1. For each of the following statements, prove that it is true or give an example to show it is false. Throughout, $A \in \mathbb{C}^{m \times m}$ unless otherwise noted.

(a) If $\lambda$ is an eigenvalue of $A$ and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.
(b) If $A$ is real and $\lambda$ is an eigenvalue of $A$, then so is $-\lambda$.
(c) If $A$ is real and $\lambda$ is an eigenvalue of $A$, then so is $\bar{\lambda}$.
(d) If $\lambda$ is an eigenvalue of $A$ and $A$ is nonsingular, then $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

2. Find the Schur factorizations of

$$A = \begin{bmatrix} 3 & 8 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 7 \\ 1 & 12 \end{bmatrix}. $$

3. Here is Gershgorin’s theorem, which holds for any $m \times m$ matrix $A$, symmetric or nonsymmetric:

Every eigenvalue of $A$ lies in at least one of the $m$ circular disks in the complex plane with centers $a_{ii}$ and radii $\sum_{j \neq i} |a_{ij}|$. Moreover, if $n$ of these disks form a connected domain that is disjoint from the other $m-n$ disks, then there are precisely $n$ eigenvalues of $A$ within this domain.

(a) Prove the first part of Gershgorin’s theorem. (Hint: Let $\lambda$ be any eigenvalue of $A$, and $x$ a corresponding eigenvector normalized so that its largest entry is 1.)
(b) Give estimates based on Gershgorin’s theorem for the eigenvalues of

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}, \quad |\varepsilon| < 1. $$

4. Suppose we have a $3 \times 3$ matrix and wish to introduce zeros by left- and/or right-multiplications by unitary matrices $Q_j$ such as Householder reflections or Givens rotations. Consider the following three matrix structures:

(a) $\begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$, \quad (b) $\begin{bmatrix} x & x & 0 \\ x & 0 & x \\ 0 & x & x \end{bmatrix}$, \quad (c) $\begin{bmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$.

For each one, decide which of the following situations holds, and justify your claim.

(i) Can be obtained by a sequence of left-multiplications by matrices $Q_j$;
(ii) Not (i), but can be obtained by a sequence of left- and right-multiplications by matrices $Q_j$;
(iii) Cannot be obtained by any sequence of left- and right-multiplications by matrices $Q_j$.

5. Let $A \in \mathbb{C}^{m \times m}$ be given, not necessarily Hermitian. Show that a number $z \in \mathbb{C}$ is a Rayleigh quotient of $A$ if and only if it is a diagonal entry of $Q^*AQ$ for some unitary matrix $Q$. Thus Rayleigh quotients are just diagonal entries of matrices, once you transform orthogonally to the right coordinate system.