Fair Capital Risk Allocation

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ABSTRACT: In this paper we develop a novel methodology for *estimation* of risk capital allocation. The methodology is rooted in the theory of risk measures. We work within a general, but tractable class of law-invariant coherent risk measures, with a particular focus on expected shortfall. We introduce the concept of *fair* capital allocations and provide explicit formulae for fair capital allocations in case when the constituents of the risky portfolio are jointly normally distributed. The main focus of the paper is on the problem of approximating fair portfolio allocations in the case of not fully known law of the portfolio constituents. We define and study the concepts of fair allocation estimators and asymptotically fair allocation estimators. A substantial part of our study is devoted to the problem of estimating fair risk allocations for expected shortfall. We study this problem under normality as well as in a nonparametric setup. We derive several estimators, and prove their fairness and/or asymptotic fairness. Last, but not least, we propose two backtesting methodologies that are oriented at assessing the performance of the allocation estimation procedure. The paper closes with a substantial numerical study of the subject.

KEYWORDS: capital allocation, fair capital allocation, asymptotic fairness, expected shortfall, risk measures, Euler principle, value-at-risk, tail-value-at-risk, backtesting capital allocation.

1 Introduction

The measurement and the management of risk is without doubt of highest importance in the financial and the insurance industries. Arguably, the theory and applications of risk measures are most useful for this purpose. For early applications in the insurance context see [Büh70, Ger74], and for a historical perspective in the financial context see [Gui16]. The seminal article [ADEH99] placed risk measurements on an axiomatic foundation paving the way to coherent risk measures which have been treated in numerous works since then. We refer to [Del00, FS11, MFE15] for an in-depth treatment of the topic.

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The application of risk measures to portfolio management naturally leads to the problem of allocating portions of the risk capital to the constituents of the portfolio, i.e. to the *risk allocation problem.* There are a number of different approaches to risk capital allocation, depending on the one hand on the class of the used risk measures, and on the other hand on the used allocation principles. The *Euler principle*, often used in risk management practice, is one example, see e.g. [Tas04, Tas07]. For coherent risk measures, the Euler principle coincides with the axiomatic approach proposed in [Kal05]. For the more general case of convex risk measures we refer to [Tsa09, MFE15] and references therein.

Risk measures as we consider them here are mathematical tools which require as inputs probability distributions of the underlying risk factors. In practical applications one is typically confronted with the fact that these probability distributions are not fully specified. For example, let X represent a P&L, which is a function of some underlying risk factors, and let ρ be the risk measure used to measure the riskiness of X, so that the desired quantity to compute is the risk $\rho(X)$. Since the probability laws of the risk factors are not fully specified, then one needs to approximate $\rho(X)$, perhaps by estimating this quantity exploiting historical data. As a consequence, the risk allocations, which are usually computed in terms of risk measures, need to be approximated, in particular by estimation.

The problem of estimation of risk has, to a great extent, been neglected in the literature. In the recent paper [PS18] a new statistical methodology for efficient estimation of risk capital $\rho(X)$ was proposed. The methodology introduced in that paper is based on the key concept, which the authors call unbiased estimation of risk also introduced in [PS18], and is based on economic principle.[1] Inspired by the ideas from [PS18], in this paper we develop a novel methodology for *estimation of capital risk allocation*.[2] We work within a general, but tractable class of coherent risk measures, the so-called weighted value-at-risk measures introduced in [Che06], with focus on the expected shortfall risk measure, which is broadly accepted in the risk management practice.

The first key concept introduced in this paper is the *fair capital risk allocation*, which builds upon the robust representation of coherent risk measures. We provide explicit formulae for fair capital allocations in case when the constituents of the portfolio are jointly normally distributed. The major focus of the paper is on the problem of approximating fair portfolio allocations when the law of the portfolio constituents is not fully known. Motivated by the concept of the fair capital allocation, we define and study the concepts of *fair allocation estimators* and *asymptotically fair allocation estimators*. A substantial portion of our study is devoted to the problem of estimating the risk allocation under expected shortfall and normality. In addition we consider a nonparametric approach to this problem. We derive several estimators, and prove their fairness and/or asymptotic fairness. Last, but not least, we propose two backtesting methodologies that are oriented at assessing the performance of the allocation estimation procedure. Finally, we perform relevant numerical studies. The results of the numerical studies that we have conducted so far are encouraging for practical use of the estimation and backtesting of the capital allocation.

This work is a first step towards developing formal methodologies for estimating and backtesting of fair capital allocation. As such, it has potential to open new theoretical and practical research avenues.

[1] The concept of unbiased estimation of risk must not be confused with the classical concept of unbiased estimator.

[2] In this paper we will occasionally write capital allocation or risk allocation in place of capital risk allocation.

2 The fair allocation principle

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be an atomless probability space, and let $\mathbb{E}$ be the expectation under $\mathbb{P}$. In what follows, all needed integrability and regularity assumptions are taken for granted.

We consider a random vector $X = (X_1, \ldots, X_d)$ whose components are interpreted as discounted future profits and losses (P&Ls). The marginal random variable X_i (margin – for short) might correspond to the ith clearing member of a central clearing counterparty (CCP), to the ith position in the portfolio, to the ith trader portfolio in a trading desk, or to the ith desk in the financial institution portfolio. In the following, we will refer to X as *portfolio* and to X_i as the ith *portfolio margin* or the ith *portfolio constituent*.

Let $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ and let $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ be a *normalized monetary risk measure*: ρ is *monotone*, i.e. $\rho(U) \leq \rho(V)$ for $U \geq V$, ρ is *cash-additive*, i.e. $\rho(U + c) = \rho(U) - c$ for all $c \in \mathbb{R}$, and ρ is *normalized*, i.e. $\rho(0) = 0$.

The riskiness of the portfolio X is measured by applying the risk measure ρ to the *aggregated portfolio P&L* denoted by

$$S := \sum_{i=1}^{d} X_i.$$

We call the quantity $\rho(S)$ the *aggregated risk*, or total risk, of the portfolio X.

Our objective is to study the issue of allocating the aggregated risk of the portfolio to the individual constituents of the portfolio. Specifically, we intend to find a vector $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$, called a *risk allocation*, such that the following *balance condition* holds

$$\rho(S) = \sum_{i=1}^{d} a_i. \tag{2.1}$$

The component a_i is interpreted as the risk contribution of X_i to the aggregated risk, and therefore $X_i + a_i$ is interpreted as the ith *secured margin* of portfolio X. Correspondingly, we call $X + a$ the *secured portfolio*, and $S + \sum_{i=1}^{d} a_i$ the *secured aggregated position*.

Stated as such, the risk allocation problem is ill–posed. Indeed, any collection of numbers $a_1, \ldots, a_d$ satisfying the balance condition (2.1) constitutes a risk allocation. In order to deal with a meaningful risk allocation problem we need to impose additional conditions, that reflect some additional and desired features of the portfolio allocation. With this in mind, we impose an additional condition on a, which we will call the *fairness condition*.

Towards this end, we require more structure on the risk measure ρ. We additionally assume that the monetary risk measure ρ is finite, law-invariant, comonotonic and coherent; see [Kus01] for details. In view of [Sha13, Theorem 2(iii)] we conclude that ρ is a *weighted value-at-risk measure*, so that it admits representation (1.1) in [Che06] for a fixed probability measure ν on $[0, 1]$. Specifically, for a continuously distributed random variable Y,

$$\rho(Y) = \rho_\nu(Y) := \int_{[0,1]} \mathrm{ES}_\alpha(Y) \nu(d\alpha), \quad Y \in L^1, \tag{2.2}$$

where ES_α is the *Expected Shortfall* (ES) risk measure (sometimes also called *tail value-at-risk* or *conditional value-at-risk*) for reference level $\alpha \in [0, 1]$. Moreover, ρ admits a robust-type representation of the form

$$\rho(Y) = \sup_{\mathbb{Q} \in \mathcal{D}} \mathbb{E}_{\mathbb{Q}}[-Y], \tag{2.3}$$

where $\mathcal{D}$ is a determining family of probability measures absolutely continuous with respect to $\mathbb{P}$. As shown in [Che06, Theorem 6.3], for any $Y \in L^1$ there exists a unique *minimal extreme measure* $\mathbb{Q}_Y \in \mathcal{D}$ such that[3]

$$\rho(Y) = \mathbb{E}_{\mathbb{Q}_Y}[-Y]. \tag{2.4}$$

Sometimes, we refer to $\mathbb{Q}_Y$ as *the worst-case scenario measure* (for position Y). We denote by Z_Y the associated Radon-Nikodym derivative $d\mathbb{Q}_Y/d\mathbb{P}$. In particular, as shown in [Che06] (cf. formula (6.2) there), if Y has a continuous distribution then we have

$$Z_Y = g(Y), \quad \text{and} \quad \rho(Y) = \mathbb{E}[-g(Y)Y], \tag{2.5}$$

for some Borel function g. For example if $\rho = \mathrm{ES}_\alpha$ is the expected shortfall at level α, then we have

$$Z_Y = \frac{1}{\alpha}\mathbb{1}_{\{Y<q_Y(\alpha)\}}, \tag{2.6}$$

where $q_Y(\alpha)$ is the α–quantile of Y.

In what follows, for simplicity, we write $\mathbb{E}_S$ instead of $\mathbb{E}_{\mathbb{Q}_S}$. The value $\mathbb{E}_S[X_i + a_i]$ represents the average performance of the secured margin $X_i + a_i$ under the extremal measure $\mathbb{Q}_S$. The following *fairness condition* selects risk allocations which are comparable under the extremal measure of the aggregated portfolio P&L.

Definition 2.1. The capital allocation $a = (a_1, \dots, a_d)$ is called *fair*, if

$$\mathbb{E}_S[X_i + a_i] = \mathbb{E}_S[X_j + a_j], \quad i, j = 1, \dots, d. \tag{2.7}$$

The economic intuition behind this definition is as follows: the worst-case-scenario $\mathbb{Q}_S$ is, in our setting, the determing scenario of the capital allocation for the portfolio through $\rho(S) = \mathbb{E}_S[-S]$ resulting from Equation (2.4). A *fair* capital allocation is meant to create secured positions $X_i + a_i$, $1 \le i \le d$, so that the averages of all secured positions with respect to the worst-case-scenario $\mathbb{Q}_S$ are all equal.

Since ρ is a monetary risk measure, the extremal measures for S and $S + c$, $c \in \mathbb{R}$, coincide. Thus, for any fair capital allocation a satisfying the balance condition in (2.1) we have

$$0 = \rho\Big(\sum_{i=1}^d (X_i + a_i)\Big) = -\mathbb{E}_S\Big[\sum_{i=1}^d (X_i + a_i)\Big] = -\sum_{i=1}^d \mathbb{E}_S[X_i + a_i], \tag{2.8}$$

and consequently the risk allocations are given by

$$a_i = -\mathbb{E}_S[X_i] = -\mathbb{E}[Z_S X_i], \quad i = 1, \dots, d. \tag{2.9}$$

In view of (2.5), we also have that

$$a_i = -\mathbb{E}\big[g\big(\sum_{k=1}^d X_k\big)X_i\big], \quad i = 1, \dots, d. \tag{2.10}$$

The concept of fairness introduced above aligns well with what has been done in some of the existing literature. In particular, the above notion of fairness implies *fairness in the sense of*

[3] Note that the set of *extreme measures*, i.e. the set of measures that satisfy (2.4), might contain more than one element. The term *minimal* corresponds to the minimal element with respect to the convex stochastic order; see [Che06] for details.

fuzzy games introduced in [Del00]. Indeed, this follows from Theorems 17 and 18 therein taking representation (2.3) into account. The fair allocation principle of Definition 2.1 has been applied in [BCF18] in the context of allocation of the total default fund among the clearing members of a CCP.

The following example illustrates the concept of fair allocation.

Example 2.2 (Mean risk allocation)**.** Consider expectation for measuring risk, i.e. $\rho(Y) = \mathbb{E}[-Y]$, in which case $\mathcal{D} = \{\mathbb{P}\}$. Then, clearly, for any $X = (X_1, \dots, X_d)$, the capital allocation $\mathbf{a} = (a_1, \dots, a_d)$ given as

$$a_i = -\mathbb{E}[X_i], \quad i = 1, \dots, d,$$

is fair.

2.1 Risk allocation under normality

As an example where explicit formulae can be obtained, we study the case of normally distributed profits and losses. In this regard, let us assume that the vector X is normally distributed under $\mathbb{P}$ with mean μ and covariance matrix Σ and fix $i \in \{1, \dots, d\}$. Then, (X_i, S) is bivariate normal, and the conditional expectation $\mathbb{E}[X_i|S]$ takes the form

$$\mathbb{E}[X_i|S] = \beta_i S + \alpha_i,$$

with $\beta_i = \frac{\mathrm{Cov}(X_i,S)}{\mathrm{Var}(S)}$, and $\alpha_i = \mu_i - \beta_i \sum_{j=1}^d \mu_j$. Since this conditional expectation is the $L^2 := L^2(\Omega, \mathcal{F}, \mathbb{P})$ orthogonal projection of X_i on the linear space spanned by S we obtain

$$X_i = \beta_i S + \alpha_i + \epsilon_i,$$

where S and ϵ_i are independent under $\mathbb{P}$, and $\mathbb{E}[\epsilon_i] = 0$. For any weighted value-at-risk measure ρ, Equation (2.9) implies that a fair capital allocation is given by

$$\begin{aligned} a_i &= -\mathbb{E}_S[X_i] = -\alpha_i - \beta_i \mathbb{E}_S[S] + \mathbb{E}_S[\varepsilon_i] \\ &= -\alpha_i + \beta_i \rho(S) + \mathbb{E}[Z_S \varepsilon_i] \\ &= -\alpha_i + \beta_i \rho(S) + \mathbb{E}[g(S)\varepsilon_i] = -\alpha_i + \beta_i \rho(S) + \mathbb{E}[g(S)]\mathbb{E}[\varepsilon_i] \\ &= -\alpha_i + \beta_i \rho(S), \end{aligned} \tag{2.11}$$

where we have used (2.5) in the fourth equality, independence of S and ϵ_i under $\mathbb{P}$ in the fifth equality, and the fact that ϵ_i has zero mean under $\mathbb{P}$, in the last equality. As expected, the total allocated risk is divided among constituents using the regression slope allocations which is typically referred to as the *covariance principle*, see [MFE15, Section 8.5].

Expected shortfall. To be more specific, we consider as an important example the expected shortfall (ES). In this regard, let $\rho = \mathrm{ES}_\alpha$ denote ES under $\mathbb{P}$ for the level $\alpha \in (0, 1)$. Then, for a continuously distributed real valued random variable Y we have

$$\mathrm{ES}_\alpha(Y) = \mathbb{E}[-Y \mid Y \leq q_Y(\alpha)], \tag{2.12}$$

where $q_Y(\alpha)$ is α-quantile of Y. Thus, since S is normally distributed, (2.12) yields

$$\mathrm{ES}_\alpha(S) = -\sum_{i=1}^d \mu_i + \frac{1}{\alpha}\sqrt{\mathrm{Var}(S)}\,\phi\big(\Phi^{-1}(\alpha)\big), \tag{2.13}$$

where ϕ and Φ are the density and the cumulative distribution function of the standard normal distribution; see [MFE15, Example 2.14]. Putting together (2.11) and (2.13) we see that the capital allocation for ES is given as

$$a_i = -\mu_i + \frac{\mathrm{Cov}(X_i, S)}{\alpha\sqrt{\mathrm{Var}(S)}}\phi(\Phi^{-1}(\alpha)), \qquad i = 1, 2, \ldots, d\,. \tag{2.14}$$

It is not difficult to see that the conditions of Proposition 2.1 in [Tas07] hold, so that this allocation is unique.

3 Fair allocation estimators

In practice, the probability distribution under $\mathbb{P}$ of X, the portfolio's P&L, is not fully specified. Since, in view of (2.5) and (2.10), we have

$$\rho(S) = -\mathbb{E}\Big[g\Big(\sum_{k=1}^{d} X_k\Big)\sum_{k=1}^{d} X_k\Big], \quad \text{and} \quad a_i = -\mathbb{E}\Big[g\Big(\sum_{k=1}^{d} X_k\Big)X_i\Big], \quad i = 1, \ldots, d, \tag{3.1}$$

then, in almost all practically relevant applications, neither the aggregated risk $\rho(S)$ nor the fair risk allocation a are known, and thus need to be estimated. Hence, appropriate estimation procedures have to be developed, in particular estimation procedures based on the historical data about realizations of the portfolio. This will involve estimating, in some way, the probability distribution of X under $\mathbb{P}$.

In the following, we set the relevant statistical framework and propose efficient procedures to deal with this estimation issue. We refer to X as to the *population*. Historical information about X is given in terms of a random sample of size n drawn from X, which we denote by $X^1, \ldots, X^n$, so that $X^1, \ldots, X^n$ are independent. Our aim is to estimate the aggregated risk $\rho(S)$ using the information contained in the sample. Towards this end we let

$$\mathbf{X}^n := \{X^j = (X_1^j, \ldots, X_d^j),\ j = 1, \ldots, n\},$$

represent the random sample, and let us denote its realization by

$$\mathbf{x}^n := \{x^j = (x_1^j, \ldots, x_d^j),\ j = 1, \ldots, n\}, \tag{3.2}$$

where x_k^j corresponds to the j-th observed (realized) value of the portfolio's kth margin.

The formal statistical setup for this situation is as follows: consider a family of probability measures $\mathcal{P} := (\mathbb{P}^\theta)_{\theta\in\Theta}$ on $(\Omega, \mathscr{F})$, where Θ denotes the parameter space. To avoid unnecessary technical difficulties, we assume that all measures in $\mathcal{P}$ are equivalent. Furthermore, we assume that for any $\theta \in \Theta$ the random sample $X^1, \ldots, X^n$ is i.i.d. under $\mathbb{P}^\theta$. Moreover, we assume that $\mathbb{P} = \mathbb{P}^{\theta_0}$ for some (unknown) parameter $\theta_0 \in \Theta$. We will denote by ρ^θ and, respectively $\mathbb{E}^\theta$, the risk measure ρ, and respectively the expectation, under the probability measure $\mathbb{P}^\theta$. Similarly to the notation $\mathbb{Q}_Y$ and Z_Y, corresponding to the reference measure $\mathbb{P}$, we will use notation $\mathbb{Q}_Y^\theta$ and Z_Y^θ with regard to the reference measure $\mathbb{P}^\theta$.

Given the random sample $\mathbf{X}^n$, the allocation a is estimated using an *allocation estimator* $\hat{A}^n = (\hat{A}_1^n, \ldots, \hat{A}_d^n)$ defined as

$$\hat{A}^n = \eta_n(\mathbf{X}^n), \tag{3.3}$$

for some measurable function $\eta_n : \mathbb{R}^{d\times n} \to \mathbb{R}^d$.

Next, we define a property that should be satisfied by any reasonable *allocation estimator*.

Definition 3.1. An allocation estimator $\hat{A}^n$ is called *fair* if, for all $\theta \in \Theta$,

$$\mathbb{E}^\theta\Big[Z^\theta_{S,\hat{A}^n}(X_i + \hat{A}^n_i)\Big] = 0, \quad i = 1, \dots, d, \tag{3.4}$$

where $Z^\theta_{S,\hat{A}^n} := Z^\theta_{S+\sum_{i=1}^d \hat{A}^n_i}$.

We emphasize that $\hat{A}^n$ is a random variable, and $Z^\theta_{S,\hat{A}^n}$ is the Radon-Nikodym derivative corresponding to $S + \sum_{i=1}^d \hat{A}^n_i$.

Intuitively, the above definition means that an allocation estimator is fair if it mimics the balanced fairness condition (2.9) for all relevant scenarios (given by probability distributions $\mathbb{P}^\theta$, $\theta \in \Theta$). In particular, the aggregated risk estimator obtained from a *fair* allocation estimator $\hat{A}$ by summation turns out to be *unbiased* in the sense of [PS18, Definition 4.1], namely, for any $\theta \in \Theta$ we get

$$\rho^\theta\Big(S + \sum_{i=1}^d \hat{A}^n_i\Big) = -\sum_{i=1}^d \mathbb{E}^\theta\Big[Z^\theta_{S,\hat{A}^n}(X_i + \hat{A}^n_i)\Big] = 0. \tag{3.5}$$

Equality (3.5) guarantees that the secured aggregated portfolio position $S + \sum_{i=1}^d \hat{A}_i$ is acceptable in the sense that it bears no risk, while Equality (3.4) ensures that the average performance of the secured marginal positions under the worst-case scenario measure for the secured portfolio S are the same and that the joint position is secured. In particular, for $d = 1$, the definitions of *fairness* and *unbiasedness* coincide.

It should be noted that (3.5) means that a fair allocation estimator charges an adequate amount of capital to secure the portfolio. This is a consequence of (3.4), which means that a fair allocation estimator applies an adequate amount of capital charge to each position constituent.

We end this section with a simple example to illustrate the concept of fairness.

Example 3.2. Consider the mean risk allocation given in Example 2.2. This leads to the family of risk measures $\rho^\theta(\cdot) = -\mathbb{E}^\theta[\,\cdot\,]$, $\theta \in \Theta$. Then, the risk allocation estimator

$$\hat{M}^n_i = -\frac{1}{n}\sum_{j=1}^n X^j_i, \qquad \text{for } i = 1, 2, \dots, d\,,$$

is a fair allocation estimator. Indeed, note that here, for each $\theta \in \Theta$, the extremal measure coincides with the original probability measure $\mathbb{P}^\theta$, i.e. $Z^\theta_{S,\hat{M}^n} \equiv 1$. Thus, for $i \in \{1, 2, \dots d\}$ we obtain

$$\mathbb{E}^\theta\Big[Z^\theta_{S,\hat{M}^n}(X_i + \hat{M}^n_i)\Big] = \mathbb{E}^\theta\Big[X_i - \frac{1}{n}\sum_{j=1}^n X^j_i\Big] = 0.$$

3.1 Estimating capital allocation under expected shortfall and normality

Following Section 2.1, we study the case where the d-dimensional random vector X is normally distributed under every $\mathbb{P}^\theta$, and we assume that the risk is measured by the expected shortfall $\mathrm{ES}^\theta_\alpha$, at a fixed level $\alpha \in (0, 1)$. In what follows, for the random sample $\mathbf{X}^n$, we will use the notation

$S^j := \sum_{i=1}^d X_i^j$, $j = 1, \dots, n$, and we set[4]

$$\begin{aligned}
\hat{\mu}_i &:= \tfrac{1}{n}\textstyle\sum_{j=1}^n X_i^j,\\
\hat{\mu}_S &:= \tfrac{1}{n}\textstyle\sum_{j=1}^n S^j = \sum_{i=1}^d \hat{\mu}_i\\
\hat{\sigma}_S^2 &:= \tfrac{1}{n-1}\textstyle\sum_{j=1}^n (S^j - \hat{\mu}_S)^2,\\
\widehat{\mathrm{Cov}}_{X_i,S} &:= \tfrac{1}{n-1}\textstyle\sum_{j=1}^n (X_i^j - \hat{\mu}_i)(S^j - \hat{\mu}_S),
\end{aligned}$$

to denote the sample mean of the ith constituent, the sample mean of the portfolio, the sample variance of the portfolio, and the sample covariance of the ith constituent and the portfolio, respectively.

Motivated by the Representation (2.11) we define the allocation estimator $\hat{B} = (\hat{B}_1, \dots, \hat{B}_d)$ as

$$\hat{B}_i := -\hat{\alpha}_i + \hat{\beta}_i\,\hat{R}(S), \qquad i = 1, \dots, d, \tag{3.6}$$

where $\hat{\beta}_i = \frac{1}{\hat{\sigma}_S^2}\widehat{\mathrm{Cov}}_{X_i,S}$ and $\hat{\alpha}_i = \hat{\mu}_i - \hat{\beta}_i\hat{\mu}_S$ are the estimators of the slope and intercept regression coefficient from the L^2 orthogonal projection of the ith margin of X onto S, and where $\hat{R}(S)$ is an unbiased risk estimator (in the sense of [PS18]) for the Expected Shortfall of the secured position S. It has been shown in [PS18, Example 5.4] that $\hat{R}(S)$ under normality can be represented as

$$\hat{R}(S) = -\hat{\mu}_S + \hat{\sigma}_S b_n, \tag{3.7}$$

where $b_n \in \mathbb{R}$ is deterministic, and depends only on the sample size n, and risk level $\alpha \in (0,1)$. Consequently, the estimator becomes

$$\hat{B}_i = -\hat{\mu}_i + \frac{\widehat{\mathrm{Cov}}_{X_i,S}}{\hat{\sigma}_S} b_n, \qquad i = 1, \dots, d.$$

Before we show that $\hat{B}$ satisfies the fairness property, we show an important conditional unbiasedness property of the estimators $\hat{\beta}_i$ and $\hat{\alpha}_i$, in the usual statistical sense. Towards this end, for $i = 1, 2, \dots, d$, we use

$$\begin{aligned}
\beta_i^\theta &:= \mathrm{Cov}^\theta(X_i, S)\cdot(\mathrm{Var}^\theta(S))^{-1},\\
\alpha_i^\theta &:= \mathbb{E}^\theta(X_i) - \beta_i^\theta \sum_{k=1}^d \mathbb{E}^\theta(X_k),
\end{aligned}$$

to denote the true regression coefficients of the L^2–orthogonal projection of ith margin of X onto S under $\mathbb{P}^\theta$, for $\theta \in \Theta$; see Section 2.1. Note that, in view of our assumption that for any $\theta \in \Theta$ the random sample $X^1, \dots, X^n$ is i.i.d. under $\mathbb{P}^\theta$, we get $\beta_i^\theta = \mathrm{Cov}^\theta(X_i^j, S^j)\cdot(\mathrm{Var}^\theta(S^j))^{-1}$ and $\alpha_i^\theta = \mathbb{E}^\theta(X_i^j) - \beta_i^\theta \sum_{k=1}^d \mathbb{E}^\theta(X_k^j)$, for $j = 1, \dots, n$.

Proposition 3.3. *For any $\theta \in \Theta$ it holds that*

$$\mathbb{E}^\theta\Big[\hat{\beta}_i \,|\, \hat{\mu}_S, \hat{\sigma}_S\Big] = \beta_i^\theta \qquad \textit{and} \qquad \mathbb{E}^\theta\Big[\hat{\alpha}_i \,|\, \hat{\mu}_S, \hat{\sigma}_S\Big] = \alpha_i^\theta, \qquad i = 1, \dots, d. \tag{3.8}$$

Proof. Recall from Section 2.1 that under normality, for $j \in \{1, \dots, n\}$, $i \in \{1, \dots, d\}$, and $\theta \in \theta$, we have

$$X_i^j = \alpha_i^\theta + \beta_i^\theta S^j + \epsilon_i^{j,\theta}, \tag{3.9}$$

[4]To ease the notation, we will drop the superscript n in the following. So, we will write $\hat{\mu}_i$ rather than $\hat{\mu}_i^n$, etc.

where $\epsilon_i^{j,\theta}$ is a zero mean Gaussian random variable independent of S^j. As a simple consequence of (3.9) we obtain that $\epsilon_i^{j,\theta}$ is independent of μ_S and σ_S under $\mathbb{P}^\theta$ for all $\theta \in \Theta$. Then, by definition,

$$\begin{aligned}\mathbb{E}^\theta\Big[\hat\beta_i \,|\, \hat\mu_S, \hat\sigma_S\Big] &= \tfrac{1}{\hat\sigma_S^2}\mathbb{E}^\theta\Big[\frac{1}{n}\sum_{j=1}^n (X_i^j - \hat\mu_i)(S^j - \hat\mu_S) \,|\, \hat\mu_S, \hat\sigma_S\Big] \\ &= \tfrac{1}{\hat\sigma_S^2}\mathbb{E}^\theta\Big[\frac{1}{n}\sum_{j=1}^n X_i^j S^j - \hat\mu_i\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\Big].\end{aligned} \tag{3.10}$$

Inserting (3.9), and using that $n^{-1}\sum_{j=1}^n (S^j)^2 = \hat\sigma_S^2 + \hat\mu_S^2$, we obtain

$$\begin{aligned}(3.10) &= \tfrac{1}{\hat\sigma_S^2}\mathbb{E}^\theta\Big[\frac{1}{n}\sum_{j=1}^n (\alpha_i^\theta + \beta_i^\theta S^j + \epsilon_i^{j,\theta})S^j - \hat\mu_i\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\Big] \\ &= \tfrac{1}{\hat\sigma_S^2}\mathbb{E}^\theta\Big[\alpha_i^\theta\hat\mu_S + \beta_i^\theta(\hat\sigma_S^2 + \hat\mu_S^2) + \frac{1}{n}\sum_{j=1}^n \epsilon_i^{j,\theta}S^j - \hat\mu_i\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\Big] \\ &= \tfrac{1}{\hat\sigma_S^2}\mathbb{E}^\theta\Big[\alpha_i^\theta\hat\mu_S + \beta_i^\theta(\hat\sigma_S^2 + \hat\mu_S^2) + \frac{1}{n}\sum_{j=1}^n \mathbb{E}^\theta\big[\epsilon_i^{j,\theta}S^j \,|\, S^j, \hat\mu_S, \hat\sigma_S\big] - \hat\mu_i\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\Big] \\ &= \tfrac{1}{\hat\sigma_S^2}\mathbb{E}^\theta\Big[\alpha_i^\theta\hat\mu_S + \beta_i^\theta(\hat\sigma_S^2 + \hat\mu_S^2) + \frac{1}{n}\sum_{j=1}^n S^j\mathbb{E}^\theta\big[\epsilon_i^{j,\theta} \,|\, S^j, \hat\mu_S, \hat\sigma_S\big] - \hat\mu_i\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\Big] \\ &= \tfrac{1}{\hat\sigma_S^2}\mathbb{E}^\theta\Big[\alpha_i^\theta\hat\mu_S + \beta_i^\theta(\hat\sigma_S^2 + \hat\mu_S^2) + \frac{1}{n}\sum_{j=1}^n S^j\mathbb{E}^\theta\big[\epsilon_i^{j,\theta}\big] - \hat\mu_i\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\Big] \\ &= \tfrac{1}{\hat\sigma_S^2}\mathbb{E}^\theta\Big[\alpha_i^\theta\hat\mu_S + \beta_i^\theta(\hat\sigma_S^2 + \hat\mu_S^2) - \hat\mu_i\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\Big] = \beta_i^\theta + \tfrac{\hat\mu_S}{\hat\sigma_S^2}\,\mathbb{E}^\theta\Big[\alpha_i^\theta + \beta_i^\theta\hat\mu_S - \hat\mu_i \,|\, \hat\mu_S, \hat\sigma_S\Big].\end{aligned}$$

We use again (3.9) and obtain

$$\hat\mu_i = \frac{1}{n}\sum_{j=1}^n X_i^j = \alpha_i^\theta + \beta_i^\theta\frac{1}{n}\sum_{j=1}^n S^j + \eta^\theta, \tag{3.11}$$

with $\eta^\theta = \sum_{j=1}^n \epsilon_i^{j,\theta}$ satisfying $\mathbb{E}^\theta[\eta^\theta \,|\, \hat\mu_S, \hat\sigma_S] = 0$, so that

$$\mathbb{E}^\theta\big[\hat\mu_i \,|\, \hat\mu_S, \hat\sigma_S\big] = \alpha_i^\theta + \beta_i^\theta\hat\mu_S, \tag{3.12}$$

and hence (3.10) $= \beta_i^\theta$ yielding our first claim. With this result and using (3.11), we obtain

$$\begin{aligned}\mathbb{E}^\theta\left[\hat\alpha_i \,|\, \hat\mu_S, \hat\sigma_S\right] &= \mathbb{E}^\theta\left[\hat\mu_i - \hat\beta_i\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\right] \\ &= \mathbb{E}^\theta\left[\hat\mu_i - \beta_i^\theta\hat\mu_S \,|\, \hat\mu_S, \hat\sigma_S\right] = \alpha_i^\theta\end{aligned}$$

which concludes the proof of (3.8). □

Proposition 3.3 shows that we can estimate the portfolio risk expressed through $\hat\mu_S$ and $\hat\sigma_S$ without impacting the statistical unbiasedness property of the regression coefficients; cf. Equation (3.7). Consequently, the risk allocation estimation procedure could be split into two independent steps. First, we estimate the aggregated portfolio risk, and then we estimate the proper allocation of the risk within portfolio constituents. Now, we use this property to show that the allocation estimator given in (3.6) satisfies the fairness property.

Theorem 3.4. *Assume that the allocation estimator $\hat{B} = (\hat{B}_1, \dots, \hat{B}_d)$ is given by (3.6) with $\hat{R}$ as in (3.7). Then, the capital allocation $\hat{B}$ is fair.*

Proof. First, we note that for any $\theta \in \Theta$ the Radon-Nikodym density $Z^\theta_{S,\hat{B}}$ is $\sigma(S + \sum_{i=1}^d \hat{B}_i)$-measurable; see [Che06, Proposition 6.2] and recall that $\hat{B}_i = -\hat{\alpha}_i + \hat{\beta}_i \hat{R}$. Moreover, since

$$\sum_{i=1}^d \hat{\beta}_i = \frac{1}{\hat{\sigma}_S^2} \sum_{i=1}^d \widehat{\mathrm{Cov}}_{X_i,S} = \frac{\hat{\sigma}_S^2}{\hat{\sigma}_S^2} = 1$$

we obtain that

$$\sum_{i=1}^d \hat{\alpha}_i = \sum_{i=1}^d \hat{\mu}_i - \hat{\mu}_S \cdot \sum_{i=1}^d \hat{\beta}_i = \hat{\mu}_S - \hat{\mu}_S = 0.$$

Consequently, as expected,

$$\sum_{i=1}^d \hat{B}_i = \hat{R} \tag{3.13}$$

and Equation (3.7) yields that $Z^\theta_{S,\hat{B}}$ is $\sigma(\hat{\mu}_S, \hat{\sigma}_S, S)$-measurable. With a view towards (3.4), we compute

$$\begin{aligned} \mathbb{E}^\theta \Big[Z^\theta_{S,\hat{B}} \hat{\alpha}_i \Big] &= \mathbb{E}^\theta \Big[Z^\theta_{S,\hat{B}} \mathbb{E}^\theta [\hat{\alpha}_i \,|\, \hat{\mu}_S, \hat{\sigma}_S, S] \Big] \\ &= \mathbb{E}^\theta \Big[Z^\theta_{S,\hat{B}} \mathbb{E}^\theta [\hat{\alpha}_i \,|\, \hat{\mu}_S, \hat{\sigma}_S] \Big] = \mathbb{E}^\theta \Big[Z^\theta_{S,\hat{B}} \alpha^\theta_i \Big], \end{aligned}$$

by Proposition 3.3. Analogously,

$$\mathbb{E}^\theta \Big[Z^\theta_{S,\hat{B}} \hat{\beta}_i \Big] = \mathbb{E}^\theta \Big[Z^\theta_{S,\hat{B}} \beta^\theta_i \Big]$$

and we obtain

$$\begin{aligned} \mathbb{E}^\theta \left[Z^\theta_{S,\hat{B}} \left(X_i + \hat{B}_i \right) \right] &= \mathbb{E}^\theta \left[Z^\theta_{S,\hat{B}} \left(X_i - \hat{\alpha}_i + \hat{\beta}_i \hat{R} \right) \right] \\ &= \mathbb{E}^\theta \left[Z^\theta_{S,\hat{B}} \left(X_i - \alpha^\theta_i + \beta^\theta_i \hat{R} \right) \right]. \end{aligned} \tag{3.14}$$

Next, using (3.5) and (3.13) yield that

$$0 = \mathbb{E}^\theta \Big[Z^\theta_{S,\hat{B}} \Big(S + \sum_{i=1}^d \hat{B}_i \Big) \Big] = \mathbb{E}^\theta \Big[Z^\theta_{S,\hat{B}} \Big(S + \hat{R} \Big) \Big]. \tag{3.15}$$

This result, together with representation (3.9) for $j = n + 1$ (recall that $X^{n+1} = X$) imply that

$$(3.14) = \mathbb{E}^\theta \left[Z^\theta_{S,\hat{B}} \left(X_i - \alpha^\theta_i - \beta^\theta_i S \right) \right] = \mathbb{E}^\theta \left[Z^\theta_{S,\hat{B}} \epsilon^\theta_i \right] = \mathbb{E}^\theta [Z^\theta_{S,\hat{B}}] \mathbb{E}^\theta [\epsilon^\theta_i] = 0, \tag{3.16}$$

where we used the fact that (ϵ^θ_i, S) is bivariate normal with uncorrelated margins, so that ϵ^θ_i is independent of S, and consequently from $Z^\theta_{S,\hat{B}}$. This concludes the proof. □

4 Asymptotic fairness

We now introduce the definition of fairness for a sequence of estimators, $(\hat{A}^n)_{n\in\mathbb{N}}$, and we define the notion of asymptotic fairness.

Definition 4.1. A sequence of allocation estimators $(\hat{A}^n)_{n\in\mathbb{N}}$ will be called *fair* at $n\in\mathbb{N}$, if $\hat{A}^n$ is fair. If fairness holds for all $n\in\mathbb{N}$, we call the sequence $(\hat{A}^n)_{n\in\mathbb{N}}$ fair. The sequence $(\hat{A}^n)_{n\in\mathbb{N}}$ is called asymptotically fair if

$$\mathbb{E}^\theta\left[Z^\theta_{S,\hat{A}^n}(X_i+\hat{A}^n_i)\right]\xrightarrow{n\to\infty}0,\quad i=1,2,\dots,d,\ \text{and}\ \theta\in\Theta. \tag{4.1}$$

In view of Theorem 3.4 it is clear that the sequence of capital allocation estimators $(\hat{B}^n)_{n\in\mathbb{N}}$ defined in (3.6), for varying n, is a fair sequence.[5]

In the rest of the section we assume that the risk allocation is done using ES with reference level α.

4.1 Asymptotic fairness of capital allocation estimators under normality

Using (2.14), we now define a sequence $\hat{C}^n=(\hat{C}^n_1,\dots,\hat{C}^n_d), n\in\mathbb{N}$, of "plug-in type" capital allocation estimators as

$$\hat{C}^n_i := -\hat{\mu}_i+\frac{\widehat{\mathrm{Cov}}_{X_i,S}}{\alpha\hat{\sigma}_S}\phi(\Phi^{-1}(\alpha)). \tag{4.2}$$

The sequence $(\hat{C}^n)_{n\in\mathbb{N}}$ is not fair, in general, but it is asymptotically fair, as proven below.

Proposition 4.2. *The sequence $(\hat{C}^n)_{n\in\mathbb{N}}$ is asymptotically fair.*

Proof. Set $\hat{F}^n := -\hat{\mu}_S+\hat{\sigma}_S\frac{\phi(\Phi^{-1}(\alpha))}{\alpha}$ and note that $\hat{C}^n_i=\hat{\alpha}^n_i+\hat{\beta}^n_i\hat{F}^n,\ i=1,2,\dots,d$.

Proceeding analogously to the proof of Theorem 3.4, with $\hat{B}$ replaced by $\hat{C}^n$ and with $\hat{R}$ replaced by $\hat{F}^n$, we see that in order to prove proposition it is enough to show that for any $\theta\in\Theta$ we have

$$\mathbb{E}^\theta\left[Z^\theta_{S+\hat{F}^n}\left(S+\hat{F}^n\right)\right]\xrightarrow{n\to\infty}0. \tag{4.3}$$

Now, note that

$$\mathbb{E}^\theta\left[Z^\theta_{S+\hat{F}^n}\left(S+\hat{F}^n\right)\right]=\rho_\theta(S+\hat{F}^n),$$

and, in the terminology of [PS18], $\hat{F}^n$ is the standard Gaussian expected shortfall plug-in estimator for S. Consequently, noting that for $d=1$ the definition of asymptotic fairness coincides with the definition of asymptotic unbiasedness given in [PS18, Definition 6.1], and using [PS18, Proposition 6.4] we conclude the proof. □

4.2 Asymptotic fairness of non-parametric capital allocation estimators

We assume throughout this section that the population X, and hence the aggregated portfolio S, are continuous random variables under any $\theta\in\Theta$. Given that the ES is used to determine the

[5] Recall that the superscript n is omitted in (3.6) for the ease of notation.

risk allocation, and taking (2.6) and (2.9) into account, we consider two natural non-parametric expected shortfall capital allocation estimators

$$\check{D}_i^n := -\frac{\sum_{k=1}^n X_i^k \mathbb{1}_{\{S^k+\widehat{\mathrm{V@R}}_\alpha^n\leq 0\}}}{n\alpha}, \quad i=1,\dots,d, \tag{4.4}$$

$$\hat{D}_i^n := -\frac{\sum_{k=1}^n X_i^k \mathbb{1}_{\{S^k+\widehat{\mathrm{V@R}}_\alpha^n\leq 0\}}}{\sum_{k=1}^n \mathbb{1}_{\{S^k+\widehat{\mathrm{V@R}}_\alpha^n\leq 0\}}}, \quad i=1,\dots,d, \tag{4.5}$$

where $\widehat{\mathrm{V@R}}_\alpha^n := -S^{(\lfloor n\alpha\rfloor+1)}$, with $S^{(j)}$ denoting the jth order statistics, and $\lfloor z\rfloor$ denoting the largest integer less or equal than z.

Proposition 4.3. *The sequences $(\hat{D}^n)_{n\in\mathbb{N}}$ and $(\check{D}^n)_{n\in\mathbb{N}}$ are asymptotically fair.*

We will show only that $\hat{D}_i^n$ is asymptotically fair. The proof for $\check{D}_i^n$ follows by similar arguments. Before we prove Proposition 4.3, let us introduce supplementary notation and a lemma that will be useful for the proof. For any $\theta\in\Theta$ we use $a^\theta=(a_1^\theta,\dots,a_n^\theta)$ to denote the true expected shortfall allocation for X under θ and so we have (cf. (2.6))

$$Z_{S,a^\theta}^\theta = \frac{1}{\alpha}\mathbb{1}_{\left\{S+\sum_{i=1}^d a_i^\theta \leq q^\theta_{S+\sum_{i=1}^d a_i^\theta}(\alpha)\right\}},$$

where $q^\theta_{S+\sum_{i=1}^d a_i^\theta}(\alpha)$ denotes the true α-quantile of $S+\sum_{i=1}^d a_i^\theta$ under $\mathbb{P}_\theta$. Similarly, we have

$$Z_{S,\hat{D}}^\theta = \frac{1}{\alpha}\mathbb{1}_{\left\{S+\sum_{i=1}^d \hat{D}_i^n \leq q^\theta_{S+\sum_{i=1}^d \hat{D}_i^n}(\alpha)\right\}}.$$

Lemma 4.4. *For any $\theta\in\Theta$ we get $Z^\theta_{S,\hat{D}^n}\xrightarrow{\mathbb{P}_\theta} Z^\theta_{S,a^\theta}$, as $n\to\infty$.*

Proof. Let us fix $\theta\in\Theta$. For brevity we use notation $r:=\sum_{i=1}^d a^\theta$ and $R_n:=\sum_{i=1}^d \hat{D}_i^n$. First we note that one can show that

$$R_n \xrightarrow{\mathbb{P}_\theta} r, \quad n\to\infty. \tag{4.6}$$

For a fixed $\epsilon\in(\frac{1}{\alpha},0)$, we get

$$\begin{aligned}
\mathbb{P}^\theta\left[|Z_{S,\hat{D}^n}-Z^\theta_{S,a^\theta}|>\epsilon\right] &= \mathbb{P}^\theta\left[|Z_{S,\hat{D}^n}-Z^\theta_{S,a^\theta}|\neq 0\right]\\
&= \mathbb{P}^\theta\left[\{S+R_n\leq q^\theta_{S+R_n}(\alpha)\}\cap\{S+r>q^\theta_{S+r}(\alpha)\}\right] \qquad (4.7)\\
&\quad + \mathbb{P}^\theta\left[\{S+R_n> q^\theta_{S+R_n}(\alpha)\}\cap\{S+r\leq q^\theta_{S+r}(\alpha)\}\right]. \qquad (4.8)
\end{aligned}$$

We want to show that (4.7) and (4.8) go to zero as $n\to\infty$. For brevity, we show the proof only for (4.7); the proof for (4.8) is analogous. For any $\epsilon_2>0$ we get

$$\begin{aligned}
(4.7) &= \mathbb{P}^\theta\left[\{q^\theta_{S+r}(\alpha)<S+r\leq q^\theta_{S+r-(r-R_n)}(\alpha)+(r-R_n)\}\right]\\
&\leq \mathbb{P}^\theta\left[\{q^\theta_{S+r}(\alpha)<S+r\leq q^\theta_{S+r-(r-R_n)}(\alpha)+|r-R_n|\}\right]\\
&\leq \mathbb{P}^\theta\left[\{|r-R_n|\geq\epsilon_2\}\right]+\mathbb{P}^\theta\left[\{q^\theta_{S+r}(\alpha)<S+r\leq q^\theta_{S+r-(r-R_n)}(\alpha)+\epsilon_2\}\right]. \qquad (4.9)
\end{aligned}$$

Using (4.6), and recalling that convergence in probability implies convergence in distribution which in turn implies convergence of quantiles (in continuity points) for $n\to\infty$ we get

$$\mathbb{P}^{\theta}\left[\{|r-R_n|\geq \epsilon_2\}\right]\to 0 \quad \text{and} \quad q^{\theta}_{S+r-(r-R_n)}(\alpha)\to q^{\theta}_{S+r}(\alpha). \tag{4.10}$$

Combining (4.9) with (4.10), noting that the choice of ϵ_2 was arbitrary, and that S is continuous, we conclude the proof. □

Now, we are ready to prove Proposition 4.3.

Proof of Proposition 4.3. Let us fix $\theta\in\Theta$ and $i\in\{1,\dots,d\}$. We want to show that

$$\mathbb{E}^{\theta}\left[Z^{\theta}_{S,\hat{D}^n}(X_i+\hat{D}^n_i)\right]\to 0,\quad n\to\infty.$$

Noting that

$$\mathbb{E}^{\theta}\left[Z^{\theta}_{S,\hat{D}^n}(X_i+\hat{D}^n_i)\right]=\mathbb{E}^{\theta}\left[Z^{\theta}_{n}(X_i+\hat{D}^n_i)\right]+\mathbb{E}^{\theta}\left[Z^{\theta}_{S,a^\theta}(X_i+\hat{D}^n_i)\right],$$

where $Z^{\theta}_n:=Z^{\theta}_{S,\hat{D}^n}-Z^{\theta}_{S,a^\theta}$, we need to prove that

$$\mathbb{E}^{\theta}\left[Z^{\theta}_{n}(X_i+\hat{D}^n_i)\right]\to 0,\quad n\to\infty, \tag{4.11}$$

and

$$\mathbb{E}^{\theta}\left[Z^{\theta}_{S,a^\theta}(X_i+\hat{D}^n_i)\right],\to 0,\quad n\to\infty. \tag{4.12}$$

We start with the proof of (4.11). Noting that for any $n\in\mathbb{N}$ we have $|Z^{\theta}_n|\leq\frac{1}{\alpha}$ and

$$\sum_{k=1}^{n}\mathbb{1}_{\{S^k+\widehat{\mathrm{V@R}}^n_\alpha\leq 0\}}=\lfloor n\alpha\rfloor+1,$$

we get

$$\begin{aligned}\left|\mathbb{E}^{\theta}\left[Z^{\theta}_{n}(X_i+\hat{D}^n_i)\right]\right| &\leq \mathbb{E}^{\theta}\left[|Z^{\theta}_n|(|X_i|+|\hat{D}^n_i|)\right]\\ &\leq \frac{1}{\alpha}\left(\mathbb{E}^{\theta}\left[\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}|X_i|\right]+\mathbb{E}^{\theta}\left[\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}|\hat{C}^n_i|\right]\right)\\ &\leq \frac{1}{\alpha}\left(\mathbb{E}^{\theta}\left[\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}|X_i|\right]+\frac{1}{\lfloor n\alpha\rfloor+1}\mathbb{E}^{\theta}\left[\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}\sum_{k=1}^{n}|X^k_i|\mathbb{1}_{\{S^k+\widehat{\mathrm{V@R}}^n_\alpha\leq 0\}}\right]\right)\\ &\leq \frac{1}{\alpha}\left(\mathbb{E}^{\theta}\left[\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}|X_i|\right]+\frac{1}{\lfloor n\alpha\rfloor+1}\mathbb{E}^{\theta}\left[\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}\sum_{k=1}^{n}|X^k_i|\right]\right)\\ &\leq \frac{1}{\alpha}\left(\mathbb{E}^{\theta}\left[\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}|X_i|\right]+\frac{n}{\lfloor n\alpha\rfloor+1}\mathbb{E}^{\theta}\left[\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}|X^1_i|\right]\right).\end{aligned} \tag{4.13}$$

Now, noting that $\mathbb{1}_{\{|Z^\theta_n|\neq 0\}}=\mathbb{1}_{\{|Z^\theta_n|<\frac{1}{2\alpha}\}}$ and using Lemma 4.4 we get

$$\mathbb{P}^{\theta}\left[|Z^{\theta}_n|\neq 0\right]\to 0,\quad n\to\infty.$$

Combining this with (4.13), noting that $|X_i|$ and $|X^1_i|$ are integrable, and $\frac{n}{\lfloor n\alpha\rfloor+1}\to\frac{1}{\alpha}$ as $n\to\infty$, we conclude the proof of (4.11).

Next, we prove (4.12). Recalling that a^θ is a true allocation for X under θ we get

$$\mathbb{E}^\theta\Big[Z^\theta_{S,a^\theta}(X_i+\hat{D}^n_i)\Big]=\mathbb{E}^\theta\Big[Z^\theta_{S,a^\theta}(X_i+a^\theta_i)\Big]+\mathbb{E}^\theta\Big[Z^\theta_{S,a^\theta}(\hat{D}^n_i-a^\theta_i)\Big]=\mathbb{E}^\theta\Big[Z^\theta_{S,a^\theta}(\hat{D}^n_i-a^\theta_i)\Big].$$

Consequently, noting that Z^θ_{S,a^θ} and $\hat{D}^n_i$ are independent under $\mathbb{P}_\theta$ we get

$$\begin{aligned}
\mathbb{E}^\theta\Big[Z^\theta_{S,a^\theta}(X_i+\hat{D}^n_i)\Big] &= \mathbb{E}^\theta\Big[Z^\theta_{S,a^\theta}\Big]\mathbb{E}^\theta\Big[\hat{D}^n_i-a^\theta_i\Big]\\
&= -\mathbb{E}^\theta\left[\frac{\sum_{k=1}^n X^k_i \mathbb{1}_{\{S^k+\hat{\mathrm{V@R}}^n_\alpha\le 0\}}}{\sum_{k=1}^n \mathbb{1}_{\{S^k+\hat{\mathrm{V@R}}^n_\alpha\le 0\}}}+a^\theta_i\right]\\
&= -\frac{1}{\lfloor n\alpha\rfloor+1}\mathbb{E}^\theta\left[\sum_{k=1}^n (X^k_i+a^\theta_i)\mathbb{1}_{\{S^k+\hat{\mathrm{V@R}}^n_\alpha\le 0\}}\right]\\
&= -\frac{n}{\lfloor n\alpha\rfloor+1}\mathbb{E}^\theta\left[(X^1_i+a^\theta_i)\mathbb{1}_{\{S^1+\hat{\mathrm{V@R}}^n_\alpha\le 0\}}\right]\\
&= -\frac{n}{\lfloor n\alpha\rfloor+1}\mathbb{E}^\theta\left[(X^1_i+a^\theta_i)\left(\mathbb{1}_{\{S^1+\hat{\mathrm{V@R}}^n_\alpha\le 0\}}-\mathbb{1}_{\{S^1\le q^\theta_S(\alpha)\}}\right)\right].
\end{aligned}$$

Note that we used the property $\mathbb{E}^\theta\Big[(X^1_i+a^\theta_i)\mathbb{1}_{\{S^1\le q^\theta_S(\alpha)\}}\Big]=0$ for the last equality. Now, noting that $\hat{\mathrm{V@R}}^n_\alpha$ is a consistent estimator of $-q^\theta_S(\alpha)$, taking similar steps as in Lemma 4.4 and the proof of (4.11), and noting that $\frac{n}{\lfloor n\alpha\rfloor+1}\to\frac{1}{\alpha}$ as $n\to\infty$, we conclude the proof of (4.12). $\square$

5 Backtesting and numerical examples

In this section we will analyze the proposed fair capital allocation methodology via some numerical examples. It goes without saying that any quantitative methodology used for measuring and allocating risk relies on an adopted formal model. It also goes without saying that actual results of risk measurement and/or risk allocation need to be tested for their adequacy. Often, testing adequacy of the results of risk measurement is done in practice using backtesting, and we will use this approach in testing the estimation procedures of fair capital allocation introduced in the previous sections.

Backtesting, applied for risk measurement in the financial context, can be summarized as follows: given a time series of capital forecasts, one compares these forecasts with the realized losses; the accumulated performance is the key ingredient of the backtesting. In particular, backtesting value-at-risk goes back to [Kup95] and recently has gained a lot of practical and theoretical interest; see [Zie16, AS14, PS18] for further details on this topic and the related literature. Similar idea can be applied to backtesting the adequacy of capital allocations.

We focus our attention on assessing the performance of a statistical capital allocation methodology when the underlying reference risk measure is expected shortfall at the fixed level $\alpha\in(0,1]$, used in computing of the values of our estimators. For this purpose we propose two backtesting frameworks:

- absolute deviation from fairness backtesting;
- risk level shifts adjustments backtesting.

The backtesting framework adopted for assessment of adequacy of estimators of capital allocations, say $\widetilde{A}=(\widetilde{A}_1,\ldots,\widetilde{A}_d)$, that were created using some capital allocation methodology,[6] uses as its

[6] We refer to such methodology as to an Internal Capital Allocation Model (ICAM).

input the observations of past P&Ls. The key ingredient to both backtesting methods is the estimation of

$$\sum_{i=1}^{d} \mathbb{E}^{\theta_0}\Big[Z^{\theta_0}_{S,\hat{A}}(X_i+\widetilde{A}_i)\Big], \tag{5.1}$$

and the estimation of

$$\mathbb{E}^{\theta_0}\Big[Z^{\theta_0}_{S,\hat{A}}(X_i+\widetilde{A}_i)\Big],\ i=1,2,\ldots,d. \tag{5.2}$$

We assume that the length of the backtesting window is m days. With each day $k=1,\ldots,m$, we associate the P&Ls X_i^k and allocation estimators $\widetilde{A}_i^k$, $i=1,\ldots,d$. The estimators $\widetilde{A}_i^k$ can be obtained in various ways. One way is to proceed in accordance to what was proposed previously in this paper. Specifically, to produce allocation estimators $\widetilde{A}_i^k$ on day k one uses market observations from the previous n days. We denote these observations as $\mathbf{X}^{k,n}=(X_i^{k-n},\ldots,X_i^k,\ i=1,\ldots,d)$. Based on these observations, and following (3.3), we compute the estimators of the allocations as $\widetilde{A}^k=\eta_n(\mathbf{X}^{k,n})$.

The realizations of X_i^k and $\widetilde{A}_i^k$ are denoted as x_i^k and $\widetilde{a}_i^k$, respectively. We set $y:=(y^1,\ldots,y^m)$, where $y^k=(y_1^k,\ldots,y_d^k)$, $k=1,\ldots,m$, and $y_i^k:=x_i^k+\widetilde{a}_i^k$, $\quad i=1,2,\ldots,d$. We also let $\xi^k:=\sum_{i=1}^d y_i^k$, $\quad k=1,2,\ldots,m$, to denote the realized aggregated secured position on day k, and we set $\xi:=(\xi^1,\ldots,\xi^m)$.

In order to proceed we introduce the following functions of $\beta\in(0,1]$,

$$G_\beta(\tilde{A}):=-\frac{\sum_{k=1}^m \xi^k \mathbb{1}_{\{\xi^k+\mathrm{V\hat{@}R}_\beta(\xi)\le 0\}}}{\sum_{k=1}^m \mathbb{1}_{\{\xi^k+\mathrm{V\hat{@}R}_\beta(\xi)\le 0\}}}, \tag{5.3}$$

and

$$G^i_\beta(\tilde{A}):=-\frac{\sum_{k=1}^m y_i^k \mathbb{1}_{\{\xi^k+\mathrm{V\hat{@}R}_\beta(\xi)\le 0\}}}{\sum_{k=1}^m \mathbb{1}_{\{\xi^k+\mathrm{V\hat{@}R}_\beta(\xi)\le 0\}}}, \quad i=1\ldots,d, \tag{5.4}$$

with $\mathrm{V\hat{@}R}_\beta$ being the empirical value-at-risk at level $\beta\in(0,1]$. Note that y_k^is are computed using as the reference risk measure ES at the fixed risk level α. If no confusion arise, we will write G_β, respectively G^i_β, instead of $G_\beta(\tilde{A})$, respectively $G^i_\beta(\tilde{A})$.

Now, similarly to the derivation of $\hat{D}_i^n$, we estimate the expectation in (5.1) as $-G_\alpha$, and we estimate (5.2) as $-G^i_\alpha$.

Deviation from fairness backtesting. If the capital allocation methodology is fair, then the obtained empirical values G^i_α, $i=1,\ldots,d$, should be close to zero, for the fixed reference level α; the bigger the obtained estimate, the bigger the potential (true) deviation from fairness for the ith margin. *The deviation from fairness backtest* assess proximity to zero of G^i_α, $i=1,\ldots,d$. A comprehensive study of properties of G^i_αs, such as 'how far from zero is an acceptable value' is beyond the scope of this manuscript. Nevertheless, the following backtesting methodology is one way to address this question.

Risk level shift backtesting. Instead of measuring the deviation from fairness directly, it is natural to find the reference risk level $\beta\in(0,1]$ that makes G^i_β closest to zero; equivalently, we want to answer the question by how much one needs to shift the reference risk level α to make the position acceptable. This approach hinges on duality-based performance measurement introduced in [PM18]. Formally, for the estimators of capital allocation $\tilde{A}$, we define

$$\Upsilon(\tilde{A}) := \inf\{\beta \in (0,1] \,:\, G_\beta(\tilde{A}) \le 0\}, \tag{5.5}$$
$$W^i_-(\tilde{A}) := \inf\{\epsilon \in [0,\alpha] \,:\, G^i_\alpha(\tilde{A}) \cdot G^i_{\alpha-\epsilon}(\tilde{A}) \le 0\}, \tag{5.6}$$
$$W^i_+(\tilde{A}) := \inf\{\epsilon \in [0,1-\alpha] \,:\, G^i_\alpha(\tilde{A}) \cdot G^i_{\alpha+\epsilon}(\tilde{A}) \le 0\}, \tag{5.7}$$

where in (5.6) we use the convention $\inf\emptyset = \alpha$, and correspondingly, in (5.7) we put $\inf\emptyset = 1-\alpha$. Similar to G^i_β, G_β, we may simple write Υ, and $W^i_\pm$. Note that G_β is a monotone decreasing function in β, while G^i_β generally speaking is not monotone. Hence, the quantities $W^\pm$ are defined as the smallest shift in the reference risk level from α, to the right or to the left, that makes the ith secured position acceptable. Thus, the closer Υ is to the initial reference risk level α the better is the total risk estimation procedure. Similarly, the closer $W^\pm$ are to zero, the better is the risk allocation procedure. One can look at W as the performance index that is dual to the ES family; see [PM18, Proposition 4.3] for more details.

Finally, by combining the left and right minimal shifts, we define the the minimal shift estimator as

$$W^i(\tilde{A}) := \begin{cases} -W^i_-, & \text{if } W^i_- < W^i_+ \\ W^i_+, & \text{if } W^i_- \ge W^i_+ \end{cases}, \quad i = 1,2,\dots,d. \tag{5.8}$$

Before moving to numerical examples, several comments on backtesting procedure are in order.

(a) It goes without saying that the results produced by the deviation from fairness and the risk level shift approaches should be compared with each other for consistency and reality check.
(b) It is worth mentioning that the two proposed backtesting methodologies can be applied to any ICAM, not necessarily those discussed in this paper.
(c) Our study of the backtesting procedure of the estimation of the risk capital allocation is preliminary. A thorough investigation of the statistical properties of G^i_α and W^i is deferred to future studies.

Next we will illustrate the performance of the capital allocation estimators $\hat{B}^n$, $\hat{C}^n$, and $\hat{D}^n$ on simulated data by applying the two backtesting procedures described above. For brevity and to ease the notation, we will write $\hat{B}^n$, $\hat{C}^n$, and $\hat{D}^n$ as $\hat{B}$, $\hat{C}$, and $\hat{D}$, respectively.

For simulations, we consider two cases of probability distributions of the P&Ls vector X - the Gaussian distribution and the Student's t-distribution. We also fix the reference level $\alpha = 0.05$. All numerical evaluations are performed using R statistical software; the source codes are available from the authors upon request.

Example 5.1 (Gaussian P&Ls)**.** We assume that the portfolio X of eight (discounted) P&Ls follows an eight dimensional Gaussian distribution $\mathcal{N}(\mu,\Sigma)$, with the (true) mean

$$\mu = (0.000786, 0.001549, 0.001660, 0.000195, 0.000650, 0.000413, -0.000401, -0.001146),$$

and the (true) variance-covariance matrix

$$\Sigma = \begin{bmatrix} 0.000226 & 0.000174 & 0.000104 & 0.000066 & 0.000069 & 0.000019 & -0.000077 & -0.000135 \\ 0.000174 & 0.000346 & 0.000135 & 0.000068 & 0.000091 & 0.000022 & -0.000082 & -0.000195 \\ 0.000104 & 0.000135 & 0.000257 & 0.000065 & 0.000084 & 0.000034 & -0.000093 & -0.000111 \\ 0.000066 & 0.000068 & 0.000065 & 0.000133 & 0.000048 & 0.000025 & -0.000058 & -0.000064 \\ 0.000069 & 0.000091 & 0.000084 & 0.000048 & 0.000137 & 0.000034 & -0.000065 & -0.000081 \\ 0.000019 & 0.000022 & 0.000034 & 0.000025 & 0.000034 & 0.000061 & -0.000022 & -0.000031 \\ -0.000077 & -0.000082 & -0.000093 & -0.000058 & -0.000065 & -0.000022 & 0.000149 & 0.000085 \\ -0.000135 & -0.000195 & -0.000111 & -0.000064 & -0.000081 & -0.000031 & 0.000085 & 0.000202 \end{bmatrix}.$$

For the purpose of obtaining the above mean vector and the variance-covariance matrix we used values of daily returns of eight stocks from $S\&P500$ index, namely: AAPL, AMZN, BA, DIS, HD, KO, JPM, and MSFT; these data were taken for the period from January 2015 till December 2018. The first six stocks represent long positions in our portfolio and the last two represent short positions; this gives the negative entries in μ and Σ. The positions in each stock are equally weighted with nominal (absolute) value \$1.

We took the learning period of $n = 500$ days, and the backtesting period of $m = 5{,}000$ days. Below, we present the results for the Gaussian plug-in estimator $\hat{C}$ and the non-parametric estimator $\hat{D}$; we omit results for estimators $\hat{B}$ and $\check{D}$, since, due to large size of the learning period, the results are almost identical to $\hat{C}$ and $\hat{D}$, respectively. Additionally, for comparison, we present results for the true allocations a; these allocations were obtained by plugging-in true mean and covariance matrix into (2.14).

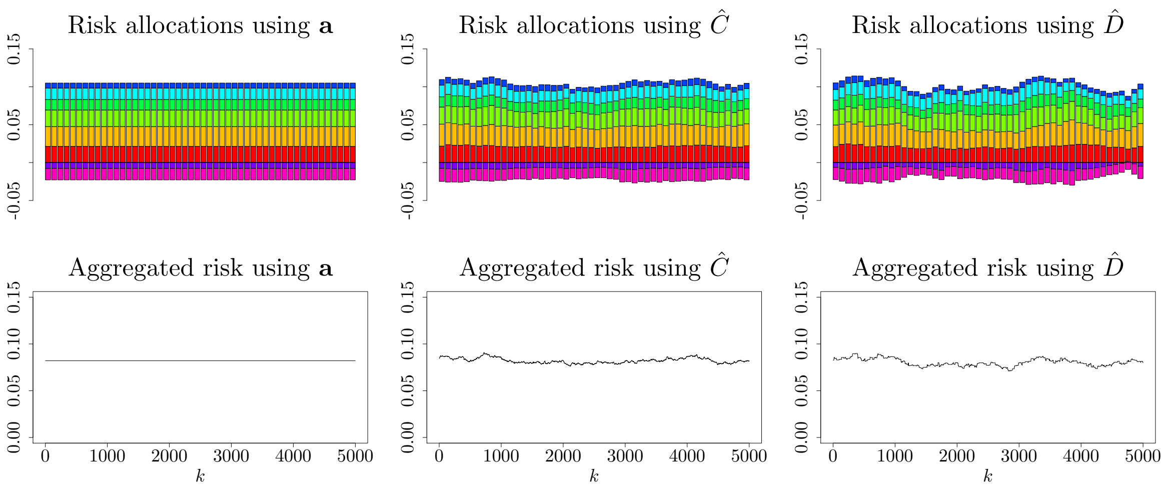

Figure 1: *Example 5.1. Top row: estimated risk allocations for portfolio constituents at each backtesting day, $k = 1, \dots, m$; the height of each colored horizontal layer represents the risk allocated to one of the constituents. Bottom row: estimated aggregated risk at each backtesting day.*

The obtained results validate, as expected, the proposed methods. In Figure 1 we present the values of the risk allocation to each constituent (top row), and the aggregated risk (bottom row). In this example, the fair risk allocation a computed with the true underlying distribution can be considered as reference for the backtesting results. The estimated risk allocations using $\hat{C}$ and $\hat{D}$ are close to the reference allocations, and as expected, the results computed using the non-parametric method $\hat{D}$ are not as close to the reference results as those obtained using $\hat{C}$ that explicitly explodes the Gaussian distribution structure of the data.

Table 1 contains the summary of the estimated backtesting measures $G_{0.05}, G^i_{0.05}, W^i$ and Υ. First, we note that the values of $G_{0.05}(a), G^i_{0.05}(a)$ and $W^i(a)$ corresponding to backtesting the fair allocation are, as expected, close to zero. In addition, $\Upsilon(a)$ is close to $\alpha = 0.05$. This indicates that the proposed backtesting methodologies are adequate. The obtained values give the benchmark for the following results produced by using $\hat{C}$ and $\hat{D}$. We note that indeed, the values of $G_{0.05}, G^i_{0.05}, W^i$ and Υ corresponding to $\hat{C}$ and $\hat{D}$ are in the same ballpark as for a, indicating that $\hat{C}$ and $\hat{D}$ are suitable risk allocation methodologies.

For convenience, we additionally present several graphical representations of the backtesting

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8		S
$G^i_{0.05}(a)$	-0.00038	0.00017	-0.00054	-0.00039	-0.00021	-0.00048	0.00076	-0.00011	$G_{0.05}(a)$	-0.00118
$W^i(a)$	-0.013	0.001	-0.002	-0.002	-0.003	-0.012	-0.017	0.001	$\Upsilon(a)$	0.047
$G^i_{0.05}(\hat{C})$	-0.00090	-0.00037	-0.00014	-0.00022	0.00043	-0.00058	0.00088	0.00008	$G_{0.05}(\hat{C})$	-0.00081
$W^i(\hat{C})$	-0.009	-0.004	-0.002	-0.002	0.004	-0.016	-0.013	-0.005	$\Upsilon(\hat{C})$	0.048
$G^i_{0.05}(\hat{D})$	0.00032	0.00110	0.00031	0.00014	-0.00003	0.00010	-0.00011	-0.00088	$G_{0.05}(\hat{D})$	0.00094
$W^i(\hat{D})$	0.003	0.005	0.002	0.005	-0.001	-0.001	0.002	0.008	$\Upsilon(\hat{D})$	0.053

Table 1: *Summary of the estimated backtesting measures for Example 5.1.*

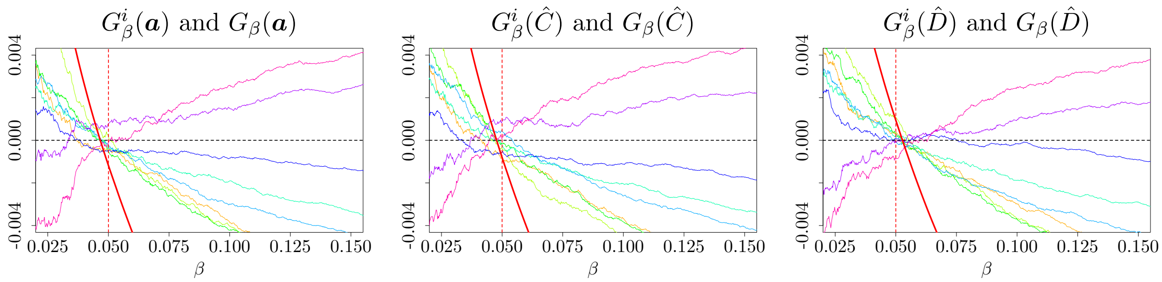

Figure 2: *Example 5.1. The bold red line represents G_β, while all other lines represent G^i_β,* $i = 1, 2, \dots, 8$.

metrics. In Figure 2 we plot G_β and G^i_β as functions of β, for the three risk allocation methods $a, \hat{C}, \hat{D}$. All these functions should take zero value around $\beta = \alpha = 0.05$, which is clearly the case. We also provide the individual values of G^i_α in Figure 3 (top row), and in Figure 3 (bottom row) we display the graphs of $G^i_\beta(\hat{D}), i = 1, \dots, 8$. Finally, Figure 4 is dedicated to risk level shift backtesting. The top row shows the values of W^i for risk allocations estimated using $a, \hat{C}$, and $\hat{D}$. The blue dots in the bottom graphs in Figure 4 depict the values of $\alpha \pm W$, all of them being close to the reference risk value $\alpha = 0.05$, which again indicates adequacy of risk allocation estimation procedure $\hat{D}$.

Example 5.2 (Student t-distributed P&Ls)**.** Similar to the previous example we consider a portfolio of eight constituents and with discounted P&L following a t-distribution with five degrees of freedom. For comparison reasons, the distribution of $(X^1, \dots, X^8)$ is modified so that it has the same mean and variance covariance structure as in Example 5.1.

First, note that there is no available counterpart of a for this setup. Second, as we will show below, since X does not follow a Gaussian distribution, one should not use $\hat{C}$ to estimate the risk allocation, and only $\hat{D}$ is an appropriate methodology in estimating risk allocation. In Figure 5, we present the estimated risk allocations computed using $\hat{C}$ and $\hat{D}$, over the entire backtesting period $k = 1, \dots, m$. It is apparent that the estimated risk allocation by these two methods are quite different. Table 2 contains the values of the estimated backtesting metrics, and for the reader's convenience $G^i_{0.05}$ and W^i are represented graphically in Figure 6. The values of $G^i_{0.05}(\hat{C})$ are of one order of magnitude further away from zero than $G^i_{0.05}(\hat{D})$, indicating that indeed risk allocation methodology $\hat{D}$ is more adequate for this experiment. We also note that magnitude of $G^i_{0.05}(\hat{D})$ in this example aligns with the benchmark values from Example 5.1. Similar arguments hold true for W^i and Υ.

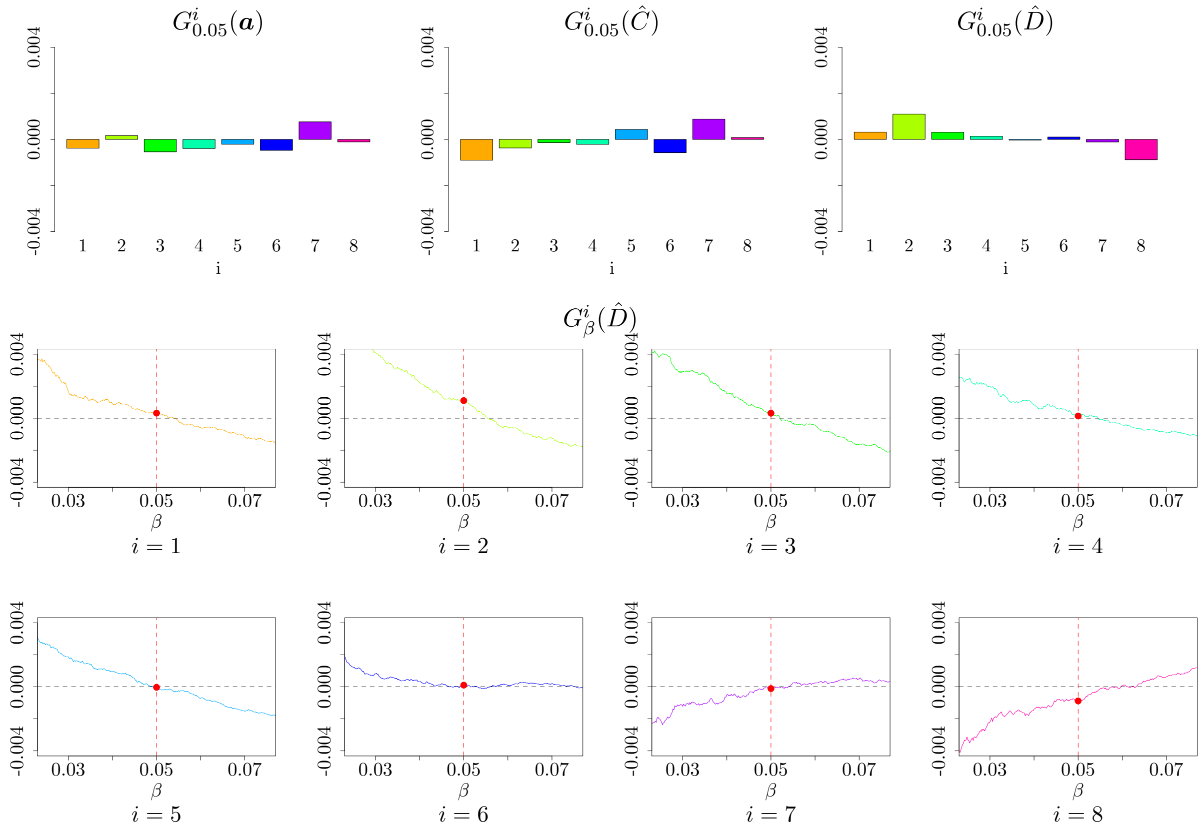

Figure 3: *Example 5.1. Graphical representation of the deviation from fairness backtesting method. The red dots in the bottom rows represent the values of $G^i_{0.05}$ using $\hat{D}$.*

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8		S
$G^i_{0.05}(\hat{C})$	0.00209	0.00363	0.00152	0.00127	0.00093	0.00088	-0.00084	-0.00076	$G_{0.05}(\hat{C})$	0.00872
$W^i(\hat{C})$	0.016	0.025	0.013	0.012	0.012	0.021	0.014	0.012	$\Upsilon(\hat{C})$	0.069
$G^i_{0.05}(\hat{D})$	0.00074	0.00062	-0.00030	0.00028	0.00049	0.00014	0.00001	0.00013	$G_{0.05}(\hat{D})$	0.00212
$W^i(\hat{D})$	0.004	0.003	-0.001	0.008	0.007	0.001	0.000	-0.001	$\Upsilon(\hat{D})$	0.054

Table 2: *Summary of the estimated backtesting measures for Example 5.2.*

Example 5.3 (Fairness and asymptotic fairness)**.** In this example we illustrate the fairness and the asymptotic fairness properties. Again, for the sake of a reference statistic which eases the presentation, we work under the normality assumption. Moreover, we consider only the first three constituents from Example 5.1, that is (X^1, X^2, X^3), because the other constituents show similar behavior. The numerical results presented below confirm that allocations a and $\hat{B}$ are fair. In addition, these results confirm that the allocations $\hat{C}$ and $\hat{D}$ are asymptotically fair even though they are not fair in this example.

Figures 7 and 8 deal with the issue of a short learning period, that is a small sample size, of $n = 250$. We see that for allocations a and $\hat{B}$ the $G^i_{0.05}$'s and W^i's are getting close to zero with increasing k, and that Υ gets close to 0.05 with increasing k, confirming that these are fair allocations. We also see that $G^i_{0.05}$'s and W^i's stay away from zero, and Υ stays away from 0.05

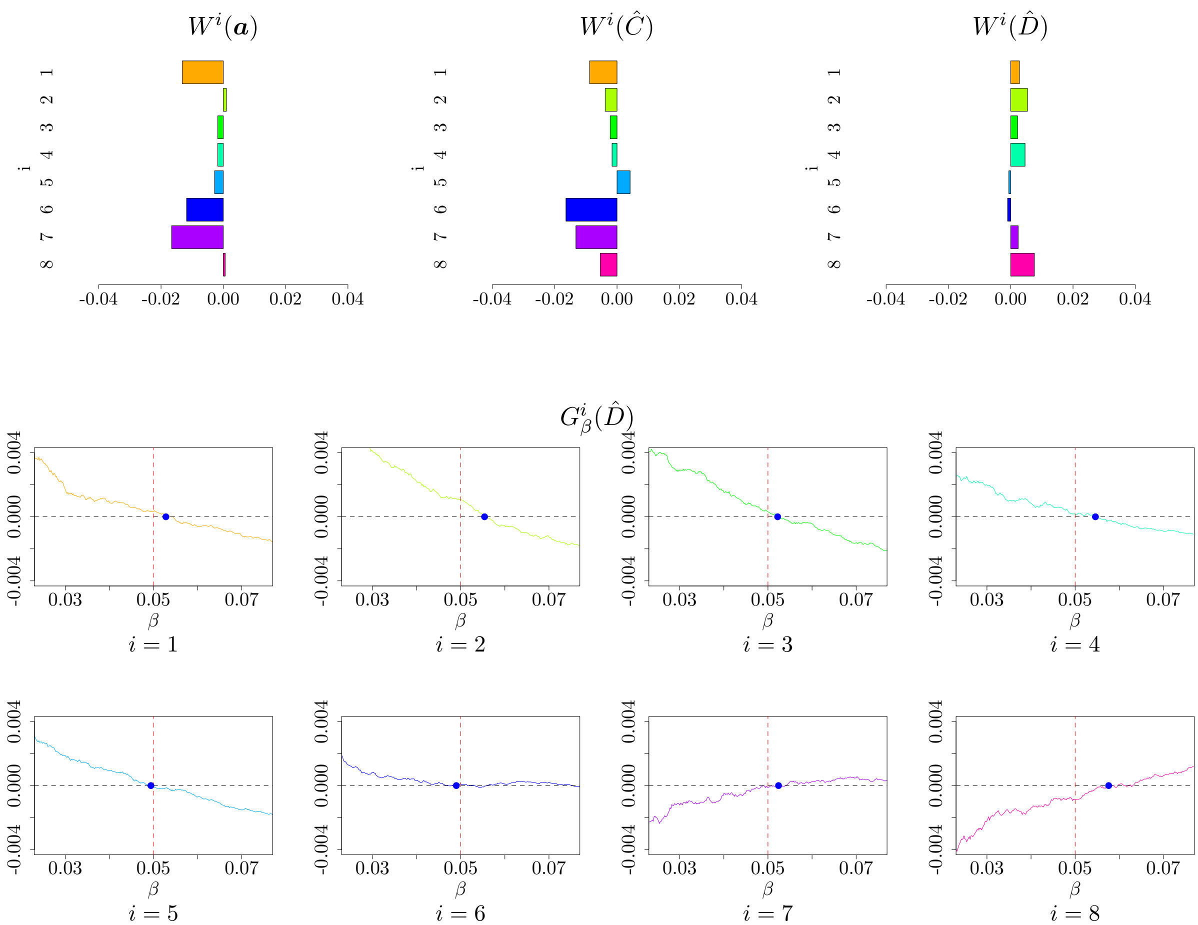

Figure 4: *Example 5.1. Graphical representation of the risk level shift backtesting method. The blue dots in the bottom rows represent the values of $W^i(\hat{D})$.*

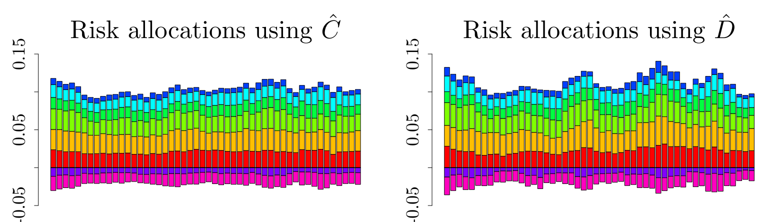

Figure 5: *Example 5.2. Estimated risk allocations for portfolio constituents at each backtesting day, $k = 1, \dots, m$; the height of each colored horizontal layer represents the risk allocated to one of the constituents.*

with increasing k for allocations $\hat{C}$ and $\hat{D}$, indicating that these are not fair allocations.

Figure 9 illustrates the asymptotic fairness of $\hat{D}^n$ with $n \to \infty$. The left panel shows that $G^i_{0.05}(\hat{D})$ get closer to zero for large k with increasing n. Similarly for the right panel, with regard to W^i and Υ.

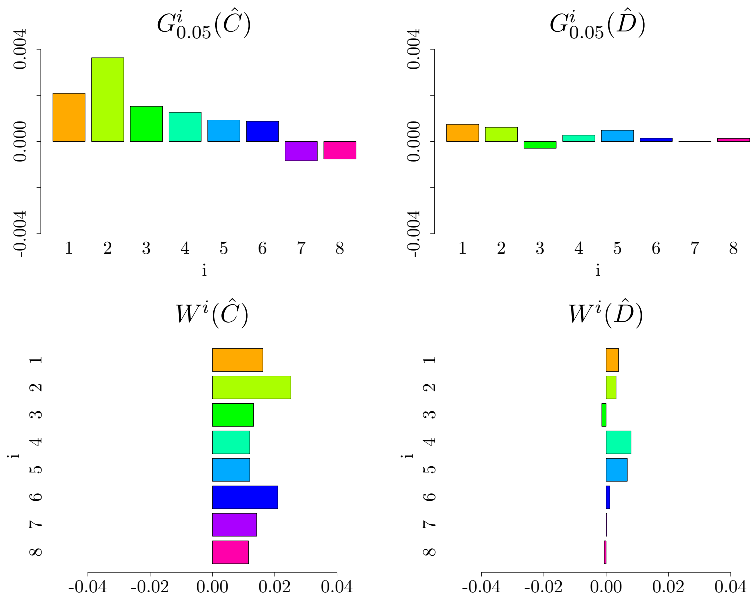

Figure 6: *Estimated backtesting measures for Example 5.2.*

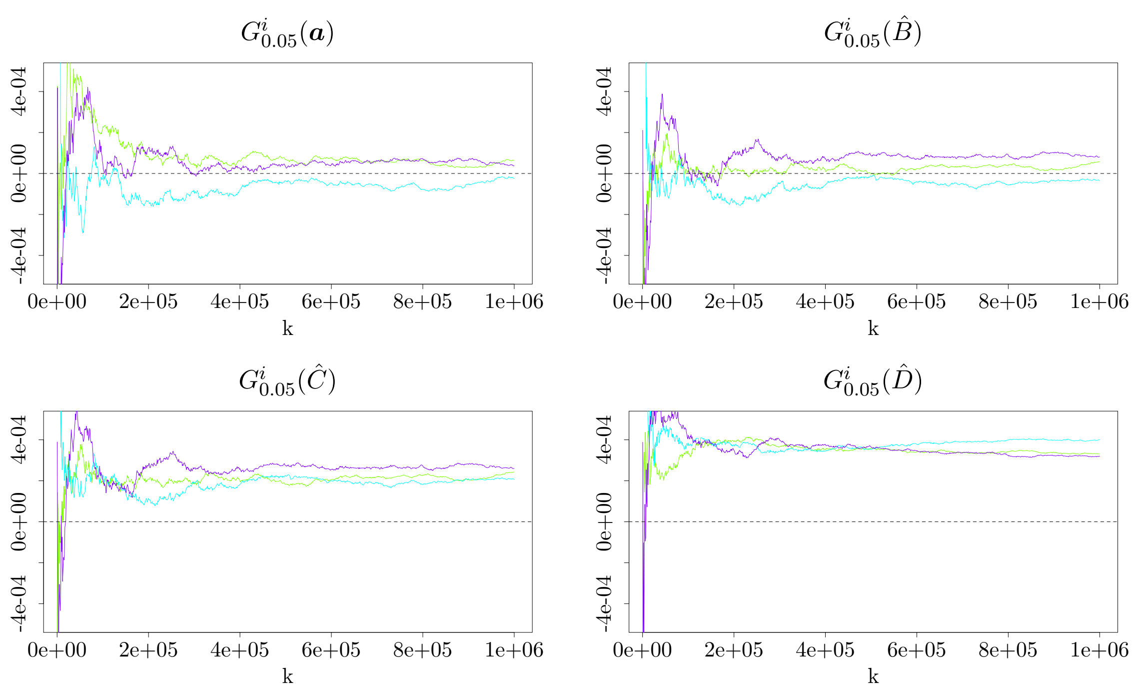

Figure 7: *Example 5.3. $G^i_{0.05}$ as function of backtesting day k, and for a fixed learning period* $n = 250$.

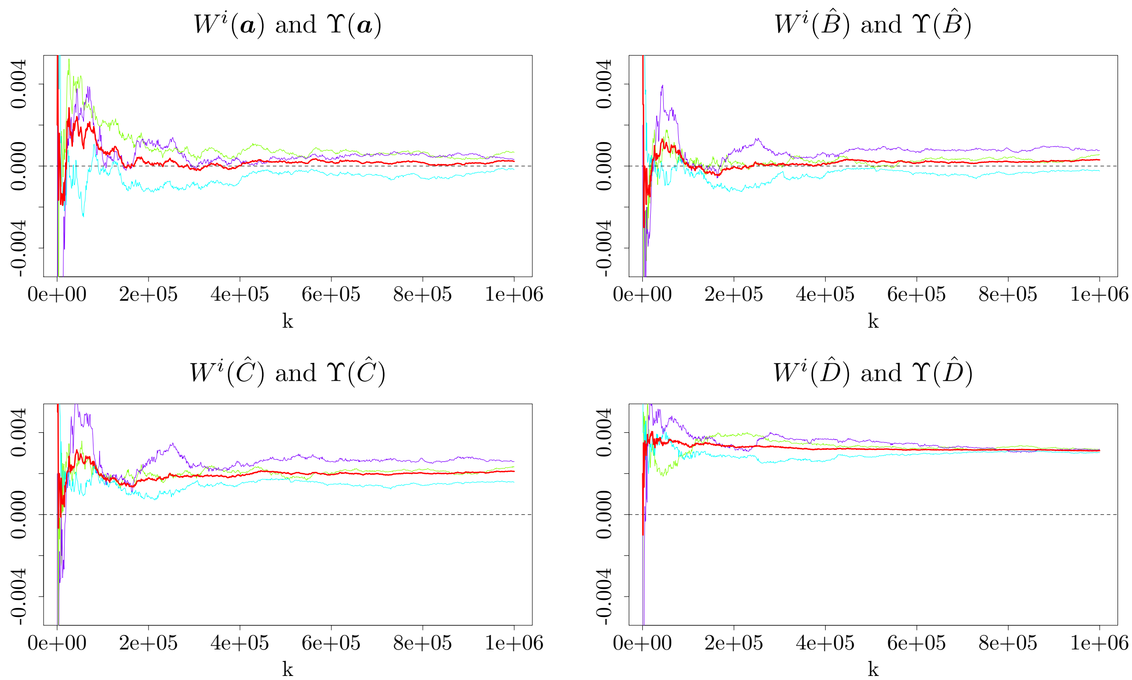

Figure 8: *Example 5.3. The red bold lines represent the values of* $\Upsilon - 0.05$*, while all other lines are represent* W^i*,* $i = 1, 2, 3$*, as functions of function of backtesting day* k*, and for a fixed learning period* $n = 250$*.*

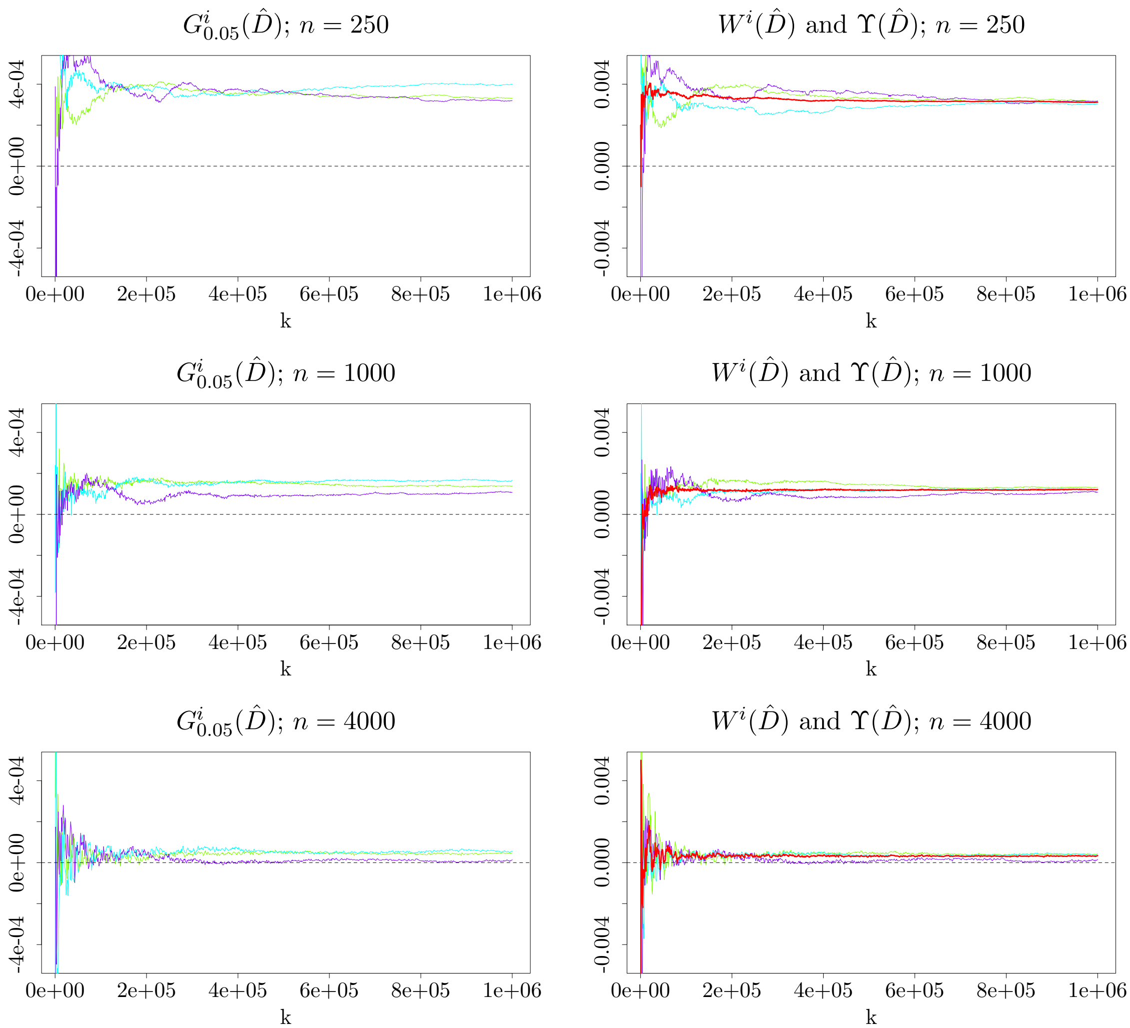

Figure 9: *Example 5.3.* $G^i_{0.05}(\hat{D})$ *as function of* k*, for different values of learning period* $n = 250$ *(top left figure),* $n = 1000$ *(left middle figure), and* $n = 4000$ *(left bottom figure). The pictures in the right column contain values of* $W^i(\hat{D})$ *and* $\Upsilon(\hat{D})$ *as functions of* k*, and for* $n = 250, 1000$ *and* 4000.

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