Math 477 — Homework Assignment 5, due Nov.9, 2006

1. For each of the following statements, prove that it is true or give an example to show it is false. Throughout, $A \in \mathbb{C}^{m\times m}$ unless otherwise noted.

   (a) If λ is an eigenvalue of A and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.

   (b) If A is real and λ is an eigenvalue of A, then so is $-\lambda$.

   (c) If A is real and λ is an eigenvalue of A, then so is $\overline{\lambda}$.

   (d) If λ is an eigenvalue of A and A is nonsingular, then λ^{-1} is an eigenvalue of A^{-1}.

2. Find the Schur factorizations of

$$A = \begin{bmatrix} 3 & 8 \\ -2 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 4 & 7 \\ 1 & 12 \end{bmatrix}.$$

3. Here is *Gerschgorin's theorem*, which holds for any $m\times m$ matrix A, symmetric or nonsymmetric:

   Every eigenvalue of A lies in at least one of the m circular disks in the complex plane with centers a_{ii} and radii $\sum_{j\neq i} |a_{ij}|$. Moreover, if n of these disks form a connected domain that is disjoint from the other $m-n$ disks, then there are precisely n eigenvalues of A within this domain.

   (a) Prove the first part of Gerschgorin's theorem. (Hint: Let λ be any eigenvalue of A, and $\boldsymbol{x}$ a corresponding eigenvector normalized so that its largest entry is 1.)

   (b) Give estimates based on Gerschgorin's theorem for the eigenvalues of

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}, \qquad |\varepsilon| < 1.$$

4. Suppose we have a 3×3 matrix and wish to introduce zeros by left- and/or right-multiplications by unitary matrices Q_j such as Householder reflections or Givens rotations. Consider the following three matrix structures:

$$\text{(a)} \begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} x & x & 0 \\ x & 0 & x \\ 0 & x & x \end{bmatrix}, \qquad \text{(c)} \begin{bmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

   For each one, decide which of the following situations holds, and justify your claim.

   (i) Can be obtained by a sequence of left-multiplications by matrices Q_j;

   (ii) Not (i), but can be obtained by a sequence of left- and right-multiplications by matrices Q_j;

   (iii) Cannot be obtained by any sequence of left- and right-multiplications by matrices Q_j.

5. Let $A \in \mathbb{C}^{m\times m}$ be given, not necessarily Hermitian. Show that a number $z \in \mathbb{C}$ is a Rayleigh quotient of A if and only if it is a diagonal entry of Q^*AQ for some unitary matrix Q. Thus Rayleigh quotients are just diagonal entries of matrices, once you transform orthogonally to the right coordinate system.