1. Determine SVDs of the following matrices. Do not use a computer, and do not use the method for hand calculations discussed in class. Use only basic properties of the SVD and note that the matrices are either diagonal matrices or rank-1 matrices:

   (a) \[
   \begin{bmatrix}
   3 & 0 \\
   0 & -2
   \end{bmatrix},
   \]
   (b) \[
   \begin{bmatrix}
   2 & 0 \\
   0 & 3
   \end{bmatrix},
   \]
   (c) \[
   \begin{bmatrix}
   0 & 2 \\
   0 & 0
   \end{bmatrix},
   \]
   (d) \[
   \begin{bmatrix}
   1 & 1 \\
   0 & 0
   \end{bmatrix},
   \]
   (e) \[
   \begin{bmatrix}
   1 & 1
   \end{bmatrix}.
   \]

2. In the discussion of matrix norms we claimed that the 2-norm of the matrix

   \[
   A = \begin{bmatrix}
   1 & 1 \\
   0 & 1
   \end{bmatrix}
   \]

is approximately 1.6180. Using the SVD, work out (the “by-hand” method is from now on allowed) the exact values of \( \sigma_{\min}(A) \) and \( \sigma_{\max}(A) \) for this matrix.

3. Consider the matrix

   \[
   A = \begin{bmatrix}
   -2 & 11 \\
   -10 & 5
   \end{bmatrix}.
   \]

   (a) Determine, on paper, a real SVD of \( A \) in the form \( A = U\Sigma V^T \). The SVD is not unique, so find the one that has the minimal number of minus signs in \( U \) and \( V \).

   (b) List the singular values, left singular vectors, and right singular vectors of \( A \). Draw a careful, labeled picture of the unit ball in \( \mathbb{R}^2 \) and its image under \( A \), together with the singular vectors, with the coordinates of their vertices labeled.

   (c) What are the \( 1-, 2-, \infty- \), and Frobenius norms of \( A \)?

   (d) Find \( A^{-1} \) not directly, but via the SVD.

   (e) Find the eigenvalues \( \lambda_1, \lambda_2 \) of \( A \).

   (f) Verify that \( \det A = \lambda_1 \lambda_2 \) and \( |\det A| = \sigma_1 \sigma_2 \).

   (g) What is the area of the ellipsoid onto which \( A \) maps the unit ball of \( \mathbb{R}^2 \)?

4. Assume \( A \) is Hermitian and positive definite, i.e., \( A \) can be uniquely factored into \( A = LL^* \) with \( L \) a lower triangular matrix with positive diagonal entries (Cholesky factorization). What is the SVD of \( A \)?

5. If \( P \) is an orthogonal projector, then \( I - 2P \) is unitary. Prove this algebraically, and give a geometric interpretation.

6. Consider the matrices

   \[
   A = \begin{bmatrix}
   1 & 0 \\
   0 & 1
   \end{bmatrix}, \quad B = \begin{bmatrix}
   1 & 2 \\
   0 & 1
   \end{bmatrix}.
   \]

Answer the following questions by hand calculation.

   (a) What us the orthogonal projector \( P \) onto range(\( A \)), and what is the image under \( P \) of the vector \([1, 2, 3]^*\)?

   (b) Same questions for \( B \).