Parameter estimation for semilinear SPDEs from local measurements

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Abstract

This work contributes to the limited literature on estimating the diffusivity or drift coefficient of nonlinear SPDEs driven by additive noise. Assuming that the solution is measured locally in space and over a finite time interval, we show that the augmented maximum likelihood estimator introduced in [AR20] retains its asymptotic properties when used for semilinear SPDEs that satisfy some abstract, and verifiable, conditions. The proofs of asymptotic results are based on splitting the solution in linear and nonlinear parts and fine regularity properties in L^p-spaces. The obtained general results are applied to particular classes of equations, including stochastic reaction-diffusion equations. The stochastic Burgers equation, as an example with first order nonlinearity, is an interesting borderline case of the general results, and is treated by a Wiener chaos expansion. We conclude with numerical examples that validate the theoretical results.

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1 Introduction

While the statistical analysis of stochastic evolution equations, and stochastic partial differential equations (SPDEs) in particular, is becoming a mature research field, there are many problems left open that broadly can be streamlined

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into two directions: a) to consider larger and more diverse classes of equations, usually dictated by specific and practically important models; b) to develop new statistical methods and techniques that are theoretically sound and practically relevant. Up until recently, most of the literature on parameter estimation for SPDEs was rooted in the so-called spectral approach by assuming that the observations are obtained in the Fourier space over some finite time interval. For details on this classical method, as well as for general historical developments in this field, we refer to the survey [Cia18]. Recently, new methods have been developed to study statistical inference problems for linear SPDEs, notably the methodology based on local measurements introduced in [AR20], as well as several approaches dedicated to discrete sampling (cf. [CH19, BT19b, BT19a, Cho19a, Cho19b, CDVK19, KU19, KT19, CK20]), data assimilation ([CCH+19, NRR19]) and Bayesian inference ([RR20], [Yan19]). Many equations of practical relevance, however, are inherently nonlinear. The few works dealing with statistical inference for nonlinear SPDEs are [CGH11, PS20], both within the spectral approach.

In this paper we consider a general class of second order *semi-linear SPDEs* of the form[1]

$$\mathrm{d}X(t) = \vartheta\Delta X(t)\,\mathrm{d}t + F(t, X(t))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t \leq T,\ X(0) = X_0,$$

defined on an appropriate Hilbert space, endowed with zero boundary conditions on a bounded domain $\Lambda \subset \mathbb{R}^d$, and where ϑ is the parameter of interest, F is a (nonlinear) function, B a linear operator, and W a cylindrical Brownian motion. The main goal of this paper is to study the *estimation of diffusivity* (drift) parameter ϑ by amalgamating the *local measurements* approach of [AR20] with the *splitting of the solution* argument of [CGH11, PS20]. The approach of the present work can be summarized as follows: as noticed in [CGH11], and consequently generalized in [PS20] (within the spectral approach), the estimator of the diffusivity coefficient ϑ for linear SPDEs retains its asymptotic properties when applied to a nonlinear SPDE, given that the nonlinear part remains smoother than the linear part. Thus, one may argue that the *augmented maximum likelihood estimator* (augmented MLE) of ϑ introduced in [AR20] for linear SPDEs, defined by

$$\widehat{\vartheta}_\delta = \frac{\int_0^T X^{\Delta}_{\delta,x_0}(t)\,\mathrm{d}X_{\delta,x_0}(t)}{\int_0^T (X^{\Delta}_{\delta,x_0}(t))^2\,\mathrm{d}t},$$

and applied to nonlinear SPDEs enjoys similar asymptotic properties, where the observables $X_{\delta,x_0}(t)$, and respectively $X^{\Delta}_{\delta,x_0}(t)$, are obtained from integrating the solution X against a kernel K_{δ,x_0}, and respectively against $\Delta K_{\delta,x_0}$, assuming that K_{δ,x_0} has support in a δ-neighborhood of a fixed spatial point x_0 (hence local measurements).

[1] The equation is strictly defined in Section 2.

In the main result of this paper, we prove that indeed, for a large class of semi-linear SPDEs, the estimator $\widehat{\vartheta}_\delta$, as $\delta \to 0$, is a consistent and asymptotically normal estimator of ϑ. This shows that spatially localized measurements contain enough information to identify the coefficient next to the highest order derivative. This is in line with the conclusion of [AR20], as well as with the literature on discrete sampling[2] listed above, but contrary to the spectral approach, where by its very nature the solution has to be observed everywhere in the physical domain. We also note that, remarkably, $\widehat{\vartheta}_\delta$ does not depend on the geometry of the domain Λ or its dimension. Moreover, the estimation procedure remains valid even when the nonlinearity F, the covariance operator B or the initial data X_0 are unknown or misspecified, as is often the case in practice.

We emphasize that the obtained results and developed methods are not simple extensions of the linear case, but fundamentally exploit fine analytical properties of the solution. Similar to [CGH11, PS20] after splitting the solution in its linear and nonlinear parts, we utilize the extra regularity of the nonlinear part (or regularity gap) to prove that the 'nonlinear bias' asymptotically vanishes.

To derive optimal statistical results, we require precise control of the spatial regularity of the solution, by using higher order fractional L^p-Sobolev type spaces. While regularity of the solutions of SPDEs is certainly well-studied in the literature, for the reader's convenience, but also for sake of completeness, we provide in Appendix B a self-contained treatment of well-posedness of SPDEs relevant to the purposes of our study.

We present abstract conditions on F, B, K and X_0, that guarantee the desired asymptotic results for $\widehat{\vartheta}_\delta$. They are verified for particular equations, including stochastic reaction-diffusion equations and the stochastic Burgers equation. We will see that equations with first order nonlinearities, such as the stochastic Burgers equation, happen to constitute the extreme case, to which the abstract asymptotic normality results do not apply. We treat this case separately, by combining the regularity analysis of the solution with its Wiener chaos expansions. We believe this will serve as the foundation in studying the statistical properties of other nonlinear stochastic evolution equations.

We also note that the operator B is not required to commute with the Laplacian Δ, which is one of the core assumptions in the spectral approach. On the other hand, we treat only the parametric case, compared to [AR20], but the extension to nonparametric ϑ is straightforward, although computationally significantly more involved.

The paper is organized as follows. In Section 2 we set the stage, starting with notations, the SPDE model, continuing with the description of the

[2]It was shown that to estimate the diffusivity coefficient in a stochastic heat equation driven by an additive noise it is enough to sample the solution at one spacial point over a finite time interval.

statistical experiment and presenting the main object of this study - the augmented MLE. Also here, we discuss the splitting of the solution argument and introduce the main structural assumptions of the model inputs. Section 3 is dedicated to the main results. The asymptotic analysis of the estimator assuming a general framework is given in Section 3.1. In Section 3.2 we investigate, still in the general setup, fine regularity properties of the solution and the corresponding perturbation process. Consequently the abstract results are applied to particular equations; see Section 3.3 and references therein for real-life applications of the considered examples. In addition to already mentioned stochastic reaction-diffusion equations and Burgers equation, we also investigate the case of linear perturbations. Discussion and illustration of theoretical results is done in Section 4. Due to the technical and lengthy nature of the proofs, to streamline the presentation, the vast majority of results is presented and proved in Appendix A. Although the well-posedness and regularity properties of the solution are at the core of our analysis, we postpone them to Appendix B.

2 Preliminaries and the main problem

2.1 Notation

Let Λ be an open and bounded set in $\mathbb{R}^d$ with smooth boundary $\partial\Lambda$ and let $\langle\cdot,\cdot\rangle$ be the inner product in $L^2(\Lambda)$. For $p>1$ and any linear operator $A: L^p(U)\to L^p(U)$, where $U\subset\mathbb{R}^d$ is open, let $\|A\|_{L^p(U)}$ denote its operator norm. For $k\in\mathbb{N}_0$, $H^k(\mathbb{R}^d)$ are the usual L^2-Sobolev spaces. Let $\Delta z=\sum_{i=1}^d\partial_i^2 z$ denote the Laplace operator on $L^p(\Lambda)$, $p>1$, with zero boundary conditions. To describe higher regularities we consider for $s\in\mathbb{R}$ the fractional Laplacians $(-\Delta)^{s/2}$ on $L^p(\Lambda)$, cf. [Yag10], and denote their domains by $W^{s,p}(\Lambda):=\{u\in L^p(\Lambda):\|u\|_{s,p}<\infty\}$, where $\|\cdot\|_{s,p}:=\|\cdot\|_{W^{s,p}(\Lambda)}:=\|(-\Delta)^{s/2}\cdot\|_{L^p(\Lambda)}$. We also set $W^s(\Lambda):=W^{s,2}(\Lambda)$ and $\|\cdot\|_s:=\|\cdot\|_{s,2}$. The spaces $W^{s,p}(\Lambda)$ differ from the Sobolev spaces as defined e.g. in [Ada75], but they are subspaces of the classical Bessel potential spaces and allow for a Sobolev embedding theorem; for details, see [Tri83], [DdMH15], [Yag10, Section 16.5].

We fix a constant $\vartheta\in\mathbb{R}_+:=(0,\infty)$, that will play the role of the parameter of interest, and denote by $(S_\vartheta(t))_{t\geq0}$ the semigroup generated by $\vartheta\Delta$ on $L^2(\Lambda)$. Moreover, Δ_0 will stand for the Laplace operator on $\mathbb{R}^d$ with domain $H^2(\mathbb{R}^d)$ and with generated semigroup $(e^{t\Delta_0})_{t\geq0}$.

Throughout this work we fix a finite time horizon $T>0$. Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{0\leq t\leq T},\mathbb{P})$ be a filtered probability space carrying a cylindrical Brownian motion W on $L^2(\Lambda)$. Informally, $\dot{W}$ is referred to as *space-time* white noise. Throughout, all equalities and inequalities, unless otherwise mentioned, will be understood in the $\mathbb{P}$-a.s. sense. As usual, we will denote by $\xrightarrow{\mathbb{P}}$ the convergence in probability, and w-lim or $\xrightarrow{d}$ will stand for the convergence in distribution. Correspondingly, $a_n = o_{\mathbb{P}}(b_n)$ means that $a_n/b_n \xrightarrow{\mathbb{P}} 0$, and $a_n = \mathcal{O}_{\mathbb{P}}(b_n)$ means that there exists $M > 0$ such that $\mathbb{P}(|a_n/b_n| > M) \to 0$.

2.2 The SPDE model

Consider the semilinear stochastic partial differential equation

$$\begin{cases} \mathrm{d}X(t) = \vartheta\Delta X(t)\,\mathrm{d}t + F(t, X(t))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t \le T, \\ X(0) = X_0 \in L^2(\Lambda), \\ X(t)|_{\partial\Lambda} = 0, \quad 0 < t \le T, \end{cases} \tag{1}$$

where $F : [0, +\infty) \times \mathcal{H} \to L^2(\Lambda)$ is a Borel measurable function with a suitable chosen Hilbert space $\mathcal{H} \subset L^2(\Lambda)$ and a linear operator $B : L^2(\Lambda) \to L^2(\Lambda)$. The initial data X_0 is assumed to be deterministic.

In what follows, we always assume that (1) has a mild solution, namely that there exists an adapted process $X = (X(t))_{0 \le t \le T}$ with values in $L^2(\Lambda)$ and such that

$$X(t) = S_\vartheta(t)X_0 + \int_0^t S_\vartheta(t-s)F(s, X(s))\,\mathrm{d}s + \int_0^t S_\vartheta(t-s)B\,\mathrm{d}W(s). \tag{2}$$

In particular, we implicitly assume that all integrals in (2) are well-defined. Sufficient conditions for the existence and uniqueness of mild solutions are well-known (cf. [DPZ14]) and will be discussed for specific equations in Section 3.3. The choice to work with mild solutions is primarily dictated by the methods we use to establish fine analytical properties of X that are needed for the statistical analysis below.

On the other hand, the statistical experiment, which will be introduced in the next section, is based only on functionals of the form $\langle X(t), z\rangle$ for some test functions z. It is therefore enough to assume that X is a weak solution to (1). That is, X is an $L^2(\Lambda)$-valued adapted process such that for any test function $z \in W^2(\Lambda)$, we have

$$\langle X(t), z\rangle = \langle X_0, z\rangle + \int_0^t \langle X(s), \vartheta\Delta z\rangle\,\mathrm{d}s + \int_0^t \langle F(s, X(s)), z\rangle\,\mathrm{d}s + \langle W(t), B^* z\rangle. \tag{3}$$

Note that the mild solution X from (2) is always a weak solution; cf. [DPZ14, Theorem 5.4]. Generally speaking, considering a weak solution will allow for a larger class of operators B, including B being the identity operator and thus (1) driven by a space-time white noise. We also believe that all results on statistical inference in this paper hold true assuming only the existence of weak solution in $L^2(\Lambda)$, and detailed proofs of this are postponed to future works.

2.3 Statistical experiment

Following the setup from [AR20], we fix a spatial point $x_0 \in \Lambda$ around which the local measurements of the solution will be performed. Throughout, we will use the following notations: for any $z \in L^2(\mathbb{R}^d)$ and $\delta > 0$,

$$\Lambda_{\delta,x_0} := \delta^{-1}(\Lambda - x_0) = \{\delta^{-1}(x - x_0) \,:\, x \in \Lambda\},$$
$$z_{\delta,x_0}(x) := \delta^{-d/2} z(\delta^{-1}(x - x_0)), \quad x \in \mathbb{R}^d,$$

and we also set $\Lambda_{0,x_0} := \mathbb{R}^d$. For $\delta > 0$, denote by Δ_{δ,x_0} the Laplace operator on $L^2(\Lambda_{\delta,x_0})$ with domain $W^{2,p}(\Lambda_{\delta,x_0}$ and by $(S_{\vartheta,\delta,x_0}(t))_{t\geq 0}$ the semigroup generated by $\vartheta\Delta_{\delta,x_0}$ on $L^2(\Lambda_{\delta,x_0})$.

The measurements are obtained with respect to a fixed function (or kernel) $K \in H^2(\mathbb{R}^d)$ with compact support in Λ_{δ,x_0} such that $K_{\delta,x_0} \in W^2(\Lambda)$. Local measurements for the solution X of (1) at x_0 with resolution level δ on the time interval $[0, T]$ are given by the real-valued processes $X_{\delta,x_0} = (X_{\delta,x_0}(t))_{0\leq t\leq T}$, and $X^{\Delta}_{\delta,x_0} = (X^{\Delta}_{\delta,x_0}(t))_{0\leq t\leq T}$, where

$$X_{\delta,x_0}(t) := \langle X(t), K_{\delta,x_0}\rangle\,, \tag{4}$$
$$X^{\Delta}_{\delta,x_0}(t) := \langle X(t), \Delta K_{\delta,x_0}\rangle\,. \tag{5}$$

Note that $X^{\Delta}_{\delta,x_0}(t) = \Delta X_{\delta,\cdot}(t)|_{x=x_0}$ by convolution, and thus, $X^{\Delta}_{\delta,x_0}(t)$ can be computed by observing $X_{\delta,x}(t)$ for x in a neighborhood of x_0.

The statistical analysis requires additional assumptions on K which will be imposed below. We will also show that the variance of the proposed estimator depends on the choice of K; cf. Theorem 3. For typical examples of K see Section 4.

2.4 The estimator

As noticed in [CGH11], and consequently used and generalized in [PS20], the estimator of the diffusivity coefficient ϑ for linear SPDEs derived within the so called spectral approach retains its asymptotic properties when applied to a nonlinear SPDE, given that the nonlinear part does not 'dominate' the linear part. Thus, for the local measurements X_{δ,x_0}, X^{Δ}_{δ,x_0}, we take as ansatz the *augmented maximum likelihood estimator* (augmented MLE) of ϑ introduced in [AR20] for linear SPDEs, which is defined by

$$\widehat{\vartheta}_\delta = \frac{\int_0^T X^{\Delta}_{\delta,x_0}(t)\,\mathrm{d}X_{\delta,x_0}(t)}{\int_0^T (X^{\Delta}_{\delta,x_0}(t))^2\,\mathrm{d}t}. \tag{6}$$

As discussed in [AR20], this estimator is closely related to, but different from, the actual MLE, which cannot be computed in closed form, even for linear equations and constant ϑ. We also note that $\widehat{\vartheta}_\delta$ makes no explicit reference to

F or B, which are generally unknown to the observer and therefore treated here as nuisance.

From (3), clearly the dynamics of X_{δ,x_0} are given by

$$\mathrm{d}X_{\delta,x_0} = \vartheta X^{\Delta}_{\delta,x_0}\,\mathrm{d}t + \langle F(t,X(t)),K_{\delta,x_0}\rangle\,\mathrm{d}t + \|B^*K_{\delta,x_0}\|\,\mathrm{d}\bar{w}(t), \tag{7}$$

where $\bar{w}(t) := \langle W(t), B^*K_{\delta,x_0}\rangle/\|B^*K_{\delta,x_0}\|$ is a scalar Brownian motion, as long as $\|B^*K_{\delta,x_0}\|$ does not vanish, which is guaranteed to be true for small $\delta > 0$ (cf. Assumption B and the discussion therein). Using (6) and (7), we obtain the error decomposition

$$\widehat{\vartheta}_\delta = \vartheta + (\mathcal{I}_\delta)^{-1}R_\delta + (\mathcal{I}_\delta)^{-1}M_\delta, \tag{8}$$

where

$$\mathcal{I}_\delta := \|B^*K_{\delta,x_0}\|^{-2}\int_0^T (X^{\Delta}_{\delta,x_0}(t))^2\,\mathrm{d}t, \qquad \text{(observed Fisher information)}$$

$$R_\delta := \|B^*K_{\delta,x_0}\|^{-2}\int_0^T X^{\Delta}_{\delta,x_0}(t)\langle F(t,X(t)),K_{\delta,x_0}\rangle\,\mathrm{d}t, \qquad \text{(nonlinear bias)}$$

$$M_\delta := \|B^*K_{\delta,x_0}\|^{-1}\int_0^T X^{\Delta}_{\delta,x_0}(t)\,\mathrm{d}\bar{w}(t). \qquad \text{(martingale part)}$$

The nonlinear bias R_δ accounts for not observing $(\langle F(t,X(t)),K_{\delta,x_0}\rangle)_{0\le t\le T}$. The observed Fisher information $\mathcal{I}_\delta$ does not correspond to the Fisher information of the statistical model, although it plays a similar role here in the sense that $\mathcal{I}_\delta \to \infty$ means 'increasing information', and hence yields consistent estimation. In view of (7), the decomposition (8) is essentially obtained from the 'whitened' process $X_{\delta,x_0}/\|B^*K_{\delta,x_0}\|$. The statistical performance of $\widehat{\vartheta}_\delta$ is therefore not affected by B, as $\delta \to 0$, as we will see below. This is in stark contrast to the regularity properties of X, which improve as B becomes more smoothing.

Using the decomposition (8), to prove consistency, it is enough to show that $(\mathcal{I}_\delta)^{-1}R_\delta$ and $(\mathcal{I}_\delta)^{-1}M_\delta$ vanish, as $\delta \to 0$, and to prove asymptotic normality, we will show that $\delta^{-1}(\mathcal{I}_\delta)^{-1}R_\delta \to 0$, while $\delta^{-1}(\mathcal{I}_\delta)^{-1}M_\delta$ converges in distribution to a Gaussian random variable.

2.5 The splitting argument and main model assumptions

In this section, we list high level structural assumptions on the model inputs that will guarantee the desired asymptotic properties of $\widehat{\vartheta}_\delta$. These assumptions will be implied by verifiable conditions on the nonlinear term F, the operator B and the initial condition X_0 in Sections 3.2 and 3.3.

Similar to [CGH11, PS20] we use the 'splitting of the solution' argument. Namely, consider the $L^2(\Lambda)$-valued process $\bar{X} = (\bar{X}(t))_{0\le t\le T}$ given by

$$\bar{X}(t) = \int_0^t S_\vartheta(t-s)B\,\mathrm{d}W(s). \tag{9}$$

Analogous to (2), $\bar{X}$ is a mild solution to the corresponding linear equation

$$\mathrm{d}\bar{X}(t) = \vartheta\Delta\bar{X}(t)\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t \leq T, \quad \bar{X}(0) = 0. \tag{10}$$

Then, the nonlinear part $\widetilde{X} := X - \bar{X}$ satisfies

$$\widetilde{X}(t) = S_\vartheta(t)X_0 + \int_0^t S_\vartheta(t-s)F(s, \bar{X}(s) + \widetilde{X}(s))\mathrm{d}s, \quad 0 \leq t \leq T, \tag{11}$$

namely, it solves the partial differential equation with random coefficients given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{X}(t) = \vartheta\Delta\widetilde{X}(t) + F(t, \bar{X}(t) + \widetilde{X}(t)), \quad 0 < t \leq T, \quad \widetilde{X}(0) = X_0. \tag{12}$$

With this at hand, the statistical properties of the local measurements in (4) and (5) can be studied separately for the linear parts $\bar{X}_{\delta,x_0}(t) := \langle \bar{X}(t), K_{\delta,x_0}\rangle$, $\bar{X}^\Delta_{\delta,x_0}(t) := \langle \bar{X}(t), \Delta K_{\delta,x_0}\rangle$ and the corresponding nonlinear parts $\widetilde{X}_{\delta,x_0}(t)$, $\widetilde{X}^\Delta_{\delta,x_0}(t)$.

Using (9), we first note that $\bar{X}_{\delta,x_0}$, $\bar{X}^\Delta_{\delta,x_0}$ are centered Gaussian processes. Following similar arguments as in [AR20], exact limits of their covariance functions, as $\delta \to 0$, will be obtained after appropriate scaling by analyzing the actions of $\vartheta\Delta$ and $S_\vartheta(t)$ on the localized functions z_{δ,x_0}; see Section A.2. These limits are non-degenerate only under certain scaling assumptions on B and K.

Assumption *B*. *There exists a constant $\gamma > d/4 - 1/2$, $\gamma \geq 0$, together with a family of linear and bounded operators $(B_{\delta,x_0},\ \delta \geq 0)$, $B_{\delta,x_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that*

$$B^*(-\Delta)^\gamma z_{\delta,x_0} = (B^*_{\delta,x_0} z)_{\delta,x_0}, \quad \delta > 0, \tag{13}$$

for any smooth function z supported in Λ_{δ,x_0}, and such that $B^*_{\delta,x_0} z \to B^*_{0,x_0} z$ in $L^2(\mathbb{R}^d)$, for $\delta \to 0$ and $z \in L^2(\mathbb{R}^d)$.

Assumption *K*. *There exists a function $\widetilde{K} \in H^{2\lceil\gamma\rceil+2}(\mathbb{R}^d)$ with compact support in Λ_{δ,x_0} such that $K = \Delta^{\lceil\gamma\rceil}\widetilde{K}$.*

Assumption *ND*. *With $\Psi(z) := \int_0^\infty \|B^*_{0,x_0} e^{s\Delta_0} z\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s$, $z \in L^2(\mathbb{R}^d)$, assume that $\|B^*_{0,x_0}(-\Delta_0)^{\lceil\gamma\rceil-\gamma}\widetilde{K}\|_{L^2(\mathbb{R}^d)} > 0$, $\Psi((-\Delta_0)^{\lceil\gamma\rceil-\gamma}\Delta\widetilde{K}) > 0$.*

In view of the error decomposition (8), the next assumption imposes mild conditions on $\widetilde{X}$ and F that allow to reduce the entire line of reasoning to the linear case; see Proposition 2. In Section 3.2 we show that these conditions are implied by spatial regularity of $\widetilde{X}$ under verifiable conditions on the nonlinearity F.

Assumption *F*. *There exists $\nu > 0$ such that*

$$\int_0^T (\widetilde{X}^{\Delta}_{\delta,x_0}(t))^2 \mathrm{d}t = o_{\mathbb{P}}(\delta^{-2+4\gamma}), \tag{14}$$

$$\int_0^T \langle F(t, X(t)), K_{\delta,x_0}\rangle^2 \,\mathrm{d}t = \mathcal{O}_{\mathbb{P}}(\delta^{2\nu-2+4\gamma}). \tag{15}$$

Assumption B essentially requires that the operator B scales locally as the fractional Laplacian $(-\Delta)^{-\gamma}$. For $\gamma > d/4 - 1/2$ this guarantees that the linear process (9) takes values in $L^2(\Lambda)$; see Proposition 30. From the scaling of the fractional Laplacian on localized functions z_{δ,x_0} (see Lemma 14) it follows that there exists at most one γ satisfying (13) with a non-degenerate operator B_{0,x_0}. Moreover, γ can be estimated from the observed data. Indeed, having a continuous path of X_{δ,x_0}, for $\delta > 0$, at our disposal, one can compute its quadratic variation, which equals $T\|B^*K_{\delta,x_0}\|^2$, cf. (3) or (7). Finally, by (32), we have that, as $\delta \to 0$, $\delta^{2\gamma}T\|B^*K_{\delta,x_0}\|^2$ converges to a non-degenerate limit, from which γ can be uniquely determined.

Clearly, Assumption B and the Banach-Steinhaus theorem imply that

$$\sup_{0<\delta\leq 1} \|B^*_\delta\| < \infty. \tag{16}$$

Assumptions K and ND are necessary to ensure non-degenerate variances for $\widehat{\vartheta}_\delta$; see Theorem 3 and the fact that

$$\begin{aligned}\Psi((-\Delta_0)^{\lceil\gamma\rceil-\gamma}\Delta\widetilde{K}) \leq& \|B^*_{0,x_0}\|^2_{L^2(\mathbb{R}^d)} \int_0^\infty \|e^{s\Delta_0}\Delta_0(-\Delta_0)^{\lceil\gamma\rceil-\gamma}\widetilde{K}\|^2_{L^2(\mathbb{R}^d)} \,\mathrm{d}s \\ =& \|B^*_{0,x_0}\|^2_{L^2(\mathbb{R}^d)} \frac{1}{2} \|(-\Delta_0)^{1/2+\lceil\gamma\rceil-\gamma}\widetilde{K}\|^2_{L^2(\mathbb{R}^d)} < \infty,\end{aligned}$$

which follows by Lemma 16 and $1/2 + \lceil\gamma\rceil - \gamma > 0$.

We also note that $X_{\delta,x_0}(t) = \Delta^{\lceil\gamma\rceil}\langle X(t), \widetilde{K}_{\delta,\cdot}\rangle|_{x=x_0}$. Thus, given that Assumption K is fulfilled and analogous to the remark after (5), the local measurements $X_{\delta,x_0}(t)$ in (4) can be obtained by observing $\langle X(t), \widetilde{K}_{\delta,x}\rangle$ for x in a neighborhood of x_0.

Next we present a few examples illustrating Assumptions B and ND.

Example 1. (i) Let γ be as in Assumption B. For a smooth function $\sigma \in C^\infty(\mathbb{R}^d)$, $\sigma(x_0) \neq 0$, define the multiplication operator $M_\sigma z = \sigma \cdot z$, and consider the linear operator $B = M_\sigma(-\Delta)^{-\gamma}$. Note that B does not commute with Δ nor with the semigroup $S(t)$, unless σ is constant. Then $B^* = (-\Delta)^{-\gamma}M_\sigma$ and according to Lemmas 14 and 18 we have

$$B^*_{\delta,x_0}z = (-\Delta_{\delta,x_0})^{-\gamma}M_{\sigma(\delta\cdot+x_0)}(-\Delta_{\delta,x_0})^{\gamma}z, \quad z \in C_c^\infty(\overline{\Lambda}_{\delta,x_0}).$$

By Lemma 18, B^*_{δ,x_0} extends to a bounded operator $B^*_{\delta,x_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ satisfying $B^*_{\delta,x_0}z \to B^*_{0,x_0}z := M_{\sigma(x_0)}z$ for $z \in L^2(\mathbb{R}^d)$. Moreover,

$$\Psi((-\Delta_0)^{\lceil\gamma\rceil-\gamma}\Delta\widetilde{K}) = \frac{\sigma^2(x_0)}{2}\|(-\Delta_0)^{1/2+\lceil\gamma\rceil-\gamma}\widetilde{K}\|^2_{L^2(\mathbb{R}^d)}.$$

Assumptions B and ND are satisfied as long as $(-\Delta_0)^{1/2+\lceil\gamma\rceil-\gamma}\widetilde{K}$ is not identically zero. For integer γ and using integration by parts the last display simplifies to $\frac{\sigma^2(x_0)}{2}\|\nabla\widetilde{K}\|^2_{L^2(\mathbb{R}^d)}$.

(ii) Let now $B=(-\Delta)^{-\gamma}M_\sigma$ for a γ as in Assumption B and $\sigma\in C(\overline{\Lambda})$, $\sigma(x_0)\neq 0$. Clearly, $B^*=M_\sigma(-\Delta)^{-\gamma}$ and we immediately obtain $B^*_{\delta,x_0}=M_\sigma$, $B^*_{0,x_0}=M_{\sigma(x_0)}$, and Ψ is as in (i).

(iii) With γ and σ as in (i) let $B=M_\sigma(-\Delta)^{-\gamma}+(-A)^{-\gamma'}$, where $\gamma<\gamma'$ and $A=\Delta-b$ for a constant $b>0$. Note that $A^*=A$ and by Lemma 13, $Az_{\delta,x_0}=\delta^{-2}(\Delta_{\delta,x_0}z-\delta^2bz)_{\delta,x_0}$. Moreover, $\|(-A)^{-\gamma'}(-\Delta)^{\gamma}z_{\delta,x_0}\|_{L^2(\mathbb{R}^d)}\to 0$. Therefore, $B^*_{\delta,x_0}z$ is as in (i), up to a small perturbation of order $o(1)$ that may depend on z, and hence B^*_{0,x_0}, Ψ are again as in (i).

3 Main results

In this section we present the main results of this paper, starting with the asymptotic properties of the augmented MLE $\widehat{\vartheta}_\delta$ pertinent to (1) in its abstract form, and then discussing refinements to Assumption F. In the second part, we consider several important classes of particular equations. Most of the examples are treated by applying the abstract results. However, we also study examples, that do not fall under the general theory, and which we treat by a different approach. Proofs of technical results are postponed to Appendix A.3.

3.1 Asymptotic analysis of the estimator

We study first the observed Fisher information. In view of the splitting argument let

$$\bar{\mathcal{I}}_\delta := \|B^*K_{\delta,x_0}\|^{-2}\int_0^T(\bar{X}^\Delta_{\delta,x_0}(t))^2\,\mathrm{d}t$$

denote the observed Fisher information corresponding to the linear part.

Proposition 2. *Assume that the Assumptions B, K and ND are satisfied. Then, as $\delta\to 0$, the following asymptotics hold true:*

(i) $\delta^2\mathbb{E}[\bar{\mathcal{I}}_\delta]\to(\vartheta\Sigma)^{-1}$, *where*

$$\Sigma := T^{-1}\|B^*_{0,x_0}(-\Delta_0)^{\lceil\gamma\rceil-\gamma}\widetilde{K}\|^2_{L^2(\mathbb{R}^d)}\Psi((-\Delta_0)^{\lceil\gamma\rceil-\gamma}\Delta\widetilde{K})^{-1};$$

(ii) $\bar{\mathcal{I}}_\delta/\mathbb{E}[\bar{\mathcal{I}}_\delta]\xrightarrow{\mathbb{P}}1.$

In addition, if Assumption F is satisfied, then:

(iii) $\mathcal{I}_\delta=\bar{\mathcal{I}}_\delta+o_{\mathbb{P}}(\delta^{-2})$;

(iv) $\mathcal{I}_\delta^{-1} R_\delta = \mathcal{O}_{\mathbb{P}}(\delta^\nu)$.

Proof. The proof is deferred to Appendix A.3. □

Now we are in the position to present our first main result.

Theorem 3. *Assume that the Assumptions B, K, ND and F are satisfied. Then the following assertions hold true:*

(i) $\widehat{\vartheta}_\delta$ *is a consistent estimator of* ϑ *and*

$$\widehat{\vartheta}_\delta = \vartheta + \mathcal{O}_{\mathbb{P}}(\delta^\nu) + \mathcal{O}_{\mathbb{P}}(\delta); \tag{17}$$

(ii) If $\nu > 1$*, then* $\widehat{\vartheta}_\delta$ *is asymptotically normal and*

$$\operatorname*{w-lim}_{\delta\to 0} \delta^{-1}(\widehat{\vartheta}_\delta - \vartheta) = \mathcal{N}\left(0, \vartheta\Sigma\right), \tag{18}$$

with Σ *as in Proposition 2(i).*

Proof. Consider the error decomposition (8) and let

$$Y_t^{(\delta)} := \|B^* K_{\delta,x_0}\|^{-1} X^{\Delta}_{\delta,x_0}(t)/\mathbb{E}[\bar{\mathcal{I}}_\delta]^{1/2}.$$

Thus, $M_\delta/\mathbb{E}[\bar{\mathcal{I}}_\delta]^{1/2} = \int_0^T Y_t^{(\delta)} d\bar{w}(t)$. By Proposition 2(i)-(iii) we obtain that

$$\mathcal{I}_\delta/\mathbb{E}[\bar{\mathcal{I}}_\delta] = (\bar{\mathcal{I}}_\delta + o_{\mathbb{P}}(\delta^{-2}))/\mathbb{E}[\bar{\mathcal{I}}_\delta] \xrightarrow{\mathbb{P}} 1,$$

such that the quadratic variation of $M_\delta/\mathbb{E}[\bar{\mathcal{I}}_\delta]^{1/2}$ satisfies $\int_0^T (Y_t^{(\delta)})^2 \,\mathrm{d}t = \mathcal{I}_\delta/\mathbb{E}[\bar{\mathcal{I}}_\delta] \xrightarrow{\mathbb{P}} 1$. From here, by a standard central limit theorem for continuous martingales (cf. [LS89, Theorem 5.5.4]), we obtain that $M_\delta/\mathbb{E}[\bar{\mathcal{I}}_\delta]^{1/2} \xrightarrow{d} \mathcal{N}(0,1)$. We also note that in view of Proposition 2(i)-(ii), $\delta^{1/2}\mathbb{E}[\bar{\mathcal{I}}_\delta]^{1/2} \to (\vartheta\Sigma)^{-1/2}$, as $\delta \to 0$. Using the above, as well as (8) and Proposition 2(iv), the identity (17) follows at once. Similarly and by employing Slutsky's Lemma, we obtain (18). The proof is complete. □

We note that the term $\mathcal{O}_{\mathbb{P}}(\delta^\nu)$ in (17), that corresponds to the nonlinear bias, will be the dominating term if $\nu < 1$, and thus the asymptotic normality result (18) does not hold. Obtaining a central limit theorem in the critical case $\nu = 1$ is usually a challenging problem, and generally speaking has to be treated on case-by-case basis; examples for this are discussed in Section 3.3.

The rate δ is minimax optimal for the model subclass when $F = 0$ and only X_{δ,x_0} is observed; cf. [AR20, Proposition 5.12]. It is therefore also minimax optimal for a general nonlinearity satisfying Assumption F with $\nu > 1$. We conjecture that the rate δ^ν for $\nu \leq 1$ is also minimax optimal.

The broad specifications of F, B and K allow for application of the asymptotic results to a wide range of SPDEs. We also emphasize that the asymptotic variance $\vartheta\Sigma$ for $\nu > 1$ depends on B only locally, while F does not appear at all. Therefore, the augmented MLE is robust to the misspecification of F and B, which practically speaking are often difficult to model exactly.

Example 4. In the setup of Example 1(i), grant Assumptions K and F for some $\nu > 1$. In this case, Σ can be computed explicitly, it is independent of $B^*_{0,x_0} = M_{\sigma(x_0)}$, and we have that

$$\operatorname*{w-lim}_{\delta\to 0} \delta^{-1}(\widehat{\vartheta}_\delta - \vartheta) = \mathcal{N}\left(0, \frac{2\vartheta\|(-\Delta_0)^{\lceil\gamma\rceil-\gamma}\widetilde{K}\|^2_{L^2(\mathbb{R}^d)}}{T\|(-\Delta_0)^{1/2+\lceil\gamma\rceil-\gamma}\widetilde{K}\|^2_{L^2(\mathbb{R}^d)}}\right).$$

Morevover, for integer γ, the asymptotic variance is equal to $2\vartheta T^{-1}\|\widetilde{K}\|^2_{L^2(\mathbb{R}^d)}\|\nabla\widetilde{K}\|^{-2}_{L^2(\mathbb{R}^d)}$.

For $\nu > 1$, there is no asymptotic bias in (18) and since the asymptotic variance depends linearly on the unknown parameter, one can easily deduce a confidence interval for ϑ as in [AR20, Corollary 5.6].

Corollary 5. *Assume that Assumptions B, K, ND and F are satisfied for $\nu > 1$. For $0 < \alpha < 1$, let*

$$I_{1-\alpha} = \left[\widehat{\vartheta}_\delta - \delta\mathcal{I}_\delta^{-1/2}q_{1-\alpha/2}, \widehat{\vartheta}_\delta + \delta\mathcal{I}_\delta^{-1/2}q_{1-\alpha/2}\right],$$

where q_β is the β-quantile of the standard normal distribution. Then, $I_{1-\alpha}$ is a confidence interval for ϑ with asymptotic coverage $1-\alpha$, as $\delta \to 0$.

Proof. By Proposition 2 we have $\mathcal{I}_\delta \to (\vartheta\Sigma)^{-1}$. Theorem 3(ii) and Slutsky's lemma then show that

$$\operatorname*{w-lim}_{\delta\to 0} \delta^{-1}\mathcal{I}_\delta^{1/2}\left(\widehat{\vartheta}_\delta - \vartheta\right) = \mathcal{N}(0,1).$$

This yields $\lim_{\delta\to 0}\mathbb{P}(\vartheta \in I_{1-\alpha}) = 1-\alpha$. □

3.2 Higher regularity of the perturbation process

We turn our attention to the problem of verifying Assumption F. Inspired by the perturbation argument of [CGH11, PS20], we study the spatial regularity of the processes $\bar{X}$ and $\widetilde{X}$. Aiming to obtain optimal regularity rates that exploit the localization under the kernel K, we consider the spaces $W^{s,p}(\Lambda)$ introduced in Section 2.

For $p \geq 2$, denote the L^p-regularity index of the linear process by $\bar{s}(p)$, where

$$\bar{s}(p) = \sup\{s \in \mathbb{R} : \bar{X} \in C([0,T]; W^{s,p}(\Lambda))\}, \quad \mathbb{P}\text{-a.s.} \tag{19}$$

Under Assumption B it can be shown (see Appendix B.1) that

$$\max(s^* - d/2 + d/p, 0) \leq \bar{s}(p) \leq s^* := 1 + 2\gamma - d/2.$$

The constant s^* should be viewed as the 'optimal expected spatial regularity' of $\bar{X}$, while $\bar{s}(p)$ depends on the geometry of the domain Λ and strict inequality may occur. Nevertheless, $\bar{s}(p) = s^*$ for rectangular domains in any dimension, and thus in particular if $d = 1$. Note that Theorem 3 and Theorem 7 below can be shown to hold also for non-smooth boundaries $\partial\Lambda$, as long as the eigenfunctions of the Laplacian are smooth on $\bar{\Lambda}$, which is true for rectangular domains.

Let us introduce the following growth condition on F, parametrized by $s, \eta \in \mathbb{R}$, $p \geq 2$, which allows to transfer the linear regularity to $\widetilde{X}$.

Assumption $A_{s,\eta,p}$. *We have $F(t,u) \equiv F(u)$, and there exist $\varepsilon > 0$ and a continuous function $g : [0,\infty) \to [0,\infty)$ such that*

$$\|F(u)\|_{s+\eta-2+\varepsilon,p} \leq g(\|u\|_{s,p}), \quad u \in W^{s,p}(\Lambda).$$

Without loss of generality, we can assume that g is non-decreasing (otherwise replace g with $x \mapsto \sup_{0\leq y\leq x} g(y)$). As we will see in the next section, the term $2-\eta$ should be understood as the order of F in the sense of a differential operator.

Proposition 6. *Let $p \geq 2$, $2 \leq p_1 \leq p$ and let $0 \leq s_1 < \bar{s}(p)$. Assume that*

$$X_0 \in W^{\bar{s}(p)+\eta,p}(\Lambda), \quad \widetilde{X} \in C([0,T]; W^{s_1,p_1}(\Lambda)),$$

and suppose that Assumption $A_{s,\eta,p'}$ holds true for some $\eta > 0$ and all $s_1 \leq s < \bar{s}(p)$, $p_1 \leq p' \leq p$. Then $\widetilde{X} \in C([0,T]; W^{\bar{s}(p)+\eta,p}(\Lambda))$. In particular, $X \in C([0,T]; W^{s,p}(\Lambda))$ for all $s < \bar{s}(p)$.

This shows that $\widetilde{X}$ is more regular in space than $\bar{X}$ with *excess regularity* η. Note that existence results for semilinear SPDEs typically provide some minimal spatial L^2-Sobolev regularity for the solution X, and thus for $\widetilde{X}$; see [LR15] or Lemma 33 below, assuming additional local Lipschitz and coercivity conditions.

Theorem 7. *Grant the assumptions of Proposition 6 and assume that $\eta > s^* - \bar{s}(p) + d/p$. Then Assumption F holds true with*

$$\nu = (\eta - (s^* - \bar{s}(p) + d/p)) \wedge 5/4.$$

Proof. Fix $\nu > 0$ as in the statement and let $r := s^* + \nu + d/p < \bar{s}(p) + \eta$, such that $\int_0^T \|\widetilde{X}(t)\|_{r,p}^2 \mathrm{d}t < \infty$ by Proposition 6. Applying Lemma 17 below with r and $q = p/(p-1)$, this means for a constant $C < \infty$

$$\begin{aligned}\int_0^T \widetilde{X}^{\Delta}_{\delta,x_0}(t)^2 \mathrm{d}t &= \int_0^T \langle \widetilde{X}(t), \delta^{-2}(\Delta K)_{\delta,x_0}\rangle^2 \mathrm{d}t \\ &\leq C\delta^{-4+2r+2d(\frac{1}{2}-\frac{1}{p})} \|(-\Delta_{\delta,x_0})^{-\frac{r-2}{2}} K\|^2_{L^q(\Lambda_{\delta,x_0})}.\end{aligned}$$

From $s^* = 1 + 2\gamma - d/2$ and $\nu \leq 5/4$, $p \geq 2$, we find that $r - 2 < 2\lceil\gamma\rceil + d/p$. The norm in the last line is therefore bounded according to Lemma 21(i). On the other hand, $-4+2r+2d(1/2-1/p)=2\nu-2+4\gamma$. As $\nu>0$, we conclude that (14) from Assumption F is fulfilled. In the same way, (15) is obtained from $\int_0^T \|F(X(t))\|_{r-2,p}^2 \mathrm{d}t<\infty$ and

$$\int_0^T \langle F(X(t)), K_{\delta,x_0}\rangle^2 dt = \int_0^T \langle (-\Delta)^{-1}F(X(t)), \delta^{-2}(\Delta K)_{\delta,x_0}\rangle^2 dt.$$

Indeed, with ε and g from Assumption $A_{s,\eta,p}$ and $0<\varepsilon'<\varepsilon$, this follows from Assumption $A_{s,\eta,p}$ with $s=\bar{s}(p)-\varepsilon'$ such that $r-2<s+\eta-2+\varepsilon$ and

$$\sup_{0\leq t\leq T}\|F(X(t))\|_{r-2,p} \leq g\left(\sup_{0\leq t\leq T}\|X(t)\|_{s,p}\right)<\infty,$$

because $X\in C([0,T];W^{s,p}(\Lambda))$ by Proposition 6. □

In the setting of Theorem 3, this result means that asymptotic normality of $\widehat{\vartheta}_\delta$ holds as soon as $\eta>s^*-\bar{s}(p)+d/p+1$, while $\widehat{\vartheta}_\delta$ is consistent if $\eta>s^*-\bar{s}(p)+d/p$. By Proposition 6 and the Sobolev embedding, this is true if $\tilde{X}$ takes values in $C^{s^*+1}(\Lambda)$ or in $C^{s^*}(\Lambda)$, respectively. If $\bar{s}(p)=s^*$ and if Proposition 6 can be applied for all $p\geq 2$, then Theorem 7 yields $\nu\geq\eta\wedge 5/4$, and therefore does not depend on the dimension d explicitly. Compared to this, the L^2-perturbation results for the spectral approach of [CGH11], [PS20] depend heavily on the dimension, with slower convergence rates for estimators of ϑ in higher dimensions. It is an interesting question if L^p-regularity for $p>2$ can improve results also for the spectral approach.

3.3 Results for particular equations

Let us apply Theorems 3 and 7 to SPDEs with specific nonlinearities. We always assume that Assumptions B, K, ND are satisfied, which already implies well-posedness of the linear part $\bar{X}$ and allows us to define the 'linear regularity gap'

$$s_{\text{gap}}=s^*-\inf_{p\geq 2}\bar{s}(p),$$

which satisfies $0\leq s_{\text{gap}}\leq d/2$; cf. Section B.1. Recall also that $s_{\text{gap}}=0$ for rectangular domains, in particular when $d=1$. For simplicity, the initial value X_0 is always assumed to satisfy $X_0\in W^{\bar{s}(p)+\eta,p}(\Lambda)$ for all $p\geq 2$ and with η to be determined, in order to apply Proposition 6.

Note that the results discussed in this section can be combined to apply to more general SPDEs by considering composite nonlinearities of the form $F(u)=a_1F_1(u)+a_2F_2(u)$ for smooth functions a_1,a_2. Indeed, if F_1 and F_2 satisfy Assumption $A_{s,\eta_1,p}$ and $A_{s,\eta_2,p}$, respectively, for $p\geq 2$ and $s,\eta_1,\eta_2\in\mathbb{R}$, then F satisfies Assumption $A_{s,\eta,p}$ with $\eta=\eta_1\wedge\eta_2$, using Lemma 18. In this sense, the results are robust under misspecification of certain lower order terms in the nonlinear part.

3.3.1 Linear perturbations

For an instructive example let us study the linear equation

$$\mathrm{d}X(t) = (\vartheta\Delta X(t) + \beta(-\Delta)^{\alpha}X(t))\mathrm{d}t + B\mathrm{d}W(t), \tag{20}$$

for a constant $\beta \in \mathbb{R}$ and $\alpha \leq 2$, such that the nonlinearity corresponds to the linear differential operator $F(u) = \beta(-\Delta)^{\alpha}u$. For applications of linear SPDEs see e.g. [Wal81], [Fra85], [Con05]. Well-posedness follows as for $\bar{X}$ under Assumption B and with X_0 sufficiently regular; cf. Section B.1. Assumption $A_{s,\eta,p}$ is easily verified to hold for all $p \geq 2$, $s \in \mathbb{R}$ and excess regularity $\eta < 2 - \alpha$ with $g(x) = x$.

It is clear that ϑ cannot be consistently estimated for $\alpha = 2$, if β is unknown. For simplicity, let as assume $s_{\text{gap}} = 0$. Then we get from Theorems 3 and 7 the following result.

Theorem 8. *Consider the mild solution to* (20) *and assume that* $s_{\text{gap}} = 0$, $\alpha < 2$. *Then:*

(i) $\widehat{\vartheta}_{\delta}$ *is a consistent estimator of* ϑ *with* $\widehat{\vartheta}_{\delta} = \vartheta + O_{\mathbb{P}}(\delta^{2-\alpha})$, *as* $\delta \to 0$.

(ii) If $\alpha \leq 1$, *then* $\widehat{\vartheta}_{\delta}$ *is a consistent and asymptotically normal estimator of* ϑ *satisfying* (18).

In the critical case $\alpha = 1$ it is *a-priori* not clear if a CLT for $\widehat{\vartheta}_{\delta}$ holds at the optimal rate δ. By an explicit computation for the bias as in [AR20, Theorem 5.3], however, this can be shown to be true, and we leave the details to the reader; cf. also the proof of Theorem 11 below. Note that the results of [AR20] are obtained for $\gamma = 0$.

3.3.2 Stochastic reaction-diffusion equations

Let us consider the equation

$$\mathrm{d}X(t,x) = (\vartheta\Delta X(t,x) + f(X(t,x)))\mathrm{d}t + B\mathrm{d}W(t,x), \quad x \in \Lambda, \tag{21}$$

where the nonlinearity $F(u)(x) = f(u(x))$ is a Nemytskii operator for a function $f : \mathbb{R} \to \mathbb{R}$. These equations are ubiquitous in physics, chemistry, biology and neuroscience, see e.g. [ASB18], [NASY62], [Fit61], [Sch72], [CA77].

Important examples are polynomial nonlinearities

$$f(x) = a_m x^m + \cdots + a_1 x + a_0, \quad x \in \mathbb{R}, \tag{22}$$

with $m \in 2\mathbb{N} + 1$ and $a_m < 0$. For a numerical example see Section 4. Theorem 36 gives sufficient conditions to guarantee that (21) is well-posed in $C([0,T]; W^{s,p}(\Lambda))$ for $d \leq 3$, $p \geq 2$ and $s > d/p$. Verification of Assumption $A_{s,\eta,p}$ is done by the following lemma.

Lemma 9. *For $0 \leq \alpha < 2$ let D_α be a differential operator of order α, i.e. $D_\alpha : W^{s+\alpha,p}(\Lambda) \to W^{s,p}(\Lambda)$ is bounded for any $p \geq 2$, $s \in \mathbb{R}$. Let $Q : \mathbb{R} \to \mathbb{R}$ be a polynomial function and assume $F(u) = D_\alpha Q(u)$. Then Assumption $A_{s,\eta,p}$ holds for any $0 \leq \eta < 2 - \alpha$, $p \geq 2$ and $s > d/p$.*

Proof. Set $\varepsilon := 2-\alpha-\eta$. Without loss of generality assume $Q(x) = x^k$ for $k \geq 0$. The case $k = 0$ is clear, so let $k > 1$. For $s > d/p$, the space $W^{s,p}(\Lambda)$ is closed under multiplication; cf. [Tri83]. This yields $\|D_\alpha(x^k)\|_{s+\eta-2+\varepsilon,p} \leq C\|x^k\|_{s,p} \leq \tilde{C}\|x\|_{s,p}^k$ for two absolute constants $C, \tilde{C} < \infty$, implying Assumption $A_{s,\eta,p}$. □

For a second class of stochastic reaction diffusion equations consider $f \in C_b^\infty(\mathbb{R})$, which is the space of smooth functions with bounded derivatives. For a concrete application see [LLB15]. In this case, well-posedness of (1) in $C([0,T]; W^{s,p}(\Lambda))$ for $p \geq 2$ and $s > d/p$ follows from Theorem 40.

Theorem 10. *Assume that (21) is well-posed in $C([0,T]; W^{s,p}(\Lambda))$ for all $p \geq 2$ and $s > d/p$, where f is either as in (22) or $f \in C_b^\infty(\mathbb{R})$. Then:*

(i) If $s_{\text{gap}} < 2$, in particular if $d \leq 3$, then $\widehat{\vartheta}_\delta$ is a consistent estimator of ϑ, and for $\nu < (2 - s_{\text{gap}}) \wedge 1$, $\widehat{\vartheta}_\delta = \vartheta + O_{\mathbb{P}}(\delta^\nu)$, as $\delta \to 0$.

(ii) If $s_{\text{gap}} < 1$, in particular if $d = 1$, then $\widehat{\vartheta}_\delta$ is a consistent and asymptotically normal estimator of ϑ satisfying (18).

Proof. It is enough to show that Assumption $A_{s,\eta,p}$ holds for all η close to 2 and all $p \geq 2$, $s > d/p$, since then the result is obtained by Theorems 3 and 7. With respect to f in (22), this follows from Lemma 9 with $\alpha = 0$, and for $f \in C_b^\infty(\mathbb{R})$ from Lemma 38(i). □

3.3.3 The stochastic Burgers equation

As a prototypical example for an SPDE with first order nonlinearity, let us consider the stochastic Burgers equation in dimension $d = 1$,

$$\mathrm{d}X(t) = (\vartheta \Delta X(t) - X(t)\partial_x X(t))\mathrm{d}t + B\mathrm{d}W(t). \tag{23}$$

This equation serves as a simple model for turbulence and is the one-dimensional analogue to the Navier-Stokes equations; for applications see e.g. the references in [HV11]. Note that the nonlinearity is given by

$$F(u) = -u\partial_x u = \partial_x \left(-\frac{1}{2}u^2\right). \tag{24}$$

It can be shown that (23) has a mild solution when $B = I$; cf. [DPDT94]. In order to obtain higher regularity of the solution let us assume Assumption B with $\gamma > \frac{1}{4}$. Theorem 43 yields $X \in C([0,T]; W^{s,p}(\Lambda))$ for all $p \geq 2$ and $1 < s < 1/2 + 2\gamma$. For $d = 1$ we find that $s_{\text{gap}} = 0$ and Lemma 9 with

$\alpha = 1$ implies Assumption $A_{s,\eta,p}$ for any $\eta < 1$. This is not enough to obtain asymptotic normality of $\widehat{\vartheta}_\delta$ using Theorems 3 and 7.

Instead, we tackle the nonlinear bias R_δ directly and show that $\delta^{-1}\mathcal{I}_\delta^{-1}R_\delta = o_{\mathbb{P}}(1)$. The proof is based on decomposing $F(X)$ using the splitting argument $X = \bar{X} + \tilde{X}$. Terms involving only $\bar{X}$ are treated by Gaussian calculus, similar to $\bar{\mathcal{I}}_\delta$ in Proposition 2. Moreover, $\bar{X}$ and $\tilde{X}$ are decoupled using the higher regularity of $\tilde{X}$ over $\bar{X}$ according to Proposition 6 and by a Wiener-chaos decomposition of $\tilde{X}(t, x_0)$. For the proof we assume $B = (-\Delta)^\gamma$ and a slightly stronger condition on the kernel K to shorten technical arguments, but this can likely be relaxed; see also the numerical study in the next section. For a proof see Appendix A.4.

Theorem 11. *Assume $B = (-\Delta)^{-\gamma}$ for $\gamma > 1/4$. Grant Assumption K and assume in addition that $\widetilde{K} = \partial_x L$ for $L \in H^{2\lceil\gamma\rceil+3}(\mathbb{R})$ having compact support. Then $\delta^{-1}\mathcal{I}_\delta^{-1}R_\delta = o_{\mathbb{P}}(1)$.*

Combining this with the discussion above, Theorems 3 and 7 yield immediately:

Theorem 12. *Assume that* (23) *is well-posed in $C([0,T]; W^{s,p}(\Lambda))$ for all $p \geq 2$ and $1 < s < 1/2 + 2\gamma$. Then the following holds:*

(i) The estimator $\widehat{\vartheta}_\delta$ is consistent with $\widehat{\vartheta}_\delta = \vartheta + O_{\mathbb{P}}(\delta^\nu)$ for any $\nu < 1$.

(ii) If the additional hypotheses from Theorem 11 are satisfied, then $\widehat{\vartheta}_\delta$ is a consistent and asymptotically normal estimator of ϑ satisfying (18).

It is straightforward to generalize this result to the stochastic Burgers equation in dimension $d \geq 2$.

4 Numerical examples

Consider for $T = 1$ and $\Lambda = (0, 1)$ the stochastic Allen-Cahn equation, which is a stochastic reaction diffusion equation of the form

$$dX(t) = (\vartheta\Delta X(t) + a_1 X(t)(a_2 - X(t))(X(t) - a_3))dt + \sigma dW(t),$$

with space-time white noise, i.e. $B = \sigma \cdot I$ and $\gamma = 0$, with zero boundary conditions and with parameters $\vartheta = 0.01$, $\sigma = 0.05$, $a_1 = 10$, $a_2 = 1$, a_3. The initial value X_0 is assumed to be smooth, equal to 1 on $[0.3, 0.7]$ and vanishing outside of $[0.3 - \varepsilon, 0.7 + \varepsilon]$ for a small $\varepsilon > 0$. The heat map for a typical realisation is presented in Figure 1 (top left).

An approximate solution is obtained by a finite difference scheme, cf. [LPS14, Example 10.31], with respect to a regular time-space grid $\{(t_k, y_j) :$

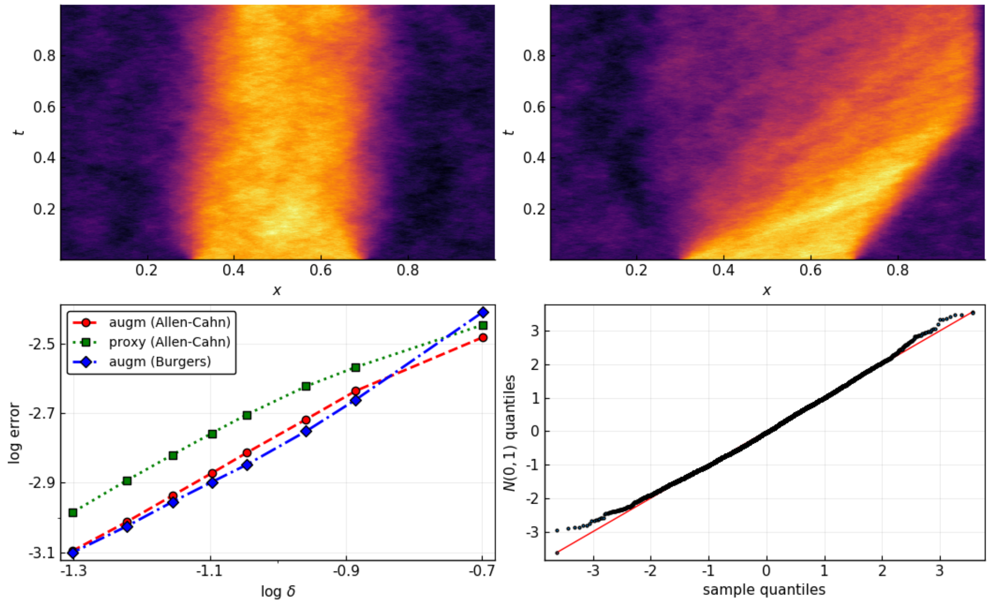

Figure 1: (top) heat maps for a typical realisation of the stochastic Allen-Cahn (left) and Burgers (right) equations; (bottom left) $\log_{10}$-$\log_{10}$ plot of root mean squared estimation errors at $x_0 = 0.4$; (bottom right) Normal Q-Q plot for Allen-Cahn at $x_0 = 0.4$, $\delta = 0.05$

$t_k = k/N, y_j = j/M, k = 0, \ldots, N, j = 0, \ldots, M\}$, with $M = 500$, $N = 10^5$. Consider as kernel the smooth bump function

$$K(x) := \widetilde{K}(x) := \exp(-\frac{12}{1-x^2}), \quad x \in (-1,1).$$

For $\delta \in [0.05, 0.2]$ and $x_0 \in (0,1)$ we then obtain approximate local measurements X_{δ,x_0}, X^{Δ}_{δ,x_0}, from which the augmented MLE $\widehat{\vartheta}_\delta$ is computed. For $x_0 < \delta$ set $K_{\delta,x_0} := K_{\delta,\delta}$ and for $x_0 > \delta$ set $K_{\delta,x_0} := K_{\delta,1-\delta}$. Note that the theoretical asymptotic variance $\vartheta\Sigma$ of Theorem 10 is available by Example 4.

For 5.000 Monte-Carlo runs, a Normal Q-Q plot for the approximate distribution of $(\vartheta\Sigma)^{-1/2}\delta^{-1}(\widehat{\vartheta}_\delta - \vartheta)$ is obtained at $x_0 = 0.4$ for $\delta = 0.05$, cf. Figure 1 (bottom right). We see that the sample distribution is already very close to the theoretical asymptotic distribution according to Theorem 10. Moreover, Figure 1 (bottom left) gives a $\log_{10}$-$\log_{10}$ plot of root mean squared estimation errors for $\delta \to 0$, demonstrating that the rate of convergence indeed approaches δ as the resolution tends to zero. For comparison, we also added results for another estimator based on observing only X_{δ,x_0}, namely the proxy MLE of [AR20], with K as described there. We see that its performance is comparable to the augmented MLE, which suggests that a similar mathematical analysis as in Theorem 3 may be possible for the proxy MLE.

At last, we consider the same steps for the stochastic Burgers equation (23) with $B = \sigma \cdot I$ for ϑ, σ as above. The heat map for a typical realisation is in Figure 1 (top right). Note that the finite difference scheme has to be adjusted, see [HV11] for details, but this adjustment does not affect the estimation of ϑ, as it is of zero differential order and therefore negligible compared to the Laplacian under scaling with δ; cf. Lemma 13. The Normal Q-Q plot remains essentially unchanged (not shown), and the error coincides for small δ with the one from the Alan-Cahn simulation. This suggests that the asymptotic variances coincide and that Theorem 11 also holds under weaker assumption, in particular for $\gamma = 0$.

Appendices

A Proofs of the main results

From now on, without loss of generality, we assume that $x_0 = 0$, or formally we replace Λ by $\Lambda - x_0$. To ease the notations, we also remove x_0 whenever necessary, for example by writing $\Lambda_\delta,\ z_\delta,\ X_\delta, B_\delta$ instead of $\Lambda_{\delta,x_0},\ z_{\delta,x_0},\ X_{\delta,x_0}, B_{\delta,x_0}$. As usual, we will denote by C a generic positive constant, which may change from line to line. In addition, $A \lesssim B$ will stand for $A \leq CB$, where C is a positive constant that may depend on T, unless otherwise stated. If not mentioned otherwise, all limits are taken as $\delta \to 0$.

Recall the notations in Section 2.3. For $2 \leq p < \infty$, $\delta > 0$, Δ_δ is the realization of the Laplacian on $L^p(\Lambda_\delta)$ with zero boundary conditions and domain $W^{2,p}(\Lambda_\delta)$, and the semigroup generated by $\vartheta\Delta_\delta$ is denoted by $(S_{\vartheta,\delta}(t))_{t\geq 0}$. In particular, $\Delta = \Delta_1$ and we write $S(t) = S_{1,1}(t)$, $S_\delta(t) = S_{1,\delta}(t)$. Note that $S_{\vartheta,\delta}(t) = S_\delta(\vartheta t)$.

A.1 On semigroups and the fractional Laplacian

The Laplacian and its semigroup satisfy a certain scaling with respect to localized functions. The proof is straightforward; see [AR20, Lemma 3.1] for details when $p = 2$, the general case is analogous.

Lemma 13. *For $2 \leq p < \infty$, $\delta > 0$:*

(i) If $z \in W^{2,p}(\Lambda_\delta)$, then $\Delta z_\delta = \delta^{-2}(\Delta_\delta z)_\delta$.

(ii) If $z \in L^p(\Lambda_\delta)$, then $S(t)z_\delta = (S_\delta(t\delta^{-2})z)_\delta$, $t \geq 0$.

In order to extend the scaling to the fractional Laplacian, recall (from [Paz83, Chapter 2.6], for example) that the fractional Laplacian can be represented as

$$(-\Delta)^{-h} = \frac{1}{\Gamma(h)} \int_0^\infty t^{h-1} S(t)\,\mathrm{d}t, \quad h > 0, \tag{25}$$

$$\begin{aligned}(-\Delta)^{h} &= -\frac{\sin \pi h}{\pi} \int_0^\infty t^{h-1} \Delta (t-\Delta)^{-1}\ \mathrm{d}t \\ &= -\Gamma(h) \frac{\sin \pi h}{\pi} \int_0^\infty (t')^{-h} \Delta S(t')\ \mathrm{d}t', \quad 0 < h < 1, \end{aligned} \tag{26}$$

where Γ is the gamma function and using the resolvent equality $(t-\Delta)^{-1} = \int_0^\infty e^{-tt'} S(t')\,\mathrm{d}t'$ in the last line.

Lemma 14. *Let $2 \le p < \infty$, $\delta > 0$. If $h > 0$ and $z \in L^p(\Lambda_\delta)$ (or $h \le 0$ and $z \in W^{-2h,p}(\Lambda_\delta)$), then $(-\Delta)^{-h} z_\delta = \delta^{2h}((-\Delta_\delta)^{-h} z)_\delta$.*

Proof. Let first $h > 0$ and $z \in L^p(\Lambda_\delta)$. In view of (25) and Lemma 13(ii) we have

$$\begin{aligned}(-\Delta)^{-h} z_\delta &= \left(\frac{1}{\Gamma(h)} \int_0^\infty t^{h-1} S_\delta(t\delta^{-2}) z\ \mathrm{d}t \right)_\delta \\ &= \delta^{2h} \left(\frac{1}{\Gamma(h)} \int_0^\infty t^{h-1} S_\delta(t) z\ \mathrm{d}t \right)_\delta = \delta^{2h} \left((-\Delta_\delta)^{-h} z \right)_\delta .\end{aligned}$$

For $h \le 0$ it is enough to consider smooth z supported in Λ_δ. Set $\tilde{h} = -h \ge 0$. By Lemma 13(i) and $(-\Delta)^{\tilde{h}} = (-\Delta)^{\tilde{h} - \lfloor \tilde{h} \rfloor} (-\Delta)^{\lfloor \tilde{h} \rfloor}$, the problem reduces to $0 \le \tilde{h} \le 1$. The result follows using (26) and Lemma 13(i,ii) from

$$(-\Delta)^{\tilde{h}} z_\delta = \delta^{-2\tilde{h}} \left(-\Gamma(\tilde{h}) \frac{\sin \pi \tilde{h}}{\pi} \int_0^\infty (t')^{-\tilde{h}} \Delta_\delta S_\delta(t') z\ \mathrm{d}t' \right)_\delta = \delta^{-2\tilde{h}} ((-\Delta_\delta)^{\tilde{h}} z)_\delta .$$

□

The proof of this lemma suggests that convergence of the operators $(-\Delta_\delta)^h$ can be obtained from the underlying semigroup.

Proposition 15. *Let $2 \le p < \infty$, $t > 0$. Then:*

(i) For any $h > 0$ there exists a universal constant $M_h < \infty$ such that $\sup_{t>0, 0<\delta\le 1} \|(-t\Delta_\delta)^h S_\delta(t)\|_{L^p(\Lambda_\delta)} \le M_h$.

(ii) If $z \in L^2(\mathbb{R}^d)$, then $S_\delta(t)(z|_{\Lambda_\delta}) \to e^{t\vartheta\Delta_0} z$ in $L^p(\mathbb{R}^d)$ as $\delta' \ge \delta \to 0$.

Proof. For $\delta = 1$, (i) is a well-known result due to the spectrum of the Laplacian being bounded away from zero; cf. [Paz83, Chapter 7.2.6]. Recall the scaling properties from Lemmas 13(ii) and 14. Using $\delta^{d(1/2-1/p)} \|z_\delta\|_{0,p} = \|z\|_{L^p(\Lambda_\delta)}$

for $z \in L^p(\Lambda_\delta)$, we then have

$$\begin{aligned}
\|(-t\Delta_\delta)^h S_\delta(t)\|_{L^p(\Lambda_\delta)} &= \sup_{\|z\|_{L^p(\Lambda_\delta)}=1} \|(-(t\delta^2)\delta^{-2}\Delta_\delta)^h S_\delta(t) z\|_{L^p(\Lambda_\delta)} \\
&= \sup_{\|z\|_{L^p(\Lambda_\delta)}=1} \|(-(t\delta^2)\delta^{-2}\Delta_\delta)^h S_\delta(t) \delta^{d(\frac{1}{2}-\frac{1}{p})} z)_\delta\|_{0,p} \\
&= \sup_{\delta^{d(1/2-1/p)}\|z_\delta\|_{0,p}=1} \|(-t\delta^2\Delta)^h S(t\delta^2)\delta^{d(\frac{1}{2}-\frac{1}{p})} z_\delta\|_{0,p} \le M_h.
\end{aligned}$$

This proves (i). Part (ii) follows from Proposition 3.5(ii) of [AR20] (with $A^*_{\vartheta,\delta,0} = \vartheta\Delta_\delta$) by replacing $L^2(\mathbb{R}^d)$ with $L^p(\mathbb{R}^d)$. □

Lemma 16. *Let $2 \le p < \infty$, $h \ge 0$ and let $z \in L^p(\mathbb{R}^d)$ have compact support in $\Lambda_{\delta'}$ for some $\delta' > 0$. Then we have for $\delta \le \delta'$:*

(i) If $z \in W^{2h,p}(\Lambda_{\delta'})$, then $(-\Delta_\delta)^h z \to (-\Delta_0)^h z$ in $L^p(\mathbb{R}^d)$ as $\delta \to 0$.

(ii) If $h < \frac{d}{2}(1-\frac{1}{p})$, then $\|(-\Delta_\delta)^{-h} z\|_{L^p(\Lambda_\delta)} \lesssim \min(\|z\|_{L^1(\mathbb{R}^d)}, \|z\|_{L^p(\mathbb{R}^d)})$.

Proof. (i). The claim is clear when $h \in \mathbb{N}_0$. For non-integer h write $h = m + h'$ with $m \in \mathbb{N}_0$ and $0 < h' < 1$. Then $z' = \Delta^m z$ has compact support and $(-\Delta_\delta)^h z = (-\Delta_\delta)^{h'} z'$. It is therefore enough to prove the claim for $0 < h < 1$. Recall the formula for the fractional Laplacian in (25). By Lemma 15(i) the integrand there is absolutely integrable uniformly in $\delta > 0$ and converges pointwise for fixed $t' > 0$ to $(t')^{-h}\Delta_0 S_0(t') z$ as $\delta \to 0$. Since the formula in (25) also holds for the fractional Laplacian Δ_0 on $\mathbb{R}^d$, this proves the result.

(ii). Since $z \in L^1(\mathbb{R}^d)$ by its compact support and because $(-\Delta_\delta)^{-h}$ is a bounded operator on $L^p(\mathbb{R}^d)$, we can assume $z \in C(\bar{\Lambda}_\delta)$. By (25) it is enough to show for $t \ge 0$ and uniformly in $\delta > 0$

$$\|S_\delta(t) z\|_{L^p(\Lambda_\delta)} \lesssim \min(1, t^{-\frac{d}{2}(1-\frac{1}{p})}) \min(\|z\|_{L^1(\mathbb{R}^d)}, \|z\|_{L^p(\mathbb{R}^d)}).$$

Proposition 15(i) already gives the bound $\|S_\delta(t) z\|_{L^p(\Lambda_\delta)} \lesssim \|z\|_{L^p(\mathbb{R}^d)}$ for $0 \le t \le 1$. For $t > 1$, Proposition 3.5(i) of [AR20] shows $|(S_\delta(t)z)(x)| \le c_1 e^{c_2 t \Delta_0}|z|(x)$, $x \in \Lambda_\delta$, with universal constants $c_1, c_2 > 0$. The result follows therefore from representing the $e^{c_2 t\Delta_0}$ as a convolution operator using the heat kernel on $\mathbb{R}^d$ such that by Young's inequality and hypercontractivity of the heat kernel

$$\|e^{c_2 t\Delta_0}|z|\|_{L^p(\mathbb{R}^d)} \lesssim \min\left(\|z\|_{L^p(\mathbb{R}^d)}, t^{-\frac{d}{2}(1-\frac{1}{p})}\|z\|_{L^1(\mathbb{R}^d)}\right).$$

□

The next lemma is a simple application of the scaling property of the fractional Laplacian and relates regularity to decay as $\delta \to 0$.

Lemma 17. *Let $u \in W^{r,p}(\Lambda)$, $z \in W^{-r,q}(\Lambda_\delta)$ for some $\delta > 0$, $r > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$|\langle u, z_\delta\rangle| \leq \delta^{r+d(\frac{1}{2}-\frac{1}{p})}\|u\|_{r,p}\|(-\Delta_\delta)^{-r/2}z\|_{L^q(\Lambda_\delta)}.$$

Proof. The Hölder inequality shows

$$|\langle u, z_\delta\rangle| = |\langle(-\Delta)^{r/2}u, (-\Delta)^{-r/2}z_\delta\rangle| \leq \|u\|_{r,p}\|z_\delta\|_{-r,q}.$$

Lemma 14 yields the identity $\|z_\delta\|_{-r,q} = \delta^r\|((-\Delta_\delta)^{-r/2}z)_\delta\|_{0,q}$ and the result follows by a change of variables. □

Lemma 18. *Let $r \geq 0$, $p > 1$ and consider the multiplication operator $M_\sigma u = \sigma \cdot u$ with $\sigma \in C^{2r'}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$ for $r' > d/(2p) + r$. Let $z \in C^\infty(\mathbb{R}^d)$ with compact support in $\Lambda_{\delta'}$ for some $\delta' > 0$ and define for $0 < \delta \leq \delta'$ the operator $G_\delta(\sigma)z := (-\Delta_\delta)^{-r}M_{\sigma(\delta\cdot)}(-\Delta_\delta)^r z$. Then:*

(i) $\sup_{0<\delta\leq\delta'}\|G_\delta(\sigma)z\|_{L^p(\mathbb{R}^d)} \lesssim 1$, and in particular, $G_\delta(\sigma)$ extends to a bounded operator $G_\delta(\sigma) : L^p(\Lambda_\delta) \to L^p(\Lambda_\delta)$.

(ii) If $\sigma \in C^{2r+1}(\mathbb{R}^d)$, then $G_\delta(\sigma)z \to G_0(\sigma)z := \sigma(0)z$ in $L^2(\mathbb{R}^d)$ for $z \in L^2(\mathbb{R}^d)$ as $\delta \to 0$.

Proof. (i). For r' as in the statement, σ induces a bounded multiplication operator M_σ on $W^{-2r,p}(\Lambda)$; cf. [Tri83, Section 2.8.2]. This means that

$$\|M_\sigma u\|_{-2r,p} \leq C\|u\|_{-2r,p}, \quad u \in W^{-2r,p}(\Lambda).$$

Therefore we have by Lemma 14

$$\begin{aligned}\|G_\delta(\sigma)z\|_{L^p(\Lambda_\delta)} &= \delta^{d(\frac{1}{2}-\frac{1}{p})}\|(-\Delta)^{-r}M_\sigma(-\Delta)^r z_\delta\|_{L^p(\Lambda)}\\ &= \delta^{d(\frac{1}{2}-\frac{1}{p})}\|M_\sigma(-\Delta)^r z_\delta\|_{-2r,p} \lesssim \delta^{d(\frac{1}{2}-\frac{1}{p})}\|(-\Delta)^r z_\delta\|_{-2r,p} \lesssim \|z\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

(ii). By (i) it is enough to consider $z \in C^\infty(\mathbb{R}^d)$ with compact support in $\Lambda_{\delta'}$. We can further restrict to $z = \Delta\widetilde{z}$ with $\widetilde{z} \in C^\infty(\mathbb{R}^d)$ also having compact support in $\Lambda_{\delta'}$. Indeed, assuming this holds, let $z \in C_c^\infty(\bar{\Lambda}_{\delta'})$. Using the Fourier transform $\mathcal{F}u$ for $u \in L^2(\mathbb{R}^d)$ define with $\varepsilon > 0$ functions

$$v_\varepsilon := \mathcal{F}^{-1}[u_\varepsilon](x) \text{ with } u_\varepsilon(\omega) := \frac{1}{\varepsilon + |i\omega|^2}\mathcal{F}z(\omega),\ \omega \in \mathbb{R}^d.$$

Note that $\mathcal{F}(\Delta v_\varepsilon)(\omega) = |i\omega|^2(\varepsilon + |i\omega|^2)^{-1}\mathcal{F}z(\omega)$ and therefore $\Delta v_\varepsilon \to z$ in $L^2(\mathbb{R}^d)$ as $\varepsilon \to 0$. By the Paley-Wiener Theorem (Rudin, Functional Analysis, Theorem II.7.22) z satisifes the exponential growth condition $|\mathcal{F}z|(\omega) \leq \gamma_N(1+|\omega|)^{-N}\exp((\delta')^{-1}|\mathrm{Im}(\omega)|)$, $\omega \in \mathbb{C}^d$, for all $N \in \mathbb{N}$ and suitable constants

γ_N. A reverse application of the same theorem shows that $u_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ is also supported in $\Lambda_{\delta'}$. Since both G_δ and G_0 are continuous, this means

$$\|(G_\delta(\sigma) - G_0(\sigma))z\|_{L^2(\mathbb{R}^d)} \lesssim \|z - \Delta v_\varepsilon\| + \|(G_\delta(\sigma) - G_0(\sigma))\Delta v_\varepsilon\|.$$

The result follows from letting first $\delta \to 0$ and then $\varepsilon \to 0$.

Assume therefore now that $z = \Delta \widetilde{z}$ with $\widetilde{z}$ as above. By Taylor's theorem and Lemma 14 we have

$$\begin{aligned}(G_\delta(\sigma)z - G_0(\sigma)z)_\delta &= (-\Delta)^{-r} M_{\sigma(\cdot)-\sigma(0)}(-\Delta)^r z_\delta \\ &= \sum_{i=1}^d \int_0^1 (-\Delta)^{-r} M_{\partial_i \sigma(h\cdot)x_i}(-\Delta)^r z_\delta \, dh.\end{aligned}$$

From $M_{\partial_i\sigma(h\cdot)x_i} = M_{\partial_i\sigma(h\cdot)}(-\Delta)^r(-\Delta)^{-r}M_{x_i}$ and (i) we find that

$$\|(G_\delta(\sigma) - G_0(\sigma))z\|_{L^2(\mathbb{R}^d)} \lesssim \|\sigma\|_{C^{2r+1}(\mathbb{R}^d)} \sum_{i=1}^d \|(-\Delta)^{-r} M_{x_i}(-\Delta)^r z_\delta\|.$$

To prove the claim it is enough to show $\|(-\Delta)^{-r}M_{x_i}(-\Delta)^r z_\delta\| \to 0$, $i = 1, \ldots, d$ as $\delta \to 0$. For this, write $r = m + r'$ with $m \in \mathbb{N}_0$ and $0 \le r' < 1$. Iterating the identity $x_i \Delta u = \Delta(x_i u) - 2\partial_i u$ for smooth u, we find $x_i \Delta^{m+1}\widetilde{z}_\delta = \Delta^{m+1}(x_i \widetilde{z}_\delta) - 2(m+1)\Delta^m \partial_i \widetilde{z}_\delta$ such that

$$\begin{aligned}&(-\Delta)^{-r}M_{x_i}(-\Delta)^r z_\delta = (-\Delta)^{-r}M_{x_i}(-\Delta)^r(\delta^2\Delta\widetilde{z}_\delta) =: J_{1,\delta} + J_{2,\delta},\\ &\text{with} \quad J_{1,\delta} := \delta^2(-1)^{m+1}(-\Delta)^{1-r'}M_{x_i}(-\Delta)^{r'}\widetilde{z}_\delta \\ &\qquad\quad\; J_{2,\delta} := \delta^2(-1)^{m+1}2(m+1)(-\Delta)^{-r'}\partial_i(-\Delta)^{r'}\widetilde{z}_\delta.\end{aligned}$$

Lemma 14 shows $\|J_{1,\delta}\| = \|(-\Delta_\delta)^{1-r'}M_{\delta x_i}(-\Delta_\delta)^{r'}\widetilde{z}\|_{L^2(\Lambda_\delta)}$. Due to the convergence $(-\Delta_\delta)^{r'}\widetilde{z} \to (-\Delta_0)^{r'}\widetilde{z}$ in $L^2(\mathbb{R}^d)$ from Lemma 16, we also have

$$\|M_{\delta x_i}(-\Delta_\delta)^{r'}\widetilde{z}\|_{L^2(\Lambda_\delta)} \le \|(-\Delta_\delta)^{r'}\widetilde{z} - (-\Delta_0)^{r'}\widetilde{z}\|_{L^2(\mathbb{R}^d)} + \|M_{\delta x_i}(-\Delta_0)^{r'}\widetilde{z}\|_{L^2(\Lambda_\delta)} \to 0,$$

using the dominated convergence theorem. Another application of Lemma 16 yields $\|J_{1,\delta}\| \to 0$. With respect to $J_{2,\delta}$, note that $\partial_i(-\Delta)^{-1/2} : L^2(\Lambda) \to L^2(\Lambda)$ and thus also its adjoint $(-\Delta)^{-1/2}\partial_i$ (extended to $L^2(\Lambda)$) are continuous. Assume first $r' < 1/2$. Then,

$$\|J_{2,\delta}\| \lesssim \delta^2\|\partial_i(-\Delta)^{r'}\widetilde{z}_\delta\| \lesssim \delta^2\|(-\Delta)^{r'+1/2}\widetilde{z}_\delta\|.$$

This vanishes as $\delta \to 0$, using Lemmas 14 and 16. For $r' \ge 1/2$, we have similarly

$$\|J_{2,\delta}\| \lesssim \delta^2\|(-\Delta)^{-1/2}\partial_i(-\Delta)^{r'}\widetilde{z}_\delta\| \lesssim \delta^2\|(-\Delta)^{r'}\widetilde{z}_\delta\| \to 0.$$

This finishes the proof.

□

A.2 Scaling of the covariance function

In this section, we study the properties of the covariance function of the Gaussian process $(t, z) \mapsto \langle \bar{X}(t), z\rangle, t \geq 0, z \in L^2(\Lambda)$, for localized functions z_δ, as well as its limit behavior when $\delta \to 0$. Repeatedly and sometimes without mentioning it explicitly, we will use properties of the fractional Laplacian and the semigroup operators $S_{\vartheta,\delta}(t) = S_\delta(\vartheta t)$, $t \geq 0$, from Section A.1.

For $t, t' \geq 0$ we use the notations

$$\begin{aligned} c(t, z, t', z') &:= \operatorname{Cov}(\langle \bar{X}(t), z\rangle, \langle \bar{X}(t'), z'\rangle), && z, z' \in L^2(\Lambda), \\ f_\delta(t, u, t', u') &:= \langle B_\delta^* S_{\vartheta,\delta}(t)u, B_\delta^* S_{\vartheta,\delta}(t')u'\rangle_{L^2(\Lambda_\delta)}, && u, u' \in L^2(\Lambda_\delta), \end{aligned}$$

and set $c(t, z) := c(t, z, t, z)$, $f_\delta(t, u) := f_\delta(t, u, t, u)$.

Lemma 19. *Grant Assumption B and let $z, z' \in L^2(\mathbb{R}^d)$ with compact support in Λ_δ, for some $\delta > 0$. Then, for $0 \leq t' \leq t \leq T$,*

$$c(t, z_\delta, t', z'_\delta) = \delta^{2+4\gamma} \int_0^{t'\delta^{-2}} f_\delta((t - t')\delta^{-2} + s, (-\Delta_\delta)^{-\gamma} z, s, (-\Delta_\delta)^{-\gamma} z')\, \mathrm{d}s.$$

Proof. Assumption B combined with Lemmas 13 and 14 imply the identities

$$\begin{aligned} B^* S_\vartheta(t - s) z_\delta &= B^*(-\Delta)^\gamma S_\vartheta(t - s)(-\Delta)^{-\gamma} z_\delta \\ &= \delta^{2\gamma} B^*(-\Delta)^\gamma (S_{\vartheta,\delta}((t - s)\delta^{-2})(-\Delta_\delta)^{-\gamma} z)_\delta \\ &= \delta^{2\gamma} (B_\delta^* S_{\vartheta,\delta}((t - s)\delta^{-2})(-\Delta_\delta)^{-\gamma} z)_\delta. \end{aligned}$$

This yields

$$\begin{aligned} c(t, z_\delta, t', z'_\delta) &= \int_0^{t'} \langle B^* S_\vartheta(t - s) z_\delta, B^* S_\vartheta(t' - s) z'_\delta\rangle\, \mathrm{d}s \\ &= \delta^{4\gamma} \int_0^{t'} f_\delta((t - s)\delta^{-2}, (-\Delta_\delta)^{-\gamma} z, (t' - s)\delta^{-2}, (-\Delta_\delta)^{-\gamma} z')\, \mathrm{d}s. \end{aligned}$$

The result follows by simple change of variables. □

Lemma 20. *Grant Assumption B and let $z, z' \in L^2(\mathbb{R}^d)$ with compact support in Λ_δ for some $\delta > 0$. Set $z^{(\delta)} := (-\Delta_\delta)^{-1/2-\gamma} z$, and $z'^{(\delta)} := (-\Delta_\delta)^{-1/2-\gamma} z'$. Then*

$$\left|c(t, z_\delta, t', z'_\delta)\right| \lesssim \delta^{2+4\gamma} \|z^{(\delta)}\|_{L^2(\Lambda_\delta)} \|z'^{(\delta)}\|_{L^2(\Lambda_\delta)}, \quad 0 \leq t' \leq t \leq T, \quad (27)$$

$$\int_0^t c(t, z_\delta, t', z'_\delta)^2\, \mathrm{d}t' \lesssim \delta^{6+8\gamma} \|(-\Delta_\delta)^{-1/2} z^{(\delta)}\|^2_{L^2(\Lambda_\delta)} \|z'^{(\delta)}\|^2_{L^2(\Lambda_\delta)}. \quad (28)$$

Proof. Recall that $\sup_{\delta>0}\|B^*_\delta\| < \infty$ from (16). By Lemma 19 and the Cauchy-Schwarz inequality, $\delta^{-2-4\gamma}\,|c(t,z_\delta,t',z'_\delta)|$ is up to a constant bounded by

$$\left(\int_0^{t'\delta^{-2}} \|S_{\vartheta,\delta}(s)S_{\vartheta,\delta}((t-t')\delta^{-2})(-\Delta_\delta)^{1/2}z^{(\delta)}\|^2_{L^2(\Lambda_\delta)}\,\mathrm{d}s\right)^{1/2} \cdot \left(\int_0^{t'\delta^{-2}} \|S_{\vartheta,\delta}(s)(-\Delta_\delta)^{1/2}z'^{(\delta)}\|^2_{L^2(\Lambda_\delta)}\,\mathrm{d}s\right)^{1/2}. \tag{29}$$

Note that $\int_0^a S_{\vartheta,\delta}(2s')u\,\mathrm{d}s' = \frac{1}{2}(I - S_{\vartheta,\delta}(2a))(-\vartheta\Delta_\delta)^{-1}u$ for $a > 0$ and $u \in L^2(\Lambda_\delta)$, which consequently implies that

$$\begin{aligned}\int_0^a \|S_{\vartheta,\delta}(s')u\|^2_{L^2(\Lambda_\delta)}\,\mathrm{d}s' &= \int_0^a \langle S_{\vartheta,\delta}(2s')u, u\rangle_{L^2(\Lambda_\delta)}\,\mathrm{d}s' \\ &= \frac{1}{2}\langle(-\vartheta\Delta_\delta)^{-1}u, u\rangle_{L^2(\Lambda_\delta)} - \frac{1}{2}\langle S_{\vartheta,\delta}(2a)(-\vartheta\Delta_\delta)^{-1}u, u\rangle_{L^2(\Lambda_\delta)} \\ &\le \frac{1}{2}\|(-\vartheta\Delta_\delta)^{-1/2}u\|^2_{L^2(\Lambda_\delta)}. \end{aligned} \tag{30}$$

Applying this to (29) yields

$$\delta^{-2-4\gamma}|c(t,z_\delta,t',z'_\delta)| \lesssim \|S_{\vartheta,\delta}((t-t')\delta^{-2})z^{(\delta)}\|_{L^2(\Lambda_\delta)}\|z'^{(\delta)}\|_{L^2(\Lambda_\delta)}. \tag{31}$$

Clearly, (27) follows from (31) by taking $t = t'$. On the other hand, by integrating (31) with respect to t', making a change of variables, and applying (30), we obtain

$$\begin{aligned}\frac{1}{\delta^{4+8\gamma}}\int_0^t c(t,z_\delta,t',z'_\delta)^2\,\mathrm{d}t' &\lesssim \delta^2\int_0^{t\delta^{-2}} \|S_{\vartheta,\delta}(t')z^{(\delta)}\|^2_{L^2(\Lambda_\delta)}\,\mathrm{d}t'\|z'^{(\delta)}\|^2_{L^2(\Lambda_\delta)} \\ &\lesssim \delta^2\|(-\Delta_\delta)^{-1/2}z^{(\delta)}\|^2_{L^2(\Lambda_\delta)}\|z'^{(\delta)}\|^2_{L^2(\Lambda_\delta)},\end{aligned}$$

which implies (28) at once. This concludes the proof. □

A.3 Proofs of technical results in Section 3

Lemma 21. *Grant Assumption K and let $1 < q < \infty$. The following holds true:*

(i) If $r < 2\lceil\gamma\rceil + d(1-\frac{1}{q})$, then $\sup_{0<\delta\le 1}\|(-\Delta_\delta)^{-r/2}K\|_{L^q(\Lambda_\delta)} < \infty$.

(ii) If $r \le 2\lceil\gamma\rceil$, then, as $\delta \to 0$, $(-\Delta_\delta)^{-r/2}K \to (-\Delta_0)^{\lceil\gamma\rceil-r/2}\widetilde{K}$ and $(-\Delta_\delta)^{-r/2}\Delta K \to (-\Delta_0)^{\lceil\gamma\rceil-r/2}\Delta\widetilde{K}$ in $L^q(\mathbb{R}^d)$.

Proof. Note that $(-\Delta_\delta)^{-r/2}K = (-\Delta_\delta)^{\lceil\gamma\rceil-r/2}\widetilde{K}$, $(-\Delta_\delta)^{-r/2}\Delta K = (-\Delta_\delta)^{\lceil\gamma\rceil-r/2}\Delta\widetilde{K}$ with $\widetilde{K}$ and $\Delta\widetilde{K}$ having compact support. The two claims follow therefore from Lemma 16.

□

Proposition 22. *Grant Assumptions B, K. Then, with Ψ from Assumption ND and as $\delta \to 0$, we have:*

(i) $\delta^{2-4\gamma}\int_0^T \mathbb{E}[\bar{X}_\delta^\Delta(t)^2]\,\mathrm{d}t \to T\vartheta^{-1}\Psi((-\Delta_0)^{\lceil\gamma\rceil-\gamma}\Delta\widetilde{K})$,

(ii) $\mathrm{Var}(\int_0^T \bar{X}_\delta^\Delta(t)^2\,\mathrm{d}t) \lesssim \delta^{-2+8\gamma}$.

Proof. (i). By Lemma 19 applied to $t = t'$ and $z_\delta = z'_\delta = \delta^{-2}(\Delta K)_\delta$ we write

$$\delta^{2-4\gamma}\int_0^T \mathbb{E}[\bar{X}_\delta^\Delta(t)^2]\,\mathrm{d}t = \int_0^T\int_0^\infty f_\delta(s,(-\Delta_\delta)^{-\gamma}\Delta K)\mathbf{1}_{\{s\leq t\delta^{-2}\}}\,\mathrm{d}s\,\mathrm{d}t.$$

Next, we set $f(s) := \|B_0^* e^{s\vartheta\Delta_0}(-\Delta_0)^{\lceil\gamma\rceil-\gamma}\Delta\widetilde{K}\|^2$, and note that $\int_0^\infty f(s)ds = \vartheta^{-1}\Psi((-\Delta_0)^{\lceil\gamma\rceil-\gamma}\Delta\widetilde{K})$, which clearly follows after substituting $s' = \vartheta s$. Recalling that $\sup_{\delta>0}\|B_\delta^*\| < \infty$ from (16), and since $\Delta K = \Delta_\delta K$, by Proposition 15(i) obtain

$$\begin{aligned}|f_\delta(s)| &\lesssim \|\Delta_\delta S_{\vartheta,\delta}(s)(-\Delta_\delta)^{-\gamma}K\|^2_{L^2(\Lambda_\delta)}\\ &\lesssim (1\wedge s^{-2})\left(\|(-\Delta_\delta)^{-\gamma}K\|^2_{L^2(\Lambda_\delta)} + \|(-\Delta_\delta)^{-\gamma}\Delta K\|^2_{L^2(\Lambda_\delta)}\right).\end{aligned}$$

Consequently, by Lemma 21, $|f_\delta(s)| \lesssim 1\wedge s^{-2}$, uniformly in $\delta > 0$, and thus $\sup_{\delta>0}|f_\delta| \in L^1([0,\infty))$. Setting $K^{(\delta)} := (-\Delta_\delta)^{-\gamma}\Delta K$, $K^{(0)} := (-\Delta_0)^{-\gamma}\Delta K$, we further have

$$\begin{aligned}\|B_\delta^* S_{\vartheta,\delta}(s)K^{(\delta)} - B_0^* e^{\vartheta s\Delta_0}K^{(0)}\|_{L^2(\mathbb{R}^d)} &\lesssim \|S_{\vartheta,\delta}(s)(K^{(0)}|_{\Lambda_\delta}) - e^{\vartheta s\Delta_0}K^{(0)}\|_{L^2(\mathbb{R}^d)}\\ &\quad + \|K^{(\delta)} - K^{(0)}\|_{L^2(\mathbb{R}^d)} + \|(B_\delta^* - B_0^*)e^{\vartheta s\Delta_0}K^{(0)})\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore, the pointwise convergence $f_\delta(s) \to f(s)$, as $\delta \to 0$, follows from Proposition 15(iii), Lemma 21, and Assumption *B*. Finally, by the dominated convergence theorem (i) is proved.

(ii). Note that the random variables $\{\bar{X}_\delta^\Delta(t)\,|\,t\geq 0\}$ are centered and jointly Gaussian. Thus, in view of Wick's formula, cf. [Jan97, Theorem 1.28], it follows that

$$\begin{aligned}\mathrm{Var}(\int_0^T \bar{X}_\delta^\Delta(t)^2 dt) &= \int_0^T\int_0^T \mathrm{Cov}(\bar{X}_\delta^\Delta(t)^2, \bar{X}_\delta^\Delta(t')^2)\,\mathrm{d}t'\,\mathrm{d}t\\ &= 4\int_0^T\int_0^t \mathrm{Cov}(\bar{X}_\delta^\Delta(t), \bar{X}_\delta^\Delta(t'))^2\,\mathrm{d}t'\,\mathrm{d}t\\ &= 4\delta^{-8}\int_0^T\int_0^t \mathbb{E}[\langle\bar{X}(t),(\Delta K)_\delta\rangle\langle\bar{X}(t'),(\Delta K)_\delta\rangle]^2\,\mathrm{d}t'\,\mathrm{d}t.\end{aligned}$$

Consequently, by Lemma 20 with $z = z' = \Delta K$, we continue

$$\mathrm{Var}(\int_0^T \bar{X}_\delta^\Delta(t)^2 dt) \lesssim \delta^{-2+8\gamma}\|(-\Delta_\delta)^{-\gamma}K\|^2_{L^2(\Lambda_\delta)}\|(-\Delta_\delta)^{1/2-\gamma}K\|^2_{L^2(\Lambda_\delta)}.$$

Invoking Lemma 21, we conclude the proof. □

Proof of Proposition 2. (i). By Assumption B, Assumption K and Lemma 14(i), it follows that

$$\delta^{-2\gamma}\|B^*K_\delta\| = \|B^*(-\Delta)^\gamma((-\Delta_\delta)^{-\gamma}K)_\delta\| = \|B_\delta^*(-\Delta_\delta)^{\lceil\gamma\rceil-\gamma}\widetilde{K}\|_{L^2(\Lambda_\delta)}.$$

Since $\sup_{0<\delta\leq 1}\|B_\delta^*\| < \infty$, cf. (16), using Lemma 21 we have

$$\delta^{-2\gamma}\|B^*K_\delta\| \to \|B_0^*(-\Delta_0)^{\lceil\gamma\rceil-\gamma}\widetilde{K}\|_{L^2(\mathbb{R}^d)}. \tag{32}$$

Noting that

$$\delta^2\mathbb{E}[\bar{\mathcal{I}}_\delta] = (\delta^{-2\gamma}\|B^*K_\delta\|)^{-2}\delta^{2-4\gamma}\int_0^T \mathbb{E}[\bar{X}_\delta^\Delta(t)^2]\,\mathrm{d}t,$$

and using (32) and Proposition 22(i), the desired result follows at once.

(ii). Convergence (32) and Proposition 22(ii) imply

$$\delta^4\mathrm{Var}(\bar{\mathcal{I}}_\delta) = (\delta^{-2\gamma}\|B^*K_\delta\|)^{-4}\delta^{4-8\gamma}\mathrm{Var}(\int_0^T \bar{X}_\delta^\Delta(t)^2\,\mathrm{d}t) \to 0.$$

From this and (i) we get that $\mathrm{Var}(\bar{\mathcal{I}}_\delta)/\mathbb{E}[\bar{\mathcal{I}}_\delta]^2 \xrightarrow{\mathbb{P}} 0$, which gives the result.

(iii). Using the decomposition $X_\delta^\Delta = \bar{X}_\delta^\Delta + \widetilde{X}_\delta^\Delta$ we write

$$\mathcal{I}_\delta - \bar{\mathcal{I}}_\delta = \|B^*K_\delta\|^{-2}\int_0^T (\widetilde{X}_\delta^\Delta(t)^2 + 2\widetilde{X}_\delta^\Delta(t)\bar{X}_\delta^\Delta(t))\mathrm{d}t.$$

Hence, by (i), (32) and the Cauchy-Schwarz inequality it is enough to have

$$\delta^{2-4\gamma}\int_0^T \widetilde{X}_\delta^\Delta(t)^2\mathrm{d}t \xrightarrow{\mathbb{P}} 0, \tag{33}$$

which is (14) from Assumption F.

(iv). The Cauchy-Schwarz inequality implies that

$$\mathcal{I}_\delta^{-1}|R_\delta| \lesssim \mathcal{I}_\delta^{-1/2}\|B^*K_\delta\|^{-1}\left(\int_0^T \langle F(t,X(t)),K_\delta\rangle^2 dt\right)^{1/2}.$$

By (32), (ii,iii) we find that $\mathcal{I}_\delta^{-1/2}\|B^*K_\delta\|^{-1} = O_{\mathbb{P}}(\delta^{1-2\gamma})$. Similar to the previous case, (15) gives the result. $\square$

Proof of Proposition 6. We start by proving a general statement. Choose ε and g as in Assumption $A_{s,\eta,p}$. Then for any $s_1 \leq s < \bar{s}(p)$, $0 < \varepsilon' < \varepsilon$ such that $s + \varepsilon' \leq \bar{s}(p)$ and $p_1 \leq p' \leq p$ the following implication holds

$$\sup_{0\leq t\leq T}\|\widetilde{X}(t)\|_{s,p'} < \infty \Rightarrow \sup_{0\leq t\leq T}\|\widetilde{X}(t)\|_{s+\eta+\varepsilon',p'} < \infty. \tag{34}$$

We proceed as in [DPDT94]. Use Proposition 15(i) for $\delta = 1$ to deduce for any $t \in [0, T]$ that

$$\|\widetilde{X}(t)\|_{s+\eta+\varepsilon',p'} \leq \|S_\vartheta(t)X_0\|_{s+\eta+\varepsilon',p'} + \int_0^t \|S_\vartheta(t-r)F(X(r))\|_{s+\eta+\varepsilon',p'}\mathrm{d}r$$
$$\lesssim \|X_0\|_{\bar{s}(p)+\eta,p} + \int_0^t (t-r)^{-1+\frac{\varepsilon-\varepsilon'}{2}}\|F(X(r))\|_{s+\eta-2+\varepsilon,p'}\mathrm{d}r.$$

Using Assumption $A_{s,\eta,p'}$ and the monotonicity of g allows upper bounding this by

$$\|X_0\|_{\bar{s}(p)+\eta,p} + \frac{2}{\varepsilon-\varepsilon'}T^{\frac{\varepsilon-\varepsilon'}{2}} g\left(\sup_{0\leq t\leq T}\|\bar{X}(t)\|_{s,p'} + \sup_{0\leq t\leq T}\|\widetilde{X}(t)\|_{s,p'}\right).$$

Since $X_0 \in W^{\bar{s}(p)+\eta,p}(\Lambda)$, $\bar{X} \in C([0,T]; W^{s,p}(\Lambda))$, we obtain (34).

Let us now prove the statement of the theorem. Applying (34) iteratively to $p' = p_1$ and all $s_1 \leq s < \bar{s}(p)$ gives $\widetilde{X} \in C([0,T]; W^{s+\eta+\varepsilon',p_1}(\Lambda))$ for all sufficiently small $\varepsilon' > 0$, and thus $\widetilde{X} \in C([0,T]; W^{s_1,p''}(\Lambda))$ by the Sobolev embedding for some suitable $p'' > p$. Repeating these steps with $p'' \geq p_1$ instead of p_1 until $p'' \geq p$ is reached, yields $\widetilde{X} \in C([0,T]; W^{\bar{s}(p)+\eta,p}(\Lambda))$ and $X \in C([0,T]; W^{s,p}(\Lambda))$. □

A.4 Proof of Theorem 11: Bias in Burgers CLT

As in Appendix B.1, let $(\lambda_k, \Phi_k)_{k\in\mathbb{N}}$ denote the eigensystem of $-\Delta$ on Λ (recall that Λ here corresponds to the shifted domain $(0,1) - x_0$, as assumed in the beginning of the proof section) such that in $d = 1$, $\lambda_k = \pi^2k^2$ and $\Phi_k = \sqrt{2}\sin(\pi k(x + x_0))$. Also, throughout this section we will assume that the assumptions of Theorem 11 are fulfilled. We frequently use that the space $W^{s,p}(\Lambda)$ is an algebra with respect to pointwise multiplication for $p \geq 2$ and $s > 1/p$; cf. proof of Lemma 9.

By Proposition 2(i-iii) and Equation (32), it is enough to show that

$$\delta^{1-4\gamma}\|B^*K_\delta\|^2R_\delta \xrightarrow{\mathbb{P}} 0, \quad \delta \to 0. \tag{35}$$

Using integration by parts, we write

$$\|B^*K_\delta\|^2R_\delta = \frac{1}{2}\int_0^T X_\delta^\Delta(t)\langle X(t)^2, \partial_x K_\delta\rangle\,\mathrm{d}t = \frac{1}{2}(U_{1,\delta} + U_{2,\delta} + U_{3,\delta}),$$

where

$$U_{1,\delta} := \int_0^T \bar{X}_\delta^\Delta(t)\langle \bar{X}(t)^2, \partial_x K_\delta\rangle \,\mathrm{d}t,$$

$$U_{2,\delta} := \int_0^T \widetilde{X}_\delta^\Delta(t)\langle X(t)^2, \partial_x K_\delta\rangle \,\mathrm{d}t + \int_0^T \bar{X}_\delta^\Delta(t)\langle \widetilde{X}(t)^2, \partial_x K_\delta\rangle \,\mathrm{d}t$$
$$+ 2\int_0^T \bar{X}_\delta^\Delta(t)\langle \bar{X}(t)(\widetilde{X}(t)-\widetilde{X}(t,0)), \partial_x K_\delta\rangle \,\mathrm{d}t =: V_{1,\delta}+V_{2,\delta}+V_{3,\delta},$$

$$U_{3,\delta} := 2\int_0^T \widetilde{X}(t,0)\bar{X}_\delta^\Delta(t)\langle \bar{X}(t), \partial_x K_\delta\rangle \,\mathrm{d}t.$$

We will treat each term separately in a series of lemmas below, and show that $\delta^{1-4\gamma}U_{j,\delta} \xrightarrow{\mathbb{P}} 0$, for $j=1,2,3$. For $U_{1,\delta}$, cf. Lemma 28, we use Gaussian calculus, while for $U_{2,\delta}$, we use the excess spatial regularity of $\widetilde{X}$ over $\bar{X}$, cf. Lemma 25. In Lemma 29, we treat $U_{3,\delta}$ by a Wiener-chaos decomposition of $\widetilde{X}(t,0)$.

Lemma 23. *Let $1<q<\infty$, $r \le 2+2\lceil\gamma\rceil$. Then, as $\delta \to 0$,*

(i) $(-\Delta_\delta)^{-r/2}\partial_x K \to (-\Delta_0)^{\lceil\gamma\rceil-r/2+1}L$ *in* $L^p(\mathbb{R}^d)$,

(ii) $(-\Delta_\delta)^{-r/2}\Phi_k(\delta\cdot)\partial_x K \to \Phi_k(0)(-\Delta_0)^{\lceil\gamma\rceil-r/2+1}L$ *in* $L^2(\mathbb{R}^d)$. *Moreover,* $\sup_{0<\delta\le 1}\|(-\Delta_\delta)^{-r/2}\Phi_k(\delta\cdot)\partial_x K\|_{L^2(\Lambda_\delta)} \lesssim \lambda_k^{r/2}$.

Proof. (i). Since $(-\Delta_\delta)^{-r/2}\partial_x K = (-\Delta_\delta)^{\lceil\gamma\rceil-r/2+1}L$, the claim follows at once by Lemma 16(i).

(ii) With $K^{(\delta)} := (-\Delta_\delta)^{-r/2}\partial_x K$ and $G_\delta(\cdot)$ as in Lemma 18, we have that $(-\Delta_\delta)^{-r/2}\Phi_k(\delta\cdot)\partial_x K = G_\delta(\Phi_k)K^{(\delta)}$. Then, the two claims follow from (i) and Lemma 18. □

Lemma 24. *For any small $\varepsilon>0$, uniformly in $0\le t\le T$, $k\ge 1$, $r \le 1+2\lceil\gamma\rceil$:*

(i) $|\widetilde{X}_\delta^\Delta(t)| \lesssim \delta^{2\gamma-\varepsilon}$, $|\bar{X}_\delta^\Delta(t)| \lesssim \delta^{2\gamma-1-\varepsilon}$,
$|\langle \widetilde{X}(t)^2, \partial_x K_\delta\rangle| \lesssim \delta^{2\gamma+1-\varepsilon}$, $|\langle X(t)^2, \partial_x K_\delta\rangle| \lesssim \delta^{2\gamma-\varepsilon}$,

(ii) $|\langle \Phi_k^2, \partial_x K_\delta\rangle| \lesssim \lambda_k^r \delta^{r-1/2-\varepsilon}$, $|\langle \widetilde{X}(t), \Phi_k\rangle| \lesssim \lambda_k^{-\gamma-3/4+\varepsilon}$.

Proof. (i) We recall that by $\gamma>1/4$ and Theorem 43, for $\varepsilon'>0$ and $p\ge 2$,

$$\bar{X} \in C([0,T]; W^{2\gamma+1/2-\varepsilon',p}(\Lambda)), \quad \widetilde{X} \in C([0,T]; W^{2\gamma+3/2-\varepsilon',p}(\Lambda)). \tag{36}$$

Since the Sobolev spaces appearing herein are also algebras with respect to pointwise multiplication, we conclude that $\widetilde{X}^2$ and, respectively, $X^2 = (\bar{X}+\widetilde{X})^2$ belong to the same spaces as $\widetilde{X}$ and, respectively, $\bar{X}$. The first two inequalities follow from Lemma 17, applied to $\delta^{-2}(\Delta K)_\delta$ with $r = 1/2+2\gamma-\varepsilon'$

and by putting $\varepsilon = \varepsilon' + 1/p$ for some small ε' and large p. The last two inequalities follow similarly, by applying Lemma 17 to $\delta^{-1}(\partial_x K)_\delta$ with $r = 1+2\gamma-\varepsilon'$ and additionally invoking Lemmas 21 and 23(i).

(ii) Using the explict form of Φ_k and λ_k, by direct computations we deduce that $\|\Phi_k^2\|_{r,p} \lesssim \lambda_k^r$. The first statement follows thus as in (i). The second one holds by (36) such that, with $r = 3/2 + 2\gamma - 2\varepsilon$, $|\langle \widetilde{X}(t), \Phi_k\rangle| \leq \|\widetilde{X}(t)\|_r \|\Phi_k\|_{-r} \lesssim \lambda_k^{-r/2}$. □

Lemma 25. *As $\delta \to 0$, we have that $\delta^{1-4\gamma} U_{2,\delta} \xrightarrow{\mathbb{P}} 0$.*

Proof. Lemma 24(i) yields $V_{1,\delta} = O_{\mathbb{P}}(\delta^{1+4\gamma-\varepsilon})$, $V_{2,\delta} = O_{\mathbb{P}}(\delta^{4\gamma-\varepsilon})$ for any small $\varepsilon > 0$. With respect to $V_{3,\delta}$ expand $\widetilde{X}(t) = \sum_{k\geq 1}\langle \widetilde{X}(t), \Phi_k\rangle \Phi_k$ such that with $g_{k,\delta}(t) := \bar{X}_\delta^\Delta(t)\langle \bar{X}(t), (\Phi_k - \Phi_k(0))\partial_x K_\delta\rangle$, we deduce

$$|V_{3,\delta}| = |2\sum_{k\geq 1}\int_0^T \langle \widetilde{X}(t), \Phi_k\rangle g_{k,\delta}(t)\,\mathrm{d}t| \lesssim \sum_{k\geq 1}\lambda_k^{-3/4-\gamma+\varepsilon}\int_0^T |g_{k,\delta}(t)|\,\mathrm{d}t, \quad (37)$$

where in the last inequality we used Lemma 24(ii). By the Cauchy-Schwarz inequality and Lemma 20 with $z = z' = \Delta K$ and for $z = z' = v^{(\delta)} := (\Phi_k(\delta\cdot) - \Phi_k(0))\partial_x K$ we have

$$\begin{aligned}\delta^{1-4\gamma}\mathbb{E}[|g_{k,\delta}(t)|] &\leq \delta^{-2-4\gamma}\mathbb{E}[\langle \bar{X}(t), (\Delta K)_\delta\rangle^2]^{1/2}\mathbb{E}[\langle \bar{X}(t), v_\delta^{(\delta)}\rangle^2]^{1/2} \\ &\lesssim \|(-\Delta_\delta)^{1/2-\gamma}K\|_{L^2(\Lambda_\delta)}^{1/2}\|(-\Delta_\delta)^{-1/2-\gamma}v^{(\delta)}\|_{L^2(\Lambda_\delta)}.\end{aligned}$$

Consequently, by Lemmas 21 and 23(ii), we get that

$$\delta^{1-4\gamma}\mathbb{E}[|g_{k,\delta}(t)|] \to 0, \qquad \sup_{0<\delta\leq 1, k\geq 1, 0\leq t\leq T}(\delta^{1-4\gamma}\lambda_k^{-\gamma}\mathbb{E}[|g_{k,\delta}(t)|]) < \infty, \quad (38)$$

which combined with (37) concludes the proof.

□

To deal with $U_{1,\delta}$ and $U_{3,\delta}$, we will prove two additional technical lemmas. For $x, x' \in \Lambda$ and $0 \leq t' \leq t \leq T$ set

$$\begin{aligned} c_{t,t'}^\Delta(x) &:= \mathbb{E}[\bar{X}_\delta^\Delta(t)\bar{X}(t', x')], & c_{t,t'}(x,x') &:= \mathbb{E}[\bar{X}(t,x)\bar{X}(t',x')], \\ c_{t,t'}^{(1)}(x,x') &:= c_{t',t}^\Delta(x)c_{t,t'}^\Delta(x'), & c_{t,t'}^{(2)}(x,x') &:= c_{t,t}^\Delta(x)c_{t',t'}^\Delta(x'). \end{aligned}$$

Lemma 26. *The following assertions hold true:*

(i) $|\langle c_{t,t}, \partial_x K_\delta\rangle| \lesssim \delta^{2\gamma-\varepsilon}$ *and* $|\int_0^t \langle c_{t,t'}, \partial_x K_\delta\rangle\,\mathrm{d}t'| \lesssim \delta^{1/2+2\gamma-\varepsilon}$,

(ii) $|\int_{\Lambda^2} c_{t,t'}(x,x')^2 \partial_x K_\delta(x)\partial_x K_\delta(x')\,\mathrm{d}x\,\mathrm{d}x'| \lesssim \delta^{4\gamma-\varepsilon}$,

(iii) $|\langle c_{t,t}^\Delta c_{t',t}^\Delta, \partial_x K_\delta\rangle| \lesssim \delta^{6\gamma-1}$ *and* $|\langle c_{t',t'}^\Delta c_{t,t'}^\Delta, \partial_x K_\delta\rangle| \lesssim \delta^{6\gamma-1}$,

(iv) for $i = 1, 2$, as $\delta \to 0$,

$$\delta^{2-8\gamma} \int_{\Lambda^2} \int_0^T \int_0^t c_{t,t'}(x,x') c^{(i)}_{t,t'}(x,x') \partial_x K_\delta(x) \partial_x K_\delta(x') \,\mathrm{d}t' \,\mathrm{d}t \,\mathrm{d}(x,x') \to 0.$$

Proof. By (10), and using the representation $W(t) = \sum_{k\geq 1} \Phi_k \beta_k(t)$, where $\beta_k, k \geq 1$, are independent standard Brownian motions, we have that

$$\bar{X}(t,x) = \sum_{k\geq 1} \lambda_k^{-\gamma} \Phi_k(x) \int_0^t e^{-(t-r)\vartheta\lambda_k} \,\mathrm{d}\beta_k(r),$$
$$\bar{X}^\Delta_\delta(t) = \sum_{k\geq 1} \lambda_k^{-\gamma} \langle \Phi_k, \Delta K_\delta \rangle \int_0^t e^{-(t-r)\vartheta\lambda_k} \,\mathrm{d}\beta_k(r).$$

Consequently, using the independence of β_k's, we obtain

$$c_{t,t'}(x,x') = \frac{1}{2\vartheta} \sum_{k\geq 1} e^{-(t-t')\vartheta\lambda_k} (e^{-2t'\vartheta\lambda_k} - 1) \lambda_k^{-1-2\gamma} \Phi_k(x) \Phi_k(x'), \tag{39}$$

$$c^\Delta_{t,t'}(x) = \frac{1}{2\vartheta} \big((S_\vartheta(2t') - I) S_\vartheta(t-t') (-\Delta)^{-2\gamma} K_\delta \big)(x). \tag{40}$$

(i) By (39) and Lemma 24(ii) with $r' = 1/2 + 2\gamma - \varepsilon'$ and $\varepsilon = \varepsilon'$, we deduce

$$|\langle c_{t,t}, \partial_x K_\delta \rangle| \lesssim \sum_{k\geq 1} \lambda_k^{-1-2\gamma} \left| \langle \Phi_k^2, \partial_x K_\delta \rangle \right| \lesssim \delta^{2\gamma - 2\varepsilon'}.$$

Analogous, the second result follows after integrating (39) with respect to t', and using Lemma 24(ii) with $r' = 1 + 2\gamma - \varepsilon'$,

$$\left| \int_0^t \langle c_{t,t'}, \partial_x K_\delta \rangle \,\mathrm{d}t' \right| \lesssim \sum_{k\geq 1} \lambda_k^{-2-2\gamma} \left| \langle \Phi_k^2, \partial_x K_\delta \rangle \right| \lesssim \delta^{1/2 + 2\gamma - 2\varepsilon'}.$$

(ii). The proof is analogous to (i).
(iii). By Lemmas 14 and 21, $\|c^\Delta_{t',t}\|_{2\gamma} \lesssim \|(-\Delta)^{-\gamma} K_\delta\| \lesssim \delta^{2\gamma}$, and consequently by the algebra property of Sobolev spaces $\|c^\Delta_{t,t} c^\Delta_{t',t}\|_{2\gamma} \lesssim \delta^{4\gamma}$. Using this and $\partial_x K_\delta = \delta^{-1} (\partial_x K)_\delta$, the desired result follows by applying Lemma 17 with $r = 2\gamma$ and $p = 2$ and consequently using Lemma 23(i).
(iv) We consider only the case $i = 1$, and one can treat the case $i = 2$ similarly. Using (39) and (40) we write

$$\delta^{2-8\gamma} \int_{\Lambda^2} \int_0^t c_{t,t'}(x,x') c^\Delta_{t,t}(x) c^\Delta_{t',t'}(x') \partial_x K_\delta(x) \partial_x K_\delta(x') \,\mathrm{d}t' \,\mathrm{d}x \,\mathrm{d}x' = \sum_{k\geq 1} a_{k,\delta}$$

with $a_{k,\delta} := \int_0^t e^{-(t-t')\vartheta\lambda_k}(e^{-2t'\vartheta\lambda_k} - 1)\lambda_k^{-1-2\gamma} b_{t,k,\delta} b_{t',k,\delta}\,\mathrm{d}t'$, and where, using Lemmas 13 and 14,

$$\begin{aligned} b_{t,k,\delta} :=& \delta^{1-4\gamma}\langle c_{t,t}^{\Delta}, \Phi_k \partial_x K_\delta\rangle = \delta^{1-4\gamma}\langle (-\Delta)^{\gamma} c_{t,t}^{\Delta}, (-\Delta)^{-\gamma}\Phi_k \partial_x K_\delta\rangle \\ =& \frac{1}{2\vartheta}\langle (S_{\vartheta,\delta}(2t\delta^{-2}) - I)(-\Delta_\delta)^{-\gamma} K, (-\Delta_\delta)^{-\gamma}\Phi_k(\delta\cdot)\partial_x K\rangle_{L^2(\Lambda_\delta)}. \end{aligned}$$

Due to Proposition 15(i) and Lemmas 21, 23(ii), note that $\sup_{\delta>0, 0\le t\le T} |b_{t,k,\delta}| \lesssim \lambda_k^{\gamma}$. Moreover, using in addition Proposition 15(iii), we also deduce that, as $\delta \to 0$,

$$b_{t,k,\delta} \to -\frac{1}{2\vartheta}\Phi_k(0)\langle (-\Delta_0)^{\lceil\gamma\rceil-\gamma}\widetilde{K}, (-\Delta_0)^{\lceil\gamma\rceil-\gamma}\partial_x \widetilde{K}\rangle_{L^2(\mathbb{R}^d)}. \tag{41}$$

Since the fractional Laplacian on $\mathbb{R}^d$ is a convolution operator and therefore commutes with the derivative ∂_x, after integration by parts, we deduce that the limit in (41) vanishes. In all, we have shown that $\sup_{\delta>0,0\le t\le T} |a_{k,\delta}| \lesssim \lambda_k^{-2-\gamma}$ and $a_{k,\delta} \to 0$, and hence, by the dominated convergence theorem the result follows. □

Lemma 27. *For any any $0 \le t, t' \le T$, we have*

(i) $|\mathbb{E}[\bar{X}_\delta^{\Delta}(t)\bar{X}_\delta^{\Delta}(t')]| \lesssim \delta^{4\gamma-1-\varepsilon}|t-t'|^{-1/2+\varepsilon}$, *for $t \neq t'$, and $\varepsilon > 0$,*

(ii) $\delta^{-4\gamma}\mathbb{E}[\langle \bar{X}(t), \partial_x K_\delta\rangle\langle \bar{X}(t'), \partial_x K_\delta\rangle] \to 0$,

(iii) $\delta^{1-4\gamma}\mathbb{E}[\bar{X}_\delta^{\Delta}(t')\langle \bar{X}(t), \partial_x K_\delta\rangle] \to 0$, *as $\delta \to 0$,.*

Proof. Using (31) with $z_\delta = z'_\delta = (\Delta K)_\delta$ yields

$$\begin{aligned} |\mathbb{E}[\bar{X}_\delta^{\Delta}(t)\bar{X}_\delta^{\Delta}(t')]| &= \delta^{-4}|\mathbb{E}[\langle \bar{X}(t), (\Delta K)_\delta\rangle\langle \bar{X}(t'), (\Delta K)_\delta\rangle]| \\ &\lesssim \delta^{4\gamma-2}\|S_{\vartheta,\delta}(|t-t'|\delta^{-2})(-\Delta_\delta)^{1/2-\gamma}K\|_{L^2(\Lambda_\delta)}\|(-\Delta_\delta)^{1/2-\gamma}K\|_{L^2(\Lambda_\delta)}. \end{aligned}$$

Then, (i) follows by Proposition 15(i) with $h = 1/2 - \varepsilon$, combined with Lemma 21, where we take $r = 1 + \gamma - \varepsilon$. Assertion (ii) follows similarly by applying (31) with $z_\delta = z'_\delta = (\partial_x K)_\delta$ and using Lemma 23(i). For (iii), in view of Lemma 19 with $B_\delta^* = I$, we have

$$\begin{aligned} A_\delta &:= \delta^{1-4\gamma}\mathbb{E}[\bar{X}_\delta^{\Delta}(t')\langle \bar{X}(t), \partial_x K_\delta\rangle] = \delta^{-2-4\gamma}\mathbb{E}[\langle \bar{X}(t'), (\Delta K)_\delta\rangle\langle \bar{X}(t), (\partial_x K)_\delta\rangle] \\ &= \int_0^{(t\wedge t')\delta^{-2}} \langle S_{\vartheta,\delta}(|t-t'|\delta^{-2} + 2s)(-\Delta_\delta)^{-\gamma}\Delta K, (-\Delta_\delta)^{-\gamma}\partial_x K\rangle_{L^2(\Lambda_\delta)}\,\mathrm{d}s \\ &= \frac{1}{2\vartheta}\langle (S_{\vartheta,\delta}((t+t')\delta^{-2}) - S_{\vartheta,\delta}(|t-t'|\delta^{-2}))(-\Delta_\delta)^{-\gamma}K, (-\Delta_\delta)^{-\gamma}\partial_x K\rangle_{L^2(\Lambda_\delta)}. \end{aligned}$$

When $t \neq t'$, then both semigroups in the above expression vanish as $\delta \to 0$, and the original claim follows. If $t = t'$, then the first semigroup vanishes as $\delta \to 0$, and by Lemmas 21 and 23(ii)

$$A_\delta \to -\frac{1}{2\vartheta}\langle (-\Delta_0)^{\lceil\gamma\rceil-\gamma}\widetilde{K}, (-\Delta_0)^{\lceil\gamma\rceil-\gamma}\partial_x \widetilde{K}\rangle_{L^2(\mathbb{R}^d)}, \quad \delta \to 0.$$

By the same arguments as in (41), we conclude that the limiting term is zero, and thus $A_\delta \to 0$. This concludes the proof. $\square$

Lemma 28. *As $\delta \to 0$, we have that $\delta^{1-4\gamma} U_{1,\delta} \xrightarrow{\mathbb{P}} 0$.*

Proof. Since $\bar{X}(t,x)$ and $\bar{X}_\delta^\Delta(t,x)$ are centered Gaussians, using Wick's formula for moments of centered Gaussians, cf. [Jan97, Theorem 1.28], we get

$$\mathbb{E}[U_{1,\delta}] = \int_\Lambda \int_0^T \mathbb{E}[\bar{X}_\delta^\Delta(t) \bar{X}(t,x)^2] \partial_x K_\delta(x) \,\mathrm{d}x \,\mathrm{d}t = 0,$$
$$\mathrm{Var}(U_{1,\delta}) = \sum_{\pi \in \Pi_2(6)} V_\pi,$$

where $\Pi_2(6)$ is the set of partitions of $\{1,\ldots,6\}$ into 2-tuples (pairs) and where

$$V_\pi = 2 \int_{\Lambda^2} \int_0^T \int_0^t \prod_{(i,j)\in\pi} \mathbb{E}[Z_i Z_j] \partial_x K_\delta(x) \partial_x K_\delta(x') \,\mathrm{d}t' \,\mathrm{d}t \,\mathrm{d}(x,x'),$$

with $Z_1 = \bar{X}_\delta^\Delta(t)$, $Z_2 = Z_3 = \bar{X}(t,x)$, $Z_4 = \bar{X}_\delta^\Delta(t')$, $Z_5 = Z_6 = \bar{X}(t',x')$.

Clearly, it is enough to show that $\delta^{2-8\gamma} V_\pi \xrightarrow{\mathbb{P}} 0$ for any $\pi \in \Pi_2(6)$. Since $Z_2 = Z_3$ and $Z_5 = Z_6$, by symmetry, it is sufficient to consider only six partitions, conveniently grouped as follows:

$$\begin{aligned}
I_1 =& \{((1,2),(3,4),(5,6)),\ ((1,5),(2,3),(4,6))\},\\
I_2 =& \{((1,2),(3,5),(4,6)),\ ((1,5),(2,4),(3,6))\},\\
I_3 =& \{((1,4),(2,3),(5,6))\}, \quad I_4 = \{((1,4),(2,5),(3,6))\}.
\end{aligned}$$

All relevant terms were already studied in Lemmas 26 and 27. For $\pi \in I_2$, we apply Lemma 26(iv) and obtain that $\delta^{2-8\gamma} V_\pi \xrightarrow{\mathbb{P}} 0$. On the other hand, $V_\pi = O_{\mathbb{P}}(\delta^{8\gamma-1-\varepsilon})$ for $\pi \in I_1$ by Lemma 26(i,iii), for $\pi \in I_3$ by Lemmas 26(i) and 27(i), and for $\pi \in I_4$ by applying Lemmas 26(ii) and 27(i).

The proof is complete. $\square$

Lemma 29. *As $\delta \to 0$, we have that $\delta^{1-4\gamma} U_{3,\delta} \xrightarrow{\mathbb{P}} 0$.*

Proof. Similar to Lemma 28, we aim to compute the mean and the variance of $U_{3,\delta}$. Since $\widetilde{X}(t,0)$ is not Gaussian, we will use study its Wiener chaos decomposition (cf. [Nua06]).

We consider the Hilbert space $\mathcal{H} := L^2([0,T]\times\Lambda)$ endowed with norm $\|z\|_{\mathcal{H}} = \int_{[0,T]\times\Lambda} z^2(t,x) \,\mathrm{d}(t,x)$, and correspondingly let $(\widetilde{W}(z))_{z\in\mathcal{H}}$ be the isonormal Gaussian process $\widetilde{W}(z) := \int_0^T z(t,\cdot) \,\mathrm{d}W(t)$. Also, let $(m_j)_{j\geq 1}$ be an orthonormal basis in $L^2([0,T])$ such that $(m_j \cdot \Phi_k)_{j,k\geq 1}$ forms an orthonormal basis in $\mathcal{H}$. We denote by $\mathcal{G}$ the sigma algebra generated by

$(\widetilde{W}(z))_{z\in\mathcal{H}}$. It is well-known (see [Nua06, Proposition 1.1.1]) that there exists a sequence of random variables $(\xi_i)_{i\geq 1}$ forming a complete orthonormal system in $L^2(\Omega,\mathcal{G},\mathbb{P})$, where each ξ_i is a linear combination of multinomials of the form $\Pi_{l=1}^M \widetilde{W}(m_{j_l}\cdot\Phi_{k_l})^{b_l}$ for some $M, j_l, k_l \in \mathbb{N}$, $a_l\in\mathbb{N}_0$.

In view of [LR15, Theorem 5.1.3 and Example 5.1.8], where we use that B is Hilbert-Schmidt for $\gamma > 1/4$, we have $\widetilde{X}\in L^2([0,T]\times\Omega; W^{1,2}(\Lambda))$, and hence $\widetilde{X}(t,0)\in L^2(\Omega,\mathcal{G},\mathbb{P})$ for $0\leq t\leq T$. This yields the chaos expansion $\widetilde{X}(t,0)=\sum_{i\geq 1} b_i(t)\xi_i$, with some deterministic $b_i\in L^2([0,T])$. For $N\in\mathbb{N}$, we put

$$U_{3,\delta,N} := 2\sum_{i=1}^N \xi_i \int_0^T b_i(t)\bar{X}_\delta^\Delta(t)\langle \bar{X}(t), \partial_x K_\delta\rangle \,\mathrm{d}t =: 2\sum_{i=1}^N \xi_i s_{i,\delta}.$$

For a fixed $\eta > 0$, choose $N\in\mathbb{N}$ sufficiently large such that

$$\int_0^T \mathbb{E}[(\widetilde{X}(t,0) - \sum_{i=1}^N b_i(t)\xi_i)^2]\,\mathrm{d}t = \sum_{i=N+1}^\infty \int_0^T b_i(t)^2\,\mathrm{d}t < \eta.$$

By the Cauchy-Schwarz inequality and Gaussianity we get that

$$\begin{aligned}\delta^{2-8\gamma}\mathbb{E}[|U_{3,\delta}-\widetilde{U}_{3,\delta}|]^2 &\lesssim \delta^{-4-8\gamma}\eta\int_0^T \mathbb{E}[\langle\bar{X}(t),(\Delta K)_\delta\rangle^2]\mathbb{E}[\langle\bar{X}(t),(\partial_x K)_\delta\rangle^2]\,\mathrm{d}t\\ &\lesssim \eta\|(-\Delta_\delta)^{1/2-\gamma}K\|^2_{L^2(\Lambda_\delta)}\|(-\Delta_\delta)^{-1/2-\gamma}\partial_x K\|^2_{L^2(\Lambda_\delta)},\end{aligned}$$

using in the last inequality we used Lemma 20(i) with $z=z'=\Delta K$ and $z=z'=\partial_x K$. Moreover, by Lemmas 21 and 23(i), the terms in the last inequality above are uniformly bounded in $\delta>0$, and hence

$$\sup_{0<\delta\leq 1}\left(\delta^{2-8\gamma}\mathbb{E}[|U_{3,\delta}-U_{3,\delta,N}|]^2\right)\lesssim\eta. \tag{42}$$

Next, we will prove that

$$\delta^{2-8\gamma}\mathbb{E}[s_{i,\delta}^2]\to 0,\quad i\in\mathbb{N}. \tag{43}$$

Analogous to Lemma 28, by Wick's formula and taking advantage of the symmetry in t,t' we obtain

$$\begin{aligned}\mathbb{E}[s_{i,\delta}^2] &= 2\int_0^T\int_0^t b_i(t)b_i(t')(\rho_{1,\delta}(t,t')+\rho_{2,\delta}(t,t')+\rho_{3,\delta}(t,t'))\,\mathrm{d}t'\,\mathrm{d}t,\\ \rho_{1,\delta}(t,t') &= \mathbb{E}[\bar{X}_\delta^\Delta(t)\langle\bar{X}(t),\partial_x K_\delta\rangle]\mathbb{E}[\bar{X}_\delta^\Delta(t')\langle\bar{X}(t'),\partial_x K_\delta\rangle],\\ \rho_{2,\delta}(t,t') &= \mathbb{E}[\bar{X}_\delta^\Delta(t')\langle\bar{X}(t),\partial_x K_\delta\rangle]\mathbb{E}[\bar{X}_\delta^\Delta(t)\langle\bar{X}(t'),\partial_x K_\delta\rangle],\\ \rho_{3,\delta}(t,t') &= \mathbb{E}[\langle\bar{X}(t),\partial_x K_\delta\rangle\langle\bar{X}(t'),\partial_x K_\delta\rangle]\mathbb{E}[\bar{X}_\delta^\Delta(t)\bar{X}_\delta^\Delta(t')].\end{aligned}$$

Clearly (43) follows from here by invoking the Cauchy-Schwarz inequality and Lemma 27(i-iii). Consequently, using (43), and applying again the Cauchy-Schwarz inequality, we deduce that $\delta^{2-8\gamma}\mathbb{E}[U_{3,\delta,N}]\to 0$ as $\delta\to 0$. Together with (42) and since η was arbitrary, we get $\delta^{1-4\gamma}U_{3,\delta}\to 0$. $\square$

B Well-Posedness and higher regularity of the solutions

In this section we provide well-posedness and higher regularity results for the linear and semilinear SPDEs relevant to our study. This is a well-established topic with a vast literature, see e.g. [DPZ14, LR15, vNVW12, Kry96]. We aim at giving a short and self-contained exhibition.

B.1 Regularity of the solution to the linear equation

We start with a result on well-posedness of the linear equation, as well as the (optimal) regularity of its solution. We recall that the Laplace operator on any smooth bounded domain $\Lambda \subset \mathbb{R}^d$ with Dirichlet boundary conditions has only point spectrum $\{-\lambda_k\}_{k\in\mathbb{N}}$, and without loss of generality can be arranged such as $0 \leq \lambda_1 \leq \ldots \leq \lambda_k \leq \ldots$. Moreover, the corresponding eigenfunctions, say $\{\Phi_k\}_{k\in\mathbb{N}}$, form a complete orthonormal system in $L^2(\Lambda)$; cf. [Shu01]. It is also well known that $\lambda_k \sim k^{2/d}$, as $k \to \infty$. Recall the optimal linear regularity $s^* = 1 + 2\gamma - d/2$ from Section 3.2.

Proposition 30. *Grant Assumption B. Then, the linear equation* (10) *has a unique mild solution* $\bar{X}$ *taking values in* $L^2(\Lambda)$*. Moreover:*

(i) $\bar{X} \in C([0,T]; W^s(\Lambda))$ *for all* $s < s^*$*, and in particular, for all* $2 \leq p < \infty$, $\bar{X} \in C([0,T]; W^{s-d/2+d/p,p}(\Lambda))$*;*

(ii) $\bar{X} \in C([0,T]; W^{s,p}(\Lambda))$ *for all* $2 \leq p < \infty$ *and* $s < s^*$*, provided that*

$$\sup_{k\geq 1} \|\Phi_k\|_{L^\infty(\Lambda)} < \infty. \tag{44}$$

Proof. Recall (9) and define for $\alpha \geq 0$ the process

$$Y_\alpha(t) := \int_0^t (t-r)^{-\alpha} S_\vartheta(t-r) B \mathrm{d}W(r), \quad 0 \leq t \leq T. \tag{45}$$

We show below for all $s \geq 0$ that

$$\mathbb{E}\left[|(-\Delta)^{\frac{s}{2}} Y_\alpha(t)(x)|^2\right] \leq C \sum_{k\geq 1} \lambda_k^{-2\gamma+s+2\alpha-1} \Phi_k^2(x), \quad x \in \Lambda. \tag{46}$$

Taking $s', \alpha = 0$ shows by Itô's isometry, with Hilbert-Schmidt norm $\|\cdot\|_2$ on $L^2(\Lambda)$, that $\int_0^t \|S_\vartheta(t-s)B\|_2^2 \,\mathrm{d}s = \mathbb{E}[\|\bar{X}(t)\|^2] < \infty$. This means that the stochastic integral in (9) is well-defined. That $\bar{X}$ is the unique mild solution to (10), follows by general theory [DPZ14, Chapter 5].

To establish the regularity of $\bar{X}$, we argue as in [DPZ14, Theorem 5.11] using the factorization method. We first show (ii). Let $s < s^*$, $p \geq 2$ and set

$E_1 = E_2 = W^{s,p}(\Lambda)$. Recall that if $Z \sim N(0,1)$, then $\mathbb{E}[|Z|^p] = c_p \mathbb{E}[Z^2]^{p/2}$ for some $c_p < \infty$. The Hölder inequality and the inequality in (46) show for $p' \geq 2$ that

$$\mathbb{E}\left[\int_0^T \|(-\Delta)^{\frac{s}{2}} Y_\alpha(t)\|_{E_2}^{p'} \mathrm{d}t\right] \lesssim \int_0^T \left(\int_\Lambda \mathbb{E}\left[|(-\Delta)^{\frac{s}{2}} Y_\alpha(t)(x)|^2\right]^{\frac{p}{2}} \mathrm{d}x\right)^{\frac{p'}{p}} \mathrm{d}t$$

$$\lesssim \left(\sum_{k\geq 1} k^{\frac{2}{d}(-2\gamma+2\alpha-1+s)}\right)^{\frac{p'}{p}},$$

where we used (44) in the last line. Since $-2\gamma + 2\alpha - 1 + s < 2\alpha - d/2$, the last line is finite for sufficiently small α. We find that Y_α has trajectories in $L^{p'}([0,T]; E_2)$. Choosing p' large enough such that $\alpha > 1/p'$ and $r = 0$ in [DPZ14, Proposition 5.9], we conclude that $\bar{X} \in C([0,T]; E_1) = C([0,T]; W^{s,p}(\Lambda))$. This proves (ii). For (i), it is enough to observe for $p' = p = 2$ that the upper bound in the last display equals $\sum_{k\geq 1} k^{\frac{2}{d}(-2\gamma+2\alpha-1+s)}$, which is finite for $s < s^*$ as just discussed. The supplement follows from the Sobolev embedding $W^s(\Lambda) \subset W^{s-d/2+d/p,p}(\Lambda)$.

We still have to prove (46). Let $B_1 := (-\Delta)^\gamma B$ and note that by Assumption B the operator[3] $B_1 : L^2(\Lambda) \to L^2(\Lambda)$ is bounded. For $x \in \Lambda$ and $f \in C(\Lambda)$, we define $\delta_x(f) = f(x)$. Then $\tilde{\delta}_x := \delta_x \circ (-\Delta)^{-\frac{d}{2}-\varepsilon}$ is a bounded linear functional on $L^2(\Lambda)$ for any $\varepsilon > 0$. Hence,

$$\begin{aligned}
\mathbb{E}\left[|(-\Delta)^{\frac{s}{2}} Y_\alpha(t)(x)|^2\right] &= \mathbb{E}\left[\left|\delta_x\left(\int_0^t (t-r)^{-\alpha}(-\Delta)^{\frac{s}{2}-\gamma} S_\vartheta(t-r) B_1 dW(r)\right)\right|^2\right] \\
&= \int_0^t (t-r)^{-2\alpha} \|B_1^*(-\Delta)^{\frac{d}{2}+\varepsilon} S_\vartheta(t-r)(-\Delta)^{\frac{s}{2}-\gamma}\tilde{\delta}_x^*\|_2^2 dr \\
&\lesssim \int_0^t (t-r)^{-2\alpha} \|(-\Delta)^{\frac{d}{2}+\varepsilon} S_\vartheta(t-r)(-\Delta)^{\frac{s}{2}-\gamma}\tilde{\delta}_x^*\|_2^2 dr \\
&= \mathbb{E}\left[\left|\delta_x\left(\int_0^t (t-r)^{-\alpha}(-\Delta)^{\frac{s}{2}-\gamma} S_\vartheta(t-r) dW(r)\right)\right|^2\right].
\end{aligned}$$

This allows us to reduce the argument to $B_1 = I$, i.e. $B = (-\Delta)^{-\gamma}$. In this case,

$$Y_\alpha(t,x) = \sum_{k=1}^\infty \lambda_k^{-\gamma} \left(\int_0^t (t-r)^{-\alpha} e^{-\lambda_k(t-r)} \mathrm{d}\beta_k(r)\right) \Phi_k(x),$$

where the $(\beta_k)_{k\in\mathbb{N}}$ are independent standard Wiener processes. The inequality

[3]With slight abuse of notations, we use the same notation for B_1 as in Assumption B, although strictly speaking they are not the same.

(46) follows then from

$$\begin{aligned}\mathbb{E}|(-\Delta)^{\frac{s}{2}}Y_\alpha(t,x)|^2 &= \sum_{k\geq 1}\lambda_k^{-2\gamma}\left(\int_0^t r^{-2\alpha}e^{-2\lambda_k r}\mathrm{d}r\right)((-\Delta)^{\frac{s}{2}}\Phi_k(x))^2\\ &\lesssim \sum_{k\geq 1}\lambda_k^{-2\gamma+2\alpha-1+s}\Phi_k^2(x).\end{aligned}$$

This concludes the proof. □

Recall the L^p-regularity index $\bar{s}$ from (19). The proposition shows that $\bar{s}\geq s^*-d/2+d/p$ for all $p\geq 2$. Choosing $\alpha=0$ in (46) also shows $\bar{s}\leq s^*$. The upper bound $\bar{s}=s^*$ is achieved if(44) holds. The condition (44) depends on the geometry of the domain Λ, but is true for rectangular domains in any dimension, in particular, for bounded intervals in $d=1$; cf. the discussion in [DPZ14, Remark 5.27].

B.2 Well-posedness and regularity of the solution to the semilinear equation

In this section we study the well-posedness and higher regularity of the solution to (12) in its mild formulation (11). We will use a classical fixed point argument, cf. [DPZ14]. In addition to Assumption $A_{s,\eta,p}$ from Section 3.2, we will make use of a local Lipschitz condition and a coercivity condition, for $p\geq 2$, and $s,s_1,s_2,\eta\geq 0$:

Assumption $A_{s,\eta,p}$. *There is $\varepsilon>0$ and a continuous function $g:[0,\infty)\to[0,\infty)$ such that for $u\in W^{s,p}(\Lambda)$:*

$$\|F(u)\|_{s+\eta-2+\varepsilon,p}\leq g(\|u\|_{s,p}). \tag{47}$$

Assumption $L_{s,\eta,p}$. *There exist $\varepsilon>0$ and a continuous function $h:[0,\infty)^2\to[0,\infty)$ such that for any $u,v\in W^{s+\eta,p}(\Lambda)$:*

$$\|F(u)-F(v)\|_{s+\eta-2+\varepsilon,p}\leq \|u-v\|_{s+\eta,p}h(\|u\|_{s,p},\|v\|_{s,p}). \tag{48}$$

Assumption C_{s_1,s_2}. *There exists a continuous function $b:[0,\infty)\to[0,\infty)$ such that for any $u\in W^{s_1}(\Lambda)$, $v\in W^{s_2}(\Lambda)$ with $F(u+v)\in W^{s_1}(\Lambda)$:*

$$\langle F(u+v),u\rangle_{W^{s_1}(\Lambda)}\leq (1+\|u\|_{s_1}^2)b(\|v\|_{s_2}). \tag{49}$$

Next we present the main result of this section.

Theorem 31. *Let $s,s_1,\eta\geq 0$, $p\geq 2$ with $s+\eta\geq s_1+2$, and suppose that*

$$X_0\in W^{s+\eta,p}(\Lambda),\quad \textit{and}\quad \bar{X}\in C([0,T];W^{s,p}(\Lambda)).$$

Assume that Assumption $A_{s',\eta,p'}$ is satisfied for $s_1 \leq s' \leq s$ and $2 \leq p' \leq p$. Furthermore suppose that Assumptions $L_{s,\eta,p}$ and $C_{s_1,s}$ are fulfilled. Then there exists a unique solution $\widetilde{X}$ to (11) *such that $\widetilde{X} \in C([0,T];W^{s+\eta,p}(\Lambda))$.*

In particular, there exists a unique mild solution $X \in C([0,T];W^{s,p}(\Lambda))$ to Equation (1)*.*

Proof. The statement follows from Proposition 6, provided that $\widetilde{X} \in C([0,T];W^{s_1}(\Lambda))$. This inclusion indeed holds true, as proved in Lemma 33 below. □

For the rest of this section, we fix $s, s_1, \eta \geq 0$ and $p \geq 2$ that satisfy the assumptions from Theorem 31. Since all the statements are pathwise, we also fix $\omega \in \Omega$. For $T', m > 0$, let

$$M(T',m) := \{u \in C([0,T'];W^{s+\eta,p}(\Lambda)) \mid \sup_{0\leq t\leq T'} \|u(t)\|_{s+\eta,p} \leq m\},$$

and define the operator $G : M(T',m) \to C([0,T'];W^{s+\eta,p}(\Lambda))$ as

$$(Gu)(t) = S_\vartheta(t)X_0 + \int_0^t S_\vartheta(t-r)F(\bar{X}+u)(r)\mathrm{d}r. \tag{50}$$

Note that $M(T',m)$ is a closed ball in a Banach space, hence complete.

Lemma 32. *Assume that Assumption $A_{s,\eta,p}$ and $L_{s,\eta,p}$ are fulfilled, $X_0 \in W^{s+\eta,p}(\Lambda)$ and let $m > \|X_0\|_{s+\eta,p}$. Then, there exists $T' > 0$ such that Equation* (11) *has a unique solution in $M(T',m)$.*

Proof. Analogous to the proof of Proposition 6(i), for any $T' > 0$, we deduce

$$\|(Gu)(t)\|_{s+\eta,p} \leq \frac{2}{\varepsilon}T'^{\frac{\varepsilon}{2}}\|X_0\|_{s+\eta,p} + \frac{2}{\varepsilon}T'^{\frac{\varepsilon}{2}} g\left(\sup_{0\leq t\leq T'}\|\bar{X}(t)\|_{s,p} + Cm\right),$$

where C is the embedding constant coming from $W^{s+\eta,p}(\Lambda) \subset W^{s,p}(\Lambda)$. Note that the above estimate holds uniformly in $t \in [0,T']$. Moreover, for sufficiently small $T' > 0$, G maps $M(T',m)$ into itself. Clearly, the claim will follow, once it is proved that T' can be chosen such that G is a contraction mapping on $M(T',m)$, which we will show next. By Proposition 15(i), and Assumption $L_{s,\eta,p}$, for any $u, v \in M(T',m)$, we have

$$\begin{aligned}
&\|(Gu-Gv)(t)\|_{s+\eta,p} \leq \int_0^t \|S_\vartheta(t-r)(F(\bar{X}-u)(r) - F(\bar{X}-v)(r))\|_{s+\eta,p}\mathrm{d}r \\
&\lesssim \int_0^t (t-r)^{-1+\frac{\varepsilon}{2}}\|F(\bar{X}-u)(r) - F(\bar{X}-v)(r)\|_{s+\eta-2+\varepsilon,p}\mathrm{d}r \\
&\leq \int_0^t (t-r)^{-1+\frac{\varepsilon}{2}}\|u(r)-v(r)\|_{s+\eta,p} h(\|\bar{X}(r)+u(r)\|_{s,p}, \|\bar{X}(r)+v(r)\|_{s,p})\mathrm{d}r \\
&\leq \frac{2}{\varepsilon}T'^{\frac{\varepsilon}{2}} \sup_{0\leq t\leq T'}\|u(t)-v(t)\|_{s+\eta,p} \sup_{0\leq t\leq T'} h\left(\|\bar{X}(t)+u(t)\|_{s,p}, \|\bar{X}(t)+v(t)\|_{s,p}\right).
\end{aligned}$$

Since $\|\bar{X}(t)+u(t)\|_{s,p} \leq \sup_{0\leq t'\leq T'}\|\bar{X}(t')\|_{s,p} + Cm$, then there exists a (random) constant $\tilde{C}$ such that

$$\sup_{0\leq t\leq T'}\|(Gu-Gv)(t)\|_{s+\eta,p} \leq \tilde{C}\frac{2}{\varepsilon}T'^{\frac{\varepsilon}{2}}\sup_{0\leq t\leq T'}\|u(t)-v(t)\|_{s+\eta,p},$$

and hence, for small enough T' the mapping G is a contraction mapping. The proof is complete. □

Lemma 33. *Assume that Assumptions $A_{s,\eta,p}$, $L_{s,\eta,p}$ and $C_{s_1,s}$ hold, with $s+\eta \geq s_1+2$, and suppose that $X_0 \in W^{s+\eta,p}(\Lambda)$, $\bar{X} \in C([0,T];W^{s,p}(\Lambda))$. Then, the solution $\widetilde{X}$ to (11) exists up to time T, and $\widetilde{X} \in C([0,T];W^{s_1,p}(\Lambda))$.*

Proof. By Lemma 32, there exists a solution $\widetilde{X} \in W^{s+\eta,p}(\Lambda) \subset W^{s_1+2}(\Lambda)$, locally in time. Let $0 < \bar{T} \leq T$ be the (random) maximal time of existence of $\widetilde{X} \in W^{s_1+2}(\Lambda)$. Whenever $\bar{T} < T$, we have $\sup_{0\leq t\leq \bar{T}}\|\widetilde{X}(t)\|_{s_1} = \infty$.

Assume $\bar{T} < T$, and set $\widetilde{X}^{(n)} := n(n-\vartheta\Delta)^{-1}\widetilde{X}$. Then, as $n \to \infty$, $\widetilde{X}^{(n)} \to \widetilde{X}$ in $C([0,\bar{T}];W^{s+\eta,p}(\Lambda))$. Furthermore,

$$\begin{aligned} R^{(n)} &:= \partial_t\widetilde{X}^{(n)} - \vartheta\Delta\widetilde{X}^{(n)} - F(\bar{X}+\widetilde{X}^{(n)}) \\ &= n(n-\vartheta\Delta)^{-1}F(\bar{X}+\widetilde{X}) - F(\bar{X}+\widetilde{X}^{(n)}) \to 0 \end{aligned}$$

in $C([0,\bar{T}];W^{s+\eta-2,p}(\Lambda))$ by $L_{s,\eta,p}$, and hence also in $C([0,\bar{T}];W^{s_1}(\Lambda))$. Now,

$$\begin{aligned} &\|\widetilde{X}^{(n)}(t)\|_{s_1}^2 \\ &\quad = 2\int_0^t \left\langle \vartheta\Delta\widetilde{X}^{(n)}(r) + F(\bar{X}+\widetilde{X}^{(n)})(r) + R^{(n)}(r), \widetilde{X}^{(n)}(r)\right\rangle_{W^{s_1}(\Lambda)} \mathrm{d}r \\ &\quad \lesssim \int_0^t \left\langle F(\bar{X}+\widetilde{X}^{(n)})(r), \widetilde{X}^{(n)}(r)\right\rangle_{W^{s_1}(\Lambda)} + \|\widetilde{X}^{(n)}(r)\|_{s_1}^2 + \|R^{(n)}(r)\|_{s_1}^2 \mathrm{d}r \\ &\quad \lesssim \int_0^t \left(1+\|\widetilde{X}^{(n)}(r)\|_{s_1}^2\right) b\left(\|\bar{X}(r)\|_s\right) + \|\widetilde{X}^{(n)}(r)\|_{s_1}^2 \\ &\qquad\qquad + \|R^{(n)}(r)\|_{s_1}^2 \mathrm{d}r, \end{aligned}$$

where we applied $C_{s_1,s}$ in the last inequality. Applying Gronwall's inequality and letting $n \to \infty$, we conclude that $\sup_{0\leq t\leq \bar{T}}\|\widetilde{X}(t)\|_{s_1}^2 < \infty$, in contradiction to $\bar{T} < T$. Hence $T = \bar{T}$ almost surely. □

In the next two sections we consider two important examples - reactions-diffusion equations and Burger's equation - and for each of them we provide simple conditions that guarantee that the conclusions from Theorem 31 are true.

B.2.1 Application to reaction-diffusion equations

As in Section 3.3.2 we consider reaction-diffusion equations whose nonlinearity is given by a function $f : \mathbb{R} \to \mathbb{R}$, namely $F(u)(x) = f(u(x))$. First, we deal with the case that f is a polynomial

$$f(x) = a_m x^m + \cdots + a_1 x + a_0, \tag{51}$$

with $a_m < 0$ and $m \in 2\mathbb{N} + 1$. We prove an auxiliary result:

Lemma 34. *Let $p \geq 1$ and let f as in* (51)*. Then:*

(i) Assumption $A_{s,\eta,p}$ is true with any $\eta < 2$, $p \geq 2$ and $s > d/p$.

(ii) Assumption $A_{s,\eta,p}$ is true for $s = 0$ with $p > d(m-1)/2$ and $\eta < 2 - d(m-1)/p$.

(iii) Assumption $L_{s,\eta,p}$ holds for any $\eta \in [0, 2)$, whenever $s > d/p$.

(iv) Assumption C_{s_1,s_2} is satisfied with $s_1 = 0$, $s_2 > d/2$.

Proof. (i). This follows from Lemma 9.
(ii). Analogous to (i), it suffices to bound $\|x^l\|_{\eta-2+\varepsilon,p}$, for $l = 2, \ldots, m$. Since $p > \frac{d}{2}(l-1)$ and $0 < \eta < 2 - \frac{d}{p}(l-1)$, by the Sobolev embedding theorem, we have $\|x^l\|_{\eta-2+\varepsilon,p} \lesssim \|x^l\|_{0,p/l} \leq \|x\|_{0,p}^l$.
(iii). This follows from $\|xy\|_{s,p} \lesssim \|x\|_{s,p}\|y\|_{s,p}$.
(iv). This is a well-known property, cf. [DPZ14, Example 7.10]. See e.g. [PS20, Proposition 2.5] for the calculation. ☐

Proposition 35. *Consider f as in* (51)*. Suppose that $X_0 \in W^{s+2,p}(\Lambda)$ and $\bar{X} \in C([0,T]; W^{s,p}(\Lambda))$ for some $p \geq 2$ and $s > 1 \vee \frac{d}{p}$. Assume that $d \leq 3$ and $p > \frac{dm}{2}$, and if $d = 3$ also assume that $m \leq 3$. Then, the assertions of Theorem 31 hold true.*

Proof. By Lemma 32 and 33 there is a solution to (11) in $L^2(\Lambda)$. As the leading coefficient of F is negative, F' is bounded from above, and it holds for sufficiently smooth Y, e.g. $Y \in W^{(3/2)\vee d}(\Lambda)$, that

$$\begin{aligned}\langle \vartheta \Delta Y + F(Y), Y\rangle_{W^{1,2}(\Lambda)} &\leq -\vartheta\|Y\|_2^2 + \langle F'(Y)\nabla Y, \nabla Y\rangle_{L^2} \\ &\lesssim -\vartheta\|Y\|_2^2 + C\|Y\|_1^2.\end{aligned}$$

Using this coercivity property, one shows as in [LR15, Lemma 4.29] using a suitable approximation sequence that $X = \overline{X} + \widetilde{X}$ (and thus $\widetilde{X}$) has in fact values in $W^1(\Lambda)$. For additional regularity, we use the Sobolev embedding theorems: If $d = 1$ or $d = 2$, then $W^1(\Lambda)$ is embedded in $L^p(\Lambda)$. By Lemma 34 (ii) and Proposition 6, $\widetilde{X} \in C([0,T]; W^{s',p}(\Lambda))$ for some $s' > d/p$ (here we use $p > dm/2$). Now conclude inductively with Lemma 34 (i). If $d = 3$ and $m = 3$, we argue similarly. $W^1(\Lambda)$ embeds into $L^6(\Lambda)$, so by Lemma 34 (ii) with $d = m = 3$, $p = 6$ and $\eta = 1/2$, $\widetilde{X}$ has values in $W^{1/2,6}(\Lambda)$, which in turn embeds into $L^q(\Lambda)$ for any $q \geq 2$. Now conclude as in the case $d \in \{1, 2\}$. □

In particular, using Proposition 30(i,ii), we have proven:

Theorem 36. *Consider f as in (51). Let $d \leq 3$. In the case $d = 3$ also assume $m \leq 3$. Grant Assumption B and let*

$$\gamma > \begin{cases} \frac{1}{4}, & d = 1, \\ 1, & d = 2, \\ \frac{3}{2}, & d = 3, \end{cases} \qquad s_d = \begin{cases} \frac{1}{2} + 2\gamma, & d = 1, \\ -1 + 2\gamma, & d = 2, \\ -2 + 2\gamma, & d = 3. \end{cases}$$

For any $p > dm/2$ and $1 \vee d/p < s < s_d$, if $X_0 \in W^{s+2,p}(\Lambda)$, then there exists a unique solution $\widetilde{X} \in C([0, T]; W^{s+2,p}(\Lambda))$ to (11), and in particular, there exists a unique mild solution $X \in C([0, T]; W^{s,p}(\Lambda))$ to (1).

Remark 37. In Theorem 36, γ is chosen such that B is always a Hilbert-Schmidt operator. In $d = 2, 3$, the condition we pose is even more restrictive as we just use minimal regularity for $\overline{X}$ from Proposition 30(i). Furthermore, note that in $d = 3$, when $d/p > 1$, w.l.o.g. we can choose p larger such that $1 \vee d/p < s_d$ is satisfied.

Next, we test the conditions for reaction terms of the form $f \in C_b^\infty(\mathbb{R})$.

Lemma 38. *Consider $f \in C_b^\infty(\mathbb{R})$. Then:*

(i) Assumption $A_{s,\eta,p}$ is true for $p \geq 2$, $s > 1$ and $\eta < 2$, and we can choose $g(x) = C(1 + |x|^{1\vee s})$ for some $C > 0$.

(ii) Assumption $L_{s,\eta,p}$ is true for $p \geq 2$, $s > d/p$ and $\eta < 2$.

(iii) Assumption C_{s_1,s_2} is true for $s_1 = 1$, $s_2 \geq 1$.

Proof. (i). With $\bar{f} = f - f(0)$, [AF92, Theorem A] gives $\|\bar{f}(u)\|_{s,p} \lesssim \|u\|_{s,p} + \|u\|_{s,p}^s$, and the claim follows easily.

(ii). First note that $f' \in C_b^\infty(\mathbb{R})$ as well. Using the algebra property of $W^{s,p}(\Lambda)$, for $u, v \in W^{s,p}(\Lambda)$ together with part (i):

$$\begin{aligned} \|f(u) - f(v)\|_{s,p} &\lesssim \int_0^1 \|f'(u + t(v - u))\|_{s,p} \mathrm{d}t \, \|u - v\|_{s,p} \\ &\lesssim \int_0^1 \left(1 + \|u + t(u - v)\|_{s,p}^{1\vee s}\right) \mathrm{d}t \, \|u - v\|_{s,p} \\ &\lesssim \left(1 + \|u\|_{s,p}^{1\vee s} + \|v\|_{s,p}^{1\vee s}\right) \|u - v\|_{s,p}, \end{aligned}$$

and the claim follows.

(iii). Making use of the boundedness of $f \in C_b^\infty(\mathbb{R}, \mathbb{R})$, we have

$$\begin{aligned}\langle f(u+v), u\rangle_{W^{1,2}(\Lambda)} &= \langle f(u+v), (-\Delta)u\rangle \\ &= \langle f'(u+v)(\nabla u + \nabla v), \nabla u\rangle \\ &\lesssim \|u\|_1^2 + \|u\|_1\|v\|_1,\end{aligned}$$

and we can choose $b(x) = 1 + 2x$. □

Proposition 39. *Consider $f \in C_b^\infty(\mathbb{R})$. Suppose that $X_0 \in W^{s+2,p}(\Lambda)$ and $\bar{X} \in C([0,T]; W^{s,p}(\Lambda))$ for some $p \geq 2$, $s > 1$. Then, the assertions of Theorem 31 hold true.*

Proof. By Lemma 32 and 33, there is a solution to (11) in $W^1(\Lambda)$. Additional $W^{s,p}(\Lambda)$-regularity for $s < 3 + 2\gamma - d/2$ and $p \geq 2$ now follows by iteratively appliying Lemma 38 and the Sobolev embedding theorem. □

We immediately get:

Theorem 40. *Consider $f \in C_b^\infty(\mathbb{R})$. Grant Assumption B and let $\gamma > 1/4$ in $d = 1$ or $\gamma > d/2$ in $d \geq 2$, respectively. Let $s_d = 1 + 2\gamma - d$ for $d \geq 2$ and $s_1 = 1/2 + 2\gamma$. For any $p \geq 2$ and $1 < s < s_d$, if $X_0 \in W^{s+2,p}(\Lambda)$, then there exists a unique solution $\widetilde{X} \in C([0,T]; W^{s+2,p}(\Lambda))$ to (11), and in particular, there exists a unique mild solution $X \in C([0,T]; W^{s,p}(\Lambda))$ to (1).*

B.2.2 Application to the stochastic Burgers equation.

Let $d = 1$ and

$$F(u) = -u\partial_x u = -\frac{1}{2}\partial_x(u^2). \tag{52}$$

Assume $X_0 \in W^{s+1,p}(\Lambda)$ and $\bar{X} \in C([0,T]; W^{s,p}(\Lambda))$ for some $s > 1$ and $p \geq 2$. Proposition 30 shows that the latter condition can be satisfied if $1+2\gamma-d/2 > 1$, i.e. $\gamma > d/4 = 1/4$, independently of $p \geq 2$.[4]

Lemma 41.

(i) Assumption $A_{s,\eta,p}$ is true for any $p \geq 2$, $s > 1/p$ and $\eta < 1$.

(ii) Assumption $A_{s,\eta,p}$ is true for $s = 0$, $p \geq 2$ with $\eta < 1 - 1/p$.

(iii) Assumption $L_{s,\eta,p}$ holds for $p \geq 2$, $s > 1/p$ and $\eta \in [0,1)$.

(iv) Assumption C_{s_1,s_2} is true for $s_1 = 0$ and $s_2 > 3/2$.

Proof. (i) - (iii) are shown as in Lemma 34. (iv) is well-known, the calculations can be found e.g. in [PS20]. □

[4] This condition can be further relaxed, e.g. [DaPrato, Debussche, Temam 1994] uses a different technique for proving coercivity in the case $\gamma = 0$.

Proposition 42. *The conclusions of Theorem 31 are applicable in this case.*

Proof. By Lemma 32 and 33 the process $\widetilde{X}$ is well-posed in $C([0,T];L^2(\Lambda))$. By Lemma 41 (ii) and Proposition 6, $\widetilde{X}$ has values in $W^{s'}(\Lambda)$ for any $s' < 1/2$, and consequently in $L^q(\Lambda)$ for any $q \geq 2$. Now conclude as in Proposition 35 by applying Lemma 41 (ii) and (i) iteratively. $\square$

Putting things together, we have:

Theorem 43. *Grant Assumption B and let $\gamma > \frac{1}{4}$. For any $p \geq 2$ and $1 < s < 1/2 + 2\gamma$, if $X_0 \in W^{s+1,p}(\Lambda)$, then there exists a unique solution $\widetilde{X} \in C([0,T];W^{s+1,p}(\Lambda))$ to (11) and a unique solution $X \in C([0,T];W^{s,p}(\Lambda))$ to (1).*

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References

[Ada75] R. A. Adams. *Sobolev spaces.* Academic Press, 1975.

[AF92] David R. Adams and Michael Frazier. Composition operators on potential spaces. *Proceedings of the American Mathematical Society*, 114(1):155–165, 1992.

[AR20] R. Altmeyer and M. Reiß. Nonparametric estimation for linear SPDEs from local measurements. *Forthcoming in Annals of Applied Probability*, 2020.

[ASB18] S. Alonso, M. Stange, and C. Beta. Modeling random crawling, membrane deformation and intracellular polarity of motile amoeboid cells. *PLOS ONE*, 13(8):1–22, 08 2018.

[BT19a] M. Bibinger and M. Trabs. On central limit theorems for power variations of the solution to the stochastic heat equation. *Forthcoming in Springer Proceedings in Mathematics & Statistics*, 2019.

[BT19b] M. Bibinger and M. Trabs. Volatility estimation for stochastic PDEs using high-frequency observations. *Forthcoming in Stochastic Processes and their Applications*, 2019.

[CA77] J. Cahn and S. Allen. A microscopic theory for domain wall motion and its experimental verification in Fe-Al alloy domain growth kinetics. *Journal de Physique Colloques*, 38(C7):51–54, 1977.

[CCH$^+$19] C. Cotter, D. Crisan, D. D. Holm, W. Pan, and I. Shevchenko. Numerically modeling stochastic lie transport in fluid dynamics. *Multiscale Modeling & Simulation*, 17(1):192–232, 2019.

[CDVK19] I. Cialenco, F. Delgado-Vences, and H.-J. Kim. Drift estimation for discretely sampled SPDEs. *Preprint, arXiv:1904.10884*, 2019.

[CGH11] I. Cialenco and N. Glatt-Holtz. Parameter estimation for the stochastically perturbed Navier-Stokes equations. *Stochastic Processes and their Applications*, 121(4):701–724, 2011.

[CH19] I. Cialenco and Y. Huang. A note on parameter estimation for discretely sampled SPDEs. *Forthcoming in Stochastics and Dynamics. DOI:10.1142/S0219493720500l*, 2019.

[Cho19a] C. Chong. High-frequency analysis of parabolic stochastic PDEs. *Forthcoming in Annals of Statistics*, 2019.

[Cho19b] C. Chong. High-frequency analysis of parabolic stochastic PDEs with multiplicative noise: Part I. *Preprint. arXiv:1908.04145*, 2019.

[Cia18] I. Cialenco. Statistical inference for SPDEs: an overview. *Statistical Inference for Stochastic Processes*, 21(2):309–329, 2018.

[CK20] I. Cialenco and H.-J Kim. Parameter estimation for discretely sampled stochastic heat equation driven by space-only noise revised. *Preprint, arXiv:2003.08920*, 2020.

[Con05] R. Cont. Modeling term structure dynamics: an infinite dimensional approach. *International Journal of Theoretical and Applied Finance*, 8(3):357–380, 2005.

[DdMH15] A. Debussche, S. de Moor, and M. Hofmanová. A regularity result for quasilinear stochastic partial differential equations of parabolic type. *SIAM Journal on Mathematical Analysis*, 47(2):1590–1614, 2015.

[DPDT94] G. Da Prato, A. Debussche, and R. Temam. Stochastic Burgers' equation. *NoDEA Nonlinear Differential Equations and Applications*, 1(4):389–402, 1994.

[DPZ14] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions.* Cambridge University Press, 2014.

[Fit61] R. Fitzhugh. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal*, 1:445–466, 1961.

[Fra85] C. Frankignoul. Sst anomalies, planetary waves and rc in the middle rectitudes. *Reviews of Geophysics*, 23(4):357–390, 1985.

[HV11] M. Hairer and J. Voss. Approximations to the stochastic burgers equation. *Journal of Nonlinear Science*, 21(6):897–920, 2011.

[Jan97] S. Janson. *Gaussian Hilbert Spaces.* Cambridge University Press, 1997.

[Kry96] N. V. Krylov. On L_p-theory of stochastic partial differential equations in the whole space. *SIAM Journal of Mathematical Analysis*, 27(2):313–340, 1996.

[KT19] Z. M. Khalil and C. Tudor. Estimation of the drift parameter for the fractional stochastic heat equation via power variation. *Modern Stochastics: Theory and Applications*, 6(4):397–417, 2019.

[KU19] Y. Kaino and M. Uchida. Parametric estimation for a parabolic linear SPDE model based on sampled data. *Preprint, arXiv:1909.13557*, 2019.

[LLB15] R. Lockley, G. Ladds, and T. Bretschneider. Image based validation of dynamical models for cell reorientation. *Cytometry Part A*, 87(6):471–480, 2015.

[LPS14] G. J. Lord, C. E. Powell, and T. Shardlow. *An Introduction to Computational Stochastic PDEs.* Cambridge University Press, 2014.

[LR15] W. Liu and M. Röckner. *Stochastic partial differential equations: an introduction.* Springer, 2015.

[LS89] R. S. Liptser and A. N. Shiryayev. *Theory of martingales.* Kluwer Academic Publishers Group, 1989.

[NASY62] A. Nagumo, S. Arimoto, and S. S. Yoshizawa. An active pulse transmission line simulating nerve axon. *Proc. IRE*, 50(10):2061–2070., 1962.

[NRR19] N. Nüsken, S. Reich, and P. J. Rozdeba. State and Parameter Estimation from Observed Signal Increments. *Entropy*, 21(5):505, 2019.

[Nua06] D. Nualart. *The Malliavin calculus and related topics.* Springer, 2006.

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer, 1983.

[PS20] G. Pasemann and W. Stannat. Drift estimation for stochastic reaction-diffusion systems. *Forthcoming in Electronic Journal of Statistics*, 2020.

[RR20] S. Reich and P. Rozdeba. Posterior contraction rates for non-parametric state and drift estimation. *Preprint, arXiv:2003.09219*, 2020.

[Sch72] F. Schlögl. Chemical reaction models for non-equilibrium phase transitions. *Zeitschrift für Physik*, 253:147–161, 1972.

[Shu01] M. A. Shubin. *Pseudodifferential operators and spectral theory.* Springer, 2001.

[Tri83] H. Triebel. *Theory of function spaces.* Birkhäuser, 1983.

[vNVW12] Jan van Neerven, Mark Veraar, and Lutz Weis. Maximal lp-regularity for stochastic evolution equations. *SIAM Journal on Mathematical Analysis*, 44(3):1372–1414, 2012.

[Wal81] J. B. Walsh. A stochastic model of neural response. *Advances in Applied Probability*, 13(2):231–281, 1981.

[Yag10] A. Yagi. *Abstract parabolic evolution equations and their applications.* Springer, 2010.

[Yan19] D. Yan. *Bayesian Inference for Gaussian Models: Inverse Problems and Evolution Equations.* PhD thesis, Universiteit Leiden, 2019.