mathcal{T}$ *and level* $\gamma >$, *satisfy*

$$\Pi_t^{ask,\gamma}(D) = \Pi_t^{bid,\gamma}(D) = \mathbb{E}^{\mathbb{Q}^*}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big].$$

Notice that, naturally, the good-deal prices no longer depend on the acceptance level γ.

Remark 4.3.8. *If for some* $t \in \{1, \ldots, T-1\}$, *we have that* $\mathcal{Q}_t^\gamma \neq \emptyset$ *and* $\mathcal{H}(t) = \{0\}$, *then we have* $\mathcal{R}(\mathcal{H}(t)) = \{\mathbb{Q} : \mathbb{Q} \sim \mathbb{P}\}$, *so* $\mathcal{Q}_t^\gamma \subseteq \mathcal{R}(\mathcal{H}(t))$. *In this case the good-deal ask and bid prices of a derivative contract* $D \in L^0$, *at time* $t \in \mathcal{T}$ *and level* $\gamma > 0$, *satisfy*

$$\Pi_t^{ask,\gamma}(D) = \sup_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big],$$

$$\Pi_t^{bid,\gamma}(D) = \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^* \,\Big|\, \mathcal{F}_t\Big].$$

Remark 4.3.9. *Let us consider the sets of extended cash flows associated with good-deal prices* $\Pi_t^{ask,\gamma}(D)$ *and* $\Pi_t^{bid,\gamma}(D)$:

$$\begin{aligned}\widehat{\mathcal{H}}(t) = &\Big\{\Big(0, \ldots, 0, \xi_t \Pi_t^{ask,\gamma}(D), H_{t+1} - \xi_t D_{t+1}^*, \ldots, H_T - \xi_t D_T^*\Big)\\ &: H \in \mathcal{H}(t),\ \xi_t \text{ is } \mathcal{F}_t\text{-measurable},\ \xi_t \geq 0\Big\},\\ \overline{\mathcal{H}}(t) = &\Big\{\Big(0, \ldots, 0, -\xi_t \Pi_t^{bid,\gamma}(D), H_{t+1} + \xi_t D_{t+1}^*, \ldots, H_T + \xi_t D_T^*\Big)\\ &: H \in \mathcal{H}(t),\ \xi_t \text{ is } \mathcal{F}_t\text{-measurable},\ \xi_t \geq 0\Big\}.\end{aligned}$$

If $\mathcal{H}(t)$ *is frictionless and complete (and therefore linear), and NGD condition holds, then as in Remark 4.3.7, we have that* $\Pi(D) := \Pi_t^{ask,\gamma}(D) = \Pi_t^{bid,\gamma}(D)$. *In this case,* *the set*

$$\widehat{\mathcal{H}}(t)+\overline{\mathcal{H}}(t)=\Big\{\Big(0,\dots,0,\xi_t\Pi_t(D),H_{t+1}-\xi_t D^*_{t+1},\dots,H_T-\xi_t D^*_T\Big) \\ : H\in\mathcal{H}(t),\ \xi_t \textit{ is } \mathcal{F}_t\textit{-measurable}\Big\}$$

is a linear space. Whenever $\Pi_t^{ask,\gamma}(D)>\Pi_t^{bid,\gamma}(D)$*, as in our general case, we have that*

$$\widehat{\mathcal{H}}(t)+\overline{\mathcal{H}}(t)=\Big\{\Big(0,\dots,0,\xi_t\Pi_t^{ask,\gamma}(D)-\phi_t\Pi_t^{bid,\gamma}(D),H_{t+1}-(\xi_t-\phi_t)D^*_{t+1}, \\ \dots,H_T-(\xi_t-\phi_t)D^*_T\Big):\ H\in\mathcal{H}(t),\ \xi_t,\phi_t \textit{ is } \mathcal{F}_t\textit{-measurable},\ \xi_t,\phi_t\geq 0\Big\}$$

is only a convex cone. This is one of the main reasons why we call our approach dynamic conic finance.

Remark 4.3.10. *Recall that, in our framework, upper and lower no-arbitrage price bounds of a derivative contract* $D\in L^0$ *are defined as*

$$\pi_t^U(D):=\sup_{\mathbb{Q}\in\mathcal{R}(\mathcal{H}(t))}\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=1}^T D^*_s\,\Big|\,\mathcal{F}_t\Big],$$

$$\pi_t^L(D):=\inf_{\mathbb{Q}\in\mathcal{R}(\mathcal{H}(t))}\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=1}^T D^*_s\,\Big|\,\mathcal{F}_t\Big].$$

Hence, it follows from Theorem 4.2.1 that if NGD is satisfied for some $\gamma>0$ *then*

$$\pi_t^L(D)\leq\Pi_t^{bid,\gamma}(D)\leq\Pi_t^{ask,\gamma}(D)\leq\pi_t^U(D).$$

As a consequence, the bid ask price interval, which is found using dynamic coherent acceptability indices, is narrower than the difference between the no-arbitrage price bounds.

4.3.2 Good-Deal Forward Ask and Bid Prices. We now define the good-deal *forward* ask and bid prices, and also prove a representation theorem similar to Theorem 4.3.1. Throughout this section, we assume that the risk-free interest rate r is deterministic.

Definition 4.3.2. *The* good-deal ask and bid forward prices*, with delivery at time T, written at time $t \in \{1, \ldots, T-1\}$, of a derivative contract $D \in L^0$, at level $\gamma > 0$ are defined as*

$$F_t^{ask,\gamma,T}(D)(\omega) := \inf\{f \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ such that } \alpha_t(\delta_t(\mathbf{1}B_T^{-1}f) + H - \delta_t^+(D^*))(\omega) \geq \gamma\}, \tag{4.20}$$

$$F_t^{bid,\gamma,T}(D)(\omega) := \sup\{f \in \mathbb{R} : \text{ there exists } H \in \mathcal{H}(t) \text{ such that } \alpha_t(-\delta_t(\mathbf{1}B_T^{-1}f) + H + \delta_t^+(D^*))(\omega) \geq \gamma\} \tag{4.21}$$

for all $\omega \in \Omega$.

Notice that the cash flow $\delta_t(\mathbf{1}B_T^{-1}f) + H - \delta_t^+(D^*)$ represents an exchange of a cash payment f at time T for a discounted cash flow D that is hedged with H. The good-deal forward ask price at level γ is the minimum amount of cash f at time T so that $\delta_t(\mathbf{1}B_T^{-1}f) + H - \delta_t^+(D^*)$ is acceptable at level γ at time t.

Let us now continue with the representation theorem for the forward ask and bid prices. This result shows that the classical relationship between the spot and forward prices is preserved in our framework, for good-deal forward ask and bid prices.

Theorem 4.3.2. *The good-deal ask and bid forward prices of a derivative contract $D \in L^0$, with delivery at time T, written at time $t \in \{1, \ldots, T-1\}$ and level $\gamma > 0$, satisfy*

$$F_t^{ask,\gamma,T}(D)(\omega) = B_T \Pi_t^{ask,\gamma}(D),$$

$$F_t^{bid,\gamma,T}(D)(\omega) = B_T \Pi_t^{bid,\gamma}(D).$$

Proof. Since B_T is deterministic, Equations (4.20) and (4.21) can written as

$$\begin{aligned} F_t^{ask,\gamma,T}(D)(\omega) &= B_T \inf\{f^* \in \mathbb{R} :\ \exists\, H \in \mathcal{H}(t) \\ &\qquad \text{such that } \alpha_t(\delta_T(\mathbf{1}f^*) + H - \delta_t^+(D^*))(\omega) \geq \gamma\}, \\ F_t^{bid,\gamma,T}(D)(\omega) &= B_T \sup\{f^* \in \mathbb{R} :\ \exists\, H \in \mathcal{H}(t) \\ &\qquad \text{such that } \alpha_t(-\delta_T(\mathbf{1}f^*) + H + \delta_t^+(D^*))(\omega) \geq \gamma\}. \end{aligned}$$

Using the translation invariance property of α (see (D6) in Definition A.1), we deduce that

$$\begin{aligned} F_t^{ask,\gamma,T}(D)(\omega) &= B_T \inf\{f^* \in \mathbb{R} :\ \exists\, H \in \mathcal{H}(t) \\ &\qquad \text{such that } \alpha_t(\delta_t(\mathbf{1}f^*) + H - \delta_t^+(D^*))(\omega) \geq \gamma\}, \\ F_t^{bid,\gamma,T}(D)(\omega) &= B_T \sup\{f^* \in \mathbb{R} :\ \exists\, H \in \mathcal{H}(t) \\ &\qquad \text{such that } \alpha_t(-\delta_t(\mathbf{1}f^*) + H + \delta_t^+(D^*))(\omega) \geq \gamma\}, \end{aligned}$$

since

$$\alpha_t(\delta_T(\mathbf{1}f^*) + H - \delta_t^+(D^*)) = \alpha_t(\delta_t(\mathbf{1}f^*) + H - \delta_t^+(D^*)).$$

Hence, by Theorem 4.3.1 we conclude that our claim holds. □

Remark 4.3.11. *If r is deterministic and the set of hedging cash flows $\mathcal{H}(t)$ forms a market that is frictionless, complete, and arbitrage-free, then $\mathcal{R}(\mathcal{H}(t))$ is a singleton, say $\{\mathbb{Q}^*\}$, and so by Theorem 4.3.2 we have that $F_t^{ask,\gamma,T}(D) = F_t^{bid,\gamma,T}(D) = B_T \mathbb{E}^{\mathbb{Q}^*}[\sum_{u=t+1}^T D_u^* \,|\, \mathcal{F}_t]$. This is compatible with the classic result that states that in a frictionless, complete, and arbitrage-free market the discounted forward price $f_t^T(D)$ of a derivative contract D, with delivery at time T, written at time $t \in \{1, \dots, T-1\}$, is given as*

$$f_t^T(D) = B_T \mathbb{E}^{\mathbb{Q}^*}\Big[\sum_{u=t+1}^{T} D_u^* \,\Big|\, \mathcal{F}_t\Big].$$

4.4 Pricing with the Dynamic Gain-Loss Ratio

In this section, we first prove some auxiliary results that hold for general DCAIs. Then, we particularize these results to a very important special case of DCAI, namely to the dynamic Gain-Loss Ratio (dGLR). Finally, we apply the pricing and hedging results developed in earlier sections using dGLR to path-dependent options. In this section, without a loss of generality, we assume that $r = 0$.

4.4.1 Characterization of DCAIs. In this section, we will prove an auxiliary result for DCAIs. For basic facts and notions regarding DCAIs, we refer to Appendix A.

From [BCZ11], we recall that for every normalized and right-continuous DCAI α there exist family $\mathcal{Q} = ((\mathcal{Q}_t^\gamma)_{t\in\mathcal{T}})_{\gamma\in(0,\infty)}$ of dynamically consistent sequences of sets of probability measures that is increasing (in γ), such that the following robust representation holds true

$$\alpha_t(D)(\omega) = \sup\left\{\gamma \in (0,\infty) : \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0\right\},\ \omega\in\Omega,\ t\in\mathcal{T},\ D\in L^0. \tag{4.22}$$

We say that a family $\mathcal{Q}$ of dynamically consistent sequences of sets of probability measures that is increasing (in γ) *corresponds* to a given normalized and right-continuous DCAI α if $\mathcal{Q}$ satisfies (4.22). Now, we will establish a characterization of families $\mathcal{Q}$ that correspond to a given normalized and right-continuous DCAI α.

Lemma 4.4.1. *Suppose that α is a normalized and right-continuous DCAI. A family $\mathcal{Q}$ corresponds to α if and only if $\mathcal{Q} \in \mathfrak{Q}^{\alpha}$, where*[16]

$$\mathfrak{Q}^{\alpha} := \Big\{\mathcal{U} \ : \ \alpha_t(D)(\omega) \geq \gamma \quad \textit{if and only if} \\ \inf_{\mathbb{Q}\in\mathcal{U}_t^{\gamma}} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0,\ \omega \in \Omega,\ \gamma \in (0,\infty),\ t \in \mathcal{T},\ D \in L^0\Big\}.$$

Proof. **Necessity:** ($\Longleftarrow$):

Let $\mathcal{U} \in \mathfrak{Q}^{\alpha}$. We fix $t \in \mathcal{T}$, $D \in L^0$, and $\omega \in \Omega$. Define the set

$$\Gamma(\mathcal{U}) := \Big\{\beta \in (0,\infty) : \inf_{\mathbb{Q}\in\mathcal{U}_t^{\beta}} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0\Big\}.$$

If $\alpha_t(D)(\omega) = \infty$, then

$$\inf_{\mathbb{Q}\in\mathcal{U}_t^{\beta}} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0, \quad \beta \in (0,\infty).$$

Therefore, $\Gamma(\mathcal{U}) = (0,\infty)$, and thus $\sup\Gamma(\mathcal{U}) = \infty$. Hence, (4.22) holds true.

If $\Gamma(\mathcal{U}) = \emptyset$, then

$$\inf_{\mathbb{Q}\in\mathcal{U}_t^{\beta}} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) < 0, \quad \beta \in (0,\infty).$$

Since $\mathcal{U} \in \mathfrak{Q}^{\alpha}$, it is true that $\alpha_t(D)(\omega) < \beta$ for all $\beta \in (0,\infty)$. However, α is nonnegative by definition, thus $\alpha_t(D)(\omega) = 0$. By convention, we are taking $\sup\emptyset = 0$, so we also have that $\sup\Gamma(\mathcal{U}) = 0$. Hence, (4.22) holds true.

If $\alpha_t(D)(\omega) = 0$, then, since $\mathcal{U} \in \mathfrak{Q}^{\alpha}$, we have that

$$\inf_{\mathbb{Q}\in\mathcal{U}_t^{\beta}} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) < 0, \quad \beta \in (0,\infty).$$

It follows that $\Gamma(\mathcal{U}) = \emptyset$, and so (4.22) holds true.

[16]We will generically denote by $\mathcal{U} = ((\mathcal{U}_t^{\gamma})_{t\in\mathcal{T}})_{\gamma\in(0,\infty)}$ a family of dynamically consistent sequences of sets of probability measures that is increasing (in γ).

Suppose $\Gamma(\mathcal{U}) \neq \emptyset$. Assume that $\alpha_t(D)(\omega) < \infty$. We first show that $\alpha_t(D)(\omega)$ is an upper bound of $\Gamma(\mathcal{U})$. Observe that if $\gamma \in \Gamma(\mathcal{U})$, then

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0.$$

Now, since $\mathcal{U} \in \mathfrak{Q}^\alpha$, we have that $\alpha_t(D)(\omega) \geq \gamma$. So $\alpha_t(D)(\omega)$ is an upper bound of $\Gamma(\mathcal{U})$. If we let $\beta' := \alpha_t(D)(\omega)$, then, because $\mathcal{U} \in \mathfrak{Q}^\alpha$, we have that

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^{\beta'}} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0.$$

Thus, $\beta' \in \Gamma(\mathcal{U})$. It follows that (4.22) holds.

Sufficiency: $(\Longrightarrow)$

Now, suppose $\mathcal{U}$ satisfies (4.22), and let $\gamma \in (0, \infty)$. If

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0,$$

then $\gamma \in \Gamma(\mathcal{U})$. By (4.22), we have that $\alpha_t(D)(\omega) \geq \gamma$.

Assume $\alpha_t(D)(\omega) \geq \gamma$. We consider the cases $\alpha_t(D)(\omega) > \gamma$ and $\alpha_t(D)(\omega) = \gamma$ separately. If $\alpha_t(D)(\omega) > \gamma$, then, since $\mathcal{U}^\gamma$ is increasing in γ, we have that

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0.$$

Next, suppose that $\alpha_t(D)(\omega) = \gamma$ and

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) < 0.$$

By Theorem A.1, the mapping

$$\gamma \longmapsto \inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega)$$

is left-continuous and monotone decreasing. Thus, by left-continuity there exists $\epsilon > 0$ so that

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^{\gamma-\epsilon}} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) < 0,$$

and by monotonicity and (4.22), we deduce that $\alpha_t(D)(\omega) \leq \gamma - \epsilon$. This implies that $\epsilon \leq 0$, which is a contradiction. Hence, we have that

$$\inf_{\mathbb{Q}\in\mathcal{U}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0,$$

which concludes the proof. □

4.4.2 Characterization of the dGLR. The Gain-Loss Ratio, first introduced in [BL00], is a performance measure that is widely used among practitioners. In the single-period case, the Gain-Loss Ratio is defined as the ratio of the expectation of returns to the expectation of negative returns:

$$\mathrm{GLR}(X) := \frac{\mathbb{E}^{\mathbb{P}}[X]}{\mathbb{E}^{\mathbb{P}}[X^-]} \qquad \text{if } \mathbb{E}^{\mathbb{P}}[X] > 0,$$

and zero otherwise[17]. As shown in [CM09], the Gain-Loss Ratio defined above is a static coherent acceptability index. Observe that the value of the GLR depends on the statistical measure $\mathbb{P}$.

In Bielecki et al. [BCZ11], a version of GLR is defined in a dynamical, multi-period setup, which is called the dynamic Gain Loss Ratio (dGLR). Let us proceed by recalling the definition of the dGLR.

Definition 4.4.1. *The dynamic Gain Loss Ratio (dGLR) for a cash flow $D \in L^0$ is defined as*

$$\mathrm{dGLR}_t(D)(\omega) := \begin{cases} \dfrac{\mathbb{E}^{\mathbb{P}}[\sum_{s=t}^{T} D_s \,|\, \mathcal{F}_t](\omega)}{\mathbb{E}^{\mathbb{P}}[(\sum_{s=t}^{T} D_s)^- \,|\, \mathcal{F}_t](\omega)}, & \text{if } \quad \mathbb{E}^{\mathbb{P}}\big[\sum_{s=t}^{T} D_s \,\big|\, \mathcal{F}_t\big](\omega) > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{4.23}$$

for all $t \in \mathcal{T}$, and $\omega \in \Omega$. By convention, $dGLR(0) = \infty$.

---

[17]Recall that a^- denotes the negative part of any real number a, i.e. $a^- = \max\{0, -a\}$

It is shown in Bielecki et al. [BCZ11] that the dGLR satisfies conditions (D1)–(D7), and therefore it is a dynamic coherent acceptability index (see Definition A.1).

Remark 4.4.1. *It is worth to remark on the interpretation of the dGLR in the context of arbitrage, which was first noticed in Bernardo and Ledoit [BL00] for the GLR (static case). Observe that*

$$\sum_{s=t}^{T} H_s(\omega) \geq 0 \text{ for all } \omega \in \Omega \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}\Big[\sum_{s=t}^{T} H_s \,\Big|\, \mathcal{F}_t\Big](\omega) > 0 \text{ for some } \omega \in \Omega$$

is equivalent to

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{s=t}^{T} H_s\Big)^{-} \,\Big|\, \mathcal{F}_t\Big](\omega) = 0 \text{ for all } \omega \in \Omega$$
$$\text{and} \quad \mathbb{E}^{\mathbb{P}}\Big[\sum_{s=t}^{T} H_s \,\Big|\, \mathcal{F}_t\Big](\omega) > 0 \text{ for some } \omega \in \Omega,$$

which is equivalent to

$$dGLR_t(H)(\omega) = \infty \quad \text{for some } \omega \in \Omega.$$

Hence, as a result of Definition 4.2.3, a cash flow $H \in \mathcal{H}(t)$ is an arbitrage opportunity at time $t \in \mathcal{T}$ if and only if $dGLR_t(H)(\omega) = \infty$ for some $\omega \in \Omega$. Equivalently, the no-arbitrage condition holds true at time $t \in \mathcal{T}$ if and only if $dGLR_t(H)$ is bounded for all $H \in \mathcal{H}(t)$. This equivalence gives an intuitive interpretation of the dGLR in terms of the no-arbitrage condition.

Let us define the family of sets of probability measures $\widehat{\mathcal{Q}} := \{\widehat{\mathcal{Q}}^\gamma,\ \gamma > 0\}$, and the family of sets of densities $\widehat{\mathcal{E}} := \{\widehat{\mathcal{E}}^\gamma,\ \gamma > 0\}$, where

$$\widehat{\mathcal{Q}}^\gamma := \Big\{\mathbb{Q} : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = c(1+\Lambda),\ c > 0,\ \Lambda \in \mathfrak{L}^\gamma,\ c\,\mathbb{E}^{\mathbb{P}}[1+\Lambda] = 1\Big\}, \tag{4.24}$$

$$\widehat{\mathcal{E}}^\gamma := \Big\{\eta := \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \ :\ \mathbb{Q} \in \mathcal{Q}^\gamma\Big\}$$
$$= \Big\{\eta := c(1+\Lambda) :\ c > 0,\ \Lambda \in \mathfrak{L}^\gamma,\ c\,\mathbb{E}^{\mathbb{P}}[1+\Lambda] = 1\Big\},$$

for all $\gamma \in (0, \infty)$, where we set

$$\mathfrak{L}^\gamma := \{\Lambda : \Lambda \text{ is an } \mathcal{F}_T\text{-measurable r.v., } 0 \le \Lambda \le \gamma\}.$$

Next, we will show that $\widehat{\mathcal{Q}}$ is an increasing family of dynamically consistent sets of probability measures corresponding to the dGLR.

Proposition 4.4.1. *The family $\widehat{\mathcal{Q}}$ is an increasing family of dynamically consistent sets of probability measures. In addition, this family corresponds to the* dGLR.

Proof. We start by observing that, for each $\gamma > 0$, the set $\widehat{\mathcal{Q}}^\gamma$ is nonempty since, in particular, we may take $\Lambda = 0$ in the definition of $\widehat{\mathcal{Q}}^\gamma$. Also, we note that $\widehat{\mathcal{Q}}^\gamma$ is increasing in γ.

For the rest of the proof we fix $\gamma > 0$. We denote by $\Upsilon^t = \{P_1^t, P_2^t, \ldots, P_{n_t}^t\}$ the unique partition of Ω at time t that generates $\mathcal{F}_t$. In order to prove our result it suffices to show that $\widehat{\mathcal{Q}}^\gamma$ is weakly consistent (see Corollary 4.1.1 in [Zha11]), which is

$$\mathbb{1}_{P_i^t} \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_t] \le \mathbb{1}_{P_i^t} \max_{\omega \in P_i^t} \left\{ \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_{t+1}](\omega) \right\}, \tag{4.25}$$

for every $t \in \{0, \ldots, T-1\}$, $P_i^t \in \Upsilon^t$, and $X \in \mathcal{F}_T$. Next, take $0 \le \Lambda \le \gamma$ and suppose that

$$\max_{\omega \in P_i^t} \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_{t+1}\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,\big|\, \mathcal{F}_{t+1}](\omega)} \le a,$$

for some $a \in \mathbb{R}$. Hence,

$$\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_{t+1}\big](\omega) \le a\mathbb{E}^{\mathbb{P}}[1+\Lambda \,|\, \mathcal{F}_{t+1}](\omega),$$

for all $\omega \in P_i^t$. Therefore, using the tower property of conditional expectations, we have that

$$\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_t\big](\omega) \le a\mathbb{E}^{\mathbb{P}}[1+\Lambda \,|\, \mathcal{F}_t](\omega),$$

for all $\omega \in P_i^t$, and, consequently

$$\max_{\omega \in P_i^t} \frac{\mathbb{E}_t^{\mathbb{P}}\big[(1+\Lambda)X\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,|\, \mathcal{F}_t](\omega)} \leq a.$$

Thus, we showed that for any $a \in \mathbb{R}$ the following implication holds,

$$\max_{\omega \in P_i^t} \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_{t+1}\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,|\, \mathcal{F}_{t+1}](\omega)} \leq a \quad \Rightarrow \quad \max_{\omega \in P_i^t} \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_t\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,|\, \mathcal{F}_t](\omega)} \leq a,$$

so that

$$\max_{\omega \in P_i^t} \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_t\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,\big|\, \mathcal{F}_t](\omega)} \leq \max_{\omega \in P_i^t} \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_{t+1}\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,\big|\, \mathcal{F}_{t+1}](\omega)}.$$

Hence, we have

$$\begin{aligned} \mathbb{1}_{P_i^t} \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_t\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,\big|\, \mathcal{F}_t](\omega)} &\leq \mathbb{1}_{P_i^t} \max_{\omega \in P_i^t} \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_t\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,\big|\, \mathcal{F}_t](\omega)} \\ &\leq \mathbb{1}_{P_i^t} \max_{\omega \in P_i^t} \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)X \,\big|\, \mathcal{F}_{t+1}\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda \,\big|\, \mathcal{F}_{t+1}](\omega)} \end{aligned}$$

for all $\omega \in \Omega$. Thus, for $\mathbb{Q} = c(1+\Lambda)\mathbb{P}$, we have that

$$\mathbb{1}_{P_i^t} \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_t](\omega) \leq \mathbb{1}_{P_i^t} \max_{\omega \in P_i^t} \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_{t+1}](\omega),$$

for all $\omega \in \Omega$. Therefore,

$$\begin{aligned} \mathbb{1}_{P_i^t} \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_t] &\leq \mathbb{1}_{P_i^t} \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \left\{ \max_{\omega \in P_i^t} \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_{t+1}](\omega) \right\} \\ &\leq \mathbb{1}_{P_i^t} \max_{\omega \in P_i^t} \left\{ \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_{t+1}](\omega) \right\}, \end{aligned}$$

which proves the weak consistency of $\widehat{\mathcal{Q}}^\gamma$.

We now show that the family $\widehat{\mathcal{Q}}$ corresponds to the dGLR. By Lemma 4.4.1, this is equivalent to show that

$$\mathrm{dGLR}_t(D)(\omega) \geq \gamma \quad \Longleftrightarrow \quad \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}^{\mathbb{Q}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \geq 0, \tag{4.26}$$

for all $\omega \in \Omega$, $t \in \mathcal{T}$ and $D \in L^0$, where for convenience we denoted $X_t^T = \sum_{u=T}^t D_u$. In the rest of the proof we fix $\omega \in \Omega$, $t \in \mathcal{T}$ and $D \in L^0$.

In order to show (4.26) we first observe that since any $\eta \in \mathcal{E}^\gamma$ is strictly positive, we may apply the abstract Bayes formula to write

$$
\begin{aligned}
\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^\gamma} \mathbb{E}^{\mathbb{Q}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \geq 0 \quad &\iff \quad \inf_{\eta\in\mathcal{E}^\gamma} \frac{\mathbb{E}^{\mathbb{P}}\big[\eta X_t^T \,\big|\, \mathcal{F}_t\big](\omega)}{\mathbb{E}^{\mathbb{P}}[\eta \,|\, \mathcal{F}_t](\omega)} \geq 0 \\
&\iff \quad \frac{\mathbb{E}^{\mathbb{P}}\big[\eta X_t^T \,\big|\, \mathcal{F}_t\big](\omega)}{\mathbb{E}^{\mathbb{P}}[\eta \,|\, \mathcal{F}_t](\omega)} \geq 0, \quad \eta \in \mathcal{E}^\gamma \\
&\iff \quad \mathbb{E}^{\mathbb{P}}\big[\eta X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \geq 0, \quad \eta \in \mathcal{E}^\gamma \\
&\iff \quad \inf_{\eta\in\mathcal{E}^\gamma} \mathbb{E}^{\mathbb{P}}\big[\eta X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \geq 0. \qquad (4.27)
\end{aligned}
$$

Next, recall that by definition of $\mathcal{E}^\gamma$ we have that

$$
\inf_{\eta\in\mathcal{E}^\gamma} \mathbb{E}^{\mathbb{P}}\big[\eta X_t^T \,\big|\, \mathcal{F}_t\big](\omega) = \inf_{\Lambda\in\mathfrak{L}^\gamma} \mathbb{E}^{\mathbb{P}}\big[(1+\Lambda) X_t^T \,\big|\, \mathcal{F}_t\big](\omega). \qquad (4.28)
$$

Observing that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda) X_t^T \,\big|\, \mathcal{F}_t\big](\omega) &= \mathbb{E}^{\mathbb{P}}\big[X_t^T + \Lambda \mathbb{1}_{\{X_t^T \leq 0\}} X_t^T + \Lambda \mathbb{1}_{\{X_t^T > 0\}} X_t^T) \,\big|\, \mathcal{F}_t\big](\omega) \\
&\geq \mathbb{E}^{\mathbb{P}}\big[X_t^T + \Lambda \mathbb{1}_{\{X_t^T \leq 0\}} X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \\
&\geq \mathbb{E}^{\mathbb{P}}\big[X_t^T + \gamma \mathbb{1}_{\{X_t^T \leq 0\}} X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \\
&= \mathbb{E}^{\mathbb{P}}\big[(1+\Lambda^*) X_t^T \,\big|\, \mathcal{F}_t\big](\omega),
\end{aligned}
$$

where $\Lambda^* := \gamma \mathbb{1}_{\{X_t^T \leq 0\}} \in \mathfrak{L}^\gamma$.

Consequently, we obtain that

$$
\inf_{\Lambda\in\mathfrak{L}^\gamma} \mathbb{E}^{\mathbb{P}}\big[(1+\Lambda) X_t^T \,\big|\, \mathcal{F}_t\big](\omega) = \mathbb{E}^{\mathbb{P}}\big[(1+\Lambda^*) X_t^T \,\big|\, \mathcal{F}_t\big](\omega).
$$

Thus, in view of (4.28), we get

$$
\begin{aligned}
\inf_{\eta\in\mathcal{E}^\gamma} \mathbb{E}^{\mathbb{P}}\big[\eta X_t^T \,\big|\, \mathcal{F}_t\big](\omega) &= \mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) + \gamma \mathbb{E}^{\mathbb{P}}\big[\mathbb{1}_{\{X_t^T \leq 0\}} X_t^T) \,\big|\, \mathcal{F}_t\big](\omega) \\
&= \mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) + \gamma \mathbb{E}^{\mathbb{P}}\big[\mathbb{1}_{\{X_t^T \leq 0\}} \big(\big(X_t^T\big)^+ - \big(X_t^T\big)^-\big) \,\big|\, \mathcal{F}_t\big](\omega) \\
&= \mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) - \gamma \mathbb{E}^{\mathbb{P}}\big[\big(X_t^T\big)^- \,\big|\, \mathcal{F}_t\big](\omega).
\end{aligned}
$$

From here and (4.27) we deduce that

$$\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^{\gamma}} \mathbb{E}^{\mathbb{Q}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \geq 0 \quad \Longleftrightarrow \quad \mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \geq \gamma \mathbb{E}^{\mathbb{P}}\big[\big(X_t^T\big)^- \,\big|\, \mathcal{F}_t\big](\omega). \tag{4.29}$$

To complete the proof of (4.26) we shall consider the following three cases: $\mathbb{E}^{\mathbb{P}}[X_t^T \,|\, \mathcal{F}_t](\omega) > 0$, $\mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) < 0$, and $\mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) = 0$.

Case 1: $\mathbb{E}^{\mathbb{P}}[X_t^T \,|\, \mathcal{F}_t](\omega) > 0$.

From the definition of the dGLR and from (4.29) we have that

$$\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^{\gamma}} \mathbb{E}^{\mathbb{Q}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \geq 0 \quad \Longleftrightarrow \quad \mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) \geq \gamma \mathbb{E}^{\mathbb{P}}\big[\big(X_t^T\big)^- \,\big|\, \mathcal{F}_t\big](\omega) \tag{4.30}$$

$$\Longleftrightarrow \quad \mathrm{dGLR}_t(D)(\omega) \geq \gamma. \tag{4.31}$$

Therefore, (4.26) holds true.

Case 2: $\mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) < 0$.

Since $\mathbb{P} \in \widehat{\mathcal{Q}}^{\gamma}$, we have that

$$\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^{\gamma}} \mathbb{E}^{\mathbb{Q}}[X_t^T \,|\, \mathcal{F}_t](\omega) \leq \mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) < 0.$$

Also, by the definition of the dGLR, we have that $\mathrm{dGLR}_t(D)(\omega) = 0$. As a result,

$$\mathrm{dGLR}_t(D)(\omega) < \gamma \quad \Longleftrightarrow \quad \inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^{\gamma}} \mathbb{E}^{\mathbb{Q}}[X_t^T \,|\, \mathcal{F}_t](\omega) < 0,$$

and so (4.26) holds true.

Case 3: $\mathbb{E}^{\mathbb{P}}\big[X_t^T \,\big|\, \mathcal{F}_t\big](\omega) = 0$.

<u>Case 3a:</u> If $\mathbb{E}^{\mathbb{P}}[(X_t^T)^- \,|\, \mathcal{F}_t](\omega) = 0$, then $\mathbb{E}^{\mathbb{P}}[(X_t^T)^+ \,|\, \mathcal{F}_t](\omega) = 0$. Since $\omega \in \Omega$ is arbitrary, we may conclude that in this case $X_t^T = 0$. Thus $\mathrm{dGLR}_t(D) = \infty$ and $\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^{\gamma}} \mathbb{E}^{\mathbb{Q}}[X_t^T \,|\, \mathcal{F}_t] = 0$, showing that (4.26) holds true.

<u>Case 3b:</u> Now, assume that $\gamma\mathbb{E}^{\mathbb{P}}[(X_t^T)^- \,|\, \mathcal{F}_t](\omega) > \mathbb{E}^{\mathbb{P}}[X_t^T \,|\, \mathcal{F}_t](\omega) = 0$. By (4.29), it follows that $\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^{\gamma}} \mathbb{E}^{\mathbb{Q}}[X_t^T \,|\, \mathcal{F}_t](\omega) < 0$. Due to the definition of the dGLR, we thus have that $\mathrm{dGLR}_t(D)(\omega) = 0$, and so (4.26) holds true in this case as well.

The proof of the proposition is complete. $\square$

The next two propositions will be needed in order to apply dGLR for pricing and hedging in the sense of Section 4.2. In the first one we show that the family $\widehat{\mathcal{E}}$ satisfies Assumption (B). In the second one, we show that, for fixed $t \in \mathcal{T}$ and $D \in L^0$, the function

$$\gamma \to \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}^{\mathbb{Q}} \Big[\sum_{s=t}^{T} D_s \,\Big|\, \mathcal{F}_t \Big],$$

satisfies Assumption (C).

Proposition 4.4.2. *For each $\gamma \in (0, \infty)$, the set of densities $\widehat{\mathcal{E}}^\gamma$ is closed and convex.*

Proof. Fix $\gamma \in (0, \infty)$. We first show that $\widehat{\mathcal{E}}^\gamma$ is closed (in $\mathbb{R}^N$).[18] Let η_k be a sequence in $\widehat{\mathcal{E}}^\gamma$ converging to some η. By the definition of $\widehat{\mathcal{E}}^\gamma$, there exist sequences Λ_k and c_k so that $\eta_k = c_k(1 + \Lambda_k)$, $c_k > 0$, $c_k = 1/\mathbb{E}^{\mathbb{P}}[1 + \Lambda_k]$, and $0 \le \Lambda_k(\omega_j) \le \gamma$ for $j = 1, \dots, N$. For each ω_j, we have that $\Lambda_k(\omega_j)$ is bounded by γ, so Λ_k is bounded. By the Bolzano-Weierstrass Theorem, there exists a subsequence Λ_{k_m} such that Λ_{k_m} converges to some Λ. This limit must satisfy $0 \le \Lambda(\omega_j) \le \gamma$ for $j = 1, \dots, N$, since a sequence converges in $\mathbb{R}^N$ if and only if it converges coordinate-wise. If Λ_{k_m} converges, then $\mathbb{E}^{\mathbb{P}}[1 + \Lambda_{k_m}]$ converges. Since $\mathbb{E}^{\mathbb{P}}[1 + \Lambda_{k_m}]$ is strictly greater than zero, we have that $1/\mathbb{E}^{\mathbb{P}}[1 + \Lambda_{k_m}]$ converges to $c := 1/\mathbb{E}^{\mathbb{P}}[1 + \Lambda]$, which means that c_{k_m} converges to c. Consequently, η_{k_m} converges to $c(1 + \Lambda)$. It follows that $\eta \in \widehat{\mathcal{E}}^\gamma$. Hence, $\widehat{\mathcal{E}}^\gamma$ is closed.

We proceed by showing that $\widehat{\mathcal{E}}^\gamma$ is convex. Let $\eta_1, \eta_2 \in \widehat{\mathcal{E}}^\gamma$ and $0 \le \lambda \le 1$. Let c_i and Λ_i correspond to η_i, in the sense of definition of $\widehat{\mathcal{E}}^\gamma$, that is, $\eta_i = c_i(1 + \Lambda_i)$, $i = 1, 2$.

[18] Clearly, we may consider $\widehat{\mathcal{E}}^\gamma$ as a subset of $\mathbb{R}^N$.

We need to show that $\lambda c_1(1+\Lambda_1)+(1-\lambda)c_2(1+\Lambda_2)\in\widehat{\mathcal{E}}^\gamma$. Define

$$\widetilde{c}:=\lambda c_1+(1-\lambda)c_2 \quad \text{and} \quad \widetilde{\Lambda}:=\frac{\lambda c_1\Lambda_1+(1-\lambda)c_2\Lambda_2}{\lambda c_1+(1-\lambda)c_2}.$$

Since

$$\lambda c_1(1+\Lambda_1)+(1-\lambda)c_2(1+\Lambda_2)=\widetilde{c}(1+\widetilde{\Lambda}),$$

it suffices to show that $0\le\widetilde{\Lambda}\le\gamma$ and $\widetilde{c}=1/\mathbb{E}^{\mathbb{P}}[1+\widetilde{\Lambda}]$. We first notice that since $0\le\Lambda_1,\Lambda_2\le\gamma$, the scalars c_1,c_2 satisfy $c_1,c_2>0$, and since $0\le\lambda\le1$, we have that

$$0\le\frac{\lambda c_1\Lambda_1+(1-\lambda)c_2\Lambda_2}{\lambda c_1+(1-\lambda)c_2}\le\gamma\frac{\lambda c_1+(1-\lambda)c_2}{\lambda c_1+(1-\lambda)c_2}=\gamma.$$

Therefore, $0\le\widetilde{\Lambda}\le\gamma$. Next, because $c_1\mathbb{E}^{\mathbb{P}}[1+\Lambda_1]=c_2\mathbb{E}^{\mathbb{P}}[1+\Lambda_2]=1$, it is true that

$$\begin{aligned}\widetilde{c}\mathbb{E}^{\mathbb{P}}[1+\widetilde{\Lambda}]&=(\lambda c_1+(1-\lambda)c_2)\mathbb{E}^{\mathbb{P}}\Big[1+\frac{\lambda c_1\Lambda_1+(1-\lambda)c_2\Lambda_2}{\lambda c_1+(1-\lambda)c_2}\Big]\\&=\lambda c_1+(1-\lambda)c_2+\lambda c_1\mathbb{E}^{\mathbb{P}}[1+\Lambda_1]+(1-\lambda)c_2\mathbb{E}^{\mathbb{P}}[1+\Lambda_2]-\lambda c_1-(1-\lambda)c_2\\&=1.\end{aligned}$$

As a result, $\widehat{\mathcal{E}}^\gamma$ is convex. □

Proposition 4.4.3. *For each $t\in\mathcal{T}, D\in L^0$ the function of $\gamma\in(0,\infty)$ defined as*

$$\rho_t^\gamma(D):=\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^\gamma}\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T}D_s\,\Big|\,\mathcal{F}_t\Big], \tag{4.32}$$

is continuous.

Proof. Let $\omega\in\Omega$. By the abstract Bayes Theorem, we have that

$$\begin{aligned}\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^\gamma}\mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^{T}D_s\,\Big|\,\mathcal{F}_t\Big](\omega)&=\inf_{\eta\in\widehat{\mathcal{E}}^\gamma}\frac{\mathbb{E}^{\mathbb{P}}\big[\eta\sum_{s=t}^{T}D_s\,\big|\,\mathcal{F}_t\big](\omega)}{\mathcal{E}^{\mathbb{P}}[\eta\,|\,\mathcal{F}_t](\omega)}\\&=\inf_{\Lambda\in\mathfrak{L}^\gamma}\frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)\sum_{s=t}^{T}D_s\,\big|\,\mathcal{F}_t\big](\omega)}{\mathbb{E}^{\mathbb{P}}[1+\Lambda\,|\,\mathcal{F}_t](\omega)}.\end{aligned}$$

The function g defined as

$$g(\Lambda)(\omega) := \frac{\mathbb{E}^{\mathbb{P}}\big[(1+\Lambda)\sum_{s=t}^{T} D_s \,\big|\, \mathcal{F}_t\,\big](\omega)}{\mathbb{E}^{\mathbb{P}}[(1+\Lambda)\,|\,\mathcal{F}_t\,](\omega)}, \quad 0 \leq \Lambda \text{ an } \mathcal{F}_T\text{-measurable r.v.}$$

is continuous in Λ. Applying Lemma B.1, we conclude that the proposition holds true. □

Remark 4.4.2. *Note that the LHS of* (4.32) *is the value of a DCRM associated with* $\widehat{\mathcal{Q}}$ *(see A.2).*

4.4.3 Pricing Barrier Options via dGLR. One of the main advantages of our dynamic framework is that good-deal ask and bid prices, as defined in Definition 4.3.1, can be computed for path-dependent options in a dynamically consistent manner. In this section, using a simple model for ask and bid prices of a stock, and choosing the dGLR as acceptability index, we compute the good-deal ask and bid prices of European-style Barrier call options in a market with transaction costs. We compare these good-deal prices with the corresponding upper and lower bounds of the no-arbitrage pricing interval.

According to Theorem 4.3.1, the good-deal ask and bid prices of a derivative contract $D \in L^0$, at level $\gamma > 0$, at time $t = 0$ satisfy

$$\Pi_0^{ask,\gamma}(D) = \sup_{\mathbb{Q}\in\widehat{\mathcal{Q}}^{\gamma}\cap\mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=1}^{T} D_s\Big],$$

$$\Pi_0^{bid,\gamma}(D) = \inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}^{\gamma}\cap\mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=1}^{T} D_s\Big].$$

Recall that $\widehat{\mathcal{Q}}$, defined in (4.24), is a dynamically consistent family of sets of probability measures that corresponds to the dGLR. Computation of the good-deal ask and bid prices of the options are carried out using the representations above. Similarly, the no-arbitrage bounds are computed using the following representations (see

Remark 4.3.10).

$$\pi_0^U(D) = \sup_{\mathbb{Q}\in\mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=1}^{T} D_s\Big],$$

$$\pi_0^L(D) = \inf_{\mathbb{Q}\in\mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=1}^{T} D_s\Big].$$

Moreover, we suppose that the bid price of the stock[19] is given in Table 4.1. The ask price process is assumed to satisfy $S^{ask} := S^{bid}(1+\lambda)$, where $\lambda \in [0,\infty)$ is

Table 4.1. Bid price paths of the stock

ω	$t=0$	$t=1$	$t=2$
ω_1	50	80	90
ω_2	50	80	70
ω_3	50	80	60
ω_4	50	40	60
ω_5	50	40	30

the *transaction costs coefficient* (see Bensaid et al. [BLPS92], and Boyle and Vorst [BV92] for details on proportional transaction costs). Also, we define the *mid price process* as $S^{mid} := (S^{ask} + S^{bid})/2$.

Note that $\widehat{\mathcal{Q}}$ is defined in terms of the reference measure $\mathbb{P}$. We assume that $\mathbb{P}$ is given as follows

$$\big(\mathbb{P}(\omega_1), \mathbb{P}(\omega_2), \mathbb{P}(\omega_3), \mathbb{P}(\omega_4), \mathbb{P}(\omega_5)\big) = (1/10, 1/8, 1/4, 1/4, 11/40).$$

[19]See Example 4.10 in Pliska [Pli97], page 134.

Moreover, in this case the filtration $\mathbb{F}$ is given by

$$\mathcal{F}_0 = \{\Omega, \emptyset\},$$
$$\mathcal{F}_1 = \big\{\Omega, \emptyset, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}\big\},$$
$$\mathcal{F}_2 = 2^{\Omega}.$$

4.4.3.1 Up-and-In Barrier Option. We price an up-and-in barrier option with barrier level 65 and strike $K = 75$. Recall that this option pays $(S_T^{mid} - 75)^+$ if $S_t^{mid} > 65$ for some $t \in \{1, 2\}$, and pays nothing otherwise.

The prices π^U, π^L, Π^{ask} and Π^{bid} for our up-and-in barrier option are given in Tables 4.2, 4.3, and 4.4, for varying transaction cost coefficients. As the γ values decrease, we see that the good-deal ask-bid intervals shrink significantly. Furthermore, as the transaction cost coefficient increases, the spread between the π^U and π^L increase. This is because it becomes more expensive to hedge the claim. Essentially, the set of risk-neutral measures $\mathcal{R}(\mathcal{H}(t))$ increases in λ. Hence, Π^{ask} and Π^{bid} converge to π^U and π^L, at higher γ values. Observe that in Table 4.2, the prices Π^{ask} and Π^{bid} begin to converge to π^U and π^L at $\gamma = 0.25$, whereas in Table 4.3 they begin to converge at around $\gamma = 0.5$, and in Table 4.4 they begin to converge after $\gamma = 0.75$.

4.4.3.2 Up-and-Out Barrier Option. Next, we price an up-and-out barrier option with barrier level 85 and strike $K = 50$. This options pays $(S_T^{mid} - 50)^+$ if $S_t^{mid} < 85$ for $t = 1, 2$, and pays 0 otherwise.

In Tables 4.5, 4.6 and 4.7, we present the no-arbitrage bounds and good-deal prices of our up-and-out barrier option for different λ values. It is easy to see that the good-deal ask-bid intervals shrink with the decreasing γ values. Since hedging the claim becomes more expensive as the transaction cost coefficient increases, we observe that the no-arbitrage price interval widens as λ increases. To be more specific, this

Table 4.2. Prices of an Up-and-In Call Option with $\lambda = 0$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	2.499842	2.287694	2.287306	1.875158
0.001	–	2.289438	2.285563	–
0.005	–	2.297186	2.277848	–
0.01	–	2.306857	2.268276	–
0.05	–	2.383699	2.194508	–
0.1	–	2.478454	2.108781	–
0.25	–	2.499841	1.887571	–
0.5	–	2.499841	1.875158	–
0.75	–	2.499841	1.875158	–
1	–	2.499842	1.875158	–
1.25	–	2.499842	1.875158	–

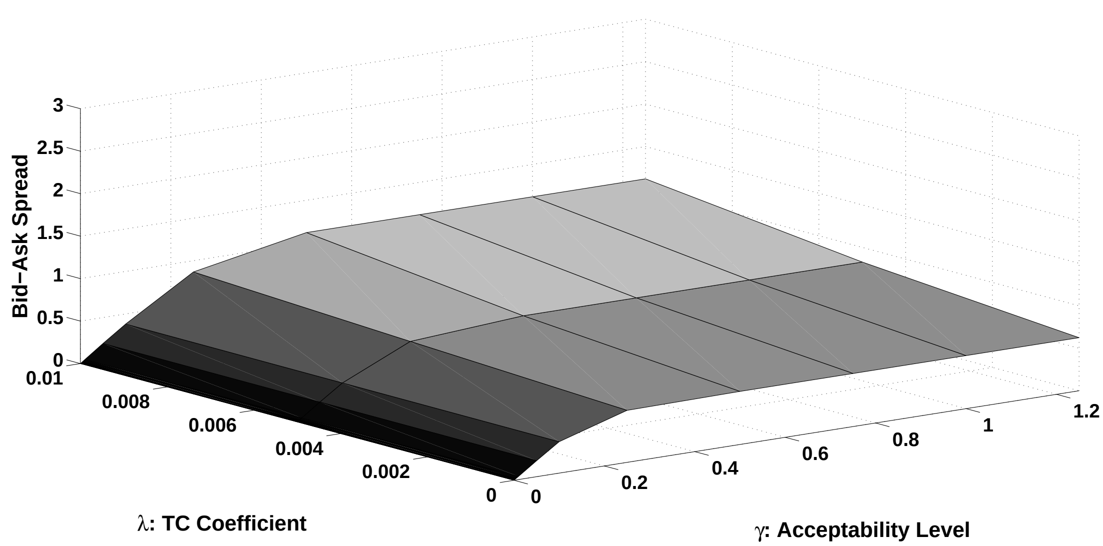

Figure 4.1. Liquidity Surface of an Up-and-In Call Option

is because the set of risk-neutral measures $\mathcal{R}(\mathcal{H}(t))$ is increasing in λ. Therefore, the more we increase the λ values, the higher γ values it takes for the good-deal ask and

Table 4.3. Prices of an Up-and-In Call Option with $\lambda = 0.005$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	2.652910	2.322009	2.321616	1.825199
0.001	–	2.323780	2.319846	–
0.005	–	2.331644	2.312015	–
0.01	–	2.341460	2.302301	–
0.05	–	2.419455	2.227425	–
0.1	–	2.515630	2.140413	–
0.25	–	2.652909	1.915884	–
0.5	–	2.652909	1.825199	–
0.75	–	2.652910	1.825199	–
1	–	2.652910	1.825199	–
1.25	–	2.652910	1.825199	–

bid prices converge to π^U and π^L. In Table 4.5, convergence of Π^{ask} and Π^{bid} to π^U and π^L happens at $\gamma = 0.75$, whereas in Table 4.6 it converges around $\gamma = 1$, and in Table 4.7 it converges after $\gamma = 1.25$.

4.4.3.3 Down-and-Out Barrier Option. In this section, we price a down-and-out barrier option with barrier level 45 and strike $K = 65$. This option pays $(S_T^{mid} - 65)^+$ if $S_t^{mid} > 45$ for $t = 1, 2$, and pays 0 otherwise.

Tables 4.8, 4.9 and 4.10 present the upper and lower no-arbitrage bounds and good-deal prices of our down-and-out barrier option for $\lambda = 0.005$ and $\lambda = 0.01$. Notice that the good-deal ask-bid intervals in the tables shrink with the increasing γ values. We find that as the transaction cost coefficient λ increases, the no-arbitrage price interval widens, since hedging the claim is more costly for higher λ values. In

Table 4.4. Prices of an Up-and-In Call Option with $\lambda = 0.01$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	2.811831	2.356325	2.355925	1.692162
0.001	–	2.358122	2.354130	–
0.005	–	2.366101	2.346183	–
0.01	–	2.376063	2.336325	–
0.05	–	2.455210	2.260343	–
0.1	–	2.552807	2.172044	–
0.25	–	2.811830	1.944198	–
0.5	–	2.811830	1.692163	–
0.75	–	2.811831	1.692163	–
1	–	2.811831	1.692162	–
1.25	–	2.811831	1.692162	–

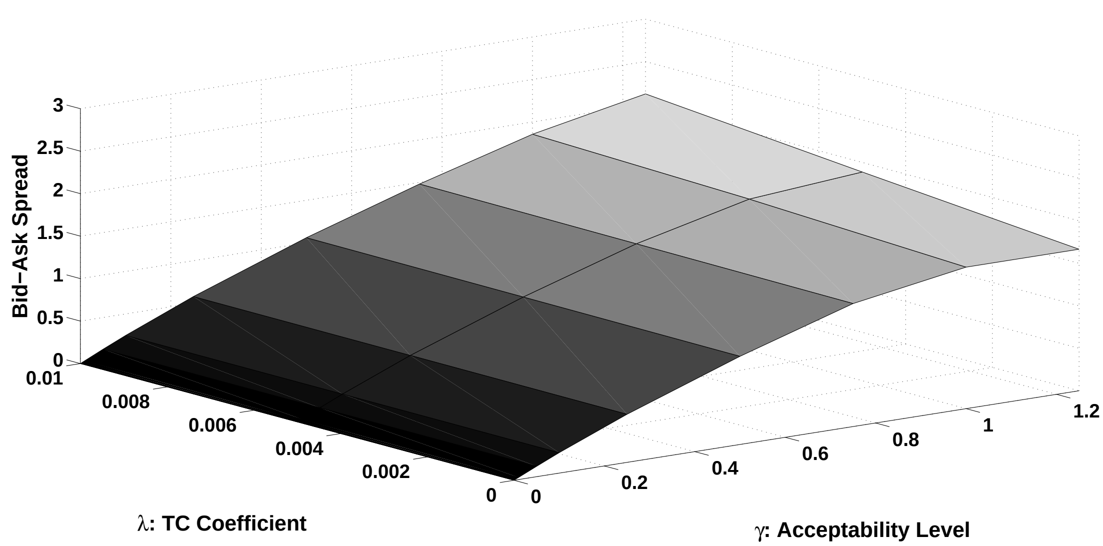

Figure 4.2. Liquidity Surface of an Up-and-Out Call Option

fact, the set of risk-neutral measures $\mathcal{R}(\mathcal{H}(t))$ is increasing in λ. Consequently, good-deal ask and bid prices converge to the upper and lower no-arbitrage bounds at higher

Table 4.5. Prices of Up-and-Out Call Option with $\lambda = 0$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	2.499578	1.400126	1.399874	0.833755
0.001	–	1.401263	1.398738	–
0.005	–	1.406314	1.393711	–
0.01	–	1.412623	1.387478	–
0.05	–	1.462869	1.339553	–
0.1	–	1.525130	1.284109	–
0.25	–	1.708359	1.142274	–
0.5	–	2.002384	0.964686	–
0.75	–	2.283052	0.834886	–
1	–	2.499578	0.833755	–
1.25	–	2.499578	0.833755	–

γ values, as λ increases, . For instance, in Table 4.8, Π^{ask} and Π^{bid} converge to π^U and π^L at $\gamma = 0.1$, whereas in Table 4.9 it converges around $\gamma = 0.25$, and in Table 4.10 it converges after $\gamma = 0.5$.

Table 4.6. Prices of Up-and-Out Call Option with $\lambda = 0.005$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	2.687791	1.415815	1.415560	0.799956
0.001	–	1.416965	1.414411	–
0.005	–	1.422073	1.409328	–
0.01	–	1.428452	1.403025	–
0.05	–	1.479260	1.354563	–
0.1	–	1.542220	1.298498	–
0.25	–	1.727502	1.155074	–
0.5	–	2.024821	0.975495	–
0.75	–	2.308634	0.844242	–
1	–	2.579841	0.799957	–
1.25	–	2.687791	0.799956	–

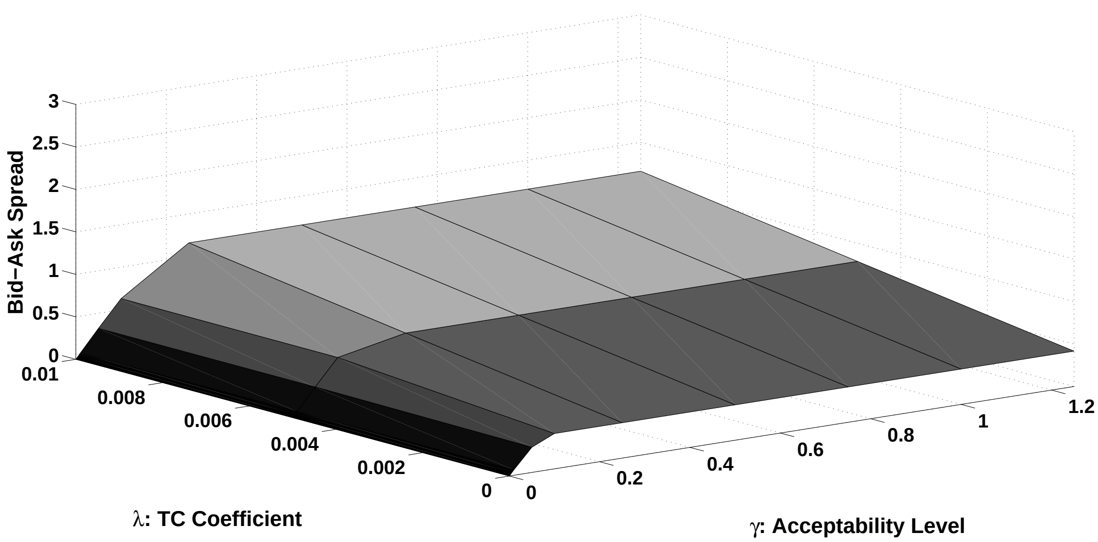

Figure 4.3. Liquidity Surface of an Down-and-Out Call Option

Table 4.7. Prices of Up-and-Out Call Option with $\lambda = 0.01$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	2.882492	1.431504	1.431246	0.750716
0.001	–	1.432667	1.430084	–
0.005	–	1.437831	1.424945	–
0.01	–	1.444281	1.418572	–
0.05	–	1.495652	1.369573	–
0.1	–	1.559309	1.312887	–
0.25	–	1.746644	1.167874	–
0.5	–	2.047259	0.986305	–
0.75	–	2.334217	0.853597	–
1	–	2.608428	0.752365	–
1.25	–	2.870724	0.750717	–

Table 4.8. Prices of Down-and-Out Call Option with $\lambda = 0$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	4.166561	4.025324	4.024676	3.750105
0.001	–	4.028239	4.021762	–
0.005	–	4.041185	4.008864	–
0.01	–	4.057338	3.992857	–
0.05	–	4.166559	3.869262	–
0.1	–	4.166561	3.750107	–
0.25	–	4.166561	3.750106	–
0.5	–	4.166561	3.750106	–
0.75	–	4.166561	3.750105	–
1	–	4.166561	3.750105	–
1.25	–	4.166561	3.750105	–

Table 4.9. Prices of Down-and-Out Call Option with $\lambda = 0.005$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	4.395436	4.067077	4.066423	3.610007
0.001	–	4.070023	4.063479	–
0.005	–	4.083103	4.050447	–
0.01	–	4.099424	4.034274	–
0.05	–	4.228856	3.909397	–
0.1	–	4.387862	3.763767	–
0.25	–	4.395434	3.610009	–
0.5	–	4.395435	3.610007	–
0.75	–	4.395436	3.610007	–
1	–	4.395436	3.610007	–
1.25	–	4.395436	3.610007	–

Table 4.10. Prices of Down-and-Out Call Option with $\lambda = 0.01$

γ	π_0^U	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	π_0^L
0.0001	4.631900	4.108831	4.108169	3.472015
0.001	–	4.111807	4.105195	–
0.005	–	4.125021	4.092030	–
0.01	–	4.141509	4.075691	–
0.05	–	4.272270	3.949531	–
0.1	–	4.432908	3.802406	–
0.25	–	4.631897	3.472017	–
0.5	–	4.631898	3.472016	–
0.75	–	4.631898	3.472016	–
1	–	4.631900	3.472015	–
1.25	–	4.631900	3.472015	–

CHAPTER 5
CONCLUSIONS AND FUTURE WORK

In this work, we studied problems in valuation and mitigation of counterparty risk modeling and pricing derivatives using dynamic coherent acceptability indices. As a conclusion, we address several open research problems and our plans for future work.

We considered the modeling of counterparty risk in the presence of bilateral margin agreements in Chapter 2. We defined an appropriate collateral process which takes various margin agreement parameters into account. The dynamics of the counterparty risk adjustment, CVA, have been found for the bilateral case. This achievement helps us to better understand and monitor the behavior of the bilateral CVA as well as the unilateral CVA and the DVA.

We observed the impact of collateral agreements on counterparty risk adjustments as well as the credit exposures such as the EPE and the ENE. We formulated the fair spread value adjustment, which we named as SVA, that indicates the additional spread value to incorporate the counterparty risk into the fair spread value. Moreover, we derive the dynamics of the fair spread and the counterparty risky spread and therefore the spread value adjustment, SVA. Finally, we presented our numerical results using a Markovian model of counterparty credit risk.

In Chapter 3, we considered the problem of collateralized CVA valuation in the presence of rating triggers in credit migrations environment. Resulting adjustment value is found as a consequence, which is then named as RVA. Moreover, we incorporated the rehypothecation risk of the collateral in our setup. We utilized the Markov copulae for modeling the rating transition probabilities, and applied to an IRS and to a CDS contract.

A natural extension of our results in Chapter 2 and in Chapter 3 is to incorporate the asymmetric funding costs, in a multi-curve environment, into the valuation of counterparty risk. Indeed, in virtue of this consideration, a new adjustment term, which is called Funding Valuation Adjustment (FVA), will take place in the CVA and the counterparty risky price computation. Within this proposed framework, the problem of dynamic hedging of counterparty risk as well as the efficient computation of the sensitivities of the CVA will need to be given increased scrutiny because of the asymmetry in the borrowing and lending rates.

On the other hand, one of the key drivers of the credit crisis in 2008, as well as the European sovereign-debt crisis in 2011 and 2012, has been the systemic risk; since the defaults of the major financial institutions often trigger the collapse of many other market participants as a result of contagion. In this regard, modeling, mitigating and hedging the counterparty credit risk by taking the systemic risks as well as the contagion effects into account is a very crucial problem. Incorporating the systemic risk into our framework remains as a future study.

In Chapter 4, we studied the problem of developing the representations of the ask and bid prices of derivatives using the theory of DCAIs in a risk-neutral setup. Our framework is constructed in discrete and finite time space, and also in finite probability space. Therefore, a major future work is the generalization of the theory of DCAIs to a general probability space and to a continuous time space. This will allow us to work in a more realistic setup, in the sense that it will be possible to calibrate our model to the real quoted market ask and bid prices. Moreover, developing dynamic versions of static acceptability indices such as AIMAX, AIMIN, AIMAXMIN and AIMINMAX is a crucial future work. As a result of this development, as well as the application of the appropriate distortion functions in dynamic setup, the ask and bid price representations will have closed-form solutions as in Cherny and Madan [CM10].

Furthermore, extending our framework to the case where we no longer have the scale invariance property (see (D4) in Definition A.1) is another important research direction. This extension, along with the generalization of DCAIs to a dynamic quasi-concave case and incorporating the theory of BSDEs in the duality results, will lead to a more realistic pricing framework, since the restrictiveness of scale invariance property has already been gained attention (see Rosazza-Gianin and Sgarra [RGS12]).

APPENDIX A

DYNAMIC COHERENT ACCEPTABILITY INDICES

In this section, we present some important definitions and results from the theory of Dynamic Coherent Acceptability Indices. For a more detailed discussion we refer to Bielecki, Cialenco, and Zhang [BCZ11].

Let us first recall the definition of a dynamic coherent acceptability index.

Definition A.1. *A dynamic coherent acceptability index (DCAI) is a function* $\alpha : \mathcal{T} \times L^0 \times \Omega \to [0, \infty]$ *that satisfies the following properties:*

(D1) Adaptiveness. *For any* $t \in \mathcal{T}$ *and* $D \in L^0$, $\alpha_t(D)$ *is* $\mathcal{F}_t$*-measurable;*

(D2) Independence of the past. *For any* $t \in \mathcal{T}$ *and* $D, D' \in L^0$, *if there exists* $A \in \mathcal{F}_t$ *such that* $\mathbb{1}_A D_s = \mathbb{1}_A D'_s$ *for all* $s \geq t$, *then* $\mathbb{1}_A \alpha_t(D) = \mathbb{1}_A \alpha_t(D')$*;*

(D3) Monotonicity. *For any* $t \in \mathcal{T}$ *and* $D, D' \in L^0$, *if* $D_s(\omega) \geq D'_s(\omega)$ *for all* $s \geq t$ *and* $\omega \in \Omega$, *then* $\alpha_t(D) \geq \alpha_t(D')$ *for all* $\omega \in \Omega$*;*

(D4) Scale invariance. $\alpha_t(\lambda D) = \alpha_t(D)$ *for all* $\lambda > 0,\ D \in L^0,\ t \in \mathcal{T}$, *and* $\omega \in \Omega$*;*

(D5) Quasi-concavity. *If* $\alpha_t(D) \geq x$ *and* $\alpha_t(D') \geq x$ *for some* $t \in \mathcal{T}$, $\omega \in \Omega$, $D, D' \in L^0$, *and* $x \in (0, \infty]$, *then* $\alpha_t(\lambda D + (1 - \lambda)D') \geq x$ *for all* $\lambda \in [0, 1]$*;*

(D6) Translation invariance. $\alpha_t(D + m\mathbb{1}_{\{t\}}) = \alpha_t(D + m\mathbb{1}_{\{s\}})$ *for every* $t \in \mathcal{T}$, $D \in L^0$, $\omega \in \Omega$, $s \geq t$ *and every* $\mathcal{F}_t$*-measurable random variable* m*;*

(D7) Dynamic consistency. *For any* $t \in [0, \dots, T - 1]$ *and* $D, D' \in L^0$, *if* $D_t(\omega) \geq 0 \geq D'_t(\omega)$ *for all* $\omega \in \Omega$, *and there exists a non-negative* $\mathcal{F}_t$*-measurable random variable* m *such that* $\alpha_{t+1}(D) \geq m(\omega) \geq \alpha_{t+1}(D')$ *for all* $\omega \in \Omega$, *then* $\alpha_t(D) \geq m(\omega) \geq \alpha_t(D')$ *for all* $\omega \in \Omega$.

We now proceed by defining of a dynamic coherent risk measure.

Definition A.2. *Dynamic coherent risk measure (DCRM) is a function* $\rho : \{0, \dots, T\} \times L^0 \times \Omega \to \mathbb{R}$ *that satisfies the following properties:*

(A1) Adaptiveness. *$\rho_t(D)$ is $\mathcal{F}_t$-measurable for all $t \in \mathcal{T}$ and $D \in L^0$;*

(A2) Independence of the past. *If $\mathbb{1}_A D_s = \mathbb{1}_A D'_s$ for some $t \in \mathcal{T}$, $D, D' \in L^0$, and $A \in \mathcal{F}_t$ and for all $s \geq t$, then $\mathbb{1}_A \rho_t(D) = \mathbb{1}_A \rho_t(D')$;*

(A3) Monotonicity. *If $D_s(\omega) \geq D'_s(\omega)$ for some $t \in \mathcal{T}$ and $D, D' \in L^0$, and for all $s \geq t$ and $\omega \in \Omega$, then $\rho_t(D) \leq \rho_t(D')$ for all $\omega \in \Omega$;*

(A4) Homogeneity. *$\rho_t(\lambda D) = \lambda \rho_t(D)$ for all $\lambda > 0$, $D \in L^0$, $t \in \mathcal{T}$, and $\omega \in \Omega$;*

(A5) Subadditivity. *$\rho_t(D + D') \leq \rho_t(D) + \rho_t(D')$ for all $t \in \mathcal{T}$, $D, D' \in L^0$, and $\omega \in \Omega$;*

(A6) Translation invariance. *$\rho_t(D + m\mathbb{1}_{\{s\}}) = \rho_t(D) - m$ for every $t \in \mathcal{T}$, $D \in L^0$, $\mathcal{F}_t$-measurable random variable m, and all $s \geq t$;*

(A7) Dynamic consistency.

$$\mathbb{1}_A(\min_{\omega \in A} \rho_{t+1}(D) - D_t) \leq \mathbb{1}_A \rho_t(D) \leq \mathbb{1}_A(\max_{\omega \in A} \rho_{t+1}(D) - D_t)\,,$$

for every $t \in \{0, 1, \ldots, T-1\}$, $D \in L^0$ and $A \in \mathcal{F}_t$.

Let us continue with an important result that provides the representation of a DCAI in terms of a family of DCRMs, and the representation of DCRM in terms of a DCAI.

Theorem A.1.

(i) If α is a normalized, right-continuous, dynamic coherent acceptability index, then there exists a left-continuous and increasing family of dynamic coherent risk measures $(\rho^\gamma)_{\gamma \in (0,\infty)}$, such that

$$\alpha_t(D)(\omega) = \sup\{\gamma \in (0,\infty) : \rho_t^\gamma(D)(\omega) \leq 0\}, \qquad \omega \in \Omega,\ t \in \mathcal{T},\ D \in L^0. \tag{A.1}$$

(ii) If $(\rho^\gamma)_{\gamma\in(0,\infty)}$ is a left-continuous and increasing family of dynamic coherent risk measures, then there exists a right-continuous and normalized dynamic coherent acceptability index α such that,

$$\rho_t^\gamma(D)(\omega) = \inf\{c\in\mathbb{R} : \alpha_t(D+\delta_t(1c))(\omega)\geq\gamma\}, \qquad \omega\in\Omega,\ t\in\mathcal{T},\ D\in L^0.$$

We assume $\inf\emptyset=\infty$ *and* $\sup\emptyset = 0$.

The proof this theorem can be found in [BCZ11].

We now state the definitions of a dynamically consistent sequence of sets of probability measures and an increasing family of sequences of sets of probability measures.

Definition A.3.

(i) A sequence of sets of probability measures $(\mathcal{Q}_t)_{t=0}^T$ absolutely continuous with respect to $\mathbb{P}$ is called dynamically consistent with respect to the filtration $(\mathcal{F}_t)_{t=0}^T$ if the sequence is of full-support and the following inequality holds true

$$\begin{aligned}\mathbb{1}_E \min_{\omega\in E}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}} \mathbb{E}^{\mathbb{Q}}[X\,|\,\mathcal{F}_{t+1}](\omega)\right\} &\leq \mathbb{1}_E \inf_{\mathbb{Q}\in\mathcal{Q}_t}\mathbb{E}^{\mathbb{Q}}[X\,|\,\mathcal{F}_t] \\ &\leq \mathbb{1}_E \max_{\omega\in E}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}} \mathbb{E}^{\mathbb{Q}}[X\,|\,\mathcal{F}_{t+1}](\omega)\right\}\end{aligned}$$

for all $t\in\{0,1,\dots,T-1\}$, $E\in\mathcal{F}_t$, and $\mathcal{F}_T$-measurable random variables X.

(ii) A family of sequences of sets of probability measures $((\mathcal{Q}_t^\gamma)_{t=0}^T)_{\gamma\in(0,\infty)}$ is called increasing if $\mathcal{Q}_t^\gamma\supseteq\mathcal{Q}_t^\beta$, for all $\gamma\geq\beta>0$ and $t\in\mathcal{T}$.

Next, we present a representation theorem for dynamic coherent risk measures in terms of dynamically consistent set of probabilities. The proof the following theorem can be found in [BCZ11].

Theorem A.2 (Robust Representation Theorem for DCRM)**.** *For $\gamma > 0$, a function $\rho^\gamma : \{0,1,\ldots,T\} \times L^0 \times \Omega \to \mathbb{R}$ is a dynamic coherent risk measure if and only if there exists a dynamically consistent family of sets of probabilities $(\mathcal{Q}_t^\gamma)_{t=0}^T$ such that,*

$$\rho_t^\gamma(D) = -\inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^T D_s \,\Big|\, \mathcal{F}_t\Big], \quad t \in \mathcal{T},\ D \in L^0. \tag{A.2}$$

This result, along with the results from Theorem A.1, which states the duality between DCAIs and DCRMs, leads to a representation theorem for dynamic coherent acceptability indices.

A direct consequence of Theorem A.1 and Theorem A.2 is the following result.

Theorem A.3.

(i) Assume that $((\mathcal{Q}_t^\gamma)_{t=0}^T)_{\gamma\in(0,\infty)}$ is an increasing family of dynamically consistent sequences of sets of probability measures. Then, the function $\alpha : \{0,1,\ldots,T\} \times L^0 \times \Omega \to [0,\infty]$ defined as follows,

$$\alpha_t(D)(\omega) = \sup\Big\{\gamma \in (0,\infty) : \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^T D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0\Big\}, \quad \omega\in\Omega,\ t \in \mathcal{T},\ D \in L^0,$$

is a normalized and right-continuous dynamic coherent acceptability index.

(ii) If α is a normalized and right-continuous dynamic coherent acceptability index, then there exists a family of dynamically consistent sequences of sets of probability measures $((\mathcal{Q}_t^\gamma)_{t=0}^T)_{\gamma\in(0,\infty)}$ such that

$$\alpha_t(D)(\omega) = \sup\Big\{\gamma \in (0,\infty) : \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}}\Big[\sum_{s=t}^T D_s \,\Big|\, \mathcal{F}_t\Big](\omega) \geq 0\Big\}, \quad \omega\in\Omega,\ t \in \mathcal{T},\ D \in L^0.$$

Here we adopt the usual convention that $\inf\emptyset = \infty$ *and* $\sup\emptyset = 0$.

The proof this theorem can also be found in [BCZ11].

APPENDIX B

TECHNICAL RESULTS

In this section we present some technical results, which are used throughout the thesis. We begin by proving two results on continuous functions.

Lemma B.1. *If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(\gamma) := \inf_{0 \le y \le \gamma} g(y)$ is continuous.*

Proof. Since g is continuous, $f(\gamma) = \min_{0 \le y \le \gamma} g(y)$. We first show that

$$\lim_{\gamma \to \gamma_0^+} \min_{\gamma_0 \le y \le \gamma} g(y) = \lim_{\gamma \to \gamma_0^-} \min_{\gamma_0 \le y \le \gamma} g(y) = g(\gamma_0).$$

Suppose $\epsilon > 0$ and $\gamma_0 \le \gamma$. Since g is continuous, for all $\epsilon' > 0$, there exists $\delta > 0$ such that $|\gamma - \gamma_0| < \delta$ implies $|g(\gamma) - g(\gamma_0)| < \epsilon'$. We notice that

$$\begin{aligned}
|g(\gamma_0) - \min_{\gamma_0 \le y \le \gamma} g(y)| &= g(\gamma_0) - \min_{\gamma_0 \le y \le \gamma} g(y) \\
&= \min_{\gamma_0 \le y \le \gamma} \{g(\gamma_0) - g(y)\} \\
&\le \min_{\gamma_0 \le y \le \gamma} \{|g(\gamma_0) - g(y)|\} \\
&= \min_{\gamma_0 \le y \le \gamma} \{|g(\gamma_0) - g(\gamma) + g(\gamma) - g(y)|\} \\
&\le \min_{\gamma_0 \le y \le \gamma} \{|g(\gamma_0) - g(\gamma)| + |g(\gamma) - g(y)|\} \\
&\le |g(\gamma_0) - g(\gamma)| + \min_{\gamma_0 \le y \le \gamma} \{|g(\gamma) - g(y)|\} \\
&\le |g(\gamma_0) - g(\gamma)| + |g(\gamma) - g(\gamma_0)| \\
&= 2|g(\gamma_0) - g(\gamma)| < 2\epsilon'.
\end{aligned}$$

Taking $\epsilon = 2\epsilon'$ shows that $\lim_{\gamma \to \gamma_0^+} \min_{\gamma_0 \le y \le \gamma} g(y) = g(\gamma_0)$.

We now show that $\lim_{\gamma \to \gamma_0^-} \min_{\gamma \le y \le \gamma_0} g(y) = g(\gamma_0)$ Again, suppose $\epsilon > 0$ and $\gamma \le \gamma_0$. Since g is continuous, for any $\epsilon' > 0$ there exists $\delta > 0$ such that $|\gamma - \gamma_0| < \delta$ implies $|g(\gamma) - g(\gamma_0)| < \epsilon'$. Notice that

$$
\begin{aligned}
|g(\gamma_0) - \min_{\gamma \le y \le \gamma_0} g(y)| &= g(\gamma_0) - \min_{\gamma \le y \le \gamma_0} g(y) \\
&= \min_{\gamma \le y \le \gamma_0} \{g(\gamma_0) - g(y)\} \\
&\le \min_{\gamma \le y \le \gamma_0} \{|g(\gamma_0) - g(y)|\} \\
&= \min_{\gamma \le y \le \gamma_0} \{|g(\gamma_0) - g(\gamma) + g(\gamma) - g(y)|\} \\
&\le \min_{\gamma \le y \le \gamma_0} \{|g(\gamma_0) - g(\gamma)| + |g(\gamma) - g(y)|\} \\
&\le |g(\gamma_0) - g(\gamma)| + \min_{\gamma \le y \le \gamma_0} \{|g(\gamma) - g(y)|\} \\
&\le |g(\gamma_0) - g(\gamma)| + |g(\gamma) - g(\gamma_0)| \\
&= 2|g(\gamma_0) - g(\gamma)| < 2\epsilon'.
\end{aligned}
$$

Taking $\epsilon = 2\epsilon'$ shows that $\lim_{\gamma \to \gamma_0^-} \min_{\gamma_0 \le y \le \gamma} g(y) = g(\gamma_0)$.

We now show that f is continuous. We need to show that

$$
\lim_{\gamma \to \gamma_0^+} f(\gamma) = \lim_{\gamma \to \gamma_0^-} f(\gamma) = f(\gamma_0).
$$

Since f is non-increasing and bounded, the limit exists. Let $0 < \gamma_0 \le \gamma < \infty$. Since $\min(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function it follows that

$$
\begin{aligned}
f(\gamma_0) - \lim_{\gamma \to \gamma_0^+} f(\gamma) &= \min_{0 \le y \le \gamma_0} g(y) - \lim_{\gamma \to \gamma_0^+} \min_{0 \le y \le \gamma} g(y) \\
&= \min_{0 \le y \le \gamma_0} g(y) - \lim_{\gamma \to \gamma_0^+} \min\left(\min_{0 \le y \le \gamma_0} g(y), \min_{\gamma_0 \le y \le \gamma} g(y) \right) \\
&= \min_{0 \le y \le \gamma_0} g(y) - \min\left(\min_{0 \le y \le \gamma_0} g(y), \lim_{\gamma \to \gamma_0^+} \min_{\gamma_0 \le y \le \gamma} g(y) \right) \\
&= \min_{0 \le y \le \gamma_0} g(y) - \min\left(\min_{0 \le y \le \gamma_0} g(y), g(\gamma_0) \right) \\
&= 0.
\end{aligned}
$$

It follows that f is right-continuous.

Now let $0 < \gamma \leq \gamma_0 < \infty$. Likewise,

$$\begin{aligned}
f(\gamma_0) - \lim_{\gamma \to \gamma_0^-} f(\gamma) &= \min_{0 \leq y \leq \gamma_0} g(y) - \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma_0} g(y) - \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \lim_{\gamma \to \gamma_0^-} \min\left(\min_{0 \leq y \leq \gamma} g(y), \min_{\gamma \leq y \leq \gamma_0} g(y) \right) - \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \min\left(\lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y), \lim_{\gamma \to \gamma_0^-} \min_{\gamma \leq y \leq \gamma_0} g(y) \right) - \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \min\left(\lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y), g(\gamma_0) \right) - \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y)
\end{aligned}$$

From the continuity of g, we see that

$$\begin{aligned}
&= \min\left(\lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y), g(\gamma_0) \right) - \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \min\left(\lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y), \lim_{\gamma \to \gamma_0^-} g(\gamma) \right) - \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \lim_{\gamma \to \gamma_0^-} \min\left(\min_{0 \leq y \leq \gamma} g(y), g(\gamma) \right) - \lim_{\gamma \to \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= 0
\end{aligned}$$

Thus, f is left-continuous, so we conclude that f is continuous. $\square$

The following lemma is an auxiliary result needed for Theorem 4.3.1.

Lemma B.2. *For any monotone increasing, continuous function $f : (0, \infty) \to \mathbb{R}$, we have that*

$$f(\gamma) \leq 0 \quad \textit{if and only if} \quad \sup\{\beta \in (0, \infty) : f(\beta) \leq 0\} \geq \gamma,$$

for any $\gamma > 0$.

Proof. Let us define the set $\Gamma := \{\beta \in (0,\infty) : f(\beta) \leq 0\}$. Assume that $f(\gamma) \leq 0$ for some $\gamma > 0$. Then, $\gamma \in \Gamma$, and therefore $\sup \Gamma \geq \gamma$.

Conversely. Suppose that $\sup \Gamma \geq \gamma$ and define $\beta^* := \sup \Gamma$. If $\sup \Gamma = \infty$, then $f(x) \leq 0$, for all $x > 0$, and in particular for $x = \gamma$. Now assume that $\beta^* \in (0,\infty)$. We first argue by contradiction that $\beta^* \in \Gamma$. If $\beta^* \notin \Gamma$, then $f(\beta^*) > 0$. Now, since f is continuous, there exists $\epsilon' > 0$ so that $0 < f(\beta^* - \epsilon')$. By the definition of the supremum of a set, we have that, for all $\epsilon > 0$, there exists $\beta^\epsilon \in \Gamma$ so that $\beta^* - \epsilon < \beta^\epsilon$. Therefore, because f is monotonically increasing, $f(\beta^* - \epsilon) \leq f(\beta^\epsilon)$. Hence, $0 < f(\beta^* - \epsilon') \leq f(\beta^\epsilon)$, which contradicts $\beta^\epsilon \in \Gamma$. We proceed by showing that $f(\gamma) \leq 0$. Since $\gamma \leq \beta^*$ and f is monotonically increasing, we have that $f(\gamma) \leq f(\beta^*)$. However, $\beta^* \in \Gamma$, so $f(\gamma) \leq f(\beta^*) \leq 0$. □

We now recall a well-known characterization of compact sets. For a proof, see Lemma I.5.6 in Dunford and Schwartz [DS58].

Lemma B.3. *A subset of a topological space is compact if and only if every family of closed sets with the finite intersection property has a nonempty intersection.*

The following theorem is an application of Hahn-Banach theorem, regarding the separation of hyperplanes.

Theorem B.1. *If $\mathcal{Z}$ and $\mathcal{C}$ are disjoint closed convex subsets of $\mathbb{R}^N$, and if $\mathcal{Z}$ is compact, then there exists a constant ϵ with $\epsilon > 0$, and a continuous linear functional $\varphi \in \mathbb{R}^N$, so that*

$$\varphi(c) \leq 0 < \epsilon < \varphi(z)$$

for all $z \in \mathcal{Z}$ and $c \in \mathcal{C}$.

Proof. By Theorem V.2.10 in Dunford and Schwartz [DS58], there exists constants a

and ϵ' with $\epsilon' > 0$, and a continuous linear functional $\varphi \in \mathbb{R}^N$, so that

$$\varphi(x) \leq a - \epsilon' < a \leq \varphi(z) \tag{B.1}$$

for all $z \in \mathcal{Z}$ and $x \in \mathcal{C}$. We now argue that $\varphi(x) \leq 0$ for all $x \in \mathcal{C}$. Suppose there exists $a_0 > 0$ and $x_0 \in \mathcal{C}$ so that $\varphi(x_0) = a_0$. Since $\mathcal{C}$ is a cone, we have that $\lambda x_0 \in \mathcal{C}$ for all $\lambda > 0$. Thus,

$$\sup_{x \in \mathcal{C}} \varphi(x) \geq \sup_{\lambda > 0} \varphi(\lambda x_0) = \sup_{\lambda > 0} \lambda a_0 = +\infty,$$

which contradicts (B.1), and hence $\varphi(x) \leq 0,\ x \in \mathcal{C}$. From here, and since φ is linear and $0 \in \mathcal{C}$, it follows that $\sup_{x \in \mathcal{C}} \varphi(x) = 0$. Thus, $a - \epsilon' \geq 0$, and hence $a > 0$. Taking $\epsilon = a$ concludes the proof. $\square$

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