# Math 590 — Homework 1 — due Wednesday, Sept.24, 2014

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1. Let $\mathcal{K}$ be a positive Hilbert–Schmidt integral operator $\mathcal{K}$ with translation invariant kernel $K(\boldsymbol{x},\boldsymbol{z}) = \widetilde{K}(\boldsymbol{x}-\boldsymbol{z})$. Use the eigenfunction basis of $\mathcal{K}$ to show that its trace is given by

$$\operatorname{trace}\mathcal{K} = \sum_{n=1}^{\infty} \lambda_n = \widetilde{K}(\mathbf{0}).$$

2. Verify that the closed form expression for the Brownian motion kernel matches its Mercer series, i.e., show that

$$\sum_{n=1}^{\infty} \frac{8}{(2n-1)^2\pi^2} \sin\left((2n-1)\frac{\pi x}{2}\right) \sin\left((2n-1)\frac{\pi z}{2}\right) = \min(x,z).$$

3. Prove properties 1–2 of reproducing kernels, i.e., show

   (a) For all $\boldsymbol{x},\boldsymbol{z} \in \Omega$
   $$\langle K(\cdot,\boldsymbol{x}), K(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_K(\Omega)} = K(\boldsymbol{x},\boldsymbol{z}) = K(\boldsymbol{z},\boldsymbol{x}).$$

   (b) For every $f \in \mathcal{H}_K(\Omega)$ and $x \in \Omega$ we have
   $$|f(\boldsymbol{x})| \le \sqrt{K(\boldsymbol{x},\boldsymbol{x})}\,\|f\|_{\mathcal{H}_K(\Omega)}.$$

4. Show that the sum of two positive definite kernels is positive definite. Do this directly from the definition, i.e., without using property 5 of reproducing kernels.

5. Show that convergence in the norm of the RKHS $\mathcal{H}_K(\Omega)$ implies pointwise convergence, i.e., if we have
$$\|f - f_n\|_{\mathcal{H}_K(\Omega)} \to 0 \quad \text{for } n \to \infty$$
then
$$|f(\boldsymbol{x}) - f_n(\boldsymbol{x})| \to 0 \quad \text{for all } \boldsymbol{x} \in \Omega.$$

6. Show that the *hat function*
$$\Phi(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$
is a strictly positive definite function on $\mathbb{R}$?