Wiener-Hopf factorization for time-inhomogeneous Markov chains and its application

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ABSTRACT: In this paper we derive the Wiener-Hopf factorization for a finite-state time-inhomogeneous Markov chain. To the best of our knowledge, this study is the first attempt to investigate the Wiener-Hopf factorization for time-inhomogeneous Markov chains. In this work we only deal with a special class of time-inhomogeneous Markovian generators, namely piece-wise constant, which allows to use an appropriately tailored randomization technique. Besides the mathematical importance of the Wiener-Hopf factorization methodology, there is also an important computational aspect: it allows for efficient computation of important functionals of Markov chains.

KEYWORDS: Wiener-Hopf factorization, inhomogeneous Markov chain, fluctuation theory, randomization method, additive functional.
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1 Introduction

In this paper we derive the Wiener-Hopf factorization (WHf) for a finite-state time-inhomogeneous Markov chain. As far as we know, our study is the first attempt to investigate the Wiener-Hopf factorization for time-inhomogeneous Markov chains. In this pioneering study we only deal with a special class of time-inhomogeneous Markovian generators, namely piece-wise constant. This allows to use an appropriately tailored randomization technique. Besides the mathematical importance of the WHf, there is an important computational aspect: this methodology allows for very efficient computation of important functionals of Markov chains.

The Wiener-Hopf factorization for finite-state Markov chains was originally derived in [BRW80] in the time-homogeneous case; see also [LMRW82] and [Wil91]. For the WHf in case of time-homogeneous Feller Markov processes we refer to [Wil08]. For some related applied work we refer to [APU03], which deals with the ruin problem, and to [Asm95, Rog94, RS94] that study fluid models. In addition, [KW90] studies the so called "noisy" Wiener-Hopf factorizations; for applications see [Asm95, Rog94, RS94, JR06, JP08, MP11, JP12, Hie14, HSZ16].

It needs to be stressed that even though the classical WHf of [BRW80] can be applied to the generator matrix, say G_t, of a time-inhomogeneous Markov chain X at every time t, these factorizations do not have any probabilistic meaning with regard to the process X. In particular, they are of no use for computing functionals such as (2.1)-(2.4) below. So, a relevant WHf for

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a time-inhomogeneous Markov chain requires a different approach than the one that would just directly apply the results of [BRW80] to each G_t, $t \geq 0$.

The paper is organized as follows. In Section 2 we provide a motivation and the setup up for our problem. In Section 3 we introduce a randomization method and we give the main results of the paper. Section 4 provides a numerical algorithm for computing our version of the WHf and its application in a specific example. Finally, we give some supporting results in the Appendix.

2 Motivation and problem set-up

Let $\mathbf{E}$ be a finite set, $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, and $X := (X_t)_{t\geq 0}$ be a *time-inhomogeneous* Markov chain on $(\Omega, \mathscr{F}, \mathbb{P})$ with state space $\mathbf{E}$ and generator function $\mathsf{G} = \{\mathsf{G}_t, t \geq 0\}$. In particular, each G_t is a $|\mathbf{E}| \times |\mathbf{E}|$ matrix. We assume that $\mathbb{P}\left(X_0 = i\right) > 0$ for each $i \in \mathbf{E}$ and we let $\mathbb{P}^i$ be the probability measure on $(\Omega, \mathscr{F})$ defined by

$$\mathbb{P}^i(A) := \mathbb{P}\left(A \,|\, X_0 = i\right), \quad A \in \mathscr{F},$$

with $\mathbb{E}^i$ denoting the associated expectation.

In this paper we assume that the generator G is piecewise constant, namely we assume that

$$\mathsf{G}_t = \begin{cases} \mathsf{G}_1, & \text{if } s_0 \leq t < s_1, \\ \mathsf{G}_2, & \text{if } s_1 \leq t < s_2, \\ \vdots & \\ \mathsf{G}_n, & \text{if } s_{n-1} \leq t < s_n, \\ \mathsf{G}_{n+1}, & \text{if } t \geq s_n, \end{cases}$$

for some $n \in \mathbb{N}$ and $0 = s_0 < s_1 < \ldots < s_n$. Without loss of generality we assume that $\mathsf{G}_1, \ldots, \mathsf{G}_{n+1}$ are not sub-Markovian. That is, the sums of row elements of G_k are all zero, for any $k = 1, \ldots, n+1$. The results of this paper carry over to the sub-Markovian case by the standard augmentation of the state space.

Next, we consider a function $v : \mathbf{E} \to \mathbb{R} \setminus \{0\}$ and we put

$$\mathbf{E}^+ := \{ i \in \mathbf{E} \,|\, v(i) > 0\} \quad \text{and} \quad \mathbf{E}^- := \{ i \in \mathbf{E} \,|\, v(i) < 0\}.$$

We also define the additive functional

$$\varphi_t := \int_0^t v(X_u)\, \mathrm{d}u, \quad t \geq 0,$$

and the first passage times

$$\tau_t^+ := \inf\{ r \geq 0 \,|\, \varphi_r > t\} \quad \text{and} \quad \tau_t^- := \inf\{ r \geq 0 \,|\, \varphi_r < -t\}.$$

The main goal of this paper is to apply the Wiener-Hopf factorization technique, which we work out in Section 3, to compute the following expectations,

$$\Pi_c^+(i, j; s_1, \ldots, s_n) := \mathbb{E}\left(e^{-c\tau_0^+} \mathbb{1}_{\{X_{\tau_0^+} = j\}} | X_0 = i \right), \quad i \in \mathbf{E}^-, \, j \in \mathbf{E}^+, \tag{2.1}$$

$$\Psi_c^+(\ell, i, j; s_1, \ldots, s_n) := \mathbb{E}\left(e^{-c\tau_\ell^+} \mathbb{1}_{\{X_{\tau_\ell^+} = j\}} | X_0 = i \right), \quad i \in \mathbf{E}^+, \, j \in \mathbf{E}^+, \, \ell > 0, \tag{2.2}$$

$$\Pi_c^-(i, j; s_1, \ldots, s_n) := \mathbb{E}\left(e^{-c\tau_0^-} \mathbb{1}_{\{X_{\tau_0^-} = j\}} | X_0 = i \right), \quad i \in \mathbf{E}^+, \, j \in \mathbf{E}^-, \tag{2.3}$$

$$\Psi_c^-(\ell, i, j; s_1, \ldots, s_n) := \mathbb{E}\left(e^{-c\tau_\ell^-} \mathbb{1}_{\{X_{\tau_\ell^-} = j\}} | X_0 = i \right), \quad i \in \mathbf{E}^-, \, j \in \mathbf{E}^-, \, \ell > 0. \tag{2.4}$$

We will focus on the computation of $\Pi_c^+(i,j;s_1,\ldots,s_n)$ and $\Psi_c^+(\ell,i,j;s_1,\ldots,s_n)$. By symmetry, analogous results can be obtained for $\Pi_c^-(i,j;s_1,\ldots,s_n)$ and $\Psi_c^-(\ell,i,j;s_1,\ldots,s_n)$. To simplify the notations, we will frequently write $\Pi_c^+(i,j)$ and $\Psi_c^+(\ell,i,j)$ in place of $\Pi_c^+(i,j;s_1,\ldots,s_n)$ and $\Psi_c^+(\ell,i,j;s_1,\ldots,s_n)$, respectively.

3 A randomization method and the Wiener-Hopf factorization

In this section we construct a *time-homogeneous* Markov chain associated to X, by randomizing the discontinuity times $s_1,\ldots,s_n$ of the generator G. This key construction will allow us to compute the expectations (2.1) and (2.2) using analogous expectations corresponding to this time-homogeneous chain. The latter expectations can be computed using Wiener-Hopf factorization theory of [BRW80].

Define $\mathbb{N}_n := \{0,\ldots,n\}$, $\widetilde{\mathbf{E}} := \mathbb{N}_n\times\mathbf{E}$ and let $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ be a complete probability space. Next, let us consider a *time-homogeneous* Markov chain, say $Z=(N,Y):=(N_t,Y_t)_{t\geq 0}$, defined on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$, taking values in $\widetilde{\mathbf{E}}$ and with generator matrix $\widetilde{\mathsf{G}}\left((n_1,j_1),(n_2,j_2)\right)_{(n_1,j_1),(n_2,j_2)\in\widetilde{\mathbf{E}}}$ given as

$$\widetilde{\mathsf{G}} = \begin{array}{c} \\ \{0\}\times\mathbf{E} \\ \{1\}\times\mathbf{E} \\ \vdots \\ \{n-1\}\times\mathbf{E} \\ \{n\}\times\mathbf{E} \end{array} \begin{array}{c} \begin{array}{ccccc} \{0\}\times\mathbf{E} & \{1\}\times\mathbf{E} & \cdots & \{n-1\}\times\mathbf{E} & \{n\}\times\mathbf{E} \end{array} \\ \begin{bmatrix} \mathsf{G}_1-q_1\mathsf{I} & q_1\mathsf{I} & \cdots & 0 & 0 \\ 0 & \mathsf{G}_2-q_2\mathsf{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathsf{G}_n-q_n\mathsf{I} & q_n\mathsf{I} \\ 0 & 0 & \cdots & 0 & \mathsf{G}_{n+1} \end{bmatrix} \end{array},$$

where $q_1,\ldots,q_n$ are positive constants and I is the identity matrix. For each $i\in\mathbf{E}$, we define the probability measure $\widetilde{\mathbb{P}}^i$ on $(\widetilde{\Omega},\widetilde{\mathcal{F}})$ by

$$\widetilde{\mathbb{P}}^i(A) := \widetilde{\mathbb{P}}\left(A\,|\,Z_0=(0,i)\right), \quad A\in\widetilde{\mathcal{F}}. \tag{3.1}$$

The next result regards the Markov property of process N.

Proposition 3.1. *For any $i\in\mathbf{E}$, the process N is a time-homogeneous Markov chain under $\widetilde{\mathbb{P}}^i$, with generator matrix given by*

$$\widetilde{\mathsf{G}}_N = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & \cdots & n-1 & n \end{array} \\ \begin{bmatrix} -q_1 & q_1 & \cdots & 0 & 0 \\ 0 & -q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -q_n & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \end{array}.$$

Proof. We will proceed in three steps.

Step 1. We start by showing that

$$\sum_{j_2\in\mathbf{E}}\left(\widetilde{\mathsf{G}}^k\right)\left((n_1,j_1),(n_2,j_2)\right) = \left(\widetilde{\mathsf{G}}_N^k\right)(n_1,n_2), \tag{3.2}$$

for any $j_1\in\mathbf{E}$, $k\in\mathbb{N}$, and $0\leq n_1,n_2\leq n$, In particular, note that the left-hand-side of (3.2) does not depend on j_1.

We will prove (3.2) by induction in k. Clearly (3.2) holds true for $k=1$. Next, assume that (3.2) holds for some $k=\ell\in\mathbb{N}$. Now, for $\ell+1$,

$$\begin{aligned}
\sum_{j_2\in\mathbf{E}}\left(\widetilde{\mathsf{G}}^{\ell+1}\right)((n_1,j_1),(n_2,j_2)) &= \sum_{j_2\in\mathbf{E}}\sum_{m=0}^{n}\sum_{j\in\mathbf{E}}\left(\widetilde{\mathsf{G}}^{\ell}\right)((n_1,j_1),(m,j))\,\widetilde{\mathsf{G}}\left((m,j),(n_2,j_2)\right)\\
&= \sum_{m=0}^{n}\sum_{j\in\mathbf{E}}\left(\widetilde{\mathsf{G}}^{\ell}\right)((n_1,j_1),(m,j))\sum_{j_2\in\mathbf{E}}\widetilde{\mathsf{G}}\left((m,j),(n_2,j_2)\right)\\
&= \sum_{m=0}^{n}\sum_{j\in\mathbf{E}}\left(\widetilde{\mathsf{G}}^{\ell}\right)((n_1,j_1),(m,j))\,\widetilde{\mathsf{G}}_N(m,n_2)\\
&= \sum_{m=0}^{n}\left(\widetilde{\mathsf{G}}_N^{\ell}\right)(n_1,m)\widetilde{\mathsf{G}}_N(m,n_2) = \left(\widetilde{\mathsf{G}}_N^{\ell+1}\right)(n_1,n_2),
\end{aligned}$$

where we used the inductive assumptions for $k=1$ and $k=\ell$ in the third and the fourth equalities, respectively. Hence (3.2) is established.

Step 2. We will show that

$$\widetilde{\mathbb{P}}^i\left(N_{t+s}=n_2\,|\,N_t=n_1\right)=\widetilde{\mathbb{P}}^i(N_{t+s}=n_2\,|\,N_t=n_1,Y_t=j)=e^{s\widetilde{\mathsf{G}}_N}(n_1,n_2), \tag{3.3}$$

for any $t,s\geq 0$, $j\in\mathbf{E}$, and $0\leq n_1\leq n_2\leq n$. In particular, note that the left-hand side of (3.3), and thus $\widetilde{\mathbb{P}}^i(N_{t+s}=n_2|N_t=n_1)$, does not depend on t. We start by checking the second equality in (3.3). For any $t,s\geq 0$, $j\in\mathbf{E}$, and $0\leq n_1\leq n_2\leq n$,

$$\begin{aligned}
\widetilde{\mathbb{P}}^i\left(N_{t+s}=n_2\,|\,N_t=n_1,Y_t=j\right) &= \sum_{k\in\mathbf{E}}\widetilde{\mathbb{P}}^i\left(N_{t+s}=n_2,Y_{t+s}=k\,|\,N_t=n_1,Y_t=j\right)\\
&= \sum_{k\in\mathbf{E}}e^{s\,\widetilde{\mathsf{G}}}((n_1,j),(n_2,k)) = \sum_{k\in\mathbf{E}}\sum_{\ell=0}^{\infty}\frac{s^\ell}{\ell!}\,\widetilde{\mathsf{G}}^\ell((n_1,j),(n_2,k))\\
&= \sum_{\ell=0}^{\infty}\frac{s^\ell}{\ell!}\sum_{k\in\mathbf{E}}\widetilde{\mathsf{G}}^\ell((n_1,j),(n_2,k))\\
&= \sum_{\ell=0}^{\infty}\frac{s^\ell}{\ell!}\,\widetilde{\mathsf{G}}_N^\ell(n_1,n_2) = e^{s\widetilde{\mathsf{G}}_N}(n_1,n_2),
\end{aligned}$$

where we used the result of Step 1 in the last two equalities. In particular, $\widetilde{\mathbb{P}}^i(N_{t+s}=n_2|N_t=n_1,Y_t=j)$ does not depend on the choice of $j\in\mathbf{E}$.

As far as the first equality in (3.2), for any $t,s\geq 0$ and $0\leq n_1\leq n_2\leq n$,

$$\begin{aligned}
\widetilde{\mathbb{P}}^i\left(N_{t+s}=n_2\,|\,N_t=n_1\right) &= \frac{\widetilde{\mathbb{P}}^i\left(N_{t+s}=n_2,N_t=n_1\right)}{\widetilde{\mathbb{P}}^i\left(N_t=n_1\right)} = \frac{\sum_{\ell\in\mathbf{E}}\widetilde{\mathbb{P}}^i\left(N_{t+s}=n_2,N_t=n_1,Y_t=j\right)}{\sum_{j\in\mathbf{E}}\widetilde{\mathbb{P}}^i\left(N_t=n_1,Y_t=j\right)}\\
&= \frac{\sum_{j\in\mathbf{E}}\widetilde{\mathbb{P}}^i\left(N_{t+s}=n_2\,|\,N_t=n_1,Y_t=j\right)\widetilde{\mathbb{P}}^i\left(N_t=n_1,Y_t=j\right)}{\sum_{j\in\mathbf{E}}\widetilde{\mathbb{P}}^i\left(N_t=n_1,Y_t=j\right)}\\
&= \frac{\sum_{j\in\mathbf{E}}\widetilde{\mathbb{P}}^i\left(N_t=n_1,Y_t=j\right)}{\sum_{j\in\mathbf{E}}\widetilde{\mathbb{P}}^i\left(N_t=n_1,Y_t=j\right)}\,e^{s\widetilde{\mathsf{G}}_N}(n_1,n_2) = e^{s\widetilde{\mathsf{G}}_N}(n_1,n_2).
\end{aligned}$$

Step 3. We are ready to complete the proof of the proposition. Towards this end we observe that, for any $m \in \mathbb{N}$, $0 = t_0 \le t_1 < \ldots < t_m$, and any $0 \le n_1 \le \ldots \le n_m \le n$,

$$
\begin{aligned}
\widetilde{\mathbb{P}}^i(N_{t_m} = n_m \mid N_{t_{m-1}} = n_{m-1}, \ldots, N_{t_1} = n_1) &= \frac{\widetilde{\mathbb{P}}^i\left(N_{t_1} = n_1, \ldots, N_{t_m} = n_m\right)}{\widetilde{\mathbb{P}}^i\left(N_{t_1} = n_1, \ldots, N_{t_{m-1}} = n_{m-1}\right)} \\
&= \frac{\sum_{j_1,\ldots,j_m \in \mathbf{E}} \widetilde{\mathbb{P}}^i\left(N_{t_1} = n_1, Y_{t_1} = j_1; \ldots; N_{t_m} = n_m, Y_{t_m} = j_m\right)}{\sum_{j_1,\ldots,j_{m-1} \in \mathbf{E}} \widetilde{\mathbb{P}}^i\left(N_{t_1} = n_1, Y_{t_1} = j_1; \ldots; N_{t_{m-1}} = n_{m-1}, Y_{t_m} = j_{m-1}\right)} \\
&= \frac{\sum_{j_1,\ldots,j_m \in \mathbf{E}} \prod_{k=1}^{m} \widetilde{\mathbb{P}}^i\left(N_{t_k} = n_k, Y_{t_k} = j_k \,|\, N_{t_{k-1}} = n_{k-1}, Y_{t_{k-1}} = j_{k-1}\right)}{\sum_{j_1,\ldots,j_{m-1} \in \mathbf{E}} \prod_{k=1}^{m-1} \widetilde{\mathbb{P}}^i\left(N_{t_k} = n_k, Y_{t_k} = j_k \,|\, N_{t_{k-1}} = n_{k-1}, Y_{t_{k-1}} = j_{k-1}\right)} \\
&= \sum_{j_m \in \mathbf{E}} \widetilde{\mathbb{P}}^i\left(N_{t_m} = n_m, Y_{t_m} = j_m \,|\, N_{t_{m-1}} = n_{m-1}, Y_{t_{m-1}} = j_{m-1}\right) \\
&= \widetilde{\mathbb{P}}^i\left(N_{t_m} = n_m \,|\, N_{t_{m-1}} = n_{m-1}, Y_{t_{m-1}} = j_{m-1}\right) \\
&= \widetilde{\mathbb{P}}^i\left(N_{t_m} = n_m \,|\, N_{t_{m-1}} = n_{m-1}\right) = e^{(t_m - t_{m-1})\widetilde{\mathsf{G}}_N}(n_{m-1}, n_m),
\end{aligned}
$$

where we used the Markov property of $Z = (N, Y)$ under $\widetilde{\mathbb{P}}^i$ in the third equality, and the result of Step 2 in the last two equalities. The proof is complete. $\square$

Let $\widetilde{\mathbb{F}}^Y = (\widetilde{\mathcal{F}}^Y_t)_{t \ge 0}$ be the filtration generated by Y, and let $\widetilde{\mathcal{F}}^Y_\infty = \sigma(\bigcup_{t \ge 0} \widetilde{\mathcal{F}}^Y_t)$. For each $i \in \mathbf{E}$, we will construct a probability measure $\overline{\mathbb{P}}^i$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}^Y_\infty)$ such that, the law of Y under $\overline{\mathbb{P}}^i$ is the same as the law of X under $\mathbb{P}^i$. Moreover, we will establish a connection between $\overline{\mathbb{P}}^i$ and $\widetilde{\mathbb{P}}^i$. For this purpose, we first let

$$S_k := \inf\left\{t \ge 0 \,|\, N_t = k\right\}, \quad k = 1, \ldots, n.$$

We will now derive the joint density of N, and $(S_1, \ldots, S_n)$ under $\widetilde{\mathbb{P}}^i$. For that, we set

$$T_1 := S_1, \quad T_k := S_k - S_{k-1}, \quad k = 2, \ldots, n. \tag{3.4}$$

It is shown in [Sys92, Section 1.1.4] that T_k's are independent and that

$$\widetilde{\mathbb{P}}^i\left(T_1 > t_1, \ldots, T_n > t_n\right) = \prod_{k=1}^{n} e^{-q_k t_k}, \quad t_1, \ldots, t_n > 0,$$

which implies that the joint density of $(T_1, \ldots, T_n)$ is given by

$$f_{T_1,\ldots,T_n}(t_1, \ldots, t_n) = \prod_{k=1}^{n} q_k\, e^{-q_k t_k}, \quad t_1, \ldots, t_n > 0. \tag{3.5}$$

Combining (3.4) and (3.5), we deduce that

$$f_{S_1,\ldots,S_n}(s_1, \ldots, s_n) = \prod_{k=1}^{n} q_k\, e^{-q_k(s_k - s_{k-1})}, \quad 0 = s_0 < s_1 < \ldots < s_n.$$

Theorem 3.2. *For any $i \in \mathbf{E}$, any $0 < s_1 < \ldots < s_n$, and any cylinder set $A \in \widetilde{\mathcal{F}}^Y_\infty$ of the form*

$$A = \left\{(Y_{u_1}, \ldots, Y_{u_m}) \in B\right\}, \quad 0 \le u_1 < u_2 < \ldots < u_m, \quad B \subseteq \mathbf{E}^m, \quad m \in \mathbb{N},$$

the limit

$$\overline{\mathbb{P}}^i\left(A; s_1, \ldots, s_n\right) := \lim_{\Delta s_k \to 0,\, k=1,\ldots,n} \frac{\widetilde{\mathbb{P}}^i\left(A,\, s_k < S_k \le s_k + \Delta s_k,\, k = 1, \ldots, n\right)}{\widetilde{\mathbb{P}}^i\left(s_k < S_k \le s_k + \Delta s_k,\, k = 1, \ldots, n\right)}, \tag{3.6}$$

exists, and can be extended to a probability measure $\overline{\mathbb{P}}^i(\cdot\,; s_1, \ldots, s_n)$ *on* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}^Y_\infty)$. *Moreover, for any* $A \in \widetilde{\mathcal{F}}^Y_\infty$, *the function* $\overline{\mathbb{P}}^i(A;\, \ldots)$ *is Borel measurable on* $\{(s_1, \ldots, s_n) \in \mathbb{R}^n \,|\, 0 < s_1 < \ldots < s_n\}$, *and*

$$\widetilde{\mathbb{P}}^i(A) = \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{P}}^i(A; s_1, \ldots, s_n) \prod_{k=1}^n \left(q_k\, e^{-q_k(s_k - s_{k-1})}\right) ds_n \cdots ds_2\, ds_1. \tag{3.7}$$

In the proof of the theorem we will use the following lemma.

Lemma 3.3. *Let us fix* $i \in \mathbf{E}$, $0 < s_1 < \ldots < s_n$, *and let* $0 = k_0 < k_1 < \ldots < k_{n+1}$ *be positive integers. In addition, let* $0 = u_0 < u_1 < \ldots < u_{k_1} \le s_1 < u_{k_1+1} < \ldots < u_{k_2} \le s_2 < \ldots \le s_n < u_{k_n+1} < \ldots < u_{k_{n+1}}$, $i_0 = i$ *and* $i_1, \ldots, i_{k_{n+1}} \in \mathbf{E}$. *Then, for any cylinder set* $A \in \widetilde{\mathcal{F}}^Y_\infty$ *of the form*

$$A = \bigcap_{j=0}^n \left\{ Y_{u_{k_j+1}} = i_{k_j+1}, \ldots, Y_{u_{k_{j+1}}} = i_{k_{j+1}} \right\} \tag{3.8}$$

we have

$$\begin{aligned} \lim_{\Delta s_\ell \to 0,\, \ell=1,\ldots,n} \frac{\widetilde{\mathbb{P}}^i\left(A,\, s_\ell < S_\ell \le s_\ell + \Delta s_\ell,\, \ell = 1, \ldots, n\right)}{\widetilde{\mathbb{P}}^i\left(s_\ell < S_\ell \le s_\ell + \Delta s_\ell,\, \ell = 1, \ldots, n\right)} &= \prod_{\ell=0}^n \left(\prod_{m=k_\ell+1}^{k_{\ell+1}} e^{(u_m - u_{m-1})\mathsf{G}_\ell}(i_{m-1}, i_m) \right) \\ &\quad \cdot \sum_{j_1, \ldots, j_n \in \mathbf{E}} \prod_{\ell=1}^n e^{(s_\ell - u_{k_\ell})\mathsf{G}_{\ell-1}}(i_{k_\ell}, j_\ell) e^{(u_{k_\ell+1} - s_\ell)\mathsf{G}_\ell}(j_\ell, i_{k_\ell+1}). \end{aligned} \tag{3.9}$$

In particular, for any $A \in \widetilde{\mathcal{F}}^Y_\infty$ *of the form* (3.8), *the above limit is Borel measurable with respect to* $(s_1, \ldots, s_n)$ *in* $\Delta_n := \{(s_1, \ldots, s_n) \in \mathbb{R}^n \,|\, 0 < s_1 < \ldots < s_n\}$.

Proof. For $\ell = 1, \ldots, n$ choose $\Delta s_\ell > 0$ so that, $s_\ell + \Delta s_\ell \le u_{k_\ell+1}$. Then,

$$\begin{aligned} &\widetilde{\mathbb{P}}^i\left(A,\, s_\ell < S_\ell \le s_\ell + \Delta s_\ell,\, \ell = 1, \ldots, n\right) \\ &\quad = \widetilde{\mathbb{P}}^i\left(Y_{u_{k_\ell+1}} = i_{k_\ell+1}, \ldots, Y_{u_{k_{\ell+1}}} = i_{k_{\ell+1}},\, \ell = 0, \ldots, n;\, N_{s_\ell} = \ell - 1, N_{s_\ell + \Delta s_\ell} = \ell,\, \ell = 1, \ldots, n\right) \\ &\quad = \sum_{j_1, \ldots, j_n, j'_1, \ldots, j'_n \in \mathbf{E}} \widetilde{\mathbb{P}}^i\Big(Z_{u_{k_\ell+1}} = (\ell, i_{k_\ell+1}),\, \ldots,\, Z_{u_{k_{\ell+1}}} = (\ell, i_{k_{\ell+1}}),\, \ell = 0, \ldots, n; \\ &\qquad\qquad Z_{s_\ell} = (\ell - 1, j_\ell),\, Z_{s_\ell + \Delta s_\ell} = (\ell, j'_\ell),\; \ell = 1, \ldots, n\Big) \\ &\quad = \sum_{j_1, \ldots, j_n, j'_1, \ldots, j'_n \in \mathbf{E}} \left[\prod_{\ell=0}^n \left(\prod_{m=k_\ell+1}^{k_{\ell+1}} e^{(u_m - u_{m-1})\widetilde{\mathsf{G}}}((\ell, i_{m-1}), (\ell, i_m)) \right) \right] \left(\prod_{\ell=1}^n e^{\Delta s_\ell \widetilde{\mathsf{G}}}\big((\ell - 1, j_\ell), (\ell, j'_\ell)\big) \right) \\ &\qquad \cdot \left(\prod_{\ell=1}^n e^{(s_\ell - u_{k_\ell})\widetilde{\mathsf{G}}}\left((\ell - 1, i_{k_\ell}), (\ell - 1, j_\ell)\right) e^{(u_{k_\ell+1} - s_\ell - \Delta s_\ell)\widetilde{\mathsf{G}}}\big((\ell, j'_\ell), (\ell, i_{k_\ell+1})\big) \right). \end{aligned}$$

In the above summation, the first product in the brackets provides the transition probabilities of the evolutions of Z between the times u_{k_ℓ} and $u_{k_{\ell+1}}$, $\ell = 0, \ldots, n$, the second product gives the transition probabilities of the evolutions of Z between the times s_ℓ and $s_\ell + \Delta s_\ell$, for each $\ell = 1, \dots, n$, and the third product denotes the transition probabilities of the evolutions of Z between the times u_{k_ℓ} and s_ℓ, and between the times $s_\ell + \Delta s_\ell$ and $u_{k_\ell+1}$, for each $\ell = 1, \dots, n$.

Next, for each $\ell = 1, \dots, n$,

$$\lim_{\Delta s_\ell \to 0} \frac{1}{\Delta s_\ell} e^{\Delta s_\ell \widetilde{\mathsf{G}}} \left((\ell-1, j_\ell), (\ell, j'_\ell)\right) = \widetilde{\mathsf{G}} \left((\ell-1, j_\ell), (\ell, j'_\ell)\right) = \begin{cases} q_\ell, & \text{if } j_\ell = j'_\ell, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned}
&\lim_{\Delta s_\ell \to 0,\, \ell=1,\dots,n} \frac{1}{\Delta s_1 \cdots \Delta s_n} \widetilde{\mathbb{P}}^i \left(A,\, s_\ell < S_\ell \le s_\ell + \Delta s_\ell,\, \ell = 1, \dots, n\right) \\
&= \prod_{\ell=0}^{n} \left(\prod_{m=k_\ell+1}^{k_{\ell+1}} e^{(u_m - u_{m-1})\widetilde{\mathsf{G}}} ((\ell, i_{m-1}), (\ell, i_m)) \right) \\
&\quad \cdot \sum_{j_1,\dots,j_n \in \mathbf{E}} \prod_{\ell=1}^{n} \Big(q_\ell\, e^{(s_\ell - u_{k_\ell})\widetilde{\mathsf{G}}} \left((\ell-1, i_{k_\ell}), (\ell-1, j_\ell)\right) e^{(u_{k_\ell+1} - s_\ell)\widetilde{\mathsf{G}}} \left((\ell, j_\ell), (\ell, i_{k_\ell+1})\right) \Big).
\end{aligned} \tag{3.10}$$

Note that, for any $j_1, j_2 \in \mathbf{E}$, and any $k \in \mathbb{N}$,

$$\begin{aligned}
\widetilde{\mathsf{G}}^k \left((\ell, j_1), (\ell, j_2)\right) &= (\mathsf{G}_\ell - q_{\ell+1}\mathsf{I})^k (j_1, j_2), \quad \ell = 0, \dots, n-1, \\
\widetilde{\mathsf{G}}^k \left((n, j_1), (n, j_2)\right) &= \mathsf{G}_n^k (j_1, j_2),
\end{aligned}$$

so that, for $t \ge 0$, we have

$$\begin{aligned}
e^{t\,\widetilde{\mathsf{G}}} \left((\ell, j_1), (\ell, j_2)\right) &= e^{t\,(\mathsf{G}_\ell - q_{\ell+1}\mathsf{I})} (j_1, j_2) = e^{-q_{\ell+1} t}\, e^{t\,\mathsf{G}_\ell} (j_1, j_2), \quad \ell = 0, \dots, n-1, \\
e^{t\,\widetilde{\mathsf{G}}} \left((n, j_1), (n, j_2)\right) &= e^{t\,\mathsf{G}_n} (j_1, j_2).
\end{aligned}$$

This, together with (3.10), implies that

$$\begin{aligned}
&\lim_{\Delta s_\ell \to 0,\, \ell=1,\dots,n} \frac{1}{\Delta s_1 \cdots \Delta s_n} \widetilde{\mathbb{P}}^i \left(A,\, s_\ell < S_\ell \le s_\ell + \Delta s_\ell,\, \ell = 1, \dots, n\right) \\
&= e^{-\sum_{\ell=1}^n q_\ell (u_{k_\ell} - u_{k_{\ell-1}})} \cdot \prod_{\ell=0}^{n} \left(\prod_{m=k_\ell+1}^{k_{\ell+1}} e^{(u_m - u_{m-1})\mathsf{G}_\ell} (i_{m-1}, i_m) \right) \\
&\quad e^{-\sum_{\ell=1}^n q_\ell (s_\ell - u_{k_\ell})} e^{-\sum_{\ell=1}^{n-1} q_\ell (u_{k_\ell+1} - s_\ell)} \sum_{j_1,\dots,j_n \in \mathbf{E}} \prod_{\ell=1}^{n} \Big(q_\ell e^{(s_\ell - u_{k_\ell})\mathsf{G}_{\ell-1}} (i_{k_\ell}, j_\ell) e^{(u_{k_\ell+1} - s_\ell)\mathsf{G}_\ell} (j_\ell, i_{k_\ell+1}) \Big) \\
&= e^{-\sum_{\ell=1}^n q_\ell (s_\ell - s_{\ell-1})} \cdot \prod_{\ell=0}^{n} \left(\prod_{m=k_\ell+1}^{k_{\ell+1}} e^{(u_m - u_{m-1})\mathsf{G}_\ell} (i_{m-1}, i_m) \right) \\
&\quad \cdot \sum_{j_1,\dots,j_n \in \mathbf{E}} \prod_{\ell=1}^{n} \Big(q_\ell\, e^{(s_\ell - u_{k_\ell})\mathsf{G}_{\ell-1}} (i_{k_\ell}, j_\ell)\, e^{(u_{k_\ell+1} - s_\ell)\mathsf{G}_\ell} (j_\ell, i_{k_\ell+1}) \Big) \\
&= \left(\prod_{\ell=1}^{n} q_\ell\, e^{-q_\ell (s_\ell - s_{\ell-1})} \right) \cdot \left[\prod_{\ell=0}^{n} \left(\prod_{m=k_\ell+1}^{k_{\ell+1}} e^{(u_m - u_{m-1})\mathsf{G}_\ell} (i_{m-1}, i_m) \right) \right] \\
&\quad \cdot \sum_{j_1,\dots,j_n \in \mathbf{E}} \prod_{\ell=1}^{n} \Big(e^{(s_\ell - u_{k_\ell})\mathsf{G}_{\ell-1}} (i_{k_\ell}, j_\ell)\, e^{(u_{k_\ell+1} - s_\ell)\mathsf{G}_\ell} (j_\ell, i_{k_\ell+1}) \Big).
\end{aligned}$$

Finally, in view of the above and the fact that

$$\lim_{\Delta s_\ell \to 0,\, \ell=1,\ldots,n} \frac{1}{\Delta s_1 \cdots \Delta s_n} \widetilde{\mathbb{P}}^i \left(s_\ell < S_\ell \le s_\ell + \Delta s_\ell,\ \ell = 1,\ldots,n \right) = \prod_{\ell=1}^{n} q_\ell \, e^{-q_\ell (s_\ell - s_{\ell-1})}, \tag{3.11}$$

we obtain (3.9). The proof is complete. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let $\mathcal{C}$ be the collection of all cylinder sets in $\widetilde{\mathcal{F}}^Y_\infty$ of the form

$$C = \{(Y_{u_1}, \ldots, Y_{u_m}) \in B\}, \quad 0 \le u_1 < u_2 < \ldots < u_m, \quad B \subseteq \mathbf{E}^m, \quad m \in \mathbb{N}.$$

Clearly, $\mathcal{C}$ is an algebra.

We first show that for any $C \in \mathcal{C}$ the limit in (3.6) exists and that an explicit formula for it can be derived. In fact, Lemma 3.3 shows that the limit in (3.6) exists, and belongs to $[0,1]$, for all the cylinder sets of the form (3.8). Thus, for a cylinder set $C \in \mathcal{C}$ an explicit formula for the limit on the right-hand side of (3.6) can be obtained as follows. First, we refine the partition $0 \le u_1 < u_2 < \ldots < u_m$ so that each subinterval of the partition $0 < s_1 < \ldots < s_n$ contains at least one of the u_i's. Clearly, since B_m is finite, A can be decomposed into a finite union of disjoint cylinder sets of the form (3.8) on the refined partition. Moreover, (3.9) provides an explicit formula for the limit in (3.6) for each of those cylinder sets of the form (3.8) on the refined partition. Finally, taking the finite sum over all those limits, we obtain the limit in (3.6) for C. In particular, for every cylinder set C, the limit in (3.6) is Borel measurable with respect to $(s_1, \ldots, s_n)$ in Δ_n.

In the second step we will demonstrate that the limit in (3.6) can be extended to a probability measure on $\sigma(\mathcal{C}) = \widetilde{\mathcal{F}}^Y_\infty$. We start from verifying the countable additivity of $\overline{\mathbb{P}}^i(\cdot\,; s_1, \ldots, s_n)$ on $\mathcal{C}$ for any fixed $0 < s_1 < \ldots < s_n$.

Since $\mathbf{E}$ is a finite set, if $(C_k)_{k\in\mathbb{N}}$ is a sequence of disjoint cylinder sets in $\mathcal{C}$ such that their union also belongs to $\mathcal{C}$, then only finite many of them are non-empty. Therefore, it suffices to verify the finite additivity of $\overline{\mathbb{P}}^i(\cdot\,; s_1, \ldots, s_n)$ on $\mathcal{C}$. Let $C_1, \ldots, C_k \in \mathcal{C}$ be disjoint cylinder sets, then there exists $m \in \mathbb{N}$ and $0 \le u_1 < u_2 < \ldots < u_m$, such that

$$C_\ell = \{(Y_{u_1}, \ldots, Y_{u_m}) \in B_\ell\} \quad \text{for some } \ B_\ell \subseteq \mathbf{E}^m, \quad \ell = 1, \ldots, k.$$

Each $\overline{\mathbb{P}}^i(C_\ell\,; s_1, \ldots, s_n)$ can be represented as

$$\overline{\mathbb{P}}^i(C_\ell\,; s_1, \ldots, s_n) = \sum_{A_\ell \in \mathcal{C}_\ell} \overline{\mathbb{P}}^i(A_\ell\,; s_1, \ldots, s_n), \quad j = 1, \ldots, k,$$

where $\mathcal{C}_\ell$, $\ell = 1, \ldots, k$, are disjoint classes of disjoint simple cylinder sets. Therefore, we have

$$\begin{aligned}
\sum_{\ell=1}^{k} \overline{\mathbb{P}}^i(C_\ell\,; s_1, \ldots, s_n) &= \sum_{\ell=1}^{k} \sum_{A_\ell \in \mathcal{C}_\ell} \overline{\mathbb{P}}^i(A_\ell\,; s_1, \ldots, s_n) \\
&= \sum_{A \in \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k} \overline{\mathbb{P}}^i(A\,; s_1, \ldots, s_n) = \overline{\mathbb{P}}^i\left(\bigcup_{\ell=1}^{k} C_\ell\,; s_1, \ldots, s_n \right).
\end{aligned}$$

Note that for any $0 < s_1 < \ldots < s_n$, $\overline{\mathbb{P}}^i(C\,; s_1, \ldots, s_n) \le 1$ for all $C \in \mathcal{C}$. By the Carathéodory extension theorem, for any $0 < s_1 < \ldots < s_n$, $\overline{\mathbb{P}}^i(\cdot\,; s_1, \ldots, s_n)$ can be uniquely extended to a probability measure on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}^Y_\infty)$.

Let $\Delta_n := \{(s_1, \ldots, s_n) \in \mathbb{R}^n \,|\, 0 < s_1 < \ldots < s_n\}$ and

$$\mathcal{D}_1 := \left\{ A \in \widetilde{\mathcal{F}}^Y_\infty \,\middle|\, \overline{\mathbb{P}}^i(A\,;\,\cdot,\cdots,\cdot) \text{ is Borel measurable on } \Delta_n \right\}.$$

We will show that $\mathcal{D}_1 = \widetilde{\mathcal{F}}^Y_\infty$. Towards this end, we first observe that (3.6) and (3.9) imply that, for any $A \in \mathcal{C}$, $\overline{\mathbb{P}}^i(A\,;\,\cdot,\cdots,\cdot)$ is Borel measurable with respect to $(s_1, \ldots, s_n)$ on Δ_n, and thus $\mathcal{D}_1 \supset \mathcal{C}$.

Next, we will show that $\mathcal{D}_1$ is a monotone class. For this, let $(A_k)_{k\in\mathbb{N}} \subset \mathcal{D}_1$ be an increasing sequence of events, so that, for any $0 < s_1 < \ldots < s_n$, we have

$$\overline{\mathbb{P}}^i\left(\bigcup_{k=1}^\infty A_k\,;\, s_1, \ldots, s_n\right) = \lim_{m\to\infty} \overline{\mathbb{P}}^i\left(A_m\,;\, s_1, \ldots, s_n\right).$$

Thus, $\overline{\mathbb{P}}^i(\cup_k A_k\,;\cdot,\cdots,\cdot)$, being a limit of a sequence of Borel measurable functions on Δ_n, is Borel measurable on Δ_n, and hence $\cup_k A_k \in \mathcal{D}_1$. Similarly, one can show that if $(A_k)_{k\in\mathbb{N}} \subset \mathcal{D}_1$ is a decreasing sequence of events, then $\cap_k A_k \in \mathcal{D}_1$. Therefore, $\mathcal{D}_1$ is a monotone class, and by the monotone class theorem $\mathcal{D}_1 = \sigma(\mathcal{C}) = \widetilde{\mathcal{F}}^Y_\infty$.

It remains to show that (3.7) holds true. In view of (3.6) and (3.11), for any cylinder set $A \in \mathcal{C}$,

$$\begin{aligned}
\overline{\mathbb{P}}^i(A\,; s_1, \ldots, s_n) &= \lim_{\Delta s_k \to 0,\, k=1,\ldots,n} \frac{\widetilde{\mathbb{P}}^i\left(A,\, s_k < S_k \le s_k + \Delta s_k,\, k = 1, \ldots, n\right)}{\widetilde{\mathbb{P}}^i\left(s_k < S_k \le s_k + \Delta s_k,\, k = 1, \ldots, n\right)} \\
&= \frac{\lim_{\Delta s_k \to 0,\, k=1,\ldots,n} (\Delta s_1 \cdots \Delta s_n)^{-1}\, \widetilde{\mathbb{P}}^i\left(A,\, s_k < S_k \le s_k + \Delta s_k,\, k = 1, \ldots, n\right)}{\lim_{\Delta s_k \to 0,\, k=1,\ldots,n} (\Delta s_1 \cdots \Delta s_n)^{-1}\, \widetilde{\mathbb{P}}^i\left(s_k < S_k \le s_k + \Delta s_k,\, k = 1, \ldots, n\right)} \\
&= \frac{\partial^n}{\partial s_1 \cdots \partial s_n} \widetilde{\mathbb{P}}^i\left(A,\, S_k \le s_k,\, k = 1, \ldots, n\right) \cdot \left(\prod_{k=1}^n q_k\, e^{-q_k(s_k - s_{k-1})}\right)^{-1}.
\end{aligned}$$

Hence, for any $A \in \mathcal{C}$,

$$\begin{aligned}
&\int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{P}}^i(A; s_1, \ldots, s_n) \prod_{k=1}^n q_k\, e^{-q_k(s_k - s_{k-1})}\, \mathrm{d}s_1 \cdots \mathrm{d}s_n \\
&\quad = \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \frac{\partial^n}{\partial s_1 \cdots \partial s_n} \widetilde{\mathbb{P}}^i\left(A,\, S_k \le s_k,\, k = 1, \ldots, n\right) \mathrm{d}s_1 \cdots \mathrm{d}s_n = \widetilde{\mathbb{P}}^i(A),
\end{aligned}$$

and thus $\mathcal{C} \subset \mathcal{D}_2$, where $\mathcal{D}_2 := \left\{ A \in \widetilde{\mathcal{F}}^Y_\infty \,\middle|\, (3.7) \text{ holds for } A \right\}$. Next, for any increasing sequence of events $(A_k)_{k\in\mathbb{N}} \subset \mathcal{D}_2$, we have that

$$\begin{aligned}
\widetilde{\mathbb{P}}^i\left(\bigcup_{k=1}^\infty A_k\right) = \lim_{k\to\infty} \widetilde{\mathbb{P}}^i(A_k) &= \lim_{k\to\infty} \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{P}}^i(A_k; s_1, \ldots, s_n) \prod_{\ell=1}^n q_\ell\, e^{-q_\ell(s_\ell - s_{\ell-1})}\, \mathrm{d}s_1 \cdots \mathrm{d}s_n \\
&= \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{P}}^i\left(\bigcup_{k=1}^\infty A_k; s_1, \ldots, s_n\right) \prod_{\ell=1}^n q_\ell\, e^{-q_\ell(s_\ell - s_{\ell-1})}\, \mathrm{d}s_1 \cdots \mathrm{d}s_n,
\end{aligned}$$

where the last equality follows from the dominated convergence theorem as well as the fact that $\overline{\mathbb{P}}^i(A_k; s_1, \ldots, s_n) \le 1$, for all $k \in \mathbb{N}$ and $0 < s_1 < \ldots < s_n$. Hence, $\cup_k A_k \in \mathcal{D}_2$. Similarly, one can show that if $(A_k)_{k\in\mathbb{N}} \subset \mathcal{D}_2$ is a decreasing sequence, then $\cap_k A_k \in \mathcal{D}_2$. Therefore, $\mathcal{D}_2$ is a monotone class, and by the monotone class theorem $\mathcal{D}_2 = \sigma(\mathcal{C}) = \widetilde{\mathcal{F}}^Y_\infty$. This completes the proof. $\square$

Next, we will prove that the law of Y under $\overline{\mathbb{P}}^i$ is the same as that of X under $\mathbb{P}^i$. As usual, $\overline{\mathbb{E}}^i(\cdot\,; s_1,\ldots,s_n)$ will denote the expectation associated with $\overline{\mathbb{P}}^i(\cdot\,; s_1,\ldots,s_n)$, for $i \in \mathbf{E}$ and $0 < s_1 < \ldots < s_n$. In the sequel, if there is no ambiguity, we will omit the parameters $s_1,\ldots,s_n$ in $\overline{\mathbb{P}}^i$ and $\overline{\mathbb{E}}^i$.

Theorem 3.4. *For any $i \in \mathbf{E}$ and $0 < s_1 < \ldots < s_n$, under $\overline{\mathbb{P}}^i$, Y is a time-inhomogeneous Markov chain with generator $\mathsf{G} = \{\mathsf{G}_t, t \geq 0\}$. In particular, X and Y have the same law under respective probability measures $\mathbb{P}^i$ and $\overline{\mathbb{P}}^i$.*

Proof. Let $u_0, u_1, \ldots, u_m$ be such that

$$0 = u_0 \leq u_1 < \ldots < u_{k_1} \leq s_1 < u_{k_1+1} < \ldots < u_{k_2} \leq s_2 < \ldots \leq s_n < u_{k_n+1} < \ldots < u_{k_{n+1}} = u_m.$$

By (3.9), for any $i_1,\ldots,i_m \in \mathbf{E}$,

$$\begin{aligned}
\overline{\mathbb{P}}^i(Y_{u_m} = i_m \mid Y_{u_{m-1}} = i_{m-1},\ldots,Y_{u_1} = i_1) &= \frac{\overline{\mathbb{P}}^i\left(Y_{u_1} = i_1,\ldots,Y_{u_m} = i_m\right)}{\overline{\mathbb{P}}^i\left(Y_{u_1} = i_1,\ldots,Y_{u_{m-1}} = i_{m-1}\right)} \\
&= \frac{\prod_{\ell=0}^{n}\left(\prod_{p=k_\ell+1}^{k_{\ell+1}} e^{(u_p-u_{p-1})\mathsf{G}_\ell}(i_{p-1},i_p)\right)}{\left[\prod_{\ell=0}^{n-1}\left(\prod_{p=k_\ell+1}^{k_{\ell+1}} e^{(u_p-u_{p-1})\mathsf{G}_\ell}(i_{p-1},i_p)\right)\right]\left(\prod_{p=k_n+1}^{k_{n+1}-1} e^{(u_p-u_{p-1})\mathsf{G}_\ell}(i_{p-1},i_p)\right)} \\
&= e^{(u_m-u_{m-1})\mathsf{G}_n}(i_{m-1},i_m).
\end{aligned}$$

On the other hand, by (3.9) again,

$$\begin{aligned}
\overline{\mathbb{P}}^i(Y_{u_m} = i_m \mid Y_{u_{m-1}} = i_{m-1}) &= \frac{\overline{\mathbb{P}}^i\left(Y_{u_m} = i_m\, Y_{u_{m-1}} = i_{m-1}\right)}{\overline{\mathbb{P}}^i\left(Y_{u_{m-1}} = i_{m-1}\right)} \\
&= \frac{\sum_{i_1,\ldots,i_{m-2}\in\mathbf{E}} \overline{\mathbb{P}}^i\left(Y_{u_1} = i_1,\ldots,Y_{u_m} = i_m\right)}{\sum_{i_1,\ldots,i_{m-2}\in\mathbf{E}} \overline{\mathbb{P}}^i\left(Y_{u_1} = i_1,\ldots,Y_{u_{m-1}} = i_{m-1}\right)} \\
&= \frac{\sum_{i_1,\ldots,i_{m-2}\in\mathbf{E}} \prod_{\ell=0}^{n}\left(\prod_{p=k_\ell+1}^{k_{\ell+1}} e^{(u_p-u_{p-1})\mathsf{G}_\ell}(i_{p-1},i_p)\right)}{\sum_{i_1,\ldots,i_{m-2}\in\mathbf{E}}\left[\prod_{\ell=0}^{n-1}\left(\prod_{p=k_\ell+1}^{k_{\ell+1}} e^{(u_p-u_{p-1})\mathsf{G}_\ell}(i_{p-1},i_p)\right)\right]\left(\prod_{p=k_n+1}^{k_{n+1}-1} e^{(u_p-u_{p-1})\mathsf{G}_\ell}(i_{p-1},i_p)\right)} \\
&= e^{(u_m-u_{m-1})\mathsf{G}_n}(i_{m-1},i_m).
\end{aligned}$$

Analogous argument carries for any $u_0 < u_1 < \ldots < u_m$, which completes the proof. $\square$

In analogy to φ_t and τ_t^+ we now define an additive functional ψ given as $\psi_t := \int_0^t v(Y_u)\,\mathrm{d}u,\ t \geq 0$, and we consider the following first passage time $\rho_t^+ := \inf\{r \geq 0 \,|\, \psi_r > t\},\ t \geq 0$.

We end this part of this section with the following corollary to Theorem 3.4.

Corollary 3.5. *For any $(s_1,\ldots,s_n)$ in Δ_n, $c > 0$, and $t > 0$,*

$$\Pi_c^+(i,j;s_1,\ldots,s_n) = \overline{\mathbb{E}}^i\left(e^{-c\rho_0^+}\mathbb{1}_{\{Y_{\rho_0^+}=j\}}; s_1,\ldots,s_n\right), \quad i \in \mathbf{E}^-,\ j \in \mathbf{E}^+, \tag{3.12}$$

$$\Psi_c^+(t,i,j;s_1,\ldots,s_n) = \overline{\mathbb{E}}^i\left(e^{-c\rho_t^+}\mathbb{1}_{\{Y_{\rho_t^+}=j\}}; s_1,\ldots,s_n\right), \quad i \in \mathbf{E}^+,\ j \in \mathbf{E}^+. \tag{3.13}$$

In particular, $\Pi_c^+(i,j;s_1,\ldots,s_n)$ and $\Psi_c^+(t,i,j;s_1,\ldots,s_n)$ are Borel measurable with respect to $(s_1,\ldots,s_n)$ in Δ_n.

3.1 Wiener-Hopf Factorization for $Z = (N, Y)$

This subsection is devoted to computing the expectations on the right-hand side in (3.12) and (3.13). This will be done by computing the corresponding expectations related to the *time-homogeneous* Markov chain $Z = (N, Y)$. The latter computation will be done using the classical Wiener-Hopf factorization results for finite state time-homogeneous Markov chains, originally derived in [BRW80].

We begin with a restatement of the classical Wiener-Hopf factorization applied to Z. Towards this end, we let $\widetilde{\mathbf{E}}^+ := \mathbb{N}_n \times \mathbf{E}^+$ and $\widetilde{\mathbf{E}}^- := \mathbb{N}_n \times \mathbf{E}^-$, and $\widetilde{v} : \widetilde{\mathbf{E}} \to \mathbb{R} \setminus \{0\}$ be a function on $\widetilde{\mathbf{E}}$ such that $\widetilde{v}(k,i) = v(i)$, for all $(k,i) \in \widetilde{\mathbf{E}}$. Next, we define the additive functional $\widetilde{\varphi}$ and the corresponding first passage times as

$$\widetilde{\varphi}_t := \int_0^t \widetilde{v}(Z_u)\,\mathrm{d}u, \quad \widetilde{\tau}_t^{\pm} := \inf\{r \geq 0 \,|\, \pm\widetilde{\varphi}_r > t\}, \quad t \geq 0.$$

Let $\widetilde{V} := diag\{\widetilde{v}(k,i) : (k,i) \in \widetilde{\mathbf{E}}\}$ (a diagonal matrix). We denote by $\widetilde{\mathsf{I}}^{\pm}$ the identity matrix of dimension $|\widetilde{\mathbf{E}}^{\pm}|$. Finally, $\mathcal{Q}(m)$ will stand for the set of $m \times m$ generator matrices (i.e., matrices with non-negative off-diagonal entries and non-positive row sums), and $\mathcal{P}(m,\ell)$ will be the set of $m \times \ell$ matrices whose rows are sub-probability vectors.

Theorem 3.6. *[BRW80, Theorem 1 & 2] Fix $c > 0$. Then,*

(i) there exists a unique quadruple of matrices $(\widetilde{\Lambda}_c^+, \widetilde{\Lambda}_c^-, \widetilde{\mathsf{G}}_c^+, \widetilde{\mathsf{G}}_c^-)$, where $\widetilde{\Lambda}_c^+ \in \mathcal{P}(|\widetilde{\mathbf{E}}^-|, |\widetilde{\mathbf{E}}^+|)$, $\widetilde{\Lambda}_c^- \in \mathcal{P}(|\widetilde{\mathbb{E}}^+|, |\widetilde{\mathbf{E}}^-|)$, $\widetilde{\mathsf{G}}_c^+ \in \mathcal{Q}(|\widetilde{\mathbf{E}}^+|)$, and $\widetilde{\mathsf{G}}_c^- \in \mathcal{Q}(|\widetilde{\mathbf{E}}^-|)$, such that

$$\widetilde{V}^{-1}\left(\widetilde{\mathsf{G}} - c\widetilde{\mathsf{I}}\right)\begin{pmatrix} \widetilde{\mathsf{I}}^+ & \widetilde{\Lambda}_c^- \\ \widetilde{\Lambda}_c^+ & \widetilde{\mathsf{I}}^- \end{pmatrix} = \begin{pmatrix} \widetilde{\mathsf{I}}^+ & \widetilde{\Lambda}_c^- \\ \widetilde{\Lambda}_c^+ & \widetilde{\mathsf{I}}^- \end{pmatrix}\begin{pmatrix} \widetilde{\mathsf{G}}_c^+ & 0 \\ 0 & -\widetilde{\mathsf{G}}_c^- \end{pmatrix}; \tag{3.14}$$

(ii) the matrices $\widetilde{\Lambda}_c^+$, $\widetilde{\Lambda}_c^-$, $\widetilde{\mathsf{G}}_c^+$, and $\widetilde{\mathsf{G}}_c^-$, admit the following probabilistic representations,

$$\widetilde{\Lambda}_c^+((k,i),(\ell,j)) = \widetilde{\mathbb{E}}\left(e^{-c\widetilde{\tau}_0^+}\mathbf{1}_{\{Z_{\widetilde{\tau}_0^+}=(\ell,j)\}}\,\middle|\, Z_0 = (k,i)\right), \quad (k,i) \in \widetilde{\mathbf{E}}^-,\ (\ell,j) \in \widetilde{\mathbf{E}}^+, \tag{3.15}$$

$$\widetilde{\Lambda}_c^-((k,i),(\ell,j)) = \widetilde{\mathbb{E}}\left(e^{-c\widetilde{\tau}_0^-}\mathbf{1}_{\{Z_{\widetilde{\tau}_0^-}=(\ell,j)\}}\,\middle|\, Z_0 = (k,i)\right), \quad (k,i) \in \widetilde{\mathbf{E}}^+,\ (\ell,j) \in \widetilde{\mathbf{E}}^-, \tag{3.16}$$

$$e^{t\,\widetilde{\mathsf{G}}_c^+}((k,i),(\ell,j)) = \widetilde{\mathbb{E}}\left(e^{-c\widetilde{\tau}_t^+}\mathbf{1}_{\{Z_{\widetilde{\tau}_t^+}=(\ell,j)\}}\,\middle|\, Z_0 = (k,i)\right), \quad (k,i) \in \widetilde{\mathbf{E}}^+,\ (\ell,j) \in \widetilde{\mathbf{E}}^+, \tag{3.17}$$

$$e^{t\,\widetilde{\mathsf{G}}_c^-}((k,i),(\ell,j)) = \widetilde{\mathbb{E}}\left(e^{-c\widetilde{\tau}_t^-}\mathbf{1}_{\{Z_{\widetilde{\tau}_t^-}=(\ell,j)\}}\,\middle|\, Z_0 = (k,i)\right), \quad (k,i) \in \widetilde{\mathbf{E}}^-,\ (\ell,j) \in \widetilde{\mathbf{E}}^-, \tag{3.18}$$

for any $t \geq 0$.

In what follows we will use the "+" part of the above formulae and only for $k = 0$. Accordingly, we define (recall (3.1))

$$\widetilde{\Pi}_c^+(i,j,\ell) := \widetilde{\Lambda}_c^+((0,i),(\ell,j)) = \widetilde{\mathbb{E}}^i\left(e^{-c\widetilde{\tau}_0^+}\mathbf{1}_{\{Z_{\widetilde{\tau}_0^+}=(\ell,j)\}}\right), \quad i \in \mathbf{E}^-,\ j \in \mathbf{E}^+,\ \ell \in \mathbb{N}, \tag{3.19}$$

$$\widetilde{\Psi}_c^+(t,i,j,\ell) := e^{t\,\widetilde{\mathsf{G}}_c^+}((0,i),(\ell,j)) = \widetilde{\mathbb{E}}^i\left(e^{-c\widetilde{\tau}_t^+}\mathbf{1}_{\{Z_{\widetilde{\tau}_t^+}=(\ell,j)\}}\right), \quad i,j \in \mathbf{E}^+,\ \ell \in \mathbb{N},\ t \geq 0. \tag{3.20}$$

Note that, for any $t \geq 0$, $\widetilde{v}(Z_t) = v(Y_t)$, which implies that $\widetilde{\varphi}_t = \psi_t$, and so $\rho_t^+ = \widetilde{\tau}_t^+$, $\rho_t^- = \widetilde{\tau}_t^-$. Hence, by taking summations over all $\ell \in \mathbb{N}$ in (3.19) and (3.20), we obtain that

$$\widetilde{\mathbb{E}}^i \left(e^{-c\rho_0^+} \mathbb{1}_{\{Y_{\rho_0^+}=j\}} \right) = \sum_{\ell=0}^{n} \widetilde{\Pi}_c^+(i,j,\ell), \quad i \in \mathbf{E}^-,\, j \in \mathbf{E}^+, \tag{3.21}$$

$$\widetilde{\mathbb{E}}^i \left(e^{-c\rho_t^+} \mathbb{1}_{\{Y_{\rho_t^+}=j\}} \right) = \sum_{\ell=0}^{n} \widetilde{\Psi}_c^+(t,i,j,\ell), \quad i,j \in \mathbf{E}^+,\, t \geq 0. \tag{3.22}$$

Observe that, in view of (3.7), if $U : \widetilde{\Omega} \to \mathbb{R}$ is an $\widetilde{\mathcal{F}}_\infty^Y$-measurable bounded random variable, then for any $i \in \mathbf{E}$,

$$\widetilde{\mathbb{E}}^i(U) = \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{E}}^i(U; s_1, \ldots, s_n) \prod_{k=1}^{n} \left(q_k\, e^{-q_k(s_k - s_{k-1})} \right) ds_n \cdots ds_2 ds_1.$$

Therefore, in light of Corollary 3.5, (3.21) and (3.22), we have that

$$\begin{aligned}\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n) &:= \sum_{\ell=0}^{n} \widetilde{\Pi}_c^+(i,j,\ell) \\ &= \int_0^\infty \int_{s_1}^\infty \ldots \int_{s_{n-1}}^\infty \Pi_c^+(i,j;s_1,\ldots,s_n) \prod_{k=1}^{n} \left(q_k e^{-q_k(s_k - s_{k-1})} \right) ds_n \cdots ds_2\, ds_1.\end{aligned}$$

$$\begin{aligned}\widehat{\Psi}_c^+(t,i,j;q_1,\ldots,q_n) &:= \sum_{\ell=0}^{n} \widetilde{\Psi}_c^+(t,i,j,\ell) \\ &= \int_0^\infty \int_{s_1}^\infty \ldots \int_{s_{n-1}}^\infty \Psi_c^+(t,i,j;s_1,\ldots,s_n) \prod_{k=1}^{n} \left(q_k e^{-q_k(s_k - s_{k-1})} \right) ds_n \cdots ds_2\, ds_1.\end{aligned}$$

By change of variables, we obtain

$$\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty \Pi_c^+(i,j;t_1,\ldots,t_1+\ldots+t_n) \prod_{k=1}^{n} \left(q_k e^{-q_k t_k} \right) dt_1 \cdots dt_n,$$

$$\widehat{\Psi}_c^+(i,j;q_1,\ldots,q_n) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty \Psi_c^+(i,j;t_1,\ldots,t_1+\ldots+t_n) \prod_{k=1}^{n} \left(q_k e^{-q_k t_k} \right) dt_1 \cdots dt_n.$$

The above two equalities together with the argument in Section 5, implies that

$$q_1^{-1} \cdots q_n^{-1} \widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n), \qquad q_1^{-1} \cdots q_n^{-1} \widehat{\Psi}_c^+(i,j;q_1,\ldots,q_n)$$

are well-defined for $q_k \in \mathbb{C}^+ := \{z \in \mathbb{C} \mid \Re(z) > 0\}, k = 1, \ldots, n$, as being the Laplace transforms of $\Pi_c^+(i,j;t_1,\ldots,t_1+\ldots+t_n)$ and $\Psi_c^+(i,j;t_1,\ldots,t_1+\ldots+t_n)$, respectively.

All the above leads to the following result, which is our main theorem, and where we make use of the inverse multivariate Laplace transform. We refer to the Appendix for the definition and the properties of the inverse multivariate Laplace transform relevant to our set-up.

Theorem 3.7. *We have that*

$$\Pi_c^+(i,j;s_1,\ldots,s_n) = \mathcal{L}^{-1}\left(q_1^{-1}\cdots q_n^{-1}\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)\right)(s_1,s_2-s_1,\ldots,s_n-s_{n-1}), \tag{3.23}$$

for any $i\in\mathbf{E}^-$, $j\in\mathbf{E}^+$, *and*

$$\Psi_c^+(t,i,j;s_1,\ldots,s_n) = \mathcal{L}^{-1}\left(q_1^{-1}\cdots q_n^{-1}\widehat{\Psi}_c^+(t,i,j;q_1,\ldots,q_n)\right)(s_1,s_2-s_1,\ldots,s_n-s_{n-1}), \tag{3.24}$$

for any $t>0$, $i,j\in\mathbf{E}^+$, *where* $\mathcal{L}^{-1}$ *is the inverse multivariate Laplace transform.*

Remark 3.8. It needs to be stressed that we can compute the values of $\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)$ and $\widehat{\Psi}_c^+(t,i,j;q_1,\ldots,q_n)$ only for positive values of q_i's. Thus, Theorem 3.7 may not be directly applied to compute $\Pi_c^+(i,j;s_1,\ldots,s_n)$ and $\Psi_c^+(t,i,j;s_1,\ldots,s_n)$. However, we can approximate these functions, as explained in Section 5.1 by using only the values of $\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)$ and $\widehat{\Psi}_c^+(t,i,j;q_1,\ldots,q_n)$ for positive values of q_i's.

4 Numerical Example

In this section we will illustrate our theoretical results with a simple, but telling example. We first describe a numerical method to approximate Π_c^+ and Ψ_c^+, and then we proceed with its application to a concrete example.

4.1 Numerical Procedure to approximate Π_c^+ and Ψ_c^+

We only consider Π_c^+. The procedure to approximate Ψ_c^+ is analogous.

According to Theorem 3.7 and Section 5.1, to approximate Π_c^+, we need to compute $\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)$ for any $q_1,\ldots,q_n>0$, and then to use the Gaver-Stehfest algorithm. Note that $\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)$ can be computed by solving (3.14) directly using the diagonalization method of [RS94]. However, because of the special structure of $\widetilde{\mathsf{G}}$, we can simplify the calculation by working on matrices of smaller dimensions. Towards this end we observe that matrices in (3.14) can be written the block form as follows,

$$\widetilde{\mathsf{G}} = \begin{array}{c} \\ (0,\mathbf{E}^+) \\ (1,\mathbf{E}^+) \\ \vdots \\ (n{-}1,\mathbf{E}^+) \\ (n,\mathbf{E}^+) \\ (0,\mathbf{E}^-) \\ (1,\mathbf{E}^-) \\ \vdots \\ (n{-}1,\mathbf{E}^-) \\ (n,\mathbf{E}^-) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0,\mathbf{E}^+) & (1,\mathbf{E}^+) & \cdots & (n,\mathbf{E}^+) & (0,\mathbf{E}^-) & (1,\mathbf{E}^-) & \cdots & (n,\mathbf{E}^-) \end{array} \\ \begin{bmatrix} \mathsf{A}_1 - q_1\mathsf{I}^+ & q_1\mathsf{I}^+ & \cdots & 0 & \mathsf{B}_1 & 0 & \cdots & 0 \\ 0 & \mathsf{A}_2 - q_2\mathsf{I}^+ & \cdots & 0 & 0 & \mathsf{B}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_n\mathsf{I}^+ & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \mathsf{A}_{n+1} & 0 & 0 & \cdots & \mathsf{B}_{n+1} \\ \mathsf{C}_1 & 0 & \cdots & 0 & \mathsf{D}_1 - q_1\mathsf{I}^- & q_1\mathsf{I}^- & \cdots & 0 \\ 0 & \mathsf{C}_2 & \cdots & 0 & 0 & \mathsf{D}_2 - q_2\mathsf{I}^- & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & q_n\mathsf{I}^- \\ 0 & 0 & \cdots & \mathsf{C}_{n+1} & 0 & 0 & \cdots & \mathsf{D}_{n+1} \end{bmatrix} \end{array}, \tag{4.1}$$

$$\widetilde{\mathsf{V}} = \begin{array}{cc} & \begin{array}{cccccccc} (0,\mathbf{E}^+) & (1,\mathbf{E}^+) & \cdots & (n,\mathbf{E}^+) & (0,\mathbf{E}^-) & (1,\mathbf{E}^-) & \cdots & (n,\mathbf{E}^-) \end{array} \\ \begin{array}{c} (0,\mathbf{E}^+) \\ (1,\mathbf{E}^+) \\ \vdots \\ (n-1,\mathbf{E}^+) \\ (n,\mathbf{E}^+) \\ (0,\mathbf{E}^-) \\ (1,\mathbf{E}^-) \\ \vdots \\ (n-1,\mathbf{E}^-) \\ (n,\mathbf{E}^-) \end{array} & \begin{bmatrix} \mathsf{V}^+ & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathsf{V}^+ & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \mathsf{V}^+ & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \mathsf{V}^- & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \mathsf{V}^- & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \mathsf{V}^- \end{bmatrix} \end{array}, \tag{4.2}$$

$$\widetilde{\Lambda}_c^+ = \begin{array}{cc} & \begin{array}{ccccc} (0,\mathbf{E}^+) & (1,\mathbf{E}^+) & \cdots & (n-1,\mathbf{E}^+) & (n,\mathbf{E}^+) \end{array} \\ \begin{array}{c} (0,\mathbf{E}^-) \\ (1,\mathbf{E}^-) \\ \vdots \\ (n-1,\mathbf{E}^-) \\ (n,\mathbf{E}^-) \end{array} & \begin{bmatrix} \widetilde{\Lambda}^+_{c,00} & \widetilde{\Lambda}^+_{c,01} & \cdots & \widetilde{\Lambda}^+_{c,0,n-1} & \widetilde{\Lambda}^+_{c,0n} \\ 0 & \widetilde{\Lambda}^+_{c,11} & \cdots & \widetilde{\Lambda}^+_{c,1,n-1} & \widetilde{\Lambda}^+_{c,1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \widetilde{\Lambda}^+_{c,n-1,n-1} & \widetilde{\Lambda}^+_{c,n-1,n} \\ 0 & 0 & \cdots & 0 & \widetilde{\Lambda}^+_{c,nn} \end{bmatrix} \end{array} \tag{4.3}$$

$$\widetilde{\Lambda}_c^- = \begin{array}{cc} & \begin{array}{ccccc} (0,\mathbf{E}^-) & (1,\mathbf{E}^-) & \cdots & (n-1,\mathbf{E}^-) & (n,\mathbf{E}^-) \end{array} \\ \begin{array}{c} (0,\mathbf{E}^+) \\ (1,\mathbf{E}^+) \\ \vdots \\ (n-1,\mathbf{E}^+) \\ (n,\mathbf{E}^+) \end{array} & \begin{bmatrix} \widetilde{\Lambda}^-_{c,00} & \widetilde{\Lambda}^-_{c,01} & \cdots & \widetilde{\Lambda}^-_{c,0,n-1} & \widetilde{\Lambda}^-_{c,0n} \\ 0 & \widetilde{\Lambda}^-_{c,11} & \cdots & \widetilde{\Lambda}^-_{c,1,n-1} & \widetilde{\Lambda}^-_{c,1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \widetilde{\Lambda}^-_{c,n-1,n-1} & \widetilde{\Lambda}^-_{c,n-1,n} \\ 0 & 0 & \cdots & 0 & \widetilde{\Lambda}^-_{c,nn} \end{bmatrix} \end{array}, \tag{4.4}$$

$$\widetilde{\mathsf{G}}_c^+ = \begin{array}{cc} & \begin{array}{ccccc} (0,\mathbf{E}^+) & (1,\mathbf{E}^+) & \cdots & (n-1,\mathbf{E}^+) & (n,\mathbf{E}^+) \end{array} \\ \begin{array}{c} (0,\mathbf{E}^+) \\ (1,\mathbf{E}^+) \\ \vdots \\ (n-1,\mathbf{E}^+) \\ (n,\mathbf{E}^+) \end{array} & \begin{bmatrix} \widetilde{\mathsf{G}}^+_{c,00} & \widetilde{\mathsf{G}}^+_{c,01} & \cdots & \widetilde{\mathsf{G}}^+_{c,0,n-1} & \widetilde{\mathsf{G}}^+_{c,0n} \\ 0 & \widetilde{\mathsf{G}}^+_{c,11} & \cdots & \widetilde{\mathsf{G}}^+_{c,1,n-1} & \widetilde{\mathsf{G}}^+_{c,1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \widetilde{\mathsf{G}}^+_{c,n-1,n-1} & \widetilde{\mathsf{G}}^+_{c,n-1,n} \\ 0 & 0 & \cdots & 0 & \widetilde{\mathsf{G}}^+_{c,nn} \end{bmatrix} \end{array}, \tag{4.5}$$

and

$$\widetilde{\mathsf{G}}_c^- = \begin{array}{cc} & \begin{array}{ccccc} (0,\mathbf{E}^-) & (1,\mathbf{E}^-) & \cdots & (n-1,\mathbf{E}^-) & (n,\mathbf{E}^-) \end{array} \\ \begin{array}{c} (0,\mathbf{E}^-) \\ (1,\mathbf{E}^-) \\ \vdots \\ (n-1,\mathbf{E}^-) \\ (n,\mathbf{E}^-) \end{array} & \begin{bmatrix} \widetilde{\mathsf{G}}^-_{c,00} & \widetilde{\mathsf{G}}^-_{c,01} & \cdots & \widetilde{\mathsf{G}}^-_{c,0,n-1} & \widetilde{\mathsf{G}}^-_{c,0n} \\ 0 & \widetilde{\mathsf{G}}^-_{c,11} & \cdots & \widetilde{\mathsf{G}}^-_{c,1,n-1} & \widetilde{\mathsf{G}}^-_{c,1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \widetilde{\mathsf{G}}^-_{c,n-1,n-1} & \widetilde{\mathsf{G}}^-_{c,n-1,n} \\ 0 & 0 & \cdots & 0 & \widetilde{\mathsf{G}}^-_{c,nn} \end{bmatrix} \end{array}. \tag{4.6}$$

Plugging (4.2)–(4.6) into (3.14) and then comparing all the block entries on both sides, we end up with the following procedure to compute the factorization recursively.

In accordance to Theorem 3.6, for any generator matrix H and any constant $c > 0$, we denote by

$$(\Lambda_c^+(\mathsf{H}), \Lambda_c^-(\mathsf{H}), \mathsf{G}_c^+(\mathsf{H}), \mathsf{G}_c^-(\mathsf{H})) \tag{4.7}$$

the unique quadruple constituting the classical Wiener-Hopf factorization (cf. [BRW80]) corresponding to H with killing rate c. In order to proceed, we let $c_k = q_k + c, \quad k \geq 1$.

We are now ready to describe the algorithm to compute $q_1^{-1} \cdots q_n^{-1} \widehat{\Pi}_c^+(i, j; q_1, \ldots, q_n)$.

Step 1. **Compute the first diagonal:** for $k = 1, \ldots, n+1$, compute

$$\widetilde{\Lambda}^+_{c,k-1,k-1} = \Lambda^+_{c_k}(\mathsf{G}_k), \tag{4.8}$$

using the diagonalization method in [RS94].

Step 2. **Compute the second diagonal:** for $k = 1, \ldots, n$, solve the following linear system

$$q_k \mathsf{I}^+ + \mathsf{B}_k \widetilde{\Lambda}^+_{c,k-1,k} = \mathsf{V}^+ \widetilde{\mathsf{G}}^+_{c,k-1,k}, \tag{4.9}$$

$$[\mathsf{D}_k - c_k \mathsf{I}^-] \widetilde{\Lambda}^+_{c,k-1,k} + q_k \widetilde{\Lambda}^+_{c,kk} = \mathsf{V}^- \widetilde{\Lambda}^+_{c,k-1,k-1} \widetilde{\mathsf{G}}^+_{c,k-1,k} + \mathsf{V}^- \widetilde{\Lambda}^+_{c,k-1,k} \widetilde{\mathsf{G}}^+_{c,kk}, \tag{4.10}$$

for $\widetilde{\Lambda}^+_{c,k-1,k}$ and $\widetilde{\mathsf{G}}^+_{c,k-1,k}$.

Step 3. **Compute the other diagonals:** for $r = 2, \ldots, n$, $k = 0, \ldots, n-r$, solve the linear system

$$\mathsf{B}_{k+1} \widetilde{\Lambda}^+_{c,k,k+r} = \mathsf{V}^+ \widetilde{\mathsf{G}}^+_{c,k,k+r}, \tag{4.11}$$

$$[\mathsf{D}_{k+1} - c_{k+1} \mathsf{I}^-] \widetilde{\Lambda}^+_{c,k,k+r} + q_{k+1} \widetilde{\Lambda}^+_{c,k+1,k+r} = \mathsf{V}^- \sum_{j=0}^{r} \widetilde{\Lambda}^+_{c,k,k+j} \widetilde{\mathsf{G}}^+_{c,k+j,k+r}, \tag{4.12}$$

for $\widetilde{\Lambda}^+_{c,k,k+r}$ and $\widetilde{\mathsf{G}}^+_{c,k,k+r}$.

Step 4. **Compute**

$$P^+(q_1, \ldots, q_n) := q_1^{-1} \cdots q_n^{-1} \widehat{\Pi}_c^+(i, j; q_1, \ldots, q_n) = q_1^{-1} \cdots q_n^{-1} \sum_{\ell=0}^{n} \widetilde{\Lambda}_{c,0\ell}.$$

for $q_1, \ldots, q_n > 0$.

Step 5. **Compute the approximate inverse Laplace transform of $P^+(q_1, \ldots, q_n)$:** use the method discussed in Section 5.1.

Remark 4.1. If $|\mathbf{E}^+| = |\mathbf{E}^-| = 1$, then the matrices in Steps 1-3 become numbers. Step 1 reduces to solving $n+1$ quadratic equations for a root in $[0, 1]$. In Step 2 and 3, for each loop, the system reduces to a system of two linear equations of two unknowns. Moreover, in this case, P^+ has a closed-form representation for $q_1, \ldots, q_n > 0$, and hence, for any $q_1, \ldots, q_n \in \mathbb{C}^+$, as mentioned in the previous section. This allows to use general numerical inverse Laplace transform methods, not necessary the Gaver-Stehfest formula from Section 5.1. In particular, one can use Talbot approximation formula (5.1) presented in Section 5, which is more efficient than the Gaver-Stehfest under fairly general assumptions (cf. [AW06]).

4.2 Application in Fluid flow problems

The Wiener-Hopf factorization for a time-homogeneous finite Markov chain was applied in [Rog94] in the context of fluid models of queues. In this section, we will apply our results to a time-inhomogeneous Markov chain fluid flow problem.

First, we briefly review the classical fluid flow problem (cf [Mit88] and [Rog94] for detailed discussion). Suppose we have a large water tank with capacity $a \in (0, \infty]$. On the top of the tank, there are $I_t \in \mathcal{I}$ pipes open at time t, with each pipe pouring water into the tank at rate r^+. At the bottom of the tank, there are $O_t \in \mathcal{O}$ taps open at time t, with each tap allowing water to flow out at rate r^-. We assume that $\mathcal{I}$ and $\mathcal{O}$ are finite sets.

Then, the volume ξ_t of water in the tank at time t satisfies

$$\frac{\mathrm{d}\xi_t}{\mathrm{d}t} = r^+ I_t - r^- O_t, \quad \text{if } 0 < \xi_t < a.$$

Moreover, if $\xi_t = 0$, i.e. if the tank is empty, then the outflow ceases. If $\xi_t = a$, i.e. if the tank is full, then water flows over the top.

Let f be a real valued function on $\mathcal{I} \times \mathcal{O}$. We assume $X_t := f(I_t, O_t),\ t \geq 0$, is a (finite state) time-inhomogeneous Markov chain, and we denote by $\mathbf{E}$ the state space of X. Let

$$v(x) := V(r^+, r^-, x), \quad x \in \mathbf{E},$$

model the water outflow/inflow, in terms of the states of X, so that

$$v(X_t) = V(r^+, r^-, f(I_t, O_t)), \quad t \geq 0$$

represents the water outflow/inflow at time t.

Let $\mathbf{E}^+$ be the set of states of X such that the water tank has greater water inflow than outflow, and let $\mathbf{E}^-$ be the set of states of X such that the water tank has greater water outflow than inflow. The integral

$$\varphi_t = \int_0^t v(X_u)\,\mathrm{d}u$$

is not exactly the water content at time t, because we should take into account those periods when the tank is full or empty. However, as noted in [Rog94], understanding φ_t, and the corresponding $\tau_t^\pm$ and $X_{\tau_t^\pm}$ allows us to easily express the quantities of interest for ξ_t in terms of Wiener-Hopf factorization, and to further compute these quantities once we compute the Wiener-Hopf factorization numerically.

We now assume that the tank has infinite capacity, $a = \infty$, and that it contains ℓ amount of water at time $t = 0$. Thus, τ_ℓ^- represents the first time after $t = 0$ that the tank goes empty. We will compute the quantity

$$\Pi_c^-(i,j) = \mathbb{E}^i \left(e^{-c\tau_0^-} \mathbb{1}_{\{X_{\tau_0^-} = j\}} \right), \quad i \in \mathbf{E}^+,\, j \in \mathbf{E}^-. \tag{4.13}$$

Towards this end, we further assume that the tank has either an aggregate water inflow at rate v^+ or an aggregate water outflow at rate v^-. In other words,

$$\mathbf{E}^+ = \{e_+\}, \quad \mathbf{E}^- = \{e_-\}, \quad v(e_+) = v^+, \quad \text{and} \quad v(e_-) = v^-.$$

Moreover, we assume that the time-inhomogeneous Markov chain X has the generator

$$\mathsf{G}_t = \begin{cases} \mathsf{G}_1, & s_0 \le t < s_1, \\ \mathsf{G}_2, & s_1 \le t < s_2, \\ \mathsf{G}_3, & t \ge s_2, \end{cases}$$

where $0 < s_1 < s_2$.

We take the following inputs: $c = 0.5, v(e_+) = 2, v(e_-) = -3, s_1 = 2, s_2 = 8$,

$$\mathsf{G}_0 = \begin{array}{c c} & \begin{array}{cc} e_+ & e_- \end{array} \\ \begin{array}{c} e_+ \\ e_- \end{array} & \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \end{array}, \quad \mathsf{G}_1 = \begin{array}{c c} & \begin{array}{cc} e_+ & e_- \end{array} \\ \begin{array}{c} e_+ \\ e_- \end{array} & \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \end{array}, \quad \mathsf{G}_2 = \begin{array}{c c} & \begin{array}{cc} e_+ & e_- \end{array} \\ \begin{array}{c} e_+ \\ e_- \end{array} & \begin{bmatrix} -5 & 5 \\ 3 & -3 \end{bmatrix} \end{array}.$$

The following table compares our result and execution time with Monte-Carlo simulation (10000 paths).

Numerical Results		
Method	Wiener-Hopf	Monte-Carlo
$\Pi_c^-(e_+, e_-)$	0.6501	0.6462
Execution time	0.15 s	3.12 s

Remark 4.2. One can also compute $\Pi_c^+(e_-, e_+)$, if it is the quantity of interest in the model. Note that if we change the labels of the states from $\{e_+, e_-\}$ to $\{e_-, e_+\}$ and modify the inputs accordingly, we can compute $\Pi_c^+(e_-, e_+)$ using the same algorithm that computes $\Pi_c^-(e_+, e_-)$.

5 Appendix: Approximation of Multivariate Inverse Laplace Transform

For the convenience of the reader, we will briefly recall the basics of Laplace transform and its inverse. Then, we will proceed with an important result regarding the approximation of the multivariate inverse Laplace transform.

Let $f : [0, \infty)^n \to [0, \infty)$ be a Borel-measurable function such that

$$\int_0^\infty \cdots \int_0^\infty f(t_1, \ldots, t_n) \prod_{k=1}^n e^{-q_k t_k} \, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

exists for any $q_1, \ldots, q_n > 0$. Then, the multivariate Laplace transform $\widehat{f}$ of f, defined by

$$\widehat{f}(q_1, \ldots, q_n) = \mathcal{L}(f)(q_1, \ldots, q_n) := \int_0^\infty \cdots \int_0^\infty f(t_1, \ldots, t_n) \prod_{k=1}^n e^{-q_k t_k} \, \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

is well-defined for any $q_k \in \mathbb{C}^+$, $k = 1, \ldots, n$, where[1] $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \Re(z) > 0\}$ with $\Re(z)$ denoting the real part of $z \in \mathbb{C}$. The inverse multivariate Laplace transform of function $g : (\mathbb{C}^+)^n \to \mathbb{C}$, is the function $\breve{g}$, such that $\mathcal{L}(\breve{g}) = g$. We will also write $\breve{g} = \mathcal{L}^{-1}(g)$. The existence and uniqueness of the inverse Laplace transform is a well understood subject (cf. [Wid41]). Although there are explicit formulas of the inverse Laplace transform for many functions, generally speaking, in many practical situations the inverse Laplace transform of a function is computed by numerical approximation

[1] We will denote by $\Re(z)$ the real part of $z \in \mathbb{C}$, and $\mathrm{i} = \sqrt{-1}$ will be used to denote the imaginary unit.

technics. We refer the reader to [AW06], and the references therein, for a unified framework for numerically inverting the Laplace transform. For sake of completeness, we present here one such method - the Talbot inversion formula - for one and two dimensional case; the multidimensional case is done by analogy.

Assume that $\widehat{f}$ is the Laplace transform of a function $f : (0, +\infty) \to \mathbb{C}$. The Talbot inversion formula to approximate f is given by

$$f_M^b(t) = \frac{2}{5t} \sum_{k=0}^{M-1} \Re \left(\gamma_k \widehat{f}(\frac{\delta_k}{t}) \right), \tag{5.1}$$

where

$$\begin{aligned} \delta_0 &= \frac{2M}{5}, \quad \delta_k = \frac{22k\pi}{5}(\cot(\frac{k\pi}{M}) + \mathrm{i}), \quad 0 < k < M, \\ \gamma_0 &= \frac{1}{2}e^{\delta_0}, \quad \gamma_k = \left(1 + \mathrm{i}\frac{k\pi}{M}(1 + \cot^2(\frac{k\pi}{M})) - \mathrm{i}\cot(\frac{k\pi}{M}) \right) e^{\delta_k}, \quad 0 < k < M. \end{aligned} \tag{5.2}$$

Analogously, given a Laplace transform $\widehat{g}$ of a complex-valued function g of two non-negative real variables, the Talbot inversion formula to compute $g(t_1, t_2)$ numerically is given by

$$g_M^b(t_1, t_2) = \frac{2}{25t_1 t_2} \sum_{k_1=0}^{M-1} \Re \left\{ \gamma_{k_1} \sum_{k_2=0}^{M-1} \left[\gamma_{k_2} \widehat{g} \left(\frac{\delta_{k_1}}{t_1}, \frac{\delta_{k_2}}{t_2} \right) + \bar{\gamma}_{k_2} \widehat{g} \left(\frac{\delta_{k_1}}{t_1}, \frac{\bar{\delta}_{k_2}}{t_2} \right) \right] \right\},$$

where $\delta_k, \gamma_k, 0 \le k < M$, are given in (5.2).

5.1 A Special Case of Numerical Inverse Laplace Transform

Let us consider a function $f : [0, \infty) \to [0, \infty)$ and its Laplace transform $\widehat{f}(q)$, for $q \in \mathbb{C}^+$. It turns out that the inverse Laplace transform of f can be approximated numerically by using only values of the function $\widehat{f}$ on the positive real line. One such approximation is the Gaver-Stehfest formula

$$f_n(t) = \frac{n \log 2}{t} \binom{2n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \widehat{f} \left(\frac{(n+k) \log 2}{t} \right). \tag{5.3}$$

For other methods and the comparison of their speeds of convergence we refer to [AW06]. Consecutive application of (5.3) leads to the multivariate Gaver-Stehfest formula.

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