FOUR DIMENSIONAL CHESS

BEN

ABSTRACT. This is a quick introduction to four dimensional chess, with the mathematics in mind.

1. FIDE CHESS

We shall assume a basic knowledge of chess, but desire to reintroduce the reader to movements of the pieces; in particular, we lead the discussion in a way that will flow more readily to multiple dimensions.

1.1. **Rook.** The FIDE Rook moves in a straight line horizontally or vertically, but may not "leap" any intervening piece. We shall find it more useful to think of the movement as a series of steps; each step is one square in a given dimension, and if the square so reached is empty, the rook may make an additional step, and so on.

1.2. **Bishop.** We make the same comment as for rooks, that it may be more convenient to think of a bishop's move as a series of steps in two dimensions (di-agonal).

1.3. **Queen.** The queen may be thought of as either:
   (1) a runner which may move in either one or two dimensions
   (2) a piece which combines the move-set of the rook and bishop

1.4. **Knight.** Certainly the knight is one of the more exotic pieces. It is a "leaper", meaning that it moves more than one square away and need not stop for intervening pieces. Furthermore, it moves one square in one dimension and two squares in another. This somewhat odd movement makes the knight perfect for math discussions such as the Knight's Tour or a variety of chess puzzles.

1.5. **Pawn.** The pawn is unique in several ways. First, it treats the two dimensions of the board differently (whereas the other pieces wouldn't notice a change in dimension names, including a rotation of the board). The pawn may only move "forward", and is the only piece which moves differently when capturing than when not capturing. The pawn also has the privilege of promotion upon reaching the forwardmost position.

2. THE CHESS BOARD

Of course the standard chess board is an 8x8 array of squares. Other boards are seen in chess variants and in puzzles like the Knight's Tour; common ones in variants include 8x9, 8x10, 5x5, and even hexagonal or triangular tilings. In a standard board, the set of squares to which a given bishop can reach (in as many moves as needed) partitions the board into two pieces, which we then color as white and black. This same coloring provides us with knights which always change colors when they move and pawns which capture to the same color and move otherwise to a new color (excepting the opening double-move).

There are many versions of three dimensional chess games, boards for which include the common 8x8x3, the almost unplayable 8x8x8, the nicely compact 5x5x5, and even the odd looking tri-dimensional chessboard first seen in Star Trek. However, there are almost no extensions into four or more dimensions. While there doesn't seem to be a large following for actually playing such games, chess has been adapted to manifolds such as cylinders, tori, and Klein bottles. However, not much analysis has been done here (in particular, the Klein bottle requires some finesse). While we are indeed working on the details of chess on manifolds, the main focus here shall be multiple dimensions.

3. Chess in 3 Dimensions

As was mentioned above, there are a good number of three-dimensional chess games, but the majority treat the new dimension as some odd transition. For instance, in almost any game played on the fairly popular 8x8x3 board, pieces have their normal moves, but may additionally "warp" up or down between the normal 8x8 boards. There are a handful of more mathematically symmetric games, usually played on a board approximately 5x5x5. This size limits the number of squares to 125, about twice that of normal chess, which is countered by the fact that the board is more compact.

Then we need to determine how pieces should behave on such a board. Instead of just having orthogonal and diagonal runners, we now can have a triagonal runner (which is frequently called a unicorn). If we think of the board constructed as a cube, then these runners move through the faces, edges, and corners of their cubes, respectively. How powerful are these pieces then? And how should the knights' and pawns' moves be extended? Should the king and queen have all three dimensions available, or do they need to be limited in power? These are all excellent questions, and all apply when we move into four dimensions, so we press on.

We note that when actually playing 3D chess, playing inside a cube is a bit cumbersome. Usually players deconstruct their board, taking it apart into several 2D boards, and laying them out in a line. Players then keep in mind that a piece has a dimension available in which it moves from its square to the corresponding square in the next board over (in either direction).

4. Chess in 4 Dimensions

4.1. **Existing Games.** There are a small handful of chess variants which claim to be four dimensional, and in some sense they are. For the dimensionally challenged of us, these games are generally laid out as a two-dimensional array of chessboards (a convenience which we adopt as well). Analogously to the 3D case, we keep in mind that we have two dimensions in which we may move between boards. For lack of preferable convention, we refer the the dimensions as small left/right, big left/right, small forward/backward, and big forward/backward, where of course a "big" direction is the move between boards.

The existing games generally are not symmetric; they treat the big directions as inherently different from the small ones. For example, a bishop might be able to move inside his small board as he sees fit, or move between boards diagonally through the array, but cannot make the diagonal move of the two directions small forward and big right. Thus, if we rotate the board through its four dimensions, the game may look very different.

4.2. **The Board.** We set our game in a 4x4x4x4 hypercube, which we visualize via a 4x4 array of 4x4 chessboards. In initial learning of the game, it may be useful to realize that we could stack the rows (columns) of boards on top of each other to form a line of adjacent 3D slices; this helps one see some of the basic movements in a more familiar 3D setting.

We shall refer to the positions which pieces occupy as "squares," despite their actual existence as hypercubes. We specify these positions with algebraic notation by first specifying the small chessboard as though their array were itself a chessboard: columns A through D and rows 1 through 4. Then we specify the square within the chessboard in a similar manner, columns a through d and rows 1 through 4. See the figures which follow.

4.3. **Runners.** In this four dimensional world, we of course have four kinds of basic runners: orthogonal, diagonal, triagonal, and quadragonal. (According to a variant piece encyclopedia, the quadragonal runner is known as a "balloon," but I have not been able to find rules for a game which includes it.) As in the 3D case, we note that these pieces will run through either the 3D "hypersurface" cubes, the 2D faces, the 1D edges, or the 0D corners (respectively) of the hypercubes which they pass through. One might intuitively realize that the orthogonal and quadragonal runners are really quite weak; if not, play around for a bit with where they can reach. The orthogonal runners can reach every square eventually, but control very little of the board at any given time. Quadragonal runners can only reach one eighth of the board! Diagonal runners can still only reach one half of the board (the standard coloring applies) but are relatively powerful, and the triagonal runners reach any square and do so in a nice fashion. Thus we set our game with two runners: the rooks, which move in either one or three dimensions; and our bishops, which move in either two or four. This fixes the bishops to one color, and balances the pieces' powers fairly nicely.

As in standard chess, a runner's movement is blocked by friendly pieces, and is halted upon capturing an opponent's piece.

4.4. **Pawns.** The layout of the board makes it convenient to define two "forward" directions, so that we may restrict pawns to moving in one of these directions. Since the board is so compact, we allow no initial double-move, and thus no en passant rule. Pawns capture by moving one square in either forward direction and one square in any lateral direction. Note that this allows pawns two non-capturing moves and eight (!) squares on which to capture. Pawns promote upon reaching the forwardmost position, i.e., the farthest position possible in both of their forward directions.

4.5. **Knights.** Knights need no exaggerated powers; their standard movement extends well into multiple dimensions. They may move two squares in any dimension, then one square in a different dimension. This movement is what I've come to call a "planar knight", since you can simply choose one of the 6 basis planes the knight occupies, then move in that plane as a normal knight. Notice that, if a knight had a larger board to work with, he could reach a total of 48 squares.

4.6. **The Royals.** The final question posed of three dimensional games was the power of the queen and king. As a first thought, we might allow both king and queen to move in any direction, queens moving as runners (a combination of the rook and bishop actually), and kings by one square only. However, if we allow a king to move in any direction by one square, he could reach a whopping 80 squares from the center of the board. That's one slippery king! The same problem occurs in three dimensional chess, but it is far more exaggerated in 4D. We have several ways of fixing this checkmating problem:

(1) Bare royals loses: A fairly common rule in chess variants, we make the loss of all units but the king and queen a loss of the game, unless in the next turn you can retaliate with a checkmate or baring of your opponent's royals. (In variants this is normally just bare king, but the queen is even harder to pin down than the king.) We've not playtested this idea yet, but I'm not so fond of it.

(2) Crippled king: We simply limit the kings' movements. In particular, allowing the king to move in only one dimension works out well. It is a bit aggravating though to have a bishop *right next* to your king and not be able to capture it.

(3) Fortress: As in Xiangqi (Chinese chess), we assign a certain block of squares for each player to be their "fortress", which the king cannot leave. We usually choose the 2x2x2x2 hypercubes which are centered in the two lateral dimensions and in the rear of the two forward dimensions. It is worth noting that the king may move (if we allow him to move in any direction) from any square in his fortress to any other square. I like this version, but it allows for perpetual check a bit too easily–I once had mate in one, but my opponent could perpetually keep my king in check inside my fortress, forcing a draw.

(4) Gladiator kings: I found this concept while perusing other almost 4D games. We add a rule such that whenever the kings come to occupy some common area (a common plane or line, for instance), they are stuck in that area. We've not played this one out yet. It sounds intriguing, but is getting a bit far away from standard chess.

(5) God pieces: We introduce a new piece/pieces with almost absurd movements. Most likely we would have no such pieces at the start of a game, but would allow pawn promotion to those pieces. If that is the case, we might add additional pawns to the startup.

(6) Multiple royals: A concept created for 3D games, we simply have more than one king, and require a player to protect *both* of them. Thus forking your opponent's kings (without your forking piece being open to capture) is a win since he/she cannot remove both of his/her kings from check.

(7) ...others?

4.7. **Initial Setup.** White rooks start at A1b1 and D1c1;
bishops at A1c1 and D1b1;
knights at B1b1 and C1c1;
queen at B1c1;
king at C1b1;
pawns at A2b2, A2c2, B2b2, B2c2, C2b2, C2c2, D2b2, and D2c2.

Black's setup is the mirror image:
rooks at A4b4 and D4c4;
bishops at A4c4 and D4b4;
knights at B4b4 and C4c4;
queen at B4c4;
king at C4b4;
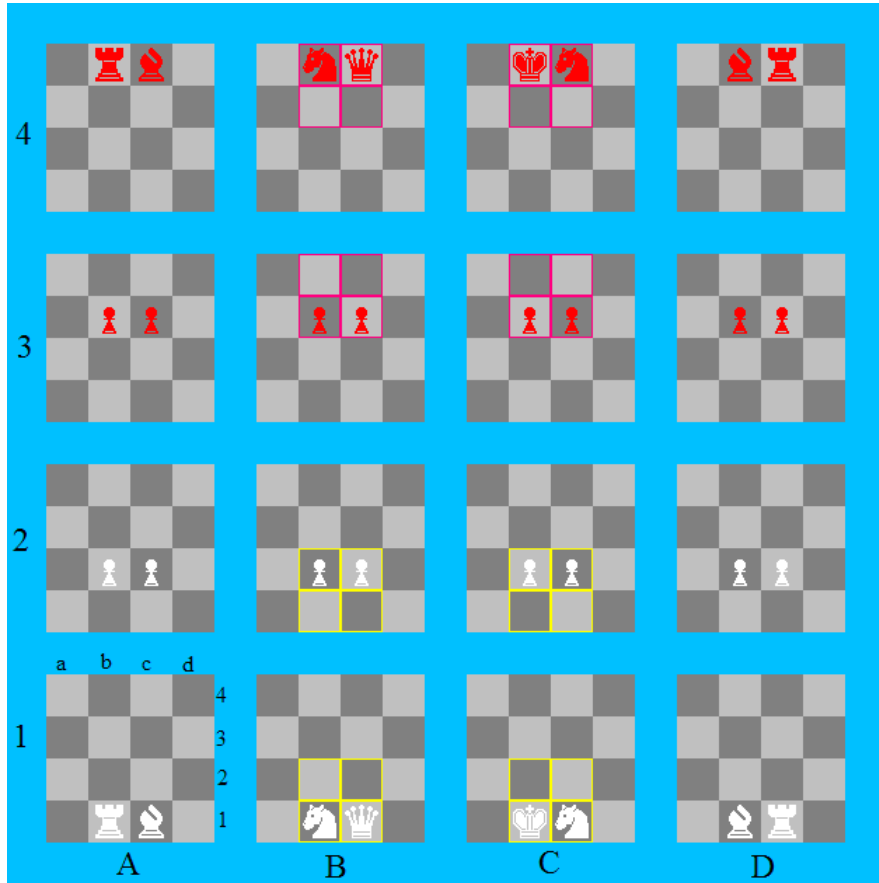pawns at A3b3, A3c3, B3b3, B3c3, C3b3, C3c3, D3b3, and D3c3.

Figure 1. The opening array. The fortresses are highlighted here.

5. Tips for Actually Playing our 4D Chess

First, some help in recognizing where pieces can and cannot move. We can think of each step of a runner's movement as a sequence of ministeps; for instance small forward, big right, and big backward corresponds to a three dimensional movement. Then, if the destination square is empty, we can make the same directional step by another small forward, big right, and big backward (assuming we don't run into the side of our board). And of course it doesn't matter which order we consider these three ministeps, and a piece which occupies one of the ministep locations does NOT block the piece's movement.
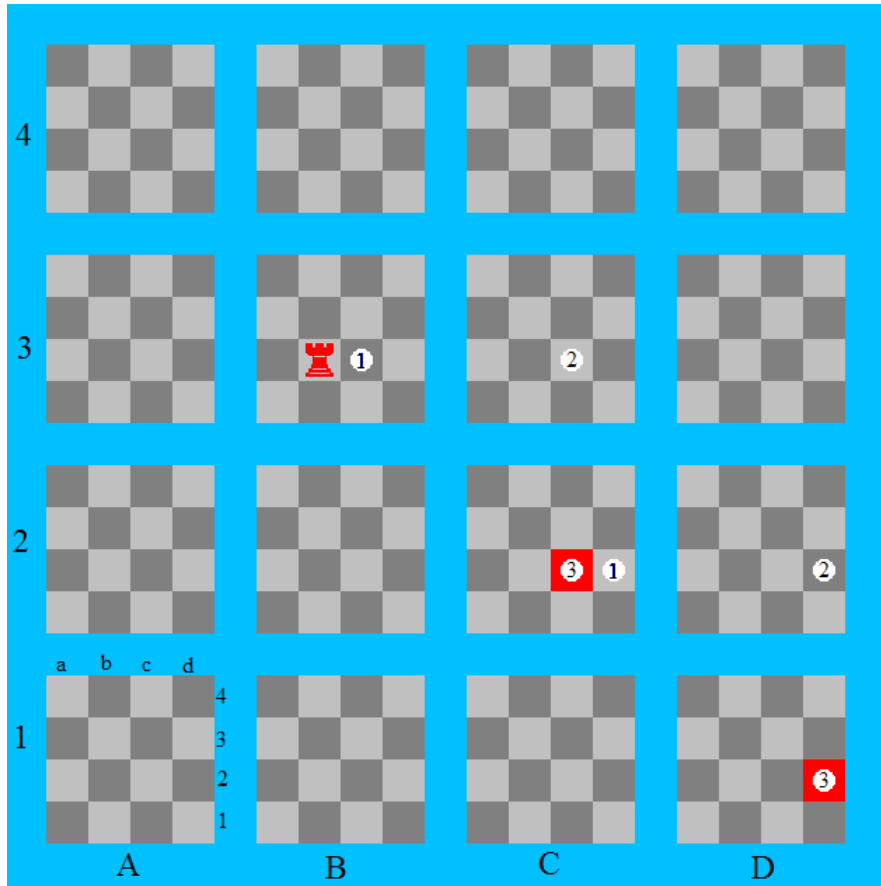
FIGURE 2. A ministep decomposition of one of the rook's paths. The rook may land on either of the red colored squares if they are both empty.
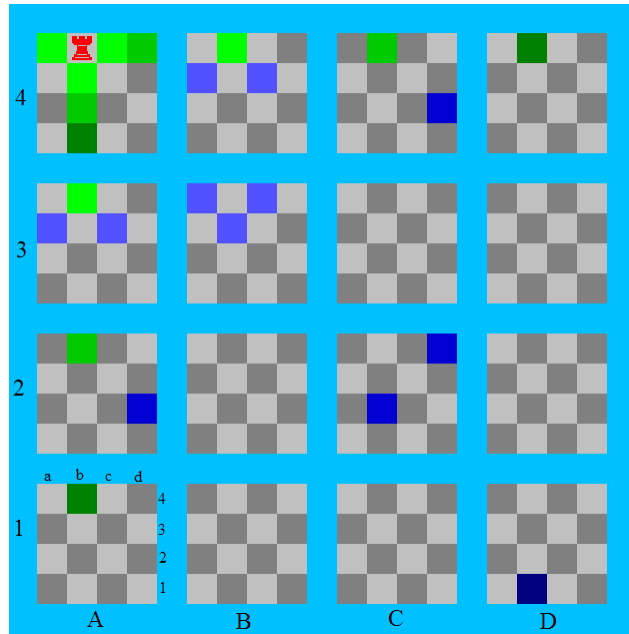
FIGURE 3. The rook's moves. Green squares are those reachable by a one-dimensional move, blue by a three-dimensional move. Lighter colors are those immediately adjacent to the rook, while darker ones are reachable after passing through a corresponding light one.
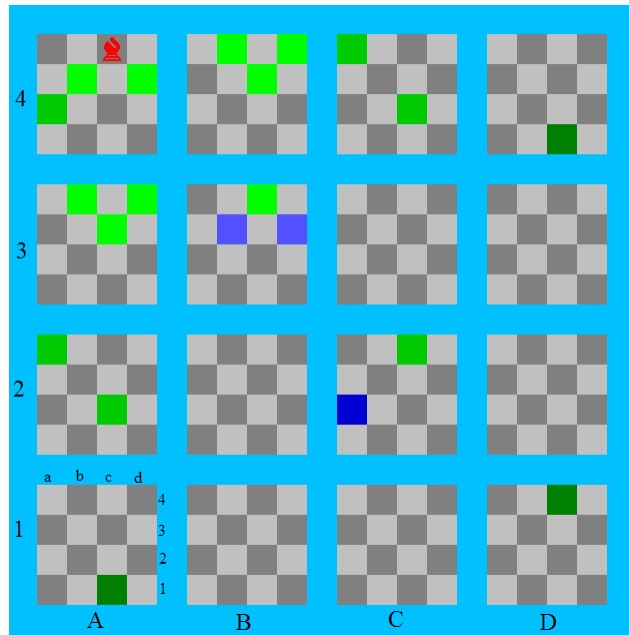

FIGURE 4. The bishop's moves. Green denotes a two-dimensional move, blue a four-dimensional one. Again darker colored squares require the corresponding light one is empty.
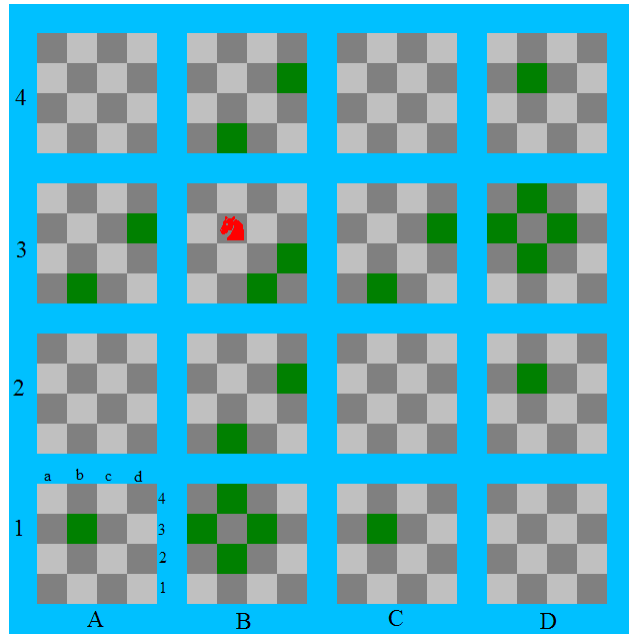
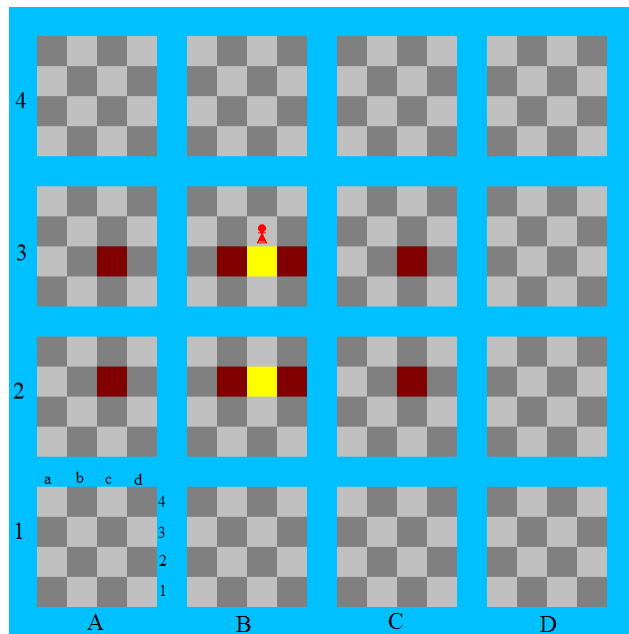FIGURE 5. The knight's moves.

FIGURE 6. The pawn's moves. The yellow squares are available only for non-capturing movement, red squares only for capturing.