Math 590 — Computer Assignment 3 — due Wednesday, Nov. 5, 2014

In this homework set you will explore the family of Matérn kernels — for simplicity in the univariate setting on the domain $\Omega = [0,1]$. As we saw in Chapter 3, the Matérn kernels are defined by

$$\kappa(\varepsilon r) = \frac{K_{d/2-\beta}(\varepsilon r)}{(\varepsilon r)^{d/2-\beta}}, \quad \beta > \frac{d}{2},$$

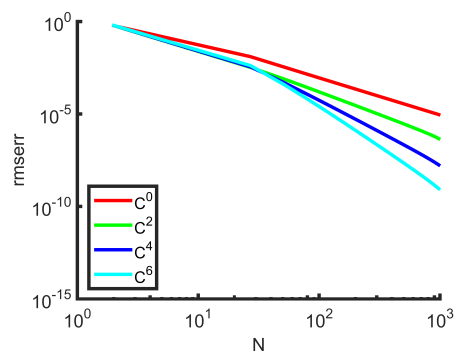
where K_ν is a modified Bessel functions of the second kind of order ν. If $\beta = \frac{d+2k+1}{2}$ then the kernels are in $C^{2k}(\Omega)$ and can be expressed as the product of a polynomial in r and an exponential function (see the slides for examples).

1. Write a MATLAB script that performs a sequence of numerical experiments interpolating data sampled from the test function $f(x) = \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ at $N = 2, 27, 52, 77, \ldots, 977$ evenly spaced points in $[0,1]$ for each of the univariate (i.e., $d = 1$) Matérn kernels corresponding to $k = 0, 1, 2, 3$. Fix the shape parameter to $\varepsilon = 5$ for all of the experiments in this exercise.

   As output of this script I expect to see a graph that displays a loglog plot of the root-mean square errors computed at $N_{\text{eval}} = 1500$ evenly spaced evaluation points ξ_i vs. N. Here the root-mean square error is defined as

   $$\text{rmserr} = \sqrt{\frac{1}{N_{\text{eval}}} \sum_{i=1}^{N_{\text{eval}}} [s(\xi_i) - f(\xi_i)]^2}.$$

   Plot the results for the four different kernels together in one graph. This should produce something similar to the following graph.

   What are the observed orders of convergence? How do they differ with the choice of kernel, i.e., with the value of k? Do they agree with the estimates obtained via sampling inequalities in Chapter 8?

2. Now perform another series of experiments similar to those in Exercise 1. However, fix the kernel to the case $d = 1$, $k = 1$ and use $\varepsilon = 1, 5, 11, 16, 21$ as values of the shape parameter. Take N and N_{eval} as before and plot the five resulting graphs together in a loglog plot of root-mean square error vs. N.

   Repeat for the case $k = 2$.

   What are the observed orders of convergence? How do they differ with the choice of ε? Do you observe anything else?

3. Repeat the experiments from Exercise 2, but now compute the root-mean square error only over the sub-interval $[0.1, 0.9]$. What orders of convergence do you observe now? Is there a theoretical explanation for what you are seeing?

4. In this problem you will investigate the "flat" $\varepsilon \to 0$ limit of Matérn kernels. Fix the Matérn kernel to the case $d = 1$, $k = 0$ and use sets of $N = 3, 5, 9, 17, 33, 65$ evenly spaced samples of the sinc function for interpolation.

   Produce a plot of the root-mean square error (again computed at $N_{\text{eval}} = 1500$ evaluation points) vs. the shape parameter ε. Your plot should consist of six different graphs (one for each value of N). For these experiments take 500 values of ε varying logarithmically from 10^{-3} to 100, i.e., `ep = logspace(-3,2,500)`.

   What interpolation method corresponds to the "flat" $\varepsilon \to 0$ limit? Can you create the corresponding horizontal error lines using this interpolation method and add them to the graph?

   For large values of ε the error graphs also become flat. How do you explain this phenomenon?

   Repeat for the case $k = 1$. How does this case differ from the $k = 0$ case?