Wiener-Hopf Factorization for Arithmetic Brownian Motion with Time-Dependent Drift and Volatility

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ABSTRACT: In this paper we obtain a Wiener-Hopf type factorization for a time-inhomogeneous arithmetic Brownian motion with deterministic time-dependent drift and volatility. To the best of our knowledge, this paper is the very first step towards realizing the objective of deriving Wiener-Hopf type factorizations for (real-valued) time-inhomogeneous Lévy processes. In particular, we argue that the classical Wiener-Hopf factorization for time-homogeneous Lévy processes quite likely does not carry over to the case of time-inhomogeneous Lévy processes.

KEYWORDS: Wiener-Hopf factorization, time-inhomogeneous Lévy process, Markov family, Feller semigroup, time-homogenization

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Contents

1 Introduction **2**

2 Setup and the Main Result **5**

2.1 Basic Setup . . . 5
2.2 Assumptions and Preliminaries . . . 6
2.3 Main Result . . . 8
2.4 Connection to Classical Wiener-Hopf Factorization for Lévy Processes . . . 11

3 Auxiliaries **12**

3.1 An Auxiliary Time-Homogeneous Markov Family . . . 12
3.2 The Feller Semigroup Property and Strong Generators of $(\mathcal{P}_\ell^{\pm})_{\ell\in\mathbb{R}_+}$ . . . 15
3.3 Additional Property of $\Gamma^{\pm}$ . . . 17

4 Proof of Main Result **22**
4.1 Proof of Proposition 2.9 22
4.2 Proof of Theorem 2.11 28
4.3 Proof of Corollary 2.14 33

5 Example **35**

A Additional Lemmas and Proofs **39**
A.1 Proof of Proposition 2.3 39
A.2 Proof of Proposition 2.4 41
A.3 Proof of Lemma 2.6 45
A.4 Proof of Lemma 3.3 45
A.5 A Technical Lemma for the Proof of (3.15) 47
A.6 Proof of Lemma 4.1 48
A.7 Proof of Lemma 4.2 48

1 Introduction

In this paper we obtain a Wiener-Hopf type factorization for a time-inhomogeneous arithmetic Brownian motion with deterministic time-dependent drift and volatility. To the best of our knowledge, this paper is the very first step towards realizing the objective of deriving Wiener-Hopf type factorizations for (real-valued) *time-inhomogeneous Lévy processes*[1]. In order to motivate this goal, we first provide a brief account of three forms of Wiener-Hopf factorizations for time-homogeneous real-valued Lévy processes based on [8, Section 11.2.1], [19, Section I.29], and [21, Section 45].

So, let $X := (X_t)_{t\in\mathbb{R}_+}$ be a time-homogeneous real-valued Lévy processes defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with $X_0 = 0$ $\mathbb{P}$-a.s., where $\mathbb{R}_+ := [0,\infty)$. We denote by $\psi(\xi)$, $\xi \in \mathbb{R}$, the characteristic exponent of X, so that $\mathbb{E}(e^{i\xi X_t}) = e^{t\psi(\xi)}$, for any $t \in \mathbb{R}_+$. For any fixed $c \in (0,\infty)$, we consider an exponentially distributed random variable $\mathbf{e}_c$ on $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathbb{E}(\mathbf{e}_c) = c^{-1}$, and we assume that $\mathbf{e}_c$ and X are independent under $\mathbb{P}$. Denote by

$$\underline{X}_t := \inf_{s\in[0,t]} X_s, \quad \overline{X}_t := \sup_{s\in[0,t]} X_s, \quad t \in \mathbb{R}_+,$$

the running maximum and running minimum processes of X, respectively. It is well known (cf. [19, Chapter I, (29.4) & (29.5)]) that

$$\overline{X}_{\mathsf{e}_c} \text{ and } X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c} \text{ are independent,} \tag{1.1}$$

and that

$$\underline{X}_{\mathsf{e}_c} \text{ and } X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c} \text{ have the same distribution.} \tag{1.2}$$

[1]By time-inhomogeneous Lévy process, we mean a continuous-time stochastic process that has the independent increments property, but not the stationary increments property. This type of processes are also known as *additive processes* (cf. [21, Definition 1.6]).

The above two properties imply that

$$\mathbb{E}\Big(e^{i\xi X_{\mathsf{e}_c}}\Big) = \mathbb{E}\Big(e^{i\xi \overline{X}_{\mathsf{e}_c}}\Big)\mathbb{E}\Big(e^{i\xi \underline{X}_{\mathsf{e}_c}}\Big), \quad \xi \in \mathbb{R}. \tag{1.3}$$

Formula (1.3) is known as the *Wiener-Hopf factorization for the real-valued Lévy process* X (cf. [8, (11.9)], [19, Chapter I. (29.2) (iii)]). It is a particular version of the so-called Pecherskii-Rogozin-Spitzer identity (see e.g. [2]).

Next, we denote by $\phi_c^+(\xi)$ (respectively, $\phi_c^-(\xi)$), $\xi \in \mathbb{R}$, the characteristic function of $\overline{X}_{\mathsf{e}_c}$ (respectively, $\underline{X}_{\mathsf{e}_c}$). Noting that

$$\mathbb{E}\Big(e^{i\xi X_{\mathsf{e}_c}}\Big) = c\,\mathbb{E}\bigg(\int_0^\infty e^{-ct}e^{i\xi X_t}\,dt\bigg) = c\int_0^\infty e^{-ct}e^{\psi(\xi)t}\,dt = \frac{c}{c-\psi(\xi)},$$

we obtain the following equivalent form of (1.3) (e.g. [8, (11.12)] and [21, (45.1)])[2]

$$\frac{c}{c-\psi(\xi)} = \phi_c^+(\xi)\phi_c^-(\xi), \quad \xi \in \mathbb{R}. \tag{1.4}$$

Now we define the following operators on $L^\infty(\mathbb{R})$

$$\begin{aligned}
\big(\mathcal{H}_c u\big)(x) &:= c\,\mathbb{E}\bigg(\int_0^\infty e^{-ct}u\big(x+X_t\big)\,dt\bigg) = \mathbb{E}\big(u\big(x+X_{\mathsf{e}_c}\big)\big),\\
\big(\mathcal{H}_c^+ u\big)(x) &:= c\,\mathbb{E}\bigg(\int_0^\infty e^{-ct}u\big(x+\overline{X}_t\big)\,dt\bigg) = \mathbb{E}\big(u\big(x+\overline{X}_{\mathsf{e}_c}\big)\big),\\
\big(\mathcal{H}_c^- u\big)(x) &:= c\,\mathbb{E}\bigg(\int_0^\infty e^{-ct}u\big(x+\underline{X}_t\big)\,dt\bigg) = \mathbb{E}\big(u\big(x+\underline{X}_{\mathsf{e}_c}\big)\big).
\end{aligned}$$

It can be shown that (1.1) and (1.2) also imply

$$\mathcal{H}_c u = \mathcal{H}_c^+\mathcal{H}_c^- u = \mathcal{H}_c^-\mathcal{H}_c^+ u, \quad u \in L^\infty(\mathbb{R}), \tag{1.5}$$

(cf. [8, (11.16)]), and that (1.5) implies (1.3) (equivalently, (1.4)); see Remark 2.15 below for a more detailed discussion. Thus we call (1.5) *the operator form* of the Wiener-Hopf factorization for the real-valued Lévy process X.

We conjuncture that when X is a time-inhomogeneous real-valued Lévy process then property (1.2) does not hold any more due to the lack of stationarity of increments. Such conjecture is strongly supported by our numerical simulations, though at this moment we do not have a formal proof for it. More precisely, we consider a Brownian motion with time-dependent drift, namely,

$$X_t = \int_0^t v(s)\,ds + W_t, \quad t \in \mathbb{R}_+,$$

for some deterministic bounded function v on $\mathbb{R}_+$. For various choices of v and $c = 1$, we use Monte Carlo method to compute $\mathbb{E}(X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c}) - \mathbb{E}(\underline{X}_{\mathsf{e}_c})$[3] with $n = 10^4$ sample paths and time

[2]There are other equivalent expressions for $\phi_c^\pm$, e.g. [21, (45.2) & (45.3)]. Moreover, it is well-known that (ϕ_c^+, ϕ_c^-) is the unique pair of characteristic functions of infinitely divisible distributions having drift 0 supported on $[0,\infty)$ and $(-\infty, 0]$, respectively, such that (1.4) holds (cf. [21, Theorem 45.2]). Those results are irrelevant to our later discussions, and are therefore omitted here.

[3]The exact formula for $\mathbb{E}(X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c}) - \mathbb{E}(\underline{X}_{\mathsf{e}_c})$ when v is non-constant is not available (even when v is piecewise constant with a single jump).

step $\Delta t = 10^{-4}$. The simulation results, summarized in Table 1, show that when v is constant, $\mathbb{E}(X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c}) = \mathbb{E}(\underline{X}_{\mathsf{e}_c})$, which is a simple consequence of (1.2). However, when v is non-constant, there is a significant gap between $\mathbb{E}(X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c})$ and $\mathbb{E}(\underline{X}_{\mathsf{e}_c})$, which is a clear contradiction to property (1.2). Therefore, there is no hope of deriving Wiener-Hopf type factorizations (in an analogous form of (1.3), (1.4), or (1.5)) for a time-inhomogeneous real-valued Lévy process X using properties like (1.1) and (1.2), and other methods are sought.

Function v	$v(s) \equiv 1$	$v(s) \equiv -1$	$v(s) = \mathbb{1}_{[0,1/2]}(s) - \mathbb{1}_{[1,3/2]}(s)$	$v(s) = \cos(s)$
$\mathbb{E}(X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c}) - \mathbb{E}(\underline{X}_{\mathsf{e}_c})$	-9.9×10^{-5}	8.1×10^{-5}	-0.1475	-0.0803

Table 1: *Simulation results for* $\mathbb{E}(X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c}) - \mathbb{E}(\underline{X}_{\mathsf{e}_c})$. *For each choice of function* v, *the expectation is computed based on* $n = 10^4$ *sample paths with time step* $\Delta t = 10^{-4}$.

In this paper, we derive a Wiener-Hopf factorization for a time-inhomogeneous diffusion process $\varphi(s, a)$ (defined as in (2.1)) with time-dependent deterministic drift and volatility coefficients. In the context of this paper this means a specific decomposition of the quantity

$$\mathbb{E}\bigg(\int_s^\tau u(\varphi_t(s, a)) h(t) \, \mathrm{d}t \bigg), \tag{1.6}$$

where τ is an arbitrary stopping time, and u and h are suitable test functions. This is the main results of our paper and it is presented in Theorem 2.11 in terms of the two passage times $\tau_\ell^\pm(s, a)$ (defined as in (2.2)) of $\varphi(s, a)$ and their functionals. In particular, when v and σ are both constants and with a special choice of h, our factorization of (1.6) recovers the operator form of the Wiener-Hopf factorization (1.5) for a Brownian motion with drift (see Corollary 2.14 and Remark 2.15 below). To the best of our knowledge, Theorem 2.11 is the very first result in the literature regarding the Wiener-Hopf factorization for time-inhomogeneous Lévy processes. Our methodology employs an in-depth analysis of the semigroups $(\mathcal{P}_\ell^\pm)_{\ell \in \mathbb{R}_+}$ associated with $\tau_\ell^\pm(s, a)$, and of their generators $\Gamma^\pm$, together with a time-homogenization technique (cf. [6, Section 3]), which does not rely on any property of $\varphi(s, a)$ analogous to (1.1) or (1.2). As a by product, we also obtain a property of $\Gamma^\pm$ in Proposition 3.6 which can be regarded as an analogue of the so-called "noisy" Wiener-Hopf factorization for time-homogeneous finite Markov chains that was studied in, for instance, [12], [14], and [18].

We need to add that the present paper is a continuation of work towards developing Wiener-Hopf type theory for time-inhomogeneous Markov processes. The previous work in this direction is presented in [3] and [4].

The rest of the paper is organized as follows. In Section 2, we first introduce the basic setup and assumptions of our model. Our main result on the Wiener-Hopf factorization of the time-inhomogeneous arithmetic Brownian motion process $\varphi(s, a)$ is presented in Section 2.3, followed by a discussion of its relation with the Wiener-Hopf factorization of time-homogeneous Lévy processes in Section 2.4. Section 3 contains some auxiliary results which are needed in the proof of the main result, including a property of $\Gamma^\pm$ that is analogous to the "noisy" Wiener-Hopf factorization, which is given in Section 3.3. Section 4 contains proofs of key results. In Section 5, we present a nontrivial example of our model for which the main assumptions are shown to be satisfied. Finally, in the appendix, we provide the proofs of some technical lemmas.

2 Setup and the Main Result

2.1 Basic Setup

Throughout this paper, we let $W := (W_t)_{t\in\mathbb{R}_+}$ be a one-dimensional standard Brownian motion defined on a complete stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is a filtration satisfying the usual conditions, and $\mathbb{R}_+ := [0,\infty)$. For any $a \in \mathbb{R}$ and $s \in \mathbb{R}_+$, we consider the following time-inhomogeneous diffusion process $\varphi(s,a) := (\varphi_t(s,a))_{t\in[s,\infty)}$, defined by

$$\varphi_t(s,a) := a + \int_s^t v(r)\,\mathrm{d}r + \int_s^t \sigma(r)\,\mathrm{d}W_r, \quad t \in [s,\infty), \tag{2.1}$$

where $v : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are $\mathcal{B}(\mathbb{R}_+)$-measurable bounded functions.

For any $s \in \mathbb{R}_+$ and $a, \ell \in \mathbb{R}$, we define the passage times of $\varphi(s,a)$ as

$$\tau_\ell^+(s,a) := \inf\big\{t \in [s,\infty) : \varphi_t(s,a) \geq \ell\big\} \quad \text{and} \quad \tau_\ell^-(s,a) := \inf\big\{t \in [s,\infty) : \varphi_t(s,a) \leq \ell\big\}, \tag{2.2}$$

with the convention $\inf \emptyset = \infty$. Both $\tau_\ell^+(s,a)$ and $\tau_\ell^-(s,a)$ are $\mathbb{F}$-stopping times since $\varphi(s,a)$ is $\mathbb{F}$-adapted and has continuous sample paths, and $\mathbb{F}$ is right-continuous (cf. [11, Chapter I, Proposition 1.28]). In view of (2.1), for any $s \in \mathbb{R}_+$ and $a, \ell \in \mathbb{R}$, we have

$$\tau_\ell^\pm(s,a) = \tau_{\ell-a}^\pm(s,0), \tag{2.3}$$

and $\tau_\ell^+(s,a) = s$ (respectively, $\tau_\ell^-(s,a) = s$) when $a \geq \ell$ (respectively, $a \leq \ell$). For notational convenience, hereafter we will write $\varphi_t(s)$ and $\tau_\ell^\pm(s)$ in place of $\varphi_t(s,0)$ and $\tau_\ell^\pm(s,0)$, respectively.

We will use the following notations for various spaces of functions.

- $L^\infty(\mathbb{R}_+)$ is the space of $\mathcal{B}(\mathbb{R}_+)$-measurable bounded real-valued functions on $\mathbb{R}_+$. If need be we extend the domain of a function $f \in L^\infty(\mathbb{R}_+)$ to include infinity, and in such case we set $f(\infty) = 0$.
- $C(\mathbb{R}_+)$ (respectively, $C(\mathbb{R})$) is the space of continuous real-valued functions on $\mathbb{R}_+$ (respectively, $\mathbb{R}$).
- $C_c(\mathbb{R}_+)$ (respectively, $C_c(\mathbb{R})$) is the space of continuous real-valued functions on $\mathbb{R}_+$ (respectively, $\mathbb{R}$) with compact support.
- $C_0(\mathbb{R}_+)$ is the space of $f \in C(\mathbb{R}_+)$ such that f vanishes at infinity.
- $C_e(\mathbb{R}_+)$ is the space of $f \in C_0(\mathbb{R}_+)$ such that f decays with exponential rate, i.e., there exist constants $K, \kappa \in (0,\infty)$, such that $|f(t)| \leq Ke^{-\kappa t}$ for all $t \in \mathbb{R}_+$.
- $C_c^1(\mathbb{R}_+)$ is the space of $f \in C_0(\mathbb{R}_+)$ such that f is continuously differentiable on $\mathbb{R}_+$ and has a compact support.
- $C_{e,\mathrm{cdl}}^{\mathrm{ac}}(\mathbb{R}_+)$ is the space of $f \in C_0(\mathbb{R}_+)$ such that there exists a càdlàg real-valued function g_f on $\mathbb{R}_+$, which decays with exponential rate, and

$$f(t) = -\int_t^\infty g_f(r)\,dr, \quad \text{for all } t \in \mathbb{R}_+. \tag{2.4}$$

- $C^k(\mathbb{R})$, $k \in \mathbb{N}$, is the space of real-valued functions on $\mathbb{R}$ which have continuous derivatives on $\mathbb{R}$ up to order k.

We conclude this subsection by introducing the following two families of operators associated with the passage times $\tau_\ell^\pm(s)$, which are key ingredients in our main result. For any $\ell \in \mathbb{R}_+$, we define $\mathcal{P}_\ell^+ : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ and $\mathcal{P}_\ell^- : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ as

$$\big(\mathcal{P}_\ell^+ f\big)(s) := \mathbb{E}\Big(f\big(\tau_\ell^+(s)\big)\Big) \quad \text{and} \quad \big(\mathcal{P}_\ell^- f\big)(s) := \mathbb{E}\Big(f\big(\tau_{-\ell}^-(s)\big)\Big), \quad s \in \mathbb{R}_+. \tag{2.5}$$

We will stipulate $(\mathcal{P}_\ell^\pm f)(\infty) = 0$ whenever we need to evaluate its value at infinity. Clearly, for any $f \in L^\infty(\mathbb{R}_+)$, $|(\mathcal{P}_\ell^\pm f)(s)| \le \|f\|_\infty < \infty$ for any $s \in \mathbb{R}_+$, so that $\mathcal{P}_\ell^\pm f \in L^\infty(\mathbb{R}_+)$.

Remark 2.1. In most of the literature on Wiener-Hopf factorization for Markov processes (cf. [1], [14], and [22]), the passage times of additive functionals with strict inequalities are considered. More precisely, in our setup, we might have investigated

$$\eta_\ell^+(s,a) := \inf\big\{t \in [s,\infty) : \varphi_t(s,a) > \ell\big\} \quad \text{and} \quad \eta_\ell^-(s,a) := \inf\big\{t \in [s,\infty) : \varphi_t(s,a) < \ell\big\}, \tag{2.6}$$

instead of $\tau_\ell^\pm(s,a)$ given as in (2.2), for any $s \in \mathbb{R}_+$ and $a, \ell \in \mathbb{R}$. Nevertheless, as shown in Proposition 2.3 below, these two types of passage times are equal to each other $\mathbb{P}$-a.s. Consequently, if we define $\mathcal{Q}_\ell^+ : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ and $\mathcal{Q}_\ell^- : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ by

$$\big(\mathcal{Q}_\ell^+ f\big)(s) := \mathbb{E}\Big(f\big(\eta_\ell^+(s)\big)\Big) \quad \text{and} \quad \big(\mathcal{Q}_\ell^- f\big)(s) := \mathbb{E}\Big(f\big(\eta_{-\ell}^-(s)\big)\Big), \quad s \in \mathbb{R}_+,$$

then $(\mathcal{P}_\ell^\pm)_{\ell\in\mathbb{R}_+}$ coincides with $(\mathcal{Q}_\ell^\pm)_{\ell\in\mathbb{R}_+}$ on $L^\infty(\mathbb{R}_+)$. Therefore, our main factorization for (1.6), given as in Theorem 2.11 below, holds for either type of passage times.

2.2 Assumptions and Preliminaries

In this section, we will introduce some assumptions and state some preliminary results in order to present our main result. We begin with the following mild assumption on the coefficient functions v and σ.

Assumption 2.2. *Throughout this paper, we assume that*

(i) v is bounded and càdlàg;

(ii) σ is càdlàg, and there exists $0 < \underline{\sigma} < \overline{\sigma} < \infty$ such that $\underline{\sigma} \le \sigma(t) \le \overline{\sigma}$, for all $t \in \mathbb{R}_+$.

Our first proposition shows that the passage times defined by (2.2) and (2.6) are the same in the $\mathbb{P}$-a.s. sense, which ensures the validity of our main result for both types of passage times (see Remark 2.1). The proof of this result is deferred to Appendix A.1.

Proposition 2.3. *Under Assumption 2.2, for any $s \in \mathbb{R}_+$ and $a, \ell \in \mathbb{R}$, $\mathbb{P}(\tau_\ell^\pm = \eta_\ell^\pm) = 1$.*

The following result, the proof of which is deferred to Appendix A.2, introduces two quantities γ^+ and γ^- that are key for our main result.

Proposition 2.4. *Suppose that Assumption 2.2 is valid.*

(i) *For any* $0 \le s < t$*, the following limit*

$$\lim_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^{\pm}(s) > t\big)$$

exists and is finite.

(ii) *For any* $s \in \mathbb{R}_+$*, define*

$$\gamma^{\pm}(s,t) := \begin{cases} \lim_{\ell\to 0+} \ell^{-1}\mathbb{P}\big(\tau_\ell^{\pm}(s) > t\big), & t \in (s,\infty), \\ 0, & t \in [0,s]. \end{cases}$$

Then $\gamma^{\pm}(s,\cdot)$ *is non-increasing and continuous on* (s,∞)*.*

(iii) *For any* $0 \le s < t$*, we have*

$$\frac{\sqrt{2}}{\sqrt{\pi\overline{\sigma}^2(t-s)}}\exp\bigg(-\frac{\overline{\sigma}^2\|v\|_\infty^2}{2\underline{\sigma}^4}(t-s)\bigg) - \frac{2\|v\|_\infty}{\underline{\sigma}^2}\Phi\bigg(-\frac{\overline{\sigma}\|v\|_\infty}{\underline{\sigma}^2}\sqrt{t-s}\bigg)$$
$$\le \gamma^{\pm}(s,t) \le \frac{\sqrt{2}}{\sqrt{\pi\underline{\sigma}^2(t-s)}}\exp\bigg(-\frac{\|v\|_\infty^2}{2\overline{\sigma}^2}(t-s)\bigg) + \frac{2\|v\|_\infty}{\underline{\sigma}^2}\Phi\bigg(\frac{\|v\|_\infty}{\underline{\sigma}}\sqrt{t-s}\bigg).$$

Our second assumption is related to the continuity of $\gamma^{\pm}(s,t)$ with respect to s.

Assumption 2.5. *The functions* v *and* σ *are such that, for every* $t \in \mathbb{R}_+$*,* $\gamma^{\pm}(\cdot,t)$ *is continuous on* $[0,t)$*.*

Denoting by $\gamma := \gamma^+ + \gamma^-$, we consider the operator Γ on $C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)$ defined by

$$\big(\Gamma f\big)(s) := \int_s^\infty g_f(r)\gamma(s,r)\,dr, \quad s \in \mathbb{R}_+, \tag{2.7}$$

where we recall (2.4) for the definition of g_f. The following lemma, the proof of which is deferred to Appendix A.3, establishes well-definedness of Γ and provides some of its basic properties.

Lemma 2.6. *Under Assumption 2.2, for every* $f \in C^{ac}_{e,cdl}(\mathbb{R}_+)$*, the integral on the right-hand side of* (2.7) *is finite for every* $s \in \mathbb{R}_+$*. Moreover, under Assumptions 2.2 and 2.5,* $\Gamma f \in C_e(\mathbb{R}_+)$*.*

Our next assumption regards the range of $\lambda - \Gamma$, which is a key in identifying Γ as a strong generator.

Assumption 2.7. *The functions* v *and* σ *are such that* $\{(\lambda-\Gamma)f : f \in C^{ac}_{e,cdl}(\mathbb{R}_+)\}$ *is dense in* $C_0(\mathbb{R}_+)$ *for some* $\lambda > 0$*.*

An example of functions v and σ, for which Assumptions 2.2, 2.5, and 2.7 are satisfied, will be presented in Section 5.

Before we proceed, we recall the definition of Feller semigroup (cf. [7, Definitions 1.1 & 1.2]).

Definition 2.8. A family of linear operators $(T_t)_{t\in\mathbb{R}_+}$ defined on $L^\infty(\mathbb{R}_+)$ is called a Feller semigroup if it is a positive contraction semigroup on $L^\infty(\mathbb{R}_+)$ which satisfies the Feller property

$$T_t f \in C_0(\mathbb{R}_+), \quad \text{for any } f \in C_0(\mathbb{R}_+), \quad t \in (0,\infty),$$

and which is strongly continuous on $C_0(\mathbb{R}_+)$, namely,

$$\lim_{t\to 0}\big\|T_t f - f\big\|_\infty = 0, \quad \text{for any } f \in C_0(\mathbb{R}_+).$$

We conclude the section by presenting a proposition, which is a key step towards establishing our main result. The proof of this proposition will be provided in Section 4.1.

Proposition 2.9. *Under Assumptions 2.2, 2.5, and 2.7, we have*

(i) the operator Γ on $C^{ac}_{e,cdl}(\mathbb{R}_+)$ is closable and the closure of Γ, denoted by $(\overline{\Gamma}, \mathscr{D}(\overline{\Gamma}))$, is the strong generator of a Feller semigroup, say, $(\mathcal{P}_\ell)_{\ell\in\mathbb{R}_+}$;

(ii) for any $f \in C_e(\mathbb{R}_+)$, the integral $\int_0^L \mathcal{P}_\ell f \, d\ell$ converge in $L^\infty(\mathbb{R}_+)$, as $L \to \infty$, to the limit denoted by $\int_0^\infty \mathcal{P}_\ell f \, d\ell$. Moreover, $\int_0^\infty \mathcal{P}_\ell f \, d\ell \in \mathscr{D}(\overline{\Gamma})$, and

$$\overline{\Gamma} \int_0^\infty \mathcal{P}_\ell f \, d\ell = -f. \tag{2.8}$$

Remark 2.10. According to Proposition 2.9 (i), the operator Γ is closable and its closure is the strong generator of a Feller semigroup. In the proof of the proposition we will first consider Γ on $C^1_c(\mathbb{R}_+) \subset C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$, which is dense in $C_0(\mathbb{R}_+)$. However, noting that each $f \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$ is differentiable a.e. on $\mathbb{R}_+$ with $f' = g_f$ a.e. on $\mathbb{R}_+$, the space $C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$ contains not only all $C^1_c(\mathbb{R}_+)$ functions (when $g_f \in C_c(\mathbb{R}_+)$), but also functions with discontinuous derivatives. In particular, it contains functions f such that $f'(t) = \mathbb{1}_{[0,T)}(t)$ a.e., for some $T \in (0, \infty)$. Those functions will turn out to be especially helpful in the study of the regularity of the semigroup generated by Γ (see Lemma 4.1 below), which is crucial in proving Proposition 2.9 part (ii).

2.3 Main Result

We now state the main result of this paper, Theorem 2.11, which provides a factorization for the expectation (1.6) in terms of the operators $(\mathcal{P}_\ell)_{\ell\in\mathbb{R}_+}$ and $(\mathcal{P}^\pm_\ell)_{\ell\in\mathbb{R}_+}$. This factorization generalizes the operator form of the Wiener-Hopf factorization (1.5) for Brownian motion with drift, and is therefore named as the *Wiener-Hopf factorization for the time-inhomogeneous arithmetic Brownian motion process* $\varphi(s, a)$. The proof of this theorem will be presented in Section 4.2.

Theorem 2.11. *Under Assumptions 2.2, 2.5, and 2.7, for any $h \in C_e(\mathbb{R}_+)$ and $u \in C(\mathbb{R})$ with*

$$\mathbb{E}\bigg(\int_s^\infty \big| u\big(\varphi_t(s,a)\big) h(t) \big| \, dt \bigg) < \infty, \tag{2.9}$$

for any $(s, a) \in \mathbb{R}_+ \times \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}\bigg(\int_s^\tau u\big(\varphi_t(s,a)\big) h(t) \sigma^2(t) \, dt \bigg) &= 2\int_0^\infty u(a+\ell) \bigg(\mathcal{P}^+_\ell \int_0^\infty \mathcal{P}_y h \, dy \bigg)(s) \, d\ell + 2\int_0^\infty u(a-\ell) \bigg(\mathcal{P}^-_\ell \int_0^\infty \mathcal{P}_y h \, dy \bigg)(s) \, d\ell \\ &\quad - 2\mathbb{E}\left(\mathbb{1}_{\{\tau<\infty\}} \int_0^\infty u\big(\varphi_\tau(s,a) + \ell\big) \bigg(\mathcal{P}^+_\ell \int_0^\infty \mathcal{P}_y h \, dy \bigg)(\tau) \, d\ell \right) \\ &\quad - 2\mathbb{E}\left(\mathbb{1}_{\{\tau<\infty\}} \int_0^\infty u\big(\varphi_\tau(s,a) - \ell\big) \bigg(\mathcal{P}^-_\ell \int_0^\infty \mathcal{P}_y h \, dy \bigg)(\tau) \, d\ell \right), \end{aligned} \tag{2.10}$$

for any $\mathbb{F}$-stopping time τ and $(s, a) \in \mathbb{R}_+ \times \mathbb{R}$.

In particular, by taking $\tau \equiv \infty$, we obtain the following corollary.

Corollary 2.12. *Under the setting of Theorem 2.11, for any* $(s,a)\in\mathbb{R}_+\times\mathbb{R}$*, we have*

$$\mathbb{E}\bigg(\int_s^\infty u\big(\varphi_t(s,a)\big)h(t)dt\bigg) = 2\int_0^\infty u(a+\ell)\bigg(\mathcal{P}^+_\ell\int_0^\infty \mathcal{P}_y h\,dy\bigg)(s)\,d\ell + 2\int_0^\infty u(a-\ell)\bigg(\mathcal{P}^-_\ell\int_0^\infty \mathcal{P}_y h\,dy\bigg)(s)\,d\ell. \tag{2.11}$$

Moreover, as a consequence of (2.10) and (2.11), for any $\mathbb{F}$*-stopping time* τ*, we have*

$$\mathbb{E}\bigg(\int_\tau^\infty u\big(\varphi_t(s,a)\big)h(t)dt\bigg) = 2\mathbb{E}\left(\mathbb{1}_{\{\tau<\infty\}}\int_0^\infty u\big(\varphi_\tau(s,a)+\ell\big)\bigg(\mathcal{P}^+_\ell\int_0^\infty \mathcal{P}_y h\,dy\bigg)(\tau)\,d\ell\right) + 2\mathbb{E}\left(\mathbb{1}_{\{\tau<\infty\}}\int_0^\infty u\big(\varphi_\tau(s,a)-\ell\big)\bigg(\mathcal{P}^-_\ell\int_0^\infty \mathcal{P}_y h\,dy\bigg)(\tau)\,d\ell\right).$$

Remark 2.13. In this remark, we will present some intuition, but not the blueprint for rigorous proof, for the formula (2.11). For simplicity, throughout this remark we fix $s\in\mathbb{R}_+$ and $a=0$,[4] The following discussion is mainly motivated by the downcrossing representation of the Brownian local time at zero (cf. [16, Theorem 6.1]) as well as the occupation density formula for local time (cf. [19, Chapter IV, (45.4)]).

(i) For any $x\in\mathbb{R}_+$ and $\ell\in(0,\infty)$, let $\tau^{+,1}_\ell(s,x):=\tau^+_{x+\ell}(s)$, and for $j\in\mathbb{N}$,

$$\tau^{-,j}_\ell(s,x) := \inf\big\{t>\tau^{+,j}_\ell(s,x):\,\varphi_t(s)=x-\ell\big\},$$
$$\tau^{+,j+1}_\ell(s,x) := \inf\big\{t>\tau^{-,j}_\ell(s,x):\,\varphi_t(s)=x+\ell\big\}.$$

The stopping time $\tau^{-,j}_\ell(s,x)$ is called the j-th downcrossing time of $[x-\ell,x+\ell]$ by the process $\varphi(s)$. From (2.5) and from the strong Markov property of $\varphi(s)$, for any $h\in L^\infty(\mathbb{R}_+)$ and $j\in\mathbb{N}$, we have

$$\mathbb{E}\Big(h\big(\tau^{-,j}_\ell(s,x)\big)\Big) = \Big(\mathcal{P}^+_{x+\ell}\mathcal{P}^-_{2\ell}\big(\mathcal{P}^+_{2\ell}\mathcal{P}^-_{2\ell}\big)^{j-1}h\Big)(s).$$

By imposing appropriate regularity assumption on h (so that the expressions in the equality below are well defined), we conjecture that

$$\mathbb{E}\bigg(\sum_{j=1}^\infty h\big(\tau^{-,j}_\ell(s,x)\big)\bigg) = \bigg(\mathcal{P}^+_{x+\ell}\mathcal{P}^-_{2\ell}\sum_{j=1}^\infty\big(\mathcal{P}^+_{2\ell}\mathcal{P}^-_{2\ell}\big)^{j-1}h\bigg)(s).$$

Note that for $f\in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$,

$$\lim_{\ell\to0+}\frac{1}{2\ell}\big(I-\mathcal{P}^+_{2\ell}\mathcal{P}^-_{2\ell}\big)f = \lim_{\ell\to0+}\frac{1}{2\ell}\big(I-\mathcal{P}^+_{2\ell}\big)f + \lim_{\ell\to0+}\frac{1}{2\ell}\mathcal{P}^+_{2\ell}\big(I-\mathcal{P}^-_{2\ell}\big)f = -\big(\Gamma^++\Gamma^-\big)f = -\overline{\Gamma}f,$$

where Γ^+ and Γ^- are the respective generators of $(\mathcal{P}^+_\ell)_{\ell\in\mathbb{R}_+}$ and $(\mathcal{P}^-_\ell)_{\ell\in\mathbb{R}_+}$ (see Propositions 3.4 and 3.5 below), and where the last equality above follows from (2.7) and Proposition 3.5. If $I-\mathcal{P}^+_{2\ell}\mathcal{P}^-_{2\ell}$ were invertible, we would have, for any $h\in C_e(\mathbb{R}_+)$,

$$\sum_{j=1}^\infty\big(\mathcal{P}^+_{2\ell}\mathcal{P}^-_{2\ell}\big)^{j-1}h = \big(I-\mathcal{P}^+_{2\ell}\mathcal{P}^-_{2\ell}\big)^{-1}h,$$

[4] Recall that for simplicity we always write $\varphi_t(s)$ and $\tau^\pm_\ell(s)$ in place of $\varphi_t(s,0)$ and $\tau^\pm_\ell(s,0)$, respectively.

so that, in view of Proposition 2.9 (ii),

$$\lim_{\ell\to0+} 2\ell\sum_{j=1}^{\infty}\big(\mathcal{P}^{+}_{2\ell}\mathcal{P}^{-}_{2\ell}\big)^{j-1}h=\overline{\Gamma}^{-1}h=-\int_0^\infty \mathcal{P}_y h\,dy.$$

Unfortunately, $I-\mathcal{P}^{+}_{2\ell}\mathcal{P}^{-}_{2\ell}$ is not invertible in general. Nevertheless, given the above discussion, we conjecture that for any $h\in C_e(\mathbb{R}_+)$,

$$\lim_{\ell\to0+} 2\ell\,\mathbb{E}\bigg(\sum_{j=1}^{\infty}h\big(\tau^{-,j}_{\ell}(s,x)\big)\bigg)=-\bigg(\mathcal{P}^{+}_{x}\int_0^\infty \mathcal{P}_y h\,dy\bigg)(s), \tag{2.12}$$

(ii) Similarly, for any $x\in(-\infty,0)$ and $\ell\in(0,-x)$, we define the downcrossing times of $[x-\ell,x+\ell]$ for $\varphi(s)$ as follows. Let $\theta^{-,1}_{\ell}(s,x):=\tau^{-}_{x-\ell}(s)$[5], and for $j\in\mathbb{N}$,

$$\theta^{+,j}_{\ell}(s,x):=\inf\big\{t>\theta^{-,j}_{\ell}(s,x):\,\varphi_t(s)=x+\ell\big\},$$
$$\theta^{-,j+1}_{\ell}(s,x):=\inf\big\{t>\theta^{+,j}_{\ell}(s,x):\,\varphi_t(s)=x-\ell\big\}.$$

For any $h\in C_e(\mathbb{R}_+)$, we conjecture that

$$\lim_{\ell\to0+} 2\ell\,\mathbb{E}\bigg(\sum_{j=1}^{\infty}h\big(\theta^{-,j}_{\ell}(s,x)\big)\bigg)=-\bigg(\mathcal{P}^{-}_{-x}\int_0^\infty \mathcal{P}_y h\,dy\bigg)(s). \tag{2.13}$$

(iii) Next, let $(L^{s,x}_t)_{t\in[s,\infty)}$ be the local time of $\varphi(s)=(\varphi_t(s))_{t\in[s,\infty)}$ at level $x\in\mathbb{R}$. By [16, Theorem 6.1], when $v(t)\equiv0$[6] and for any $T\in(0,\infty)$, we have

$$\lim_{\ell\to0+} 4\ell\sum_{j=1}^{\infty}\mathbb{1}_{\{\tau^{-,j}_{\ell}(s,0)\le T\}}=L^{s,0}_T,\quad \mathbb{P}-\text{a.s.} \tag{2.14}$$

When v is a function on $\mathbb{R}_+$ more general then an indicator function, then, assuming suitable conditions (such as Assumptions 2.2, 2.5, and 2.7), we conjecture that, for $h\in C_0(\mathbb{R}_+)$ satisfying certain additional regularity conditions,

$$\lim_{\ell\to0+} 4\ell\,\mathbb{E}\bigg(\sum_{j=1}^{\infty}h\big(\tau^{-,j}_{\ell}(s,x)\big)\bigg)=\mathbb{E}\bigg(\int_s^\infty h(t)\,dL^{s,x}_t\bigg),\quad x\in\mathbb{R}_+, \tag{2.15}$$

$$\lim_{\ell\to0+} 4\ell\,\mathbb{E}\bigg(\sum_{j=1}^{\infty}h\big(\theta^{-,j}_{\ell}(s,x)\big)\bigg)=\mathbb{E}\bigg(\int_s^\infty h(t)\,dL^{s,x}_t\bigg),\quad x\in(-\infty,0). \tag{2.16}$$

Both (2.15) and (2.16) reduce to (2.14) in case of v being an indicator function. By comparing (2.12) and (2.13) with (2.15) and (2.16), respectively, we deduce that, for any $h\in C_e(\mathbb{R}_+)$ (with possibly additional regularity conditions),

$$\mathbb{E}\bigg(\int_s^\infty h(t)\,dL^{s,x}_t\bigg)=\begin{cases}-2\bigg(\mathcal{P}^{+}_{x}\displaystyle\int_0^\infty \mathcal{P}_y h\,dy\bigg)(s), & x\in\mathbb{R}_+\\[2ex] -2\bigg(\mathcal{P}^{-}_{-x}\displaystyle\int_0^\infty \mathcal{P}_y h\,dy\bigg)(s), & x\in(-\infty,0)\end{cases}. \tag{2.17}$$

[5]When $[x,-\ell,x+\ell]\subset(-\infty,0)$, the first down crossing time of $[x-\ell,x+\ell]$ for $\varphi(s)$ is simply the its hitting time at level $x-\ell$.

[6]In this case $\varphi_t(s)=W_t-W_s$, $t\in[s,\infty)$, which has the same law as a standard Brownian motion.

(iv) Finally, we recall the well-known occupation density formula (cf. [19, Chapter IV, (45.4)]) which states that, for any positive Borel-measurable functions u (on $\mathbb{R}$) and h (on $\mathbb{R}_+$) (and eventually for any Borel-measurable functions u and h such that $\int_s^\infty u(\varphi_t(s,0))h(t)\,dt < \infty$ $\mathbb{P}$-a.s.), we have

$$\int_s^\infty u(\varphi_t(s))h(t)\sigma^2(t)\,dt = \int_{-\infty}^\infty u(x)\left(\int_s^\infty h(t)\,dL_t^{s,x}\right)dx, \quad \mathbb{P}-\text{a.s.} \tag{2.18}$$

Combining (2.17) with (2.18), we obtain (2.11) with suitable choices of u and h.

(v) Conversely, once we establish (2.11), then (2.17) is expected to be shown as a consequence of (2.11) and (2.18). In addition, we conjecture that (2.17) remains valid when $h(t) = \mathbb{1}_{[0,T]}(t)$, for any $T \in (0,\infty)$. In this case, (2.17) provides a formula for the expected local time of $\varphi(s)$ at level x up to any finite terminal T.

2.4 Connection to Classical Wiener-Hopf Factorization for Lévy Processes

In this section we will establish a connection between our main result and the classical Wiener-Hopf factorization for real-valued Lévy processes. More precisely, in the following corollary, we recover from (2.11) the operator form of the Wiener-Hopf factorization (1.5) in the setup of a time-homogenous Brownian motion with drift. The proof of the corollary is deferred to Section 4.3.

Corollary 2.14. *Suppose that $v(t) \equiv v \in \mathbb{R}$ and $\sigma(t) \equiv \sigma \in (0,\infty)$, for all $t \in \mathbb{R}_+$. For any $c \in (0,\infty)$ and $u \in C(\mathbb{R})$ with*

$$\mathbb{E}\left(\int_0^\infty e^{-ct}\big|u(\varphi_t(0,a))\big|\,dt\right) < \infty, \quad \textit{for any } a \in \mathbb{R}, \tag{2.19}$$

we define, for any $a \in \mathbb{R}$,

$$\begin{aligned}
(\mathcal{E}_c u)(a) &:= c\,\mathbb{E}\left(\int_0^\infty e^{-ct}u(\varphi_t(0,a))\,dt\right),\\
(\mathcal{E}_c^+ u)(a) &:= c\,\mathbb{E}\left(\int_0^\infty e^{-ct}u(\overline{\varphi}_t(0,a))\,dt\right),\\
(\mathcal{E}_c^- u)(a) &:= c\,\mathbb{E}\left(\int_0^\infty e^{-ct}u(\underline{\varphi}_t(0,a))\,dt\right),
\end{aligned}$$

where for any $t \in \mathbb{R}_+$, $\overline{\varphi}_t(0,a) := \sup_{r\in[0,t]}\varphi_r(0,a)$ and $\underline{\varphi}_t(0,a) := \inf_{r\in[0,t]}\varphi_r(0,a)$. Then, we have

$$\mathcal{E}_c u = \mathcal{E}_c^+\mathcal{E}_c^- u = \mathcal{E}_c^-\mathcal{E}_c^+ u. \tag{2.20}$$

Remark 2.15. In this remark, we will show that formula (2.20) is a special case of (1.5). Towards this end, we first show that (1.1) and (1.2) imply (1.5). For any $a \in \mathbb{R}$ and $u \in L^\infty(\mathbb{R})$ we have

$$\begin{aligned}
(\mathcal{H}_c u)(a) &= \mathbb{E}\big(u(a + X_{\mathsf{e}_c})\big) = \mathbb{E}\big(u(a + \overline{X}_{\mathsf{e}_c} + X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c})\big)\\
&= \mathbb{E}\Big(\mathbb{E}\Big(u\big(a + \overline{X}_{\mathsf{e}_c} + X_{\mathsf{e}_c} - \overline{X}_{\mathsf{e}_c}\big)\,\Big|\,\overline{X}_{\mathsf{e}_c}\Big)\Big) = \mathbb{E}\Big(\big(\mathcal{H}_c^- u\big)\big(a + \overline{X}_{\mathsf{e}_c}\big)\Big) = \big(\mathcal{H}_c^+\mathcal{H}_c^- u\big)(a),
\end{aligned}$$

where we have used (1.1) and (1.2) in the last second equality above. The second equality in (1.5) can be proved in a similar manner. Clearly, for $u \in C_b(\mathbb{R})$, (2.20) is a special case of (1.5).

We will now demonstrate that (1.5) implies (1.3) (equivalently, (1.4)). Indeed, by (1.5), we have

$$\big(\mathcal{H}_c \sin(\xi\,\cdot)\big)(a) = \big(\mathcal{H}_c^+\big(\mathcal{H}_c^- \sin(\xi\,\cdot)\big)\big)(a), \quad \big(\mathcal{H}_c \cos(\xi\,\cdot)\big)(a) = \big(\mathcal{H}_c^+\big(\mathcal{H}_c^- \cos(\xi\,\cdot)\big)\big)(a), \quad a, \xi \in \mathbb{R}.$$

Thus, by the Euler formula, as well as the linearity of $\mathcal{H}_c$ and $\mathcal{H}_c^{\pm}$, we see that (1.5) holds true for $u(x) = e^{i\xi x}$, $x \in \mathbb{R}$. It follows immediately that for all $\xi \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\Big(e^{i\xi X_{\mathrm{e}_c}}\Big) &= \big(\mathcal{H}_c\, e^{i\xi\,\cdot}\big)(0) = \big(\mathcal{H}_c^+\mathcal{H}_c^-\, e^{i\xi\,\cdot}\big)(0) = \bigg(\mathcal{H}_c^+\bigg(c\,\mathbb{E}\bigg(\int_0^\infty e^{-ct} e^{i\xi(\cdot\,+\underline{X}_t)}\,dt\bigg)\bigg)\bigg)(0) \\
&= \big(\mathcal{H}_c^+ e^{i\xi\,\cdot}\big)(0)\cdot c\,\mathbb{E}\bigg(\int_0^\infty e^{-ct} e^{i\xi\underline{X}_t}\,dt\bigg) \\
&= c\,\mathbb{E}\bigg(\int_0^\infty e^{-ct} e^{i\xi\overline{X}_t}\,dt\bigg)\cdot c\,\mathbb{E}\bigg(\int_0^\infty e^{-ct} e^{i\xi\underline{X}_t}\,dt\bigg) = \mathbb{E}\Big(e^{i\xi\overline{X}_{\mathrm{e}_c}}\Big)\mathbb{E}\Big(e^{i\xi\underline{X}_{\mathrm{e}_c}}\Big).
\end{aligned}$$

In particular, when X is an arithmetic Brownian motion, (2.20) implies (1.3) (equivalently, (1.4)).

Remark 2.16. In case when $v(\cdot) \equiv v \in \mathbb{R}$ and $\sigma(\cdot) \equiv \sigma \in (0,\infty)$, Assumption 2.2 holds trivially. Moreover, the validity of Assumptions 2.5 and 2.7 in this case follows from a straightforward adaptation of the proof in Section 5, where those two assumptions are verified when both v and σ are piecewise constant.

3 Auxiliaries

In this section, we will present some auxiliary results needed for the proof of our main result.

3.1 An Auxiliary Time-Homogeneous Markov Family

Here we will introduce a time-homogeneous Markov family $\widetilde{\mathcal{M}}$ by applying standard time homogenization techniques to the time-inhomogeneous Markov family $\widehat{\mathcal{M}}$ to be defined below and associated with $\{\varphi(s,a), (s,a) \in \mathbb{R}_+ \times \mathbb{R}\}$. Similar construction was done in [4, Section 4.1]. Hereafter, we denote by $\mathscr{Z} := \mathbb{R}_+ \times \mathbb{R}$, $\overline{\mathscr{Z}} := \mathscr{Z} \cup \{(\infty,\infty)\}$, $\overline{\mathbb{R}}_+ := [0,\infty]$, and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.

Let $\widehat{\Omega} = C(\mathbb{R}_+)$ which is the space of real-valued continuous functions on $\mathbb{R}_+$. We stipulate $\widehat{\omega}(\infty) = \infty$ for every $\widehat{\omega} \in \widehat{\Omega}$. One can construct a *standard* canonical[7] time-inhomogeneous Markov family (cf. [10, Section I.6, Definition 6])

$$\widehat{\mathcal{M}} := \Big\{\Big(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{\mathbb{F}}_s, (\widehat{\varphi}_t)_{t\in[s,\infty]}, \widehat{\mathbb{P}}_{s,a}\Big),\ (s,a) \in \overline{\mathscr{Z}}\Big\}$$

with transition function $\widehat{P}$ given by[8]

$$\widehat{P}(s,a,t,A) := \mathbb{P}\big(\varphi_t(s,a) \in A\big), \quad t \in [s,\infty], \quad (s,a) \in \overline{\mathscr{Z}}, \quad A \in \mathcal{B}(\overline{\mathbb{R}}). \tag{3.1}$$

A routine check verifies that $\widehat{P}$ is a Feller transition function.[9] This allows us to apply [10, Section I.6, Theorem 3] in order to prove existence of such Markov family $\widehat{\mathcal{M}}$.[10] In particular, it holds that,

[7] By canonical, we mean $\widehat{\varphi}_t(\widehat{\omega}) = \widehat{\omega}(t)$ for all $t \in \overline{\mathbb{R}}_+$.

[8] We stipulate $\varphi_\infty(s,a) \equiv \infty$, for any $(s,a) \in \overline{\mathscr{Z}}$.

[9] We refer to [10, Section I.6, Page 78] for the definition of the Feller transition function.

[10] The details of construction of the family $\widehat{\mathcal{M}}$ and its properties take much space, and therefore are not given here. They can be obtained from the authors upon request.

for $0 \le t \le t'$ and $A \in \mathcal{B}(\overline{\mathbb{R}})$,

$$\widehat{P}(t,y,t',A) = \widehat{\mathbb{P}}_{t,y}\big(\widehat{\varphi}_{t'} \in A\big). \tag{3.2}$$

Moreover, by investigating the finite dimensional distributions of $\widehat{\varphi} := (\widehat{\varphi}_t)_{t\in[s,\infty]}$ under $\widehat{\mathbb{P}}_{s,a}$, and since $\widehat{\varphi}$ admits continuous sample paths, it can be shown that, for any $(s,a) \in \overline{\mathscr{Z}}$,

$$\text{the law of } \widehat{\varphi} \text{ under } \widehat{\mathbb{P}}_{s,a} \;=\; \text{the law of } \varphi(s,a) \text{ under } \mathbb{P}. \tag{3.3}$$

We will verify that two-dimensional distributions agree. Let $t' > t \ge s$ and $\xi, \xi' \in \mathbb{R}$. The Markov property of $\widehat{\varphi}$ under $\widehat{\mathbb{P}}_{s,a}$ and (3.2) imply that

$$\begin{aligned}
\widehat{\mathbb{E}}_{s,a}\Big(e^{i\xi\widehat{\varphi}_t + i\xi'\widehat{\varphi}_{t'}}\Big) &= \widehat{\mathbb{E}}_{s,a}\Big(e^{i\xi\widehat{\varphi}_t}\,\widehat{\mathbb{E}}_{t,\widehat{\varphi}_t}\Big(e^{i\xi'\widehat{\varphi}_{t'}}\Big)\Big) = \int_{-\infty}^{\infty} \frac{e^{i\xi x}\,\widehat{\mathbb{E}}_{t,x}\Big(e^{i\xi'\widehat{\varphi}_{t'}}\Big)}{\sqrt{2\pi\int_s^t \sigma^2(r)dr}} \exp\bigg(-\frac{(x-a-\int_s^t v(r)dr)^2}{2\int_s^t \sigma^2(r)dr}\bigg)dx \\
&= \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\sqrt{2\pi\int_s^t \sigma^2(r)dr}} \exp\bigg(i\xi'\bigg(x + \int_t^{t'} v(r)\,dr\bigg) - \frac{1}{2}(\xi')^2\int_t^{t'}\sigma^2(r)\,dr - \frac{(x-a-\int_s^t v(r)dr)^2}{2\int_s^t \sigma^2(r)dr}\bigg)\,dx \\
&= \exp\left(i(\xi+\xi')\bigg(a + \int_s^t v(r)\,dr\bigg) - \frac{1}{2}(\xi+\xi')\int_t^{t'}\sigma^2(r)\,dr + i\xi'\int_t^{t'} v(r)\,dr - \frac{1}{2}(\xi')^2\int_t^{t'}\sigma^2(r)\,dr\right).
\end{aligned}$$

On the other hand, the definition of $\varphi(s,a)$ implies that

$$\begin{aligned}
&\mathbb{E}\Big(e^{i\xi\varphi_t(s,a)}\,e^{i\xi'\varphi_{t'}(s,a)}\Big) = \mathbb{E}\Big(e^{i(\xi+\xi')\varphi_t(s,a)}\,e^{i\xi'(\varphi_{t'}(s,a)-\varphi_t(s,a))}\Big) \\
&\quad = \exp\left(i(\xi+\xi')\bigg(a + \int_s^t v(r)\,dr\bigg) - \frac{1}{2}(\xi+\xi')\int_t^{t'}\sigma^2(r)\,dr + i\xi'\int_t^{t'} v(r)\,dr - \frac{1}{2}(\xi')^2\int_t^{t'}\sigma^2(r)\,dr\right),
\end{aligned}$$

which demonstrates that two-dimensional distributions of $\widehat{\varphi}$ and $\varphi(s,a)$ agree. The above argument generalizes to any finite dimensional distributions by induction.

Considering the standard Markov family $\widehat{\mathcal{M}}$, for any $s \in \mathbb{R}_+$ and $\ell \in \mathbb{R}$, we define

$$\widehat{\tau}_\ell^+(s) := \inf\big\{t \in [s,\infty] : \widehat{\varphi}_t \ge \ell\big\} \quad \text{and} \quad \widehat{\tau}_\ell^-(s) := \inf\big\{t \in [s,\infty] : \widehat{\varphi}_t \le \ell\big\}, \tag{3.4}$$

which are both $\widehat{\mathbb{F}}_s$-stopping times in light of the continuity of $\widehat{\varphi}$ and the right-continuity of the filtration $\widehat{\mathbb{F}}_s$. In what follows, when no confusion arises, we will omit the variable s in $\widehat{\tau}_\ell^\pm(s)$. The following proposition is an immediate consequence of (2.2), (3.3), and (3.4), so its proof is skipped.

Proposition 3.1. *For any $f \in L^\infty(\overline{\mathbb{R}}_+)$, $(s,a) \in \mathscr{Z}$, and $\ell \in \mathbb{R}$, we have*

$$\widehat{\mathbb{E}}_{s,a}\Big(f\big(\widehat{\tau}_\ell^\pm\big)\Big) = \mathbb{E}\Big(f\big(\tau_\ell^\pm(s,a)\big)\Big).$$

Next, we will transform the time-inhomogeneous Markov family $\widehat{\mathcal{M}}$ into a *time-homogeneous* Markov family

$$\widetilde{\mathcal{M}} := \Big\{\Big(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{F}}, (Z_t)_{t\in\overline{\mathbb{R}}_+}, (\theta_r)_{r\in\mathbb{R}_+}, \widetilde{\mathbb{P}}_{s,a}\Big),\, (s,a) \in \overline{\mathscr{Z}}\Big\}$$

following the setup in [6, Section 3]. The construction of $\widetilde{\mathcal{M}}$ proceeds as follows.

- We let $\widetilde{\Omega} := \overline{\mathbb{R}}_+ \times \widehat{\Omega}$ to be the new sample space, with elements $\widetilde{\omega} = (s, \omega)$, where $s \in \overline{\mathbb{R}}_+$ and $\omega \in \Omega$. On $\widetilde{\Omega}$ we consider the σ-field

$$\widetilde{\mathscr{F}} := \left\{ \widetilde{A} \subset \widetilde{\Omega} : \ \widetilde{A}_s \in \widehat{\mathscr{F}}^s_\infty \ \text{ for any } s \in \overline{\mathbb{R}}_+ \right\},$$

where $\widetilde{A}_s := \{\omega \in \Omega : (s, \omega) \in \widetilde{A}\}$ and $\widehat{\mathscr{F}}^s_\infty := \sigma(\bigcup_{t \geq s} \widehat{\mathscr{F}}^s_t)$.

- We let $\overline{\mathscr{Z}} = \mathscr{Z} \cup \{(\infty, \infty)\}$ be the new state space, with elements $z = (s, a)$. On $\mathscr{Z} = \mathbb{R}_+ \times \mathbb{R}$ we consider the σ-field

$$\widetilde{\mathcal{B}}(\mathscr{Z}) := \left\{ \widetilde{B} \subset \mathscr{Z} : \ \widetilde{B}_s \in \mathcal{B}(\mathbb{R}) \ \text{ for any } s \in \mathbb{R}_+ \right\},$$

where $\widetilde{B}_s := \{a \in \mathbb{R} : (s, a) \in \widetilde{B}\}$. Let $\widetilde{\mathcal{B}}(\overline{\mathscr{Z}}) := \sigma(\widetilde{\mathcal{B}}(\mathscr{Z}) \cup \{(\infty, \infty)\})$.

- We consider a family of probability measures $\{\widetilde{\mathbb{P}}_{(s,a)}, (s, a) \in \overline{\mathscr{Z}}\}$, where, for $(s, a) \in \overline{\mathscr{Z}}$,

$$\widetilde{\mathbb{P}}_{s,a}\big(\widetilde{A}\big) := \widehat{\mathbb{P}}_{s,a}\big(\widetilde{A}_s\big), \quad \widetilde{A} \in \widetilde{\mathscr{F}}. \tag{3.5}$$

Frequently, for convenience, we will write $\widetilde{\mathbb{P}}_z\big(\widetilde{A}\big)$ in place of $\widetilde{\mathbb{P}}_{(s,a)}\big(\widetilde{A}\big)$ where $z = (s, a)$.

- We consider the process $Z := (Z_t)_{t \in \overline{\mathbb{R}}_+}$ on $(\widetilde{\Omega}, \widetilde{\mathscr{F}})$, where, for $t \in \overline{\mathbb{R}}_+$,

$$Z_t(\widetilde{\omega}) := \big(s + t, \widehat{\varphi}_{s+t}(\omega)\big), \quad \widetilde{\omega} = (s, \omega) \in \widetilde{\Omega}. \tag{3.6}$$

Hereafter, we denote the two components of Z by Z^1 and Z^2, respectively.

- On $(\widetilde{\Omega}, \widetilde{\mathscr{F}})$, we define $\widetilde{\mathbb{F}} := (\widetilde{\mathscr{F}}_t)_{t \in \overline{\mathbb{R}}_+}$, where $\widetilde{\mathscr{F}}_t := \widetilde{\mathscr{G}}_{t+}$ (with the convention $\widetilde{\mathscr{G}}_{\infty+} = \widetilde{\mathscr{G}}_\infty$), and $(\widetilde{\mathscr{G}}_t)_{t \in \overline{\mathbb{R}}_+}$ is the completion of the natural filtration generated by Z with respect to the set of probability measures $\{\widetilde{\mathbb{P}}_z, z \in \overline{\mathscr{Z}}\}$ (cf. [10, Chapter I, Page 43]).

- Finally, for any $r \in \mathbb{R}_+$, we consider the shift operator $\theta_r : \widetilde{\Omega} \to \widetilde{\Omega}$ defined by

$$\theta_r \, \widetilde{\omega} = (u + r, \omega_{\cdot + r}), \quad \widetilde{\omega} = (u, \omega) \in \widetilde{\Omega}.$$

It follows that $Z_t \circ \theta_r = Z_{t+r}$, for any $t, r \in \mathbb{R}_+$.

We define a transition function $\widetilde{P}$ on $\overline{\mathscr{Z}} \times \overline{\mathbb{R}}_+ \times \widetilde{\mathcal{B}}(\overline{\mathscr{Z}})$ by

$$\widetilde{P}\big(z, t, \widetilde{B}\big) := \widetilde{\mathbb{P}}_z\big(Z_t \in \widetilde{B}\big), \quad z = (s, a) \in \overline{\mathscr{Z}}, \quad t \in \overline{\mathbb{R}}_+, \quad \widetilde{B} \in \widetilde{\mathcal{B}}(\overline{\mathscr{Z}}).$$

In view of (3.2) and (3.5) we have

$$\widetilde{P}\big(z, t, \widetilde{B}\big) = \widehat{\mathbb{P}}_{s,a}\Big(\widehat{\varphi}_{t+s} \in \widetilde{B}_{s+t}\Big) = \widehat{P}\big(s, a, s + t, \widetilde{B}_{s+t}\big). \tag{3.7}$$

Recall that the transition function $\widehat{P}$, defined in (3.1), is a Feller transition function. This, together with [6, Theorem 3.2], implies that $\widetilde{P}$ is also a Feller transition function. In light of the continuity of the sample paths of Z, and invoking [10, Section I.4, Theorem 7], we conclude that $\widetilde{\mathcal{M}}$ is a *time-homogeneous strong* Markov family.

For $\ell \in \mathbb{R}$, we define

$$\widetilde{\tau}_\ell^+ := \inf\big\{t \in \overline{\mathbb{R}}_+ : \, Z_t^2 \geq \ell\big\} \quad \text{and} \quad \widetilde{\tau}_\ell^- := \inf\big\{t \in \overline{\mathbb{R}}_+ : Z_t^2 \leq \ell\big\}.$$

Both $\widetilde{\tau}_\ell^+$ and $\widetilde{\tau}_\ell^-$ are $\widetilde{\mathbb{F}}$-stopping times since Z^2 has continuous sample paths and since $\widetilde{\mathbb{F}}$ is right-continuous. By Proposition 3.1, (3.5), and (3.6), for any $f \in L^\infty(\mathbb{R}_+)$, $(s,a) \in \mathscr{Z}$, and $\ell \in \mathbb{R}$,

$$\widetilde{\mathbb{E}}_{s,a}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^\pm}\Big)\Big) = \mathbb{E}\Big(f\big(\tau_\ell^\pm(s,a)\big)\Big), \tag{3.8}$$

which, together with (2.3), implies that

$$\widetilde{\mathbb{E}}_{s,a}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^\pm}\Big)\Big) = \widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}_{\ell-a}^\pm}\Big)\Big). \tag{3.9}$$

We conclude this section by presenting a couple of lemmas which will be needed in the sequel. The first one provides an important identity related to the strong Markov property of Z. It is a simple adaption of [4, Lemma 4.2], and the proof is therefore omitted here.

Lemma 3.2. *Let $\widetilde{\tau}$ is an $\widetilde{\mathbb{F}}$-stopping time and let $f \in L^\infty(\mathbb{R}_+)$. Then, for any $(s,a) \in \mathscr{Z}$ and $\ell \in \mathbb{R}$, we have*

$$\mathbb{1}_{\{\widetilde{\tau} \leq \widetilde{\tau}_\ell^\pm\}} \widetilde{\mathbb{E}}_{s,a}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^\pm}\Big)\Big|\widetilde{\mathscr{F}}_{\widetilde{\tau}}\Big) = \mathbb{1}_{\{\widetilde{\tau} \leq \widetilde{\tau}_\ell^\pm\}} \widetilde{\mathbb{E}}_{Z^1_{\widetilde{\tau}}, Z^2_{\widetilde{\tau}}}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^\pm}\Big)\Big),$$

where we clarify that $\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\tau}}, Z^2_{\widetilde{\tau}}}(\,\cdot\,)$ reads $\widetilde{\mathbb{E}}_{t,b}(\,\cdot\,)|_{(t,b)=(Z^1_{\widetilde{\tau}}, Z^2_{\widetilde{\tau}})}$.

The next lemma, the proof of which is deferred to Appendix A.4, provides the regularity of $\widetilde{\mathbb{E}}_{s,0}(f(\widetilde{\tau}_\ell^\pm))$ with respect to different variables.

Lemma 3.3. *Under Assumption 2.2, for any $f \in C_0(\mathbb{R}_+)$, we have*

(i) $\ell \mapsto \widetilde{\mathbb{E}}_{s,0}(f(Z^1_{\widetilde{\tau}_\ell^\pm}))$ is uniformly continuous on $\mathbb{R}_+$, uniformly in $s \in \mathbb{R}_+$;

(ii) for each $\ell \in \mathbb{R}_+$, $s \mapsto \widetilde{\mathbb{E}}_{s,0}(f(Z^1_{\widetilde{\tau}_\ell^\pm}))$ belongs to $C_0(\mathbb{R}_+)$.

3.2 The Feller Semigroup Property and Strong Generators of $(\mathcal{P}_\ell^\pm)_{\ell \in \mathbb{R}_+}$

In this section, we will investigate the Feller semigroup property (recall Definition 2.8) of $(\mathcal{P}_\ell^\pm)_{\ell \in \mathbb{R}_+}$ defined by (2.5). Moreover, we will characterize the strong generators of $(\mathcal{P}_\ell^\pm)_{\ell \in \mathbb{R}_+}$ on $C^{\text{ac}}_{\text{e,cdl}}(\mathbb{R}_+)$, which is a dense subset of their domains.

In view of (2.5) and (3.8) we can rewrite $(\mathcal{P}_\ell^\pm)_{\ell \in \mathbb{R}_+}$ in terms of the time-homogeneous Markov family $\widetilde{\mathcal{M}}$ as follows: for any $f \in L^\infty(\mathbb{R}_+)$,

$$\big(\mathcal{P}_\ell^\pm f\big)(s) = \widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^\pm}\Big)\Big), \quad s \in \mathbb{R}_+. \tag{3.10}$$

This representation will be conveniently used later, starting with the following proposition.

Proposition 3.4. *Under Assumption 2.2, $(\mathcal{P}_\ell^\pm)_{\ell \in \mathbb{R}_+}$ is a Feller semigroup.*

Proof. We will only verify the Feller semigroup property for $(\mathcal{P}_\ell^+)_{\ell\in\mathbb{R}_+}$, as the "−" case can be proved in an analogous way.

We first verify the semigroup property of $(\mathcal{P}_\ell^+)_{\ell\in\mathbb{R}_+}$. To this end, we fix any $f \in L^\infty(\mathbb{R}_+)$ and $s \in \mathbb{R}_+$. By (2.2) and (2.5), we have $(\mathcal{P}_0^+ f)(s) = \mathbb{E}(f(\tau_0^+(s))) = f(s)$. Next, for any $\ell \in \mathbb{R}_+$ and $h > 0$, by (3.10) and (A.15), we obtain that

$$\begin{aligned}(\mathcal{P}_{\ell+h}^+ f)(s) &= \widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell+h}}\Big)\Big) = \widetilde{\mathbb{E}}_{s,0}\bigg(\mathbb{1}_{\{\widetilde{\tau}^+_\ell<\infty\}}\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\tau}^+_\ell},0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_h}\Big)\Big)\bigg) = \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_\ell<\infty\}}\mathcal{P}_h^+ f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\\ &= \widetilde{\mathbb{E}}_{s,0}\Big(\mathcal{P}_h^+ f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) = \big(\mathcal{P}_\ell^+\mathcal{P}_h^+ f\big)(s),\end{aligned}$$

where we use our convention that $g(\infty) = 0$ for any $g \in L^\infty(\mathbb{R}_+)$. Hence, $(\mathcal{P}_\ell^+)_{\ell\in\mathbb{R}_+}$ is a semigroup on $L^\infty(\mathbb{R}_+)$. The positivity and contraction properties of $\mathcal{P}_\ell^+$ on $L^\infty(\mathbb{R}_+)$, for any $\ell \in \mathbb{R}_+$, follow immediately from its definition (2.5).

Finally, the Feller property of $(\mathcal{P}_\ell^+)_{\ell\in\mathbb{R}_+}$ follows immediately from Lemma 3.3 (ii) and from (3.10), while the strong continuity on $C_0(\mathbb{R}_+)$ is a direct consequence of Lemma 3.3 (i). The proof of the proposition is now complete. □

Let $(\Gamma^+, \mathscr{D}(\Gamma^+))$ (respectively, $(\Gamma^-, \mathscr{D}(\Gamma^-))$) be the strong generator of $(\mathcal{P}_\ell^+)_{\ell\in\mathbb{R}_+}$ (respectively, $(\mathcal{P}_\ell^-)_{\ell\in\mathbb{R}_+}$).[11] Since $(\mathcal{P}_\ell^\pm)_{\ell\in\mathbb{R}_+}$ are Feller semigroups it holds that $\mathscr{D}(\Gamma^\pm) \subset C_0(\mathbb{R}_+)$.

The next proposition provides an integral representation for $\Gamma^\pm$ on $C^{\mathrm{ac}}_{\mathrm{e,cdl}}(\mathbb{R}_+)$.

Proposition 3.5. *Under Assumptions 2.2 and 2.5, we have $C^{ac}_{e,cdl}(\mathbb{R}_+) \subset \mathscr{D}(\Gamma^\pm)$. Moreover, for any $f \in C^{ac}_{e,cdl}(\mathbb{R}_+)$,*

$$\big(\Gamma^\pm f\big)(s) = \int_0^\infty g_f(t)\gamma^\pm(s,t)\,dt, \quad s \in \mathbb{R}_+, \tag{3.11}$$

where the integral on the right-hand side is finite.

Proof. We will only present the proof for the "+" case, as the "−" case can be proved in an analogous way. For any $f \in C^{\mathrm{ac}}_{\mathrm{e,cdl}}(\mathbb{R}_+)$, using arguments similar to those in the proof of Lemma 2.6, we deduce that the integral on the right-hand side of (3.11) is finite for every $s \in \mathbb{R}_+$, and belongs to $C_e(\mathbb{R}_+)$.

In view of [7, Theorem 1.33] and Proposition 3.4, in order to prove $C^{\mathrm{ac}}_{\mathrm{e,cdl}}(\mathbb{R}_+) \subset \mathscr{D}(\Gamma^+)$ and (3.11), we only need to show that, for any $f \in C^{\mathrm{ac}}_{\mathrm{e,cdl}}(\mathbb{R}_+)$ and $s \in \mathbb{R}_+$,

$$\lim_{\ell\to 0+}\frac{1}{\ell}\big(\big(\mathcal{P}_\ell^+ f\big)(s) - f(s)\big) = \int_0^\infty g_f(t)\gamma^+(s,t)\,dt, \tag{3.12}$$

Towards this end, for any $\ell \in (0,1)$, an application of integration by parts yields

$$\begin{aligned}\frac{1}{\ell}\big(\big(\mathcal{P}_\ell^+ f\big)(s) - f(s)\big) &= \frac{1}{\ell}\Big(\mathbb{E}\big(f\big(\tau_\ell^+(s)\big)\big) - f(s)\Big) = \frac{1}{\ell}\bigg(-\int_s^\infty g_f(t)\,\mathbb{P}\big(\tau_\ell^+(s) \le t\big)\,dt + \int_s^\infty g_f(t)\,dt\bigg)\\ &= \int_s^\infty g_f(t)\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t\big)\,dt.\end{aligned}$$

[11] Although the strong and weak generators are the same for Feller semigroups, cf. [17, Theorem 2.1.3] , we use the convention of referring to a generator as the strong generator.

In light of Lemma A.1 and (A.7), there exists $b \in (0,\ell)$, such that

$$\begin{aligned}
\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t\big) &\le \frac{1}{\ell}\left(\Phi\left(\frac{\ell+\|v\|_\infty(t-s)}{\underline{\sigma}\sqrt{t-s}}\right) - e^{-2\|v\|_\infty \ell/\underline{\sigma}^2}\,\Phi\left(-\frac{\ell-\|v\|_\infty(t-s)}{\underline{\sigma}\sqrt{t-s}}\right)\right)\\
&= \frac{2\,e^{-(b+\|v\|_\infty(t-s))^2/(2\underline{\sigma}^2(t-s))}}{\underline{\sigma}\sqrt{2\pi(t-s)}} + \frac{2\|v\|_\infty}{\underline{\sigma}^2} e^{-2\|v\|_\infty b/\underline{\sigma}^2}\,\Phi\left(-\frac{\ell-\|v\|_\infty(t-s)}{\underline{\sigma}\sqrt{t-s}}\right)\\
&\le \frac{2}{\underline{\sigma}\sqrt{2\pi(t-s)}} + \frac{2\|v\|_\infty}{\underline{\sigma}^2}.
\end{aligned}$$

Therefore, (3.12) follows immediately from Proposition 2.4 (ii), the exponential decay of g_f, and the dominated convergence, which completes the proof of the proposition. □

3.3 Additional Property of $\Gamma^\pm$

In this section, we will establish an operator equation for $\Gamma^\pm$, which is crucial for the proof of our main result. Let $\mathscr{D}((\Gamma^+)^2)$ (respectively, $\mathscr{D}((\Gamma^-)^2)$) be the largest possible domain on which $\Gamma^+ \circ \Gamma^+$ (respectively, $\Gamma^- \circ \Gamma^-$) is well-defined, namely,

$$\mathscr{D}\big((\Gamma^+)^2\big) := \big\{f \in \mathscr{D}(\Gamma^+) : \Gamma^+ f \in \mathscr{D}(\Gamma^+)\big\} \quad \text{and} \quad \mathscr{D}\big((\Gamma^-)^2\big) := \big\{f \in \mathscr{D}(\Gamma^-) : \Gamma^- f \in \mathscr{D}(\Gamma^-)\big\}.$$

It is well known that both $\mathscr{D}((\Gamma^+)^2)$ and $\mathscr{D}((\Gamma^-)^2)$ are dense in $C_0(\mathbb{R}_+)$ (cf. [17, Chpater I, Theorem 2.7]).

Proposition 3.6. *Under Assumption 2.2, for any $f \in \mathscr{D}((\Gamma^\pm)^2)$, f is right-differentiable on $\mathbb{R}_+$ and left-differentiable on $(0,\infty)$. Moreover, we have*

$$\begin{aligned}
f'_+(s) \mp v(s)\big(\Gamma^\pm f\big)(s) + \frac{1}{2}\sigma^2(s)\big((\Gamma^\pm)^2 f\big)(s) = 0, \quad s \in \mathbb{R}_+,\\
f'_-(s) \mp v(s-)\big(\Gamma^\pm f\big)(s) + \frac{1}{2}\sigma^2(s-)\big((\Gamma^\pm)^2 f\big)(s) = 0, \quad s \in (0,\infty).
\end{aligned}$$

In particular, f is differentiable on $D(v,\sigma) := \{s \in \mathbb{R}_+ : v \text{ and } \sigma \text{ are continuous at } s\}$, and

$$f'(s) \mp v(s)\big(\Gamma^\pm f\big)(s) + \frac{1}{2}\sigma^2(s)\big((\Gamma^\pm)^2 f\big)(s) = 0, \quad s \in D(v,\sigma). \tag{3.13}$$

Proof. We will only provide the proof for the "+" case, as the "−" case can be proved in an analogous way. We fix $f \in \mathscr{D}((\Gamma^+)^2)$ for the rest of the proof.

We begin by observing that in order to prove the proposition it suffices to show that for any $\ell \in (0,\infty)$

$$\lim_{\delta\to 0+} \frac{1}{\delta}\Big(\big(\mathcal{P}_\ell^+ f\big)(s+\delta) - \big(\mathcal{P}_\ell^+ f\big)(s)\Big) = v(s)\big(\mathcal{P}_\ell^+ \Gamma^+ f\big)(s) - \frac{1}{2}\sigma^2(s)\big(\mathcal{P}_\ell^+ (\Gamma^+)^2 f\big)(s), \quad s \in \mathbb{R}_+. \tag{3.14}$$

To see this, we first note that for $f \in \mathscr{D}((\Gamma^+)^2)$ we have $\Gamma^+ f \in \mathscr{D}(\Gamma^+) \subset C_0(\mathbb{R}_+)$ and $(\Gamma^+)^2 f \in C_0(\mathbb{R}_+)$, and so, by the Feller property of $\mathcal{P}^+$, we have $\mathcal{P}_\ell^+\Gamma^+ f$, $\mathcal{P}_\ell^+(\Gamma^+)^2 f \in C_0(\mathbb{R}_+)$. This, together with Assumption 2.2, implies that the function on the right-hand side of (3.14) is càdlàg and bounded on $\mathbb{R}_+$. Therefore, by Lemma A.3, we obtain that

$$\big(\mathcal{P}_\ell^+ f\big)(s) - \big(\mathcal{P}_\ell^+ f\big)(0) = \int_0^s \Big(v(r)\big(\mathcal{P}_\ell^+\Gamma^+ f\big)(r) - \frac{1}{2}\sigma^2(r)\big(\mathcal{P}_\ell^+(\Gamma^+)^2 f\big)(r)\Big)\,dr, \quad s \in \mathbb{R}_+. \tag{3.15}$$

By letting $\ell \to 0+$ in (3.15) and using the strong continuity of $(\mathcal{P}^+_\ell)_{\ell\in\mathbb{R}_+}$, we deduce that

$$f(s) - f(0) = \int_0^s \Big(v(r)\big(\Gamma^+ f\big)(r) - \frac{1}{2}\sigma^2(r)\big((\Gamma^+)^2 f\big)(r)\Big) dr, \quad s \in \mathbb{R}_+.$$

The statement of proposition follows immediately from the above identity, Assumption 2.2, and a routine proof of elementary calculus.

It remains to prove (3.14). We will fix $\ell \in (0,\infty)$ and $s \in \mathbb{R}_+$ for the rest of the proof. To begin with, for any $\delta > 0$, by Lemma 3.2 and (3.10), we have

$$\begin{aligned}
\frac{1}{\delta}\Big(\big(\mathcal{P}^+_\ell f\big)(s+\delta) - \big(\mathcal{P}^+_\ell f\big)(s)\Big) &= \frac{1}{\delta}\bigg(\widetilde{\mathbb{E}}_{s+\delta,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - \widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\bigg) \\
&= \frac{1}{\delta}\bigg(\widetilde{\mathbb{E}}_{s+\delta,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_\ell \ge \delta\}} f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_\ell < \delta\}} f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\bigg) \\
&= \frac{1}{\delta}\bigg(\widetilde{\mathbb{E}}_{s+\delta,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_\ell \ge \delta\}} \widetilde{\mathbb{E}}_{Z^1_\delta, Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\Big) - \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_\ell < \delta\}} f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\bigg) \\
&= \frac{1}{\delta}\,\bigg(\widetilde{\mathbb{E}}_{s+\delta,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - \widetilde{\mathbb{E}}_{s,0}\Big(\widetilde{\mathbb{E}}_{s+\delta, Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\Big)\bigg) \\
&\quad + \frac{1}{\delta}\,\widetilde{\mathbb{E}}_{s,0}\bigg(\mathbb{1}_{\{\widetilde{\tau}^+_\ell < \delta\}}\bigg(\widetilde{\mathbb{E}}_{s+\delta, Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\bigg)\bigg).
\end{aligned} \tag{3.16}$$

For the second term above, it follows from (3.8), Lemma A.1 and (A.7) that, as $\delta \to 0+$,

$$\begin{aligned}
\frac{1}{\delta}\,\widetilde{\mathbb{E}}_{s,0}\bigg(\mathbb{1}_{\{\widetilde{\tau}^+_\ell < \delta\}}\bigg|\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\bigg|\bigg) &\le \frac{2\|f\|_\infty}{\delta}\,\widetilde{\mathbb{P}}_{s,0}\big(\widetilde{\tau}^+_\ell \le \delta\big) = \frac{2\|f\|_\infty}{\delta}\,\mathbb{P}\big(\tau^+_\ell(s) \le \delta + s\big) \\
\le \frac{2\|f\|_\infty}{\delta}\,\bigg(1 - \Phi\bigg(\frac{\ell - \|v\|_\infty \underline{\sigma}^{-2}\overline{\sigma}^2\delta}{\overline{\sigma}\sqrt{\delta}}\bigg) &+ e^{2\|v\|_\infty \ell/\underline{\sigma}^2}\,\Phi\bigg(-\,\frac{\ell + \|v\|_\infty \underline{\sigma}^{-2}\overline{\sigma}^2\delta}{\overline{\sigma}\sqrt{\delta}}\bigg)\bigg) \to 0.
\end{aligned} \tag{3.17}$$

In order to compute the limit of the first term in (3.16), as $\delta \to 0+$, ideally we wish to apply Itô's formula directly for the function

$$\psi(a) := \widetilde{\mathbb{E}}_{s+\delta,a}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) = \widetilde{\mathbb{E}}_{s+\delta,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell-a}}\Big)\Big) = \begin{cases} \big(\mathcal{P}^+_{\ell-a} f\big)(s+\delta), & a \le \ell, \\ f(s+\delta), & a > \ell, \end{cases}$$

where we have used (3.9) in the second equality, and (3.10) in the second equality. However, ψ may not be differentiable at $a = \ell$, so some alternative approach has to be sought. To this end, we define an auxiliary function $h : \mathscr{X} \to \mathbb{R}$ (recall $\mathscr{X} = \mathbb{R}_+ \times \mathbb{R}$) as

$$h(t,r) := \begin{cases} \big(\mathcal{P}^+_r \Gamma^+ f\big)(t), & (t,r) \in \mathbb{R}^2_+, \\ 2\big(\Gamma^+ f\big)(t) - \big(\mathcal{P}^+_{-r}\Gamma^+ f\big)(t), & (t,r) \in \mathbb{R}_+ \times (-\infty, 0). \end{cases} \tag{3.18}$$

Since $f \in \mathscr{D}((\Gamma^+)^2)$, $\Gamma^+ f \in \mathscr{D}(\Gamma^+) \subset C_0(\mathbb{R}_+)$, which, together with Proposition 3.4, implies that h is continuous on $\mathscr{X}$. The contraction property of $\mathcal{P}^+_\ell$, for any $\ell \in \mathbb{R}_+$, ensures that

$$\|h\|_\infty \le 3\big\|\Gamma^+ f\big\|_\infty. \tag{3.19}$$

Moreover, by [9, Chapter 1, Proposotion 1.5 (b)],

$$\frac{\partial}{\partial r} h(t,r) = \big(\mathcal{P}^+_{|r|}(\Gamma^+)^2 f\big)(t), \quad (t,r) \in \mathscr{X}. \tag{3.20}$$

The choice of $f \in \mathscr{D}((\Gamma^+)^2)$ (so that $(\Gamma^+)^2 f \in C_0(\mathbb{R}_+)$) together with Proposition 3.4 again implies that $\partial h/\partial r$ is continuous on $\mathscr{Z}$. Next, we define $H : \mathscr{Z} \to \mathbb{R}$ as

$$H(t,r) := f(t) + \int_0^r h(t,y)\,dy, \quad (t,r) \in \mathscr{Z}. \tag{3.21}$$

It follows immediately from (3.19) that

$$\big|H(t,r)\big| \le \|f\|_\infty + 3\big\|\Gamma^+ f\big\|_\infty |r|, \quad (t,r) \in \mathscr{Z}. \tag{3.22}$$

Note that for any $t \in \mathbb{R}_+$, $H(t,\cdot) \in C^2(\mathbb{R})$, and by (3.10) and (3.18), as well as by [9, Chapter 1, Proposition 1.5 (c)],

$$H(t,r) = \big(\mathcal{P}_r^+ f\big)(t) = \widetilde{\mathbb{E}}_{t,0}\Big(f\Big(Z^1_{\widetilde{\tau}_r^+}\Big)\Big), \quad (t,r) \in \mathbb{R}_+^2.$$

Together with (3.9), we obtain that

$$\begin{aligned}
\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big) &= \mathbb{1}_{\{Z^2_\delta \le \ell\}}\,\widetilde{\mathbb{E}}_{t,b}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big)\Big|_{(t,b)=(s+\delta,Z^2_\delta)} + \mathbb{1}_{\{Z^2_\delta > \ell\}}\,\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(g\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big) \\
&= \mathbb{1}_{\{Z^2_\delta \le \ell\}}\,\widetilde{\mathbb{E}}_{t,0}\Big(f\Big(Z^1_{\widetilde{\tau}_{\ell-b}^+}\Big)\Big)\Big|_{(t,b)=(s+\delta,Z^2_\delta)} + \mathbb{1}_{\{Z^2_\delta > \ell\}}\,\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(g\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big) \\
&= \mathbb{1}_{\{Z^2_\delta \le \ell\}}\Big(\mathcal{P}^+_{\ell-Z^2_\delta} f\Big)(s+\delta) + \mathbb{1}_{\{Z^2_\delta > \ell\}}\,\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big) \\
&= \mathbb{1}_{\{Z^2_\delta \le \ell\}} H\big(s+\delta,\ell-Z^2_\delta\big) + \mathbb{1}_{\{Z^2_\delta > \ell\}}\,\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big).
\end{aligned}$$

Hence, the first term in (3.16) can be further decomposed as

$$\begin{aligned}
&\frac{1}{\delta}\bigg(\widetilde{\mathbb{E}}_{s+\delta,0}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big) - \widetilde{\mathbb{E}}_{s,0}\Big(\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big)\Big)\bigg) \\
&\quad = \frac{1}{\delta}\bigg(H(s+\delta,\ell) - \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{Z^2_\delta \le \ell\}} H\big(s+\delta,\ell-Z^2_\delta\big) + \mathbb{1}_{\{Z^2_\delta > \ell\}}\,\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big)\Big)\bigg) \\
&\quad = \frac{1}{\delta}\,\widetilde{\mathbb{E}}_{s,0}\Big(H(s+\delta,\ell) - H\big(s+\delta,\ell-Z^2_\delta\big)\Big) \\
&\qquad + \frac{1}{\delta}\,\widetilde{\mathbb{E}}_{s,0}\bigg(\mathbb{1}_{\{Z^2_\delta > \ell\}}\Big(H\big(s+\delta,\ell-Z^2_\delta\big) - \widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big)\Big)\bigg)
\end{aligned} \tag{3.23}$$

For the first term in (3.23), by (3.1) and (3.7), we have

$$\begin{aligned}
\widetilde{\mathbb{E}}_{s,0}\Big(H(s+\delta,\ell) - H\big(s+\delta,\ell-Z^2_\delta\big)\Big) &= \widehat{\mathbb{E}}_{s,0}\Big(H(s+\delta,\ell) - H\big(s+\delta,\ell-\widehat{\varphi}_{s+\delta}\big)\Big) \\
&= \mathbb{E}\Big(H(s+\delta,\ell) - H\big(s+\delta,\ell-\varphi_{s+\delta}(s)\big)\Big).
\end{aligned}$$

Recalling $H(t,\cdot) \in C^2(\mathbb{R})$ for any $t \in \mathbb{R}_+$, by (2.1), (3.20), (3.21), and Itô's formula, we deduce that

$$\begin{aligned}
&\frac{1}{\delta}\,\widetilde{\mathbb{E}}_{s,0}\Big(H(s+\delta,\ell) - H\big(s+\delta,\ell-Z^2_\delta\big)\Big) \\
&\quad = \frac{1}{\delta}\,\mathbb{E}\bigg(\int_s^{s+\delta} h\big(s+\delta,\ell-\varphi_t(s)\big)v(t)\,dt - \frac{1}{2}\int_s^{s+\delta}\frac{\partial}{\partial r}h\big(s+\delta,\ell-\varphi_t(s)\big)\sigma^2(t)\,dt\bigg) \\
&\quad = \mathbb{E}\bigg(\frac{1}{\delta}\int_s^{s+\delta} h\big(s+\delta,\ell-\varphi_t(s)\big)v(t)\,dt - \frac{1}{2\delta}\int_s^{s+\delta}\Big(\mathcal{P}^+_{|\ell-\varphi_t(s)|}(\Gamma^+)^2 f\Big)(s+\delta)\sigma^2(t)\,dt\bigg).
\end{aligned}$$

Note that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\varphi_\cdot(s)(\omega)$ is continuous on $[s,\infty)$, and so there exists $\delta_0 = \delta_0(\omega) \in (0,1)$ such that $|\varphi_t(s)(\omega)| \le \ell/2$ for all $t \in [s, s+\delta_0]$. Using the (joint) continuity of h on $\mathscr{Z}$ (in particular, the uniform continuity of h on $[s,s+1]\times[\ell/2,3\ell/2]$, the continuity of sample paths of $\varphi(s)$, and the right-continuity of v, we obtain that, as $\delta \to 0+$,

$$\begin{aligned}&\left|\frac{1}{\delta}\int_s^{s+\delta} h\big(s+\delta, \ell-\varphi_t(s)(\omega)\big)v(t)\,dt - h(s,\ell)v(s)\right|\\ &\le \left|\frac{1}{\delta}\int_s^{s+\delta} h\big(s, \ell-\varphi_t(s)(\omega)\big)v(t)dt - h(s,\ell)v(s)\right| + \sup_{\substack{(t,r),(t',r')\in[s,s+1]\times[\ell/2,3\ell/2]\\ |t-t'|\le\delta,\,|r-r'|\le\delta}} \big|h(t,r)-h(t',r')\big|\|v\|_\infty \to 0.\end{aligned}$$

Similarly, noting that $(r,t)\mapsto(\mathcal{P}_r^+(\Gamma^+)^2 f)(t)$ is jointly continuous on $\mathbb{R}_+^2$ (since $(\mathcal{P}_\ell^+)_{\ell\in\mathbb{R}_+}$ is strongly continuous and $f \in \mathscr{D}((\Gamma^+)^2)$), we also have, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{\delta\to0+}\frac{1}{2\delta}\int_s^{s+\delta}\Big(\mathcal{P}^+_{|\ell-\varphi_t(s)(\omega)|}(\Gamma^+)^2 f\Big)(s+\delta)\sigma^2(t)\,dt = \frac{1}{2}\big(\mathcal{P}_\ell^+(\Gamma^+)^2 f\big)(s)\sigma^2(s).$$

Therefore, by (3.19) and the contraction property of $\mathcal{P}_\ell^+$, for any $\ell\in\mathbb{R}_+$, the dominated convergence theorem implies that

$$\begin{aligned}\lim_{\delta\to0+}\frac{1}{\delta}\,\widetilde{\mathbb{E}}_{s,0}\Big(H(s+\delta,\ell) - H\big(s+\delta,\ell-Z_\delta^2\big)\Big) &= v(s)h(s,\ell) - \frac{1}{2}\sigma^2(s)\big(\mathcal{P}_\ell^+(\Gamma^+)^2 f\big)(s)\\ &= v(s)\big(\mathcal{P}_\ell^+\Gamma^+ f\big)(s) - \frac{1}{2}\sigma^2(s)\big(\mathcal{P}_\ell^+(\Gamma^+)^2 f\big)(s), \quad (3.24)\end{aligned}$$

where the second equality follows from (3.18).

As for the second term in (3.23), by (3.1), (3.7), and (3.22), we first have

$$\begin{aligned}&\left|\widetilde{\mathbb{E}}_{s,0}\bigg(\mathbb{1}_{\{Z_\delta^2>\ell\}}\Big(H\big(s+\delta,\ell-Z_\delta^2\big) - \widetilde{\mathbb{E}}_{s+\delta,Z_\delta^2}\Big(f\Big(Z^1_{\widetilde{\tau}_\ell^+}\Big)\Big)\Big)\bigg)\right|\\ &\le \widetilde{\mathbb{E}}_{s,0}\bigg(\mathbb{1}_{\{Z_\delta^2>\ell\}}\Big(2\|f\|_\infty + 3\big\|\Gamma^+ f\big\|_\infty\big|\ell - Z_\delta^2\big|\Big)\bigg) = \widehat{\mathbb{E}}_{s,0}\bigg(\mathbb{1}_{\{\widehat{\varphi}_{s+\delta}>\ell\}}\Big(2\|f\|_\infty + 3\big\|\Gamma^+ f\big\|_\infty\big|\ell-\widehat{\varphi}_{s+\delta}\big|\Big)\bigg)\\ &= 2\|f\|_\infty\,\mathbb{P}\big(\varphi_{s+\delta}(s)>\ell\big) + 3\big\|\Gamma^+ f\big\|_\infty\mathbb{E}\Big(\mathbb{1}_{\{\varphi_{s+\delta}(s)>\ell\}}\big|\ell-\varphi_{s+\delta}(s)\big|\Big).\end{aligned}$$

Noting that, under $\mathbb{P}$, $\varphi_{s+\delta}(s)$ has a univariate normal distribution $N(\int_s^{s+\delta}v(r)dr, \int_s^{s+\delta}\sigma^2(r)dr)$, we deduce that

$$\mathbb{P}\big(\varphi_{s+\delta}(s)>\ell\big) \le 1 - \Phi\bigg(\frac{\ell - \|v\|_\infty\delta}{\overline{\sigma}\sqrt{\delta}}\bigg),$$

where Φ denotes the standard univariate normal distribution function, and that

$$\mathbb{E}\Big(\mathbb{1}_{\{\varphi_{s+\delta}(s)>\ell\}}\big|\ell-\varphi_{s+\delta}(s)\big|\Big) \le \int_\ell^\infty \frac{1}{\sqrt{2\pi\delta}\,\underline{\sigma}}\exp\left(-\frac{1}{2\overline{\sigma}^2\delta}\bigg(z - \int_s^{s+\delta}v(r)dr\bigg)^2\right)(z-\ell)dz.$$

Therefore, for any $\delta\in(0,\ell/\|v\|_\infty)$, we obtain that

$$\begin{aligned}&\frac{1}{\delta}\left|\widetilde{\mathbb{E}}_{s,0}\left(\mathbb{1}_{\{Z^2_\delta>\ell\}}\left(H\big(s+\delta,\ell-Z^2_\delta\big)-\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\right)\right)\right|\\&\leq\frac{2\|f\|_\infty}{\delta}\left(1-\Phi\left(\frac{\ell-\|v\|_\infty\delta}{\overline{\sigma}\sqrt{\delta}}\right)\right)+\frac{3\|\Gamma^+f\|_\infty}{\delta}\int_\ell^\infty\frac{1}{\sqrt{2\pi\delta}\,\underline{\sigma}}\exp\left(-\frac{\big(z-\|v\|_\infty\delta\big)^2}{2\overline{\sigma}^2\delta}\right)(z-\ell)dz\\&\leq\frac{2\|f\|_\infty}{\delta}\left(1-\Phi\left(\frac{\ell-\|v\|_\infty\delta}{\overline{\sigma}\sqrt{\delta}}\right)\right)+\frac{3\overline{\sigma}\|\Gamma^+f\|_\infty}{\underline{\sigma}\delta}\int_\ell^\infty\frac{1}{\sqrt{2\pi\delta}\,\overline{\sigma}}\exp\left(-\frac{\big(z-\|v\|_\infty\delta\big)^2}{2\overline{\sigma}^2\delta}\right)\big(\|v\|_\infty\delta-\ell\big)dz\\&\quad+\frac{3\|\Gamma^+f\|_\infty}{\delta}\int_\ell^\infty\frac{1}{\sqrt{2\pi\delta}\,\underline{\sigma}}\exp\left(-\frac{\big(z-\|v\|_\infty\delta\big)^2}{2\overline{\sigma}^2\delta}\right)\big(z-\|v\|_\infty\delta\big)dz\\&\leq\left(\frac{2\|f\|_\infty}{\delta}+\frac{3\overline{\sigma}\|\Gamma^+f\|_\infty}{\underline{\sigma}\delta}\right)\left(1-\Phi\left(\frac{\ell-\|v\|_\infty\delta}{\overline{\sigma}\sqrt{\delta}}\right)\right)+\frac{3\|\Gamma^+f\|_\infty\overline{\sigma}^2}{\sqrt{2\pi\delta}\underline{\sigma}}\exp\left(-\frac{\big(\ell-\|v\|_\infty\delta\big)^2}{2\overline{\sigma}^2\delta}\right)\to0,\end{aligned}\tag{3.25}$$

as $\delta\to0+$.

By combining (3.23), (3.24), and (3.25), we obtain that

$$\lim_{\delta\to0+}\frac{1}{\delta}\left(\widetilde{\mathbb{E}}_{s+\delta,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)-\widetilde{\mathbb{E}}_{s,0}\left(\widetilde{\mathbb{E}}_{s+\delta,Z^2_\delta}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\right)\right)=v(s)\big(\mathcal{P}^+_\ell\Gamma^+f\big)(s)-\frac{1}{2}\sigma^2(s)\big(\mathcal{P}^+_\ell(\Gamma^+)^2f\big)(s),$$

which, together with (3.17), implies (3.14). The proof of the proposition is complete. ☐

Remark 3.7. Proposition 3.6 can be regarded as an analogue of the so-called "noisy" Wiener-Hopf (NWH) factorization that was studied in e.g. [12], [14], and [18]. In order to describe the NWH consider the following Markov-modulated process on $(\Omega,\mathscr{F},\mathbb{P})$

$$\psi_t:=\int_0^t v\big(Y_s\big)\,ds+\int_0^t\sigma\big(Y_s\big)\,dW_s,\quad t\in\mathbb{R}_+,\tag{3.26}$$

where $Y:=(Y_t)_{t\in\mathbb{R}_+}$ is a continuous-time time-homogeneous Markov chain, independent of W, with finite state space $\mathbf{E}$ and (possibly submarkovian) generator matrix $\mathsf{\Lambda}$, and $v:\mathbf{E}\to\mathbb{R}$ and $\sigma:\mathbf{E}\to(0,\infty)$ are deterministic functions. Define the passage times

$$\zeta^+_\ell:=\inf\big\{t\in\mathbb{R}_+:\psi_t>\ell\big\}\quad\text{and}\quad\zeta^-_\ell:=\inf\big\{t\in\mathbb{R}_+:\psi_t<-\ell\big\},\quad\ell\in\mathbb{R}_+.\tag{3.27}$$

It was shown in [12], [14], and [18], respectively under different technical assumptions, that there exists a unique pair of $|\mathbf{E}|\times|\mathbf{E}|$ generator matrices $(\mathsf{Q}^+,\mathsf{Q}^-)$ such that

$$\mathsf{\Lambda}\mp\mathsf{V}\mathsf{Q}^\pm+\frac{1}{2}\mathsf{\Sigma}^2\big(\mathsf{Q}^\pm\big)^2=0,\tag{3.28}$$

where $\mathsf{V}:=\operatorname{diag}\{v(i),i\in\mathbf{E}\}$ and $\mathsf{\Sigma}:=\operatorname{diag}\{\sigma(i),i\in\mathbf{E}\}$. Moreover, $\mathsf{Q}^\pm$ admits the following probabilistic interpretation

$$e^{\ell\mathsf{Q}^\pm}(i,j)=\mathbb{P}\Big(Y_{\zeta^\pm_\ell}=j\,\Big|\,Y_0=i\Big),\quad i,j\in\mathbf{E},\quad\ell\in\mathbb{R}_+.$$

That is, $\mathsf{Q}^\pm$ is the generator matrix of the time-changed Markov chain $(Y_{\zeta^\pm_\ell})_{\ell\in\mathbb{R}_+}$.

Proposition 3.6 generalizes the above result in the following manner. We first replace the time-homogenous finite-state Markov chain in (3.26) with the time-homogeneous deterministic Markov process $Y_t \equiv s+t$, $t \in \mathbb{R}_+$, for some $s \in \mathbb{R}_+$. Clearly, the generator of Y is the first-order differential operator and its state space is $\mathbb{R}_+$. We still define the passage times $\zeta_\ell^\pm$ as in (3.27). Then for each $s \in \mathbb{R}_+$, the time-changed process $(Y_{\zeta_\ell^\pm})_{\ell\in\mathbb{R}_+}$ is given by $Y_{\zeta_\ell^\pm} = s + \zeta_\ell^\pm$, $\ell \in \mathbb{R}_+$, which has the same law as $\tau_\ell^\pm(s)$ (given as in (2.2)) under $\mathbb{P}$. Therefore, the equation (3.13) is analogous to (3.28) with Λ replaced by the first-order differential operator, and $\mathcal{Q}^\pm$ replaced by the generator of the time-changed process $(Y_{\zeta_\ell^\pm})_{\ell\in\mathbb{R}_+}$, which coincides with the generator $\Gamma^\pm$ of $(\tau_\ell^\pm)_{\ell\in\mathbb{R}_+}$.

4 Proof of Main Result

In this section, we will present the proof of our main result on the Wiener-Hopf factorization for the time-inhomogeneous diffusion process φ. Towards this end we first provide, in Section 4.1, the proof of Proposition 2.9. The proof of Theorem 2.11 is then presented in Section 4.2, followed by the proof of Corollary 2.14 that is given in Section 4.3.

4.1 Proof of Proposition 2.9

We begin with the proof of Proposition 2.9 (i). Our proof is based on the version of Hille-Yosida theorem as stated in [7, Theorem 1.30].

Proof of Proposition 2.9 (i). The proof is divided into the following two steps.

Step 1. In this step we will establish the *positive maximum principle* for Γ. Given our setup, Γ is said to satisfy the positive maximum principle if for any $f \in C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)$ and $s_0 \in \mathbb{R}_+$ with $f(s_0) = \sup_{s\in\mathbb{R}_+} f(s) \geq 0$, we have $(\Gamma f)(s_0) \leq 0$.

Throughout this step, we fix any $f \in C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)$ and $s_0 \in \mathbb{R}_+$ such that $f(s_0) = \sup_{s\in\mathbb{R}_+} f(s)$. Since g_f (recalling (2.4)) vanishes at infinity with exponential rate, so does f. Hence, it is necessary to have $f(s_0) \geq 0$. The proof is further divided into the following three steps.

Step 1.1. Assume first that there exist $J \in \mathbb{N}$ and $0 \leq s_0 < s_1 < \cdots < s_J < \infty$ such that $\text{supp}(f) \subset [0, s_J]$, that g_f is nonpositive on $[s_{j-1}, s_j)$ when j is odd, and that g_f is nonnegative on $[s_{j-1}, s_j)$ when j is even (if $J \geq 2$). We will show that, for any $j = 1, \ldots, J$,

$$\int_{s_0}^{s_j} g_f(r)\gamma(s_0, r)\, dr \leq 0. \tag{4.1}$$

In particular, we have $(\Gamma f)(s_0) = \int_{s_0}^{s_J} g_f(r)\gamma(s_0, r)dr \leq 0$.

To begin with, when $j = 1$, since γ is nonnegative and g_f is nonpositive on $[s_0, s_1)$, clearly we have $\int_{s_0}^{s_1} g_f(r)\gamma(s_0, r)\, dr \leq 0$. Moreover, when $J \geq 2$ and for $j = 2$, since $\gamma(s_0, \cdot)$ is nonnegative and non-increasing on (s_0, ∞), g_f is nonpositive (respectively, nonnegative) on $[s_0, s_1)$ (respectively, $[s_1, s_2)$), and since s_0 is a maximum point of f, we deduce that

$$\begin{aligned}\int_{s_0}^{s_2} g_f(r)\gamma(s_0, r)\, dr &= \int_{s_0}^{s_1} g_f(r)\gamma(s_0, r)\, dr + \int_{s_1}^{s_2} g_f(r)\gamma(s_0, r)\, dr \\ &\leq \gamma(s_0, s_1) \int_{s_0}^{s_2} g_f(r)\, dr = \gamma(s_0, s_1)\big(f(s_2) - f(s_0)\big) \leq 0. \end{aligned} \tag{4.2}$$

To proceed with the proof of (4.1) for $j = 3, \dots, J$ when $J \geq 3$, we will first prove by induction that, for any $j = 2, \dots, J$,[12]

$$\int_{s_0}^{s_j} g_f(r)\gamma(s_0, r)\, dr \leq \gamma\Big(s_0, s_{2\lfloor (j-1)/2\rfloor+1}\Big) \int_0^{s_j} g_f(r)\, dr, \tag{4.3}$$

The case when $j = 2$ has been verified in the first inequality of (4.2). Now assume that (4.3) holds for $j = 2, \dots, n$ for some $n \in \mathbb{N}$ with $n < J$. If n is odd so that g_f is nonnegative on $[s_n, s_{n+1})$, since $\gamma(s_0, \cdot)$ is nonnegative and non-increasing on (s_0, ∞), by the induction hypothesis we have

$$\int_{s_0}^{s_{n+1}} g_f(r)\gamma(s_0, r)\, dr \leq \gamma(s_0, s_n)\int_0^{s_n} g_f(r)\, dr + \gamma(s_0, s_n)\int_{s_n}^{s_{n+1}} g_f(r)\, dr = \gamma(s_0, s_n)\int_{s_0}^{s_{n+1}} g_f(r)\, dr.$$

Similarly, if n is even so that g_f is nonpositive on $[s_n, s_{n+1})$, we have

$$\int_{s_0}^{s_{n+1}} g_f(r)\gamma(s_0, r) dr \leq \gamma(s_0, s_{n-1})\int_0^{s_n} g_f(r) dr + \gamma(s_0, s_{n+1})\int_{s_n}^{s_{n+1}} g_f(r) dr \leq \gamma(s_0, s_{n+1})\int_0^{s_{n+1}} g_f(r) dr.$$

The proof of (4.3) for any $j = 2, \dots, J$ is complete by induction.

Returning to the proof of (4.1), for any $j = 3, \dots, J$, since γ is nonnegative and s_0 is a maximum point of f, by (4.3) we obtain that

$$\int_{s_0}^{s_j} g_f(r)\gamma(s_0, r)\, dr \leq \gamma\Big(s_0, s_{2\lfloor (j-1)/2\rfloor+1}\Big) \int_0^{s_j} g_f(r)\, dr = \gamma\Big(s_0, s_{2\lfloor (j-1)/2\rfloor+1}\Big)\big(f(s_j) - f(s_0)\big) \leq 0,$$

which completes the proof of (4.1) for any $j = 1, \dots, J$.

We conclude this step by noting that the arguments above do not depend on the values of f on $[0, s_0]$ nor on the càdlàg property of g_f.

Step 1.2. Next, we assume that $f \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$ has a compact support, i.e., there exists $T \in (s_0, \infty)$ such that $\mathrm{supp}(f) \subset [0, T]$. We will construct a sequence of Borel measurable functions $(h_n)_{n\in\mathbb{N}}$ on $[s_0, T]$ which satisfy the following properties:

(a) $\|h_n\|_\infty \leq \|g_f\|_\infty$, for any $n \in \mathbb{N}$;

(b) there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ of $(h_n)_{n\in\mathbb{N}}$ which converges to $g_f|_{[s_0,T]}$ Leb-a.e., where Leb denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$;

(c) for each $n \in \mathbb{N}$, $H_n(s_0) = \sup_{r\in[s_0,T]} H_n(r)$, where

$$H_n(t) := -\int_t^T h_n(r) dr, \quad t \in [s_0, T]; \tag{4.4}$$

(d) for each $n \in \mathbb{N}$, there are $J_n \in \mathbb{N}$ and $s_0 < s_1^n < \cdots < s_{J_n}^n = T$, such that h_n is non-positive on $[s_{j-1}^n, s_j^n)$ when j is odd, and non-negative when j is even (if $J_n \geq 2$).

Once such sequence is constructed, by applying the result of Step 1.1 to each h_n, we obtain that

$$\int_{s_0}^T h_n(r)\gamma(s_0, r)\, dr \leq 0, \quad n \in \mathbb{N}.$$

[12] Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Using properties (a) and (b) above as well as the boundedness of g_f, we deduce from dominated convergence that

$$\big(\Gamma f\big)(s_0) = \int_{s_0}^{T} g_f(r)\gamma(s_0,r)\,dr = \lim_{k\to\infty}\int_{s_0}^{T} h_{n_k}(r)\gamma(s_0,r)\,dr \le 0.$$

We begin the construction with introducing some notations. For any $n\in\mathbb{N}$, we set $A_n := \{r\in[s_0,T] : |g_f(r)| > 1/n\}$, which is clearly a nondecreasing sequence of Borel sets, and we denote its limit by $A := \lim_{n\to\infty} A_n = \cup_{n\in\mathbb{N}} A_n = \{r\in[s_0,T] : |g_f(r)| > 0\}$. We also define

$$g_n(t) := g_f(t)\mathbb{1}_{A_n}(t), \quad G_n(t) = -\int_t^T g_n(r)\,dr, \quad t\in[s_0,T], \quad n\in\mathbb{N}. \tag{4.5}$$

We first claim that for each $n\in\mathbb{N}$, there exist $K_n\in\mathbb{N}$ and $s_0 = t_0^n < t_1^n < \cdots < t_{K_n}^n = T$ such that g_n is either nonpositive or nonnegative on each subinterval $[t_{k-1}^n, t_k^n)$. The proof of this claim is done by contradiction. Assume that the claim is false, namely, that there exists some $N\in\mathbb{N}$, and for any finite partition of $[s_0,T)$, we can find two points x and y in at least one of the subintervals such that $g_N(x) > 0$ and $g_N(y) < 0$. From the definition of A_N, and since g_N is supported on A_N on which it coincides with g_f, we have $g_f(x) > 1/N$ and $g_f(y) < -1/N$. Now consider the sequence of uniform partitions of $[s_0,T]$ with mesh $(T-s_0)/2^m$, $m\in\mathbb{N}$. The previous discussion shows that, for any $m\in\mathbb{N}$, there exists a subinterval of the uniform partition, denoted by $[a_m,b_m)$, and $x_m, y_m, z_m \in [a_m,b_m)$ with $x_m < y_m < z_m$, such that

$$\big|g_f(x_m)\big| > \frac{1}{N},\ \ \big|g_f(y_m)\big| > \frac{1}{N},\ \ \big|g_f(z_m)\big| > \frac{1}{N},\ \ g_f(x_m)g_f(y_m) < 0,\ \ g_f(x_m)g_f(z_m) > 0. \tag{4.6}$$

Since both sequences of endpoints $(a_m)_{m\in\mathbb{N}}$ and $(b_m)_{m\in\mathbb{N}}$ are bounded with $b_m - a_m = (T-s_0)/2^m \to 0$, as $m\to\infty$, there exists a subsequence $(m_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ and $t\in[s_0,T]$ such that $a_{m_k}\to t$ and $b_{m_k}\to t$, as $k\to\infty$. It follows that $x_{m_k}\to t$, $y_{m_k}\to t$, and $z_{m_k}\to t$, as $k\to\infty$. Note that for each $k\in\mathbb{N}$, we must have either $x_{m_k} < y_{m_k} < t$ or $t \le y_{m_k} < z_{m_k}$. Hence, either there exists a further subsequence $(m_j')_{j\in\mathbb{N}}\subset(m_k)_{k\in\mathbb{N}}$ such that both $(x_{m_j'})_{j\in\mathbb{N}}$ and $(y_{m_j'})_{j\in\mathbb{N}}$ converge to t from below, or there exists another further subsequence $(m_j'')_{j\in\mathbb{N}}\subset(m_k)_{k\in\mathbb{N}}$ such that both $(y_{m_j''})_{j\in\mathbb{N}}$ and $(z_{m_j''})_{j\in\mathbb{N}}$ converge to t from above, as $k\to\infty$. In view of (4.6), we have found two sequences of real numbers both of which converge to t from above (or below), such that the values of g_f along these two sequences have different signs but with magnitudes greater than $1/N$. Therefore, in either case the corresponding two sequences of the values of g_f cannot have the same limit, which is clearly a contradiction to the càdlàg property of g_f at t.

Next, we will construct a Borel set $D_n\subset A_n$ for every $n\in\mathbb{N}$, such that the sequence of functions

$$h_n(t) := g_f(t)\mathbb{1}_{A_n\setminus D_n}(t) = g_n(t)\mathbb{1}_{A_n\setminus D_n}(t), \quad t\in[s_0,T], \quad n\in\mathbb{N}, \tag{4.7}$$

satisfies properties (c) and (d). Note that such sequence automatically satisfies property (a). Toward this end, we fix any $n\in\mathbb{N}$ in the following discussion. By (4.5), invoking the property of g_n described right below (4.5), we see that $\sup_{t\in[s_0,T]} G_n(t)$ must be achieved at at least one of the partition points $t_0^n = s_0, t_1^n, \ldots, t_{K_n-1}^n, t_{K_n}^n = T$, and we denote the smallest such point by $t_*^{(n)}$. By the definition of G_n in (4.5), we see that g_n must be nonnegative on the subinterval with right-end point t_*^n. Moreover, we further divide those subintervals $[t_{k-1}^n, t_k^n)$, on which g_n is nonnegative, into disjoint intervals, denoted by $I_1^n, \dots, I_{M_n}^n$ for some $M_n \in \mathbb{N}$, such that $\sup I_{m-1}^n \le \inf I_m^n$ and that

$$\mathrm{Leb}(I_m^n) \le 1/(n\|g_f\|_\infty). \tag{4.8}$$

Then, there exists $m_*^n \in \mathbb{Z}_+$ such that $\sup J_{m_*^n}^n = t_*^n$ (with convention $\sup \emptyset = s_0$), and we have[13]

$$\int_{\bigcup_{m=0}^{m_*^n} I_m^n} g_n(r)\, dr \ge \int_{s_0}^{t_*^n} g_n(r)\, dr = G_n(t_*^n) - G_n(s_0) = \sup_{t\in[s_0,T]} G_n(t) - G_n(s_0).$$

Also, we let $\underline{m}^n$ be the smallest nonnegative integer such that

$$\int_{\bigcup_{m=0}^{\underline{m}^n} I_m^n} g_n(r)\, dr \ge G_n(t_*^n) - G_n(s_0) = \sup_{t\in[s_0,T]} G_n(t) - G_n(s_0), \tag{4.9}$$

and define

$$D_n := \bigcup_{m=0}^{\underline{m}^n} I_m^n \cap A_n. \tag{4.10}$$

Together with the definitions of A_n and g_n, we have $g_n(t) > 1/n$ for any $t \in D_n$. Combining (4.4), (4.5), (4.7), (4.9), and (4.10), we deduce that

$$\begin{aligned} H_n(s_0) &= -\int_{s_0}^T h_n(r)\, dr = -\int_{A_n\setminus D_n} g_n(r)\, dr = -\int_{A_n} g_n(r)\, dr + \int_{D_n} g_n(r)\, dr \\ &= -\int_{s_0}^T g_n(r)\, dr + \int_{\bigcup_{m=0}^{\underline{m}^n} I_m^n} g_n(r)\, dr \ge G_n(s_0) + G_n(t_*^n) - G_n(s_0) = G_n(t_*^n). \end{aligned} \tag{4.11}$$

Noting that $\underline{m}^n \le m_*^n$, by letting $t_D^n := \sup D_n$, we have $t_D^n \le t_*^n$. Moreover, since $\cup_{m=0}^{\underline{m}^n} I_m^n$ contains all the points in $[s_0, t_D^n]$ at which g_n is positive, by (4.7) we see that h_n is nonpositive on $[s_0, t_D^n]$, and thus H_n, defined by (4.4), is non-increasing on $[s_0, t_D^n]$. In addition, it follows from (4.7) that $h_n(t) = g_n(t)$ for all $t \in (t_D^n, T]$, which, together with (4.4) and (4.5), implies that $H_n(t) = G_n(t)$ for all $t \in [t_D^n, T]$. Therefore, we obtain that

$$\sup_{t\in[s_0,T]} H_n(t) = \max\left\{ \sup_{t\in[s_0,t_D^n]} H_n(t),\ \sup_{t\in(t_D^n,T]} H_n(t) \right\} = \max\left\{H_n(s_0),\, G_n(t_*^n)\right\} = H_n(s_0), \tag{4.12}$$

where the last equality follows from (4.11). Hence, we have shown that $(h_n)_{n\in\mathbb{N}}$ defined as in (4.7) satisfies property (c). As for property (d), recalling that as shown above, for each $n \in \mathbb{N}$, there exist $K_n \in \mathbb{N}$ and and $s_0 = t_0^n < t_1^n < \cdots < t_{K_n}^n = T$ such that g_n is either nonpositive or nonnegative on each subinterval $[t_{k-1}^n, t_k^n)$. By the construction of D_n and h_n, we see that the same property holds for each h_n. That is, for each $n \in \mathbb{N}$, there exists $J_n \in \mathbb{N}$ and $s_0 = s_0^n < s_1^n < \cdots < s_{J_n}^n = T$ such that h_n is either nonpositive or nonnegative on each $[s_{j-1}^n, s_j^n)$. By merging all the consecutive subintervals on which h_n has the same sign, we can always assume that h_n has alternating signs on $[s_{j-1}^n, s_j^n)$, $j = 1, \dots, J_n$. Moreover, in view of (4.4) and (4.12), h_n must be nonpositive on $[s_0^n, s_1^n)$. Therefore, we obtain a finite partition $s_0 = s_0^n < s_1^n < \cdots < s_{J_n}^n = T$ such that h_n is nonpositive on $[s_{j-1}^n, s_j^n)$ when j is odd and nonnegative when j is even, which is indeed property (d).

[13] We let $I_0^n = \emptyset$.

It remains to show that the functions $(h_n)_{n\in\mathbb{N}}$, defined as in (4.7) (with D_n given by (4.10)), satisfy property (b). Toward this end, by (4.5), (4.8), (4.9), (4.10), and since g_n is nonnegative on each I^n_m, if $\underline{m}^n \geq 1$, we have

$$\int_{D_n} g_n(r)\,dr = \int_{\bigcup_{m=0}^{\underline{m}^n} I^n_m} g_n(r)\,dr \leq \int_{\bigcup_{i=0}^{\underline{m}^n-1} I^n_m} g_n(r)\,dr + \frac{1}{n} \leq \sup_{t\in[s_0,T]} G_n(t) - G_n(s_0) + \frac{1}{n}, \tag{4.13}$$

where the last inequality is due to the fact that $\underline{m}_n$ is the smallest nonnegative integer such that (4.9) holds true. When $\underline{m}^n = 0$, clearly $D_n = I^n_0 = \emptyset$ and (4.13) holds trivially. Moreover, recalling $f \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$ and $f(s_0) = \sup_{s\in\mathbb{R}_+} f(s)$, by (4.5) we have

$$\begin{aligned}
&\sup_{t\in[s_0,T]} G_n(t) - G_n(s_0) = \sup_{t\in[s_0,T]} G_n(t) - \sup_{t\in[s_0,T]} f(t) + f(s_0) - G_n(s_0) \leq 2 \sup_{t\in[s_0,T]} \big|G_n(t) - f(t)\big| \\
&\leq 2 \sup_{t\in[s_0,T]} \int_t^T \big|g_n(r) - g_f(r)\big|\,dr = 2 \sup_{t\in[s_0,T]} \int_{[t,T]\cap A} \big|g_f(r)\mathbb{1}_{A_n}(r) - g_f(r)\big|\,dr \leq 2\|g_f\|_\infty \,\mathrm{Leb}(A\setminus A_n).
\end{aligned}$$

Together with (4.13), we obtain that

$$\int_{D_n} g_n(r)\,dr \leq \frac{1}{n} + 2\|g_f\|_\infty\,\mathrm{Leb}(A\setminus A_n) \to 0, \quad \text{as } n \to 0.$$

Finally, since $g_n > 1/n$ on D_n, we deduce that $\lim_{n\to\infty}\mathrm{Leb}(D_n) = 0$, and together with (4.7), we conclude that, as $n\to\infty$,

$$\mathrm{Leb}\big(\{t\in[s_0,T] : g_f(r) \neq h_n(t)\}\big) = \mathrm{Leb}\big((A\setminus A_n)\cup D_n\big) \leq \mathrm{Leb}(A\setminus A_n) + \mathrm{Leb}(D_n) \to 0,$$

which clearly implies property (b).

Step 1.3. Finally, we consider any arbitrary $f \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$. For any $T\in(s_0,\infty)$, let $f_T(t) := \mathbb{1}_{[0,T]}(t)(f(t)-f(T))$. Clearly, $f_T \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$ with

$$\mathrm{supp}(f_T) \subset [0,T]; \quad f_T(s_0) = \sup_{t\in[s_0,T]} f_T(t), \quad f_T(t) = -\int_t^\infty g_f(r)\mathbb{1}_{[0,T)}(r)\,dr, \quad t\in\mathbb{R}_+,$$

where $g_f\mathbb{1}_{[0,T)}$ is càdlàg on $\mathbb{R}_+$. Applying the result of Step 1.2 to f_T, we obtain that

$$\begin{aligned}
\big(\Gamma f\big)(s_0) = \int_{s_0}^\infty g_f(r)\gamma(s_0,r)\,dr &= \int_{s_0}^T g_f(r)\gamma(s_0,r)\,dr + \int_T^\infty g_f(r)\gamma(s_0,r)\,dr \\
&\leq \int_T^\infty g_f(r)\gamma(s_0,r)\,dr \to 0, \quad \text{as } T\to\infty,
\end{aligned}$$

which completes the proof of the positive maximum principle for f.

Step 2. We now complete the proof of Proposition 2.9 (i). In view of Lemma 2.6, we have $\Gamma f \in C_0(\mathbb{R}_+)$ for all $f \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$, which is dense in $C_0(\mathbb{R}_+)$. Together with the positive maximum principle for Γ verified in Step 1 above as well as Assumption 2.7, the statement of Proposition 2.9 (i) is a direct consequence of the Hille-Yosida theorem (cf. [7, Theorem 1.30]). □

Let $(\mathcal{P}_\ell)_{\ell\in\mathbb{R}_+}$ be the Feller semigroup established in Proposition 2.9 (i). In view of Riesz representation theorem (cf. [20, Theorem 6.19]), for any $\ell, s \in \mathbb{R}_+$, there exists a measure $\mu_{\ell,s}$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$\big(\mathcal{P}_\ell f\big)(s) = \int_0^\infty f(t)\,\mu_{\ell,s}(dt), \quad f \in C_0(\mathbb{R}_+). \tag{4.14}$$

Since $\mathcal{P}_\ell$ is a contraction operator the measure $\mu_{\ell,s}$ is a sub-probability.

The proof of Proposition 2.9 (ii) requires the following technical lemma, the proof of which is deferred to Appendix A.6.

Lemma 4.1. *Under Assumptions 2.2, 2.5 and 2.7, for any $\varepsilon > 0$, there is a constant $c = c(\varepsilon, \|v\|_\infty, \overline{\sigma}, \underline{\sigma}) \in (0,\infty)$, such that for any $T \in (0,\infty)$ and $\ell \in \mathbb{R}_+$,*

$$\big\|\mathcal{P}_\ell \mathbb{1}_{[0,T]}\big\|_\infty \le 2e^{-(c/(2T\wedge 1))\varepsilon\ell}.$$

Proof of Proposition 2.9 (ii). Let $f \in C_e(\mathbb{R}_+)$, i.e., there exist $K, \kappa \in (0,\infty)$, such that $|f(t)| \le Ke^{-\kappa t}$ for all $t \in \mathbb{R}_+$. By (4.14), Lemma 4.1 (with $\varepsilon = 1$), and Fubini's theorem, there exists $c_1 = c_1(\|v\|_\infty, \overline{\sigma}, \underline{\sigma}) \in (0,\infty)$, such that for any $\ell, s \in \mathbb{R}_+$,

$$\begin{aligned}
\big|(\mathcal{P}_\ell f)(s)\big| &\le \int_0^\infty |f(t)|\,\mu_{\ell,s}(dt) \le K\int_0^\infty e^{-\kappa t}\,\mu_{\ell,s}(dt) = K\int_0^\infty\Big(\int_t^\infty \kappa e^{-\kappa r}\,dr\Big)\mu_{\ell,s}(dt)\\
&= K\kappa\int_0^\infty e^{-\kappa r}\Big(\int_0^r \mu_{\ell,s}(dt)\Big)dr \le K\kappa\int_0^\infty e^{-\kappa r}\Big(\int_0^\infty \widetilde{f}_r(t)\,\mu_{\ell,s}(dt)\Big)dr = K\kappa\int_0^\infty e^{-\kappa r}\big(\mathcal{P}_\ell\widetilde{f}_r\big)(s)\,dr\\
&\le K\kappa\int_0^\infty e^{-\kappa r}\big(\mathcal{P}_\ell\mathbb{1}_{[0,2r]}\big)(s)\,dr \le 2K\kappa\int_0^\infty e^{-\kappa r-(c_1/(4r\wedge 1))\ell}\,dr,
\end{aligned}$$

where $\widetilde{f}_r$ is some function in $C_0(\mathbb{R}_+)$ with $\mathbb{1}_{[0,r]} \le \widetilde{f}_r \le \mathbb{1}_{[0,2r]}$. Hence, by Fubini's theorem again, for any $L \ge c_1^2$, we obtain that

$$\begin{aligned}
\Big\|\int_L^\infty \mathcal{P}_\ell f\,d\ell\Big\|_\infty &\le \int_L^\infty 2K\kappa\Big(\int_0^\infty e^{-\kappa r-(c_1/(4r\wedge 1))\ell}\,dr\Big)d\ell \le \frac{2K\kappa}{c_1}\int_0^\infty \big((4r)\vee c_1\big)e^{-\kappa r-(c_1/(4r\wedge 1))L}\,dr\\
&\le \frac{2K\kappa}{c_1}\Big(\sqrt{L}\,e^{-c_1\sqrt{L}}\int_0^{\sqrt{L}/4} e^{-\kappa r}\,dr + 4\int_{\sqrt{L}/4}^\infty re^{-\kappa r}\,dr\Big)\\
&\le \frac{2K\kappa}{c_1}\Big(\frac{L}{4}\,e^{-c_1\sqrt{L}} + \frac{\sqrt{L}}{\kappa}\,e^{-\kappa\sqrt{L}/4} + \frac{4e^{-\kappa\sqrt{L}/4}}{\kappa^2}\Big),
\end{aligned}$$

which shows the convergence of $\int_0^\infty \mathcal{P}_\ell f\,d\ell$ in $L^\infty(\mathbb{R}_+)$. Moreover, by Proposition 2.9 and [9, Chapter 1, Proposition 1.5 (a)], for any $L \in (0,\infty)$, we have $\int_0^L \mathcal{P}_\ell f\,d\ell \in \mathscr{D}(\overline{\Gamma})$ and

$$\mathcal{P}_L f - f = \overline{\Gamma}\int_0^L \mathcal{P}_\ell f\,d\ell.$$

Hence, by (4.14) and Lemma 4.1, we obtain that, for any $T \in (0,\infty)$,

$$\begin{aligned}
\lim_{L\to\infty}\Big\|\overline{\Gamma}\int_0^L \mathcal{P}_\ell f\,d\ell - (-f)\Big\|_\infty &= \lim_{L\to\infty}\big\|\mathcal{P}_L f\big\|_\infty \le \lim_{L\to\infty}\sup_{s\in\mathbb{R}_+}\Big|\int_0^T f(t)\,\mu_{L,s}(dt)\Big| + \sup_{t\in[T,\infty)}|f(t)|\\
&\le 2\|f\|_\infty\lim_{L\to\infty} e^{-(c/(2T)\wedge 1)\varepsilon L} + \sup_{t\in[T,\infty)}|f(t)| = \sup_{t\in[T,\infty)}|f(t)|.
\end{aligned}$$

Since $T \in (0,\infty)$ is arbitrary and $f \in C_e(\mathbb{R}_+)$, by taking $T \to \infty$ on the right-hand side of the last equality above, we deduce that $\overline{\Gamma}\int_0^L \mathcal{P}_\ell f\,d\ell$ converges to $-f$ in $L^\infty(\mathbb{R}_+)$, as $L \to \infty$. Finally, since $\overline{\Gamma}$ is a closed operator, we conclude that $\int_{\mathbb{R}_+}\mathcal{P}_\ell f\,d\ell \in \mathscr{D}(\overline{\Gamma})$ and that $\overline{\Gamma}\int_{\mathbb{R}_+}\mathcal{P}_\ell f\,d\ell = -f$, which completes the proof of the proposition. ∎

4.2 Proof of Theorem 2.11

We are now ready to present the proof of our main result Theorem 2.11. We will fix any $(s,a)\in\mathscr{Z}$ throughout the proof, which will proceeds in the following two steps.

Step 1. In this step, we will prove (2.10) for any $u\in C(\mathbb{R})$, $h\in C_e(\mathbb{R}_+)$, and any $\mathbb{F}$-stopping time τ, under the following additional assumptions

(a) $u\in C_c(\mathbb{R})$ with $\operatorname{supp}(u)\subset[-M,M]$, for some $M\in(0,\infty)$;

(b) $h\in C_e(\mathbb{R}_+)$ is of the form $h=\Gamma f$, for some $f\in C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)$;

(c) there exists $T\in(0,\infty)$, such that $\tau\le T$ $\mathbb{P}$-a.s..

With the above choices of u and h, (2.9) is clearly satisfied. Also, since $f\in C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)\subset C_e(\mathbb{R}_+)$, we obtain from Lemma 2.6 that $\Gamma f\in C_e(\mathbb{R}_+)$, which, together with Proposition 2.9 (i) and [9, Chapter 1, Proposition 1.5 (a) & (c)], implies that, for any $L\in(0,\infty)$,

$$\int_0^L \mathcal{P}_\ell\,\Gamma f\,d\ell=\int_0^L \mathcal{P}_\ell\,\overline{\Gamma} f\,dy=\overline{\Gamma}\int_0^L \mathcal{P}_\ell f\,d\ell.$$

By Proposition 2.9 (ii) and the closedness of $\overline{\Gamma}$, we obtain that

$$\int_0^\infty \mathcal{P}_\ell\,\Gamma f\,d\ell=\lim_{L\to\infty}\int_0^L \mathcal{P}_\ell\,\Gamma f\,d\ell=\lim_{L\to\infty}\overline{\Gamma}\int_0^L \mathcal{P}_\ell f\,d\ell=\overline{\Gamma}\int_0^\infty \mathcal{P}_\ell f\,d\ell=-f.$$

Hence, under the additional assumptions (a)−(c) above, (2.10) can be written as

$$\begin{aligned}\mathbb{E}\bigg(\int_s^\tau u\big(\varphi_t(s,a)\big)\big(\Gamma f\big)(t)\,dt\bigg)&=-2\int_0^\infty u(a+\ell)\big(\mathcal{P}^+_\ell f\big)(s)\,d\ell-2\int_0^\infty u(a-\ell)\big(\mathcal{P}^-_\ell f\big)(s)\,d\ell\\&\quad+2\,\mathbb{E}\bigg(\int_0^\infty u\big(\varphi_\tau(s,a)+\ell\big)\big(\mathcal{P}^+_\ell f\big)(\tau)\,d\ell+\int_0^\infty u\big(\varphi_\tau(s,a)-\ell\big)\big(\mathcal{P}^-_\ell f\big)(\tau)\,d\ell\bigg).\end{aligned}\tag{4.15}$$

To proceed the proof of (4.15), we first introduce some auxiliary functions. To begin with, let

$$F(t,x):=2\int_0^\infty u(x+\ell)\big(\mathcal{P}^+_\ell f\big)(t)\,d\ell+2\int_0^\infty u(x-\ell)\big(\mathcal{P}^-_\ell f\big)(t)\,d\ell,\quad (t,x)\in\mathscr{Z}.\tag{4.16}$$

Also, for any $\varepsilon>0$, we define

$$f^+_\varepsilon(t):=\frac{1}{\varepsilon}\int_0^\varepsilon\big(\mathcal{P}^+_\ell f\big)(t)\,d\ell,\quad f^-_\varepsilon(t):=\frac{1}{\varepsilon}\int_0^\varepsilon\big(\mathcal{P}^-_\ell f\big)(t)\,d\ell,\quad t\in\mathbb{R}_+,\tag{4.17}$$

and

$$F_\varepsilon(t,x):=2\int_0^\infty u(x+\ell)\big(\mathcal{P}^+_\ell f^+_\varepsilon\big)(t)\,d\ell+2\int_0^\infty u(x-\ell)\big(\mathcal{P}^-_\ell f^-_\varepsilon\big)(t)\,d\ell,\quad (t,x)\in\mathscr{Z}.\tag{4.18}$$

Clearly, $\|f^\pm_\varepsilon\|_\infty\le\|f\|_\infty$, and so by the contraction property of $(\mathcal{P}^\pm_\ell)_{\ell\in\mathbb{R}_+}$ (see Proposition 3.4),

$$\big\|F_\varepsilon\big\|_\infty\le 4M\|u\|_\infty\big\|\mathcal{P}^+_\ell f^+_\varepsilon\big\|_\infty+4M\|u\|_\infty\big\|\mathcal{P}^-_\ell f^-_\varepsilon\big\|_\infty\le 8M\|u\|_\infty\|f\|_\infty.$$

Moreover, from the strong continuity of $(\mathcal{P}^{\pm}_{\ell})_{\ell\in\mathbb{R}_+}$ (see Proposition 3.4), we have

$$\big\|f^{\pm}_{\varepsilon}-f\big\|_{\infty}\leq\frac{1}{\varepsilon}\int_0^{\varepsilon}\big\|\mathcal{P}^{\pm}_{\ell}f-f\big\|_{\infty}d\ell\to 0,\quad\text{as }\ \varepsilon\to 0+,\tag{4.19}$$

and hence

$$\begin{aligned}\big\|F_{\varepsilon}-F\big\|_{\infty}&\leq 2\sup_{x\in\mathbb{R}}\int_0^{\infty}\big|u(x+\ell)\big|\big\|\mathcal{P}^{+}_{\ell}\big(f^{+}_{\varepsilon}-f\big)\big\|_{\infty}\,d\ell+2\sup_{x\in\mathbb{R}}\int_0^{\infty}\big|u(x-\ell)\big|\big\|\mathcal{P}^{-}_{\ell}\big(f^{-}_{\varepsilon}-f\big)\big\|_{\infty}\,d\ell\\&\leq 4M\|u\|_{\infty}\Big(\big\|f^{+}_{\varepsilon}-f\big\|_{\infty}+\big\|f^{-}_{\varepsilon}-f\big\|_{\infty}\Big)\to 0,\quad\text{as }\ \varepsilon\to 0+\,.\end{aligned}\tag{4.20}$$

In addition, since $f\in C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)\subset\mathscr{D}(\Gamma^{\pm})$, by (4.17) and [9, Chapter 1, Proposition 1.5 (a) & (c)], we have $f^{\pm}_{\varepsilon}\in\mathscr{D}(\Gamma^{\pm})$ and

$$\Gamma^{\pm}f^{\pm}_{\varepsilon}=\frac{1}{\varepsilon}\,\Gamma^{\pm}\!\int_0^{\varepsilon}\mathcal{P}^{\pm}_{\ell}f\,d\ell=\frac{1}{\varepsilon}\int_0^{\varepsilon}\mathcal{P}^{\pm}_{\ell}\Gamma^{\pm}f\,d\ell\in\mathscr{D}(\Gamma^{\pm}),\tag{4.21}$$

i.e., $f^{\pm}_{\varepsilon}\in\mathscr{D}((\Gamma^{\pm})^2)$. Hence, with similar reasoning as in (4.19), we obtain that, as $\varepsilon\to 0+$,

$$\big\|\Gamma^{\pm}f^{\pm}_{\varepsilon}-\Gamma^{\pm}f\big\|_{\infty}=\left\|\frac{1}{\varepsilon}\int_0^{\varepsilon}\mathcal{P}^{\pm}_{\ell}\Gamma^{\pm}f\,d\ell-\Gamma^{\pm}f\right\|_{\infty}\leq\frac{1}{\varepsilon}\int_0^{\varepsilon}\big\|\mathcal{P}^{\pm}_{\ell}\Gamma^{\pm}f-\Gamma^{\pm}f\big\|_{\infty}d\ell\to 0.\tag{4.22}$$

In addition to the assumptions (a)$-$(c) above, we first provide the proof of (4.15) when $u\in C^1_c(\mathbb{R})$. Such choice of u, together with (4.18), the boundedness of $\mathcal{P}^{\pm}_{\ell}f^{\pm}_{\varepsilon}$, and dominated convergence, ensures that $F_{\varepsilon}(t,\cdot)$ is differentiable on $\mathbb{R}$, for every $t\in\mathbb{R}_+$, and that

$$\frac{\partial}{\partial x}F_{\varepsilon}(t,x)=2\int_0^{\infty}u'(x+\ell)\big(\mathcal{P}^{+}_{\ell}f^{+}_{\varepsilon}\big)(t)\,d\ell+2\int_0^{\infty}u'(x-\ell)\big(\mathcal{P}^{-}_{\ell}f^{-}_{\varepsilon}\big)(t)\,d\ell,\quad(t,x)\in\mathscr{Z}.\tag{4.23}$$

By [9, Chapter 1, Proposition 1.5 (b)] and integration by parts, and noting that u is compactly supported, we deduce that, for any $(t,x)\in\mathscr{Z}$,

$$\begin{aligned}\frac{\partial}{\partial x}F_{\varepsilon}(t,x)&=2u(x)\big(f^{-}_{\varepsilon}(t)-f^{+}_{\varepsilon}(t)\big)-2\int_0^{\infty}u(x+\ell)\frac{d}{d\ell}\big(\mathcal{P}^{+}_{\ell}f^{+}_{\varepsilon}\big)(t)\,d\ell+2\int_0^{\infty}u(x-\ell)\frac{d}{d\ell}\big(\mathcal{P}^{-}_{\ell}f^{-}_{\varepsilon}\big)(t)\,d\ell\\&=2u(x)\big(f^{-}_{\varepsilon}(t)-f^{+}_{\varepsilon}(t)\big)-2\int_0^{\infty}u(x+\ell)\big(\mathcal{P}^{+}_{\ell}\,\Gamma^{+}f^{+}_{\varepsilon}\big)(t)\,d\ell+2\int_0^{\infty}u(x-\ell)\big(\mathcal{P}^{-}_{\ell}\,\Gamma^{-}f^{-}_{\varepsilon}\big)(t)\,d\ell.\end{aligned}\tag{4.24}$$

A similar argument shows that $\partial F(t,\cdot)/\partial x$ is differentiable on $\mathbb{R}$, for every $t\in\mathbb{R}_+$, and that

$$\begin{aligned}\frac{\partial^2}{\partial x^2}F_{\varepsilon}(t,x)&=2\int_0^{\infty}u(x+\ell)\big(\mathcal{P}^{+}_{\ell}(\Gamma^{+})^2f^{+}_{\varepsilon}\big)(t)\,d\ell+2\int_0^{\infty}u(x-\ell)\big(\mathcal{P}^{-}_{\ell}(\Gamma^{-})^2f^{-}_{\varepsilon}\big)(t)\,d\ell\\&\quad+2u(x)\Big(\big(\Gamma^{+}f^{+}_{\varepsilon}\big)(t)+\big(\Gamma^{-}f^{-}_{\varepsilon}\big)(t)\Big)+2u'(x)\big(f^{-}_{\varepsilon}(t)-f^{+}_{\varepsilon}(t)\big).\end{aligned}\tag{4.25}$$

The dominated convergence, together with the facts that $\mathcal{P}^{\pm}_{\ell}(\Gamma^{\pm})^2f^{\pm}_{\varepsilon},\Gamma^{\pm}f^{\pm}_{\varepsilon},f^{\pm}_{\varepsilon}\in C_0(\mathbb{R}_+)$ (since $f^{\pm}_{\varepsilon}\in\mathscr{D}((\Gamma^{\pm})^2)$ as shown by (4.21)) and that $u\in C^1_c(\mathbb{R})$, ensures that $\partial^2F/\partial x^2\in C(\mathscr{Z})$.

Moreover, since $f^{\pm}_{\varepsilon}\in\mathscr{D}((\Gamma^{\pm})^2)$ (so that $\Gamma^{\pm}f^{\pm}_{\varepsilon}\in\mathscr{D}(\Gamma^{\pm})$), we obtain from [9, Chapter 1, Proposition 1.5 (b)] that, for any $\ell\in\mathbb{R}_+$, $\mathcal{P}^{\pm}_{\ell}f^{\pm}_{\varepsilon}\in\mathscr{D}(\Gamma^{\pm})$ and $\Gamma^{\pm}\mathcal{P}^{\pm}_{\ell}f^{\pm}_{\varepsilon}=\mathcal{P}^{\pm}_{\ell}\Gamma^{\pm}f^{\pm}_{\varepsilon}\in\mathscr{D}(\Gamma^{\pm})$, i.e., $\mathcal{P}^{\pm}_{\ell}f^{\pm}_{\varepsilon}\in\mathscr{D}((\Gamma^{\pm})^2)$. It follows from Proposition 3.6 that $\mathcal{P}^{\pm}_{\ell}f^{\pm}_{\varepsilon}$ is right-differentiable on $\mathbb{R}_+$ and

$$\big(\mathcal{P}^{\pm}_{\ell}f^{\pm}_{\varepsilon}\big)'_{+}(t)=\pm v(t)\big(\mathcal{P}^{\pm}_{\ell}\,\Gamma^{\pm}f^{\pm}_{\varepsilon}\big)(t)-\frac{1}{2}\sigma^2(t)\big(\mathcal{P}^{\pm}_{\ell}(\Gamma^{\pm})^2f^{\pm}_{\varepsilon}\big)(t),\quad t\in\mathbb{R}_+,$$

where the right-hand side above is càdlàg and bounded in light of Assumption 2.2 and the fact that $\mathcal{P}^{\pm}_{\ell}\,\Gamma^{\pm}f^{\pm}_{\varepsilon}, \mathcal{P}^{\pm}_{\ell}(\Gamma^{\pm})^2 f^{\pm}_{\varepsilon} \in C_0(\mathbb{R}_+)$. This, together with (4.18) and the dominated convergence argument, implies that for any $x\in\mathbb{R}$, $F_\varepsilon(\cdot,x)$ is right-differentiable on $\mathbb{R}_+$ with

$$\begin{aligned}\frac{\partial_+}{\partial t}F_\varepsilon(t,x) &= 2\int_0^\infty u(x+\ell)\big(\mathcal{P}^+_\ell f^+_\varepsilon\big)'_+(t)\,d\ell + 2\int_0^\infty u(x-\ell)\big(\mathcal{P}^-_\ell f^-_\varepsilon\big)'_+(t)\,d\ell\\ &= 2\int_0^\infty u(x+\ell)\Big(v(t)\big(\mathcal{P}^+_\ell\,\Gamma^+ f^+_\varepsilon\big)(t) - \frac{1}{2}\sigma^2(t)\big(\mathcal{P}^+_\ell(\Gamma^+)^2 f^+_\varepsilon\big)(t)\Big)d\ell\\ &\quad - 2\int_0^\infty u(x-\ell)\Big(v(t)\big(\mathcal{P}^-_\ell\,\Gamma^- f^-_\varepsilon\big)(t) + \frac{1}{2}\sigma^2(t)\big(\mathcal{P}^-_\ell(\Gamma^-)^2 f^-_\varepsilon\big)(t)\Big)d\ell. \qquad (4.26)\end{aligned}$$

Since $\mathcal{P}^{\pm}_{\ell}\,\Gamma^{\pm}f^{\pm}_{\varepsilon}, \mathcal{P}^{\pm}_{\ell}(\Gamma^{\pm})^2 f^{\pm}_{\varepsilon} \in C_0(\mathbb{R}_+)$ and $u\in C_c(\mathbb{R})$, we obtain from Assumption 2.2 and the dominated convergence argument again, that $\partial_+F_\varepsilon(\cdot,x)/\partial t$ is càdlàg and bounded on $\mathbb{R}_+$, for any $x\in\mathbb{R}$. In view of Lemma A.3, this implies that $\partial_+F_\varepsilon(\cdot,x)/\partial t$ is absolutely continuous on $\mathbb{R}_+$, for any $x\in\mathbb{R}$. Hence, by integration by parts, for any $L\in(0,\infty)$ and any continuously differentiable test function ρ on $[0,L]$ with $\rho(0)=\rho(L)=0$, we obtain that

$$0 = F_\varepsilon(L,x)\rho(L) - F_\varepsilon(0,x)\rho(0) = \int_0^L \frac{\partial_+}{\partial t}F_\varepsilon(t,x)\rho(t)\,dt + \int_0^L F_\varepsilon(t,x)\rho'(t)\,dt, \quad x\in\mathbb{R},$$

that is, $\partial_+F_\varepsilon(\cdot,x)/\partial t$ is the generalized derivative of $F_\varepsilon(\cdot,x)$ on $[0,L]$ (cf. [15, Section 2.1, Definition 1]).

For any $n\in\mathbb{N}$, let $\tau_n(s,a) := \tau^+_{a+n}(s,a)\wedge\tau^-_{a-n}(s,a)$. By Itô formula with generalized derivatives (cf. [15, Section 2.10, Theorem 1]), and using (4.24), (4.25), and (4.26), we deduce that

$$\begin{aligned}&F_\varepsilon\big(\tau\wedge\tau_n(s,a), \varphi_{\tau\wedge\tau_n(s,a)}(s,a)\big) - F_\varepsilon(s,a)\\ &\quad= \int_s^{\tau\wedge\tau_n(s,a)}\Big(\frac{\partial_+}{\partial t}F_\varepsilon\big(t,\varphi_t(s,a)\big) + v(t)\frac{\partial}{\partial x}F_\varepsilon\big(t,\varphi_t(s,a)\big) + \frac{1}{2}\sigma^2(t)\frac{\partial^2}{\partial x^2}F_\varepsilon\big(t,\varphi_t(s,a)\big)\Big)dt\\ &\qquad+ \int_s^{\tau\wedge\tau_n(s,a)}\frac{\partial}{\partial x}F_\varepsilon\big(t,\varphi_t(s,a)\big)\sigma(t)\,dW_t\\ &\quad= \int_s^{\tau\wedge\tau_n(s,a)}\Big(u\big(\varphi_t(s,a)\big)\big(\big(\Gamma^+f^+_\varepsilon\big)(t) + \big(\Gamma^-f^-_\varepsilon\big)(t)\big) + u'\big(\varphi_t(s,a)\big)\big(f^-_\varepsilon(t) - f^+_\varepsilon(t)\big)\Big)\sigma^2(t)\,dt\\ &\qquad+ \int_s^{\tau\wedge\tau_n(s,a)}\frac{\partial}{\partial x}F_\varepsilon\big(t,\varphi_t(s,a)\big)\sigma(t)\,dW_t.\end{aligned}$$

In view of Assumption 2.2, (4.17), and (4.23), and recalling the contraction property of $\mathcal{P}^{\pm}_\ell$, $\ell\in\mathbb{R}_+$, and the fact that $u\in C^1_c(\mathbb{R})$, we deduce that $\|\sigma F_\varepsilon\|_\infty \le 8M\|u'\|_\infty\|f\|_\infty < \infty$. Hence, by taking expectation on both sides of the above equality, we obtain that

$$\begin{aligned}&\mathbb{E}\Big(F_\varepsilon\big(\tau\wedge\tau_n(s,a), \varphi_{\tau\wedge\tau_n(s,a)}(s,a)\big)\Big) - F_\varepsilon(s,a)\\ &\quad= \mathbb{E}\bigg(\int_s^{\tau\wedge\tau_n(s,a)}\Big(u\big(\varphi_t(s,a)\big)\big(\big(\Gamma^+f^+_\varepsilon\big)(t)+\big(\Gamma^-f^-_\varepsilon\big)(t)\big)+u'\big(\varphi_t(s,a)\big)\big(f^-_\varepsilon(t)-f^+_\varepsilon(t)\big)\Big)\sigma^2(t)\,dt\bigg). \qquad (4.27)\end{aligned}$$

Moreover, by (2.3), Lemma A.1, and (A.7), we have

$$\mathbb{P}\big(\tau_n(s,a)\le T\big) \le \mathbb{P}\big(\tau^+_n(s)\le T\big) + \mathbb{P}\big(\tau^-_{-n}(s)\le T\big) \le 2\left(1-\mathbb{P}\left(\sup_{r\in[0,\overline{\sigma}^2(T-s)]}\left(\frac{\|v\|_\infty r}{\underline{\sigma}^2}+W_r\right)<n\right)\right)\to 0,$$

as $n \to \infty$. Recalling that $\tau \le T$ $\mathbb{P}$-a.s., this implies that $\tau \wedge \tau_n(s,a) \to \tau$ in probability, as $n \to \infty$. Hence, by dominated convergence and using the boundedness of F_ε, $f_\varepsilon^\pm$, $\Gamma^\pm f_\varepsilon^\pm$, u, u', τ, and σ^2, we can take $n \to \infty$ on both sides of (4.27) to deduce that

$$\begin{aligned}&\mathbb{E}\big(F_\varepsilon\big(\tau,\varphi_\tau(s,a)\big)\big) - F_\varepsilon(s,a)\\ &\quad = \mathbb{E}\bigg(\int_s^\tau \Big(u\big(\varphi_t(s,a)\big)\big(\big(\Gamma^+ f_\varepsilon^+\big)(t) + \big(\Gamma^- f_\varepsilon^-\big)(t)\big) + u'\big(\varphi_t(s,a)\big)\big(f_\varepsilon^-(t) - f_\varepsilon^+(t)\big)\Big)\sigma^2(t)\,dt\bigg).\end{aligned}$$

Finally, by taking $\varepsilon \to 0+$ on both sides of the above equality, and using (4.19), (4.20), (4.22), as well as dominated convergence, we obtain that

$$\mathbb{E}\big(F\big(\tau,\varphi_\tau(s,a)\big)\big) - F(s,a) = \mathbb{E}\bigg(\int_s^\tau u\big(\varphi_t(s,a)\big)\Big(\big(\Gamma^+ f\big)(t) + \big(\Gamma^- f\big)(t)\Big)\sigma^2(t)\,dt\bigg),$$

which is indeed (4.15) in light of (4.16).

As for the validity of (4.15) for $u \in C_c(\mathbb{R})$, we note that $C_c^1(\mathbb{R})$ is dense in $C_c(\mathbb{R})$ with the sup-norm. Hence, there exist $(u_n)_{n\in\mathbb{N}} \subset C_c^1(\mathbb{R})$ such that $\|u_n - u\|_\infty \to 0$, as $n \to \infty$, and (4.15) holds true for each u_n. Therefore, the validity of (4.15) for $u \in C_c(\mathbb{R})$ follows from dominated convergence and the uniform boundedness of $(\|u_n\|_\infty)_{n\in\mathbb{N}}$.

Step 2. Next, we will prove (2.10) for any $h \in C_e(\mathbb{R}_+)$ and any $\mathbb{F}$-stopping time τ, while u is still assumed to satisfy condition (a) in Step 1. The exponential decay of h and the boundedness of u ensure that (2.9) is satisfied in this case.

To this end, we first observe from Proposition 2.9 (ii) that $\int_0^\infty (\mathcal{P}_\ell h) d\ell \in \mathscr{D}(\overline{\Gamma})$. By Proposition 2.9 (i), the graph of $\overline{\Gamma}$ is the closure of that of Γ. Hence, there exist $(f_n)_{n\in\mathbb{N}} \subset C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)$, so that

$$\lim_{n\to\infty} \bigg\| f_n - \int_0^\infty \mathcal{P}_\ell h\,d\ell \bigg\|_\infty = 0 \quad \text{and} \quad \lim_{n\to\infty} \bigg\| \Gamma f_n - \overline{\Gamma} \int_0^\infty \mathcal{P}_\ell h\,d\ell \bigg\|_\infty = 0.$$

It follows from (2.8) that

$$\lim_{n\to\infty} \big\|\Gamma f_n + h\big\|_\infty = 0 \tag{4.28}$$

By the contraction property of $\mathcal{P}_\ell^\pm$, $\ell \in \mathbb{R}_+$, we also have

$$\lim_{n\to\infty} \sup_{\ell\in\mathbb{R}_+} \bigg\| \mathcal{P}_\ell^\pm f_n - \mathcal{P}_\ell^\pm \int_0^\infty \mathcal{P}_y h\,dy \big\|_\infty \le \lim_{n\to\infty} \bigg\| f_n - \int_0^\infty \mathcal{P}_y h\,dy \bigg\|_\infty = 0. \tag{4.29}$$

By (4.15), for any $n \in \mathbb{N}$ and $T \in \mathbb{R}_+$, with $\tau_T := \tau \wedge T$, we have

$$\begin{aligned}\mathbb{E}\bigg(\int_s^{\tau_T} u\big(\varphi_t(s,a)\big)\big(\Gamma f_n\big)(t)\sigma^2(t)\,dt\bigg) &= -2\int_0^\infty u(a+\ell)\big(\mathcal{P}_\ell^+ f_n\big)(s)\,d\ell - 2\int_0^\infty u(a-\ell)\big(\mathcal{P}_\ell^- f_n\big)(s)\,d\ell\\ &\quad + 2\,\mathbb{E}\bigg(\int_0^\infty u\big(\varphi_{\tau_T}(s,a)+\ell\big)\big(\mathcal{P}_\ell^+ f_n\big)(\tau_T)\,d\ell + \int_0^\infty u\big(\varphi_{\tau_T}(s,a)-\ell\big)\big(\mathcal{P}_\ell^- f_n\big)(\tau_T)\,d\ell\bigg).\end{aligned}$$

Letting $n \to \infty$ in the above equality and, by (4.28), (4.29), the assumption that $u \in C_c^1(\mathbb{R}_+)$, and the dominated convergence, we obtain that

$$\begin{aligned}
\mathbb{E}\bigg(\int_s^{\tau_T} u\big(\varphi_t(s,a)\big) h(t)\sigma^2(t)\,dt\bigg) &= -\lim_{n\to\infty} \mathbb{E}\bigg(\int_s^{\tau_T} u\big(\varphi_t(s,a)\big)\big(\Gamma f_n\big)(t)\sigma^2(t)\,dt\bigg)\\
&= 2\lim_{n\to\infty}\int_0^\infty u(a+\ell)\big(\mathcal{P}_\ell^+ f_n\big)(s)\,d\ell + 2\lim_{n\to\infty}\int_0^\infty u(a-\ell)\big(\mathcal{P}_\ell^- f_n\big)(s)\,d\ell\\
&\quad - 2\lim_{n\to\infty}\mathbb{E}\bigg(\int_0^\infty u\big(\varphi_{\tau_T}(s,a)+\ell\big)\big(\mathcal{P}_\ell^+ f_n\big)(\tau_T)\,d\ell + \int_0^\infty u\big(\varphi_{\tau_T}(s,a)-\ell\big)\big(\mathcal{P}_\ell^- f_n\big)(\tau_T)\,d\ell\bigg)\\
&= 2\int_0^\infty u(a+\ell)\bigg(\mathcal{P}_\ell^+\int_{\mathbb{R}_+}\mathcal{P}_y h\,dy\bigg)(s)\,d\ell + 2\int_0^\infty u(a-\ell)\bigg(\mathcal{P}_\ell^-\int_{\mathbb{R}_+}\mathcal{P}_y h\,dy\bigg)(s)\,d\ell\\
&\quad - 2\,\mathbb{E}\bigg(\int_0^\infty u\big(\varphi_{\tau_T}(s,a)+\ell\big)\bigg(\mathcal{P}_\ell^+\int_0^\infty\mathcal{P}_y h\,dy\bigg)(\tau_T)\,d\ell + \int_0^\infty u\big(\varphi_{\tau_T}(s,a)-\ell\big)\bigg(\mathcal{P}_\ell^-\int_0^\infty\mathcal{P}_y h\,dy\bigg)(\tau_T)\,d\ell\bigg).
\end{aligned}$$

Moreover, since $\int_{\mathbb{R}_+}\mathcal{P}_y h\,dy \in \mathscr{D}(\overline{\Gamma})$, for any $\ell\in\mathbb{R}_+$, we have $\mathcal{P}_\ell^+\int_{\mathbb{R}_+}\mathcal{P}_y h\,dy \in C_0(\mathbb{R}_+)$ and $(\mathcal{P}_\ell^+\int_{\mathbb{R}_+}\mathcal{P}_y h\,dy)(\infty)=0$. With the help of the continuity of sample paths of $\varphi(s,a)$, the contraction property of $(\mathcal{P}_\ell^\pm)_{\ell\in\mathbb{R}_+}$, the assumption that $u\in C_c^1(\mathbb{R})$, (2.9), and dominated convergence, we conclude by taking $T\to\infty$ in the above equality that

$$\begin{aligned}
\mathbb{E}\bigg(\int_s^\tau u\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg) &= 2\int_0^\infty u(a+\ell)\bigg(\mathcal{P}_\ell^+\int_0^\infty\mathcal{P}_y h\,dy\bigg)(s)\,d\ell + 2\int_0^\infty u(a-\ell)\bigg(\mathcal{P}_\ell^-\int_0^\infty\mathcal{P}_y h\,dy\bigg)(s)\,d\ell\\
-2\,\mathbb{E}\bigg(\mathbb{1}_{\{\tau<\infty\}}\bigg(\int_0^\infty u\big(\varphi_\tau(s,a)+\ell\big)&\bigg(\mathcal{P}_\ell^+\int_0^\infty\mathcal{P}_y h\,dy\bigg)(\tau)d\ell + \int_0^\infty u\big(\varphi_\tau(s,a)-\ell\big)\bigg(\mathcal{P}_\ell^-\int_0^\infty\mathcal{P}_y h\,dy\bigg)(\tau)d\ell\bigg)\bigg).
\end{aligned}$$

Step 3. Finally, we will complete the proof of (2.10) for any $\mathbb{F}$-stopping time τ, $h\in C_e(\mathbb{R}_+)$, and $u\in C(\mathbb{R})$, which satisfy the condition (2.9). Without loss of generality, we assume that both u and h are nonnegative. For general u and h it is sufficient to take $u=u^+-u^-$, and $h=h^+-h^-$ and the result follows from the linearity of integral and the operators $(\mathcal{P}_\ell)_{\ell\in\mathbb{R}_+}$ and $(\mathcal{P}_\ell^\pm)_{\ell\in\mathbb{R}_+}$.

Let now $(u_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of nonnegative functions in $C_c(\mathbb{R})$ such that, for any $n\in\mathbb{N}$, $u_n(x)=u(x)$ for all $x\in[-n,n]$, and that $\operatorname{supp}(u_n)\subset[-n-1,n+1]$. From the result in Step 2, for every $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\mathbb{E}\bigg(\int_s^\tau u_n\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg)\\
&\quad = 2\int_0^\infty u_n(a+\ell)\bigg(\mathcal{P}_\ell^+\int_0^\infty\mathcal{P}_y h\,dy\bigg)(s)\,d\ell + 2\int_0^\infty u_n(a-\ell)\bigg(\mathcal{P}_\ell^-\int_0^\infty\mathcal{P}_y h\,dy\bigg)(s)\,d\ell\\
&\qquad - 2\,\mathbb{E}\bigg(\mathbb{1}_{\{\tau<\infty\}}\int_0^\infty u_n\big(\varphi_\tau(s,a)+\ell\big)\bigg(\mathcal{P}_\ell^+\int_0^\infty\mathcal{P}_y h\,dy\bigg)(\tau)\,d\ell\bigg)\\
&\qquad - 2\,\mathbb{E}\bigg(\mathbb{1}_{\{\tau<\infty\}}\int_0^\infty u_n\big(\varphi_\tau(s,a)-\ell\big)\bigg(\mathcal{P}_\ell^-\int_0^\infty\mathcal{P}_y h\,dy\bigg)(\tau)\,d\ell\bigg).
\end{aligned} \tag{4.30}$$

In particular, for $\tau\equiv\infty$, we have

$$\begin{aligned}
&\mathbb{E}\bigg(\int_s^\infty u_n\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg)\\
&\quad = 2\int_0^\infty u_n(a+\ell)\bigg(\mathcal{P}_\ell^+\int_0^\infty\mathcal{P}_y h\,dy\bigg)(s)\,d\ell + 2\int_0^\infty u_n(a-\ell)\bigg(\mathcal{P}_\ell^-\int_0^\infty\mathcal{P}_y h\,dy\bigg)(s)\,d\ell.
\end{aligned} \tag{4.31}$$

By (2.9) and the monotone convergence, we deduce that

$$\lim_{n\to\infty}\mathbb{E}\bigg(\int_s^\tau u_n\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg)=\mathbb{E}\bigg(\int_s^\tau u\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg)<\infty,$$
$$\lim_{n\to\infty}\mathbb{E}\bigg(\int_s^\infty u_n\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg)=\mathbb{E}\bigg(\int_s^\infty u\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg)<\infty.$$

Together with (4.30) and (4.31), we obtain that

$$\lim_{n\to\infty}\mathbb{E}\bigg(\mathbb{1}_{\{\tau<\infty\}}\int_0^\infty\bigg(u_n\big(\varphi_\tau(s,a)+\ell\big)\bigg(\mathcal{P}_\ell^+\int_0^\infty\mathcal{P}_y h\,dy\bigg)(\tau)+u_n\big(\varphi_\tau(s,a)-\ell\big)\bigg(\mathcal{P}_\ell^-\int_0^\infty\mathcal{P}_y h\,dy\bigg)(\tau)\bigg)\,d\ell\bigg)$$
$$=\lim_{n\to\infty}\mathbb{E}\bigg(\int_s^\infty u_n\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg)-\lim_{n\to\infty}\mathbb{E}\bigg(\int_s^\tau u_n\big(\varphi_t(s,a)\big)h(t)\sigma^2(t)\,dt\bigg)<\infty.$$

Therefore, we can pass the limit, as $n\to\infty$, for each term on either side of (4.30), which leads to (2.10) by monotone convergence. The proof of Theorem 2.11 is now complete.

4.3 Proof of Corollary 2.14

In this section, we will present a proof of Corollary 2.14. We start with the following technical lemma, the proof of which is deferred to Appendix A.7.

Lemma 4.2. *Under the setting of Corollary 2.14, for any $k,\ell\in\mathbb{R}_+$, $\mathcal{P}_k^-$ and $\mathcal{P}_\ell^+$ commute. In particular, $(\mathcal{P}_\ell^+\mathcal{P}_\ell^-)_{\ell\in\mathbb{R}_+}$ is a Feller semigroup, and $\mathcal{P}_\ell^+\mathcal{P}_\ell^-=\mathcal{P}_\ell$ on $L^\infty(\mathbb{R}_+)$, for any $\ell\in\mathbb{R}_+$, where $(\mathcal{P}_\ell)_{\ell\in\mathbb{R}_+}$ is given as in Proposition 2.9 (i). Moreover, for any $f\in C^{ac}_{e,cdl}(\mathbb{R}_+)$ and $\ell\in\mathbb{R}_+$, we have $\mathcal{P}_\ell^\pm f\in C^{ac}_{e,cdl}(\mathbb{R}_+)$, and*

$$\Gamma^+\mathcal{P}_\ell^-f=\mathcal{P}_\ell^-\Gamma^+f,\quad \Gamma^-\mathcal{P}_\ell^+f=\mathcal{P}_\ell^+\Gamma^-f.$$

Proof of Corollary 2.14. We will only present the proof of (2.20) in the case when $u\in C_c^1(\mathbb{R})$. The result for general $u\in C(\mathbb{R})$ satisfying (2.19) follows from an approximation argument similar to those in the proof of Theorem 2.11 (see the last paragraph in Step 1, and Step 3 therein). In what follows, we fix any $c\in(0,\infty)$, $a\in\mathbb{R}$, and we set $h(t)=e^{-ct}$, $t\in\mathbb{R}_+$.

To begin with, by Proposition 2.9 (ii) and the contraction property of $\mathcal{P}_\ell^\pm$, $\ell\in\mathbb{R}_+$ (since $(\mathcal{P}_\ell^\pm)_{\ell\in\mathbb{R}_+}$ is a Feller semigroup shown as in Proposition 3.4), we see that both $\int_0^L\mathcal{P}_y h dy$ and $\mathcal{P}_\ell^\pm\int_0^L\mathcal{P}_y h dy$ converge in $L^\infty(\mathbb{R}_+)$, as $L\to\infty$. It follows from [9, Chapter 1, Lemma 1.4 (b)], Proposition 2.9 (ii), and Lemma 4.2 that, for any $\ell\in\mathbb{R}_+$,

$$\mathcal{P}_\ell^+\int_0^\infty\mathcal{P}_y h\,dy=\mathcal{P}_\ell^+\lim_{L\to\infty}\int_0^L\mathcal{P}_y h\,dy=\lim_{L\to\infty}\mathcal{P}_\ell^+\int_0^L\mathcal{P}_y h\,dy=\lim_{L\to\infty}\int_0^L\mathcal{P}_\ell^+\mathcal{P}_y^+\mathcal{P}_y^-h\,dy=\int_0^\infty\mathcal{P}_{\ell+y}^+\mathcal{P}_y^-h\,dy,$$

and similarly,

$$\mathcal{P}_\ell^-\int_0^\infty\mathcal{P}_y h\,dy=\int_0^\infty\mathcal{P}_y^+\mathcal{P}_{\ell+y}^-h\,dy.$$

Hence, by Corollary 2.12 and Fubini's theorem, we have

$$
\begin{aligned}
\big(\mathcal{E}_c u\big)(a) &= c\sigma^{-2}\,\mathbb{E}\bigg(\int_0^\infty u\big(\varphi_t(0,a)\big)h(t)\sigma^2\,dt\bigg)\\
&= 2c\sigma^{-2}\int_0^\infty u(a+\ell)\bigg(\mathcal{P}_\ell^+\int_0^\infty \mathcal{P}_y h\,dy\bigg)(0)\,d\ell + 2c\sigma^{-2}\int_0^\infty u(a-\ell)\bigg(\mathcal{P}_\ell^-\int_0^\infty \mathcal{P}_y h\,dy\bigg)(0)\,d\ell\\
&= 2c\sigma^{-2}\int_0^\infty u(a+\ell)\bigg(\int_0^\infty \mathcal{P}_{\ell+y}^+\mathcal{P}_y^- h\,dy\bigg)(0)\,d\ell + 2c\sigma^{-2}\int_0^\infty u(a-\ell)\bigg(\int_0^\infty \mathcal{P}_y^+\mathcal{P}_{\ell+y}^- h\,dy\bigg)(0)\,d\ell\\
&= 2c\sigma^{-2}\sigma^{-2}\int_0^\infty\int_y^\infty u(a+x-y)\big(\mathcal{P}_x^+\mathcal{P}_y^- h\big)(0)\,dx\,dy + 2c\sigma^{-2}\int_0^\infty\int_0^y u(a+x-y)\big(\mathcal{P}_x^+\mathcal{P}_y^- h\big)(0)\,dx\,dy\\
&= 2c\sigma^{-2}\int_0^\infty\int_0^\infty u(a+x-y)\big(\mathcal{P}_x^+\mathcal{P}_y^- h\big)(0)\,dx\,dy. \qquad (4.32)
\end{aligned}
$$

Due to the time-homogeneity of $\varphi(a)$, we see that, for any $\ell\in\mathbb{R}$ and $s\in\mathbb{R}_+$, $\tau_\ell^\pm(s)-s$ has the identical law as $\tau_\ell^\pm(0)$ under $\mathbb{P}$, and so

$$
\big(\mathcal{P}_\ell^\pm h\big)(s) = \mathbb{E}\big(e^{-c\tau_\ell^\pm(s)}\big) = \mathbb{E}\big(e^{-c(s+\tau_\ell^\pm(0))}\big) = e^{-cs}\,\mathbb{E}\big(e^{-c\tau_\ell^\pm(0)}\big) = h(s)\big(\mathcal{P}_\ell^\pm h\big)(0), \quad \ell,s\in\mathbb{R}_+.
$$

It follows that, for any $x,y\in\mathbb{R}_+$,

$$
\big(\mathcal{P}_x^+\mathcal{P}_y^- h\big)(0) = \big(\mathcal{P}_x^+ h\big)(0)\cdot\big(\mathcal{P}_y^- h\big)(0), \qquad (4.33)
$$

and that (noting that $h(t)=e^{-ct}\in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)\subset\mathscr{D}(\Gamma^\pm)$)

$$
\big(\Gamma^\pm h\big)(s) = \lim_{\ell\to 0+}\frac{1}{\ell}\big(\mathcal{P}_\ell^\pm h - h\big)(s) = h(s)\cdot\lim_{\ell\to 0+}\frac{1}{\ell}\big(\mathcal{P}_\ell^\pm h - h\big)(0) = h(s)\big(\Gamma^\pm h\big)(0). \qquad (4.34)
$$

Combining (4.34) with [9, Chapter 1, Proposition 1.5 (b)] leads to, for any $\ell\in\mathbb{R}_+$,

$$
\frac{\partial}{\partial\ell}\big(\mathcal{P}_\ell^\pm h\big)(s) = \big(\mathcal{P}_\ell^\pm\Gamma^\pm h\big)(s) = \big(\Gamma^\pm h\big)(0)\cdot\big(\mathcal{P}_\ell^\pm h\big)(s), \quad s\in\mathbb{R}_+, \qquad (4.35)
$$

and thus

$$
\big(\mathcal{P}_\ell^\pm h\big)(s) = h(s)\,e^{\ell(\Gamma^\pm h)(0)}, \quad s\in\mathbb{R}_+. \qquad (4.36)
$$

By combining (4.32), (4.33), and (4.36), we obtain that

$$
\big(\mathcal{E}_c u\big)(a) = 2\int_0^\infty\int_0^\infty u(a+x-y)\,e^{x(\Gamma^+h)(0)+y(\Gamma^-h)(0)}\,dx\,dy. \qquad (4.37)
$$

Next, we will investigate the expression $\mathcal{E}_c^+\mathcal{E}_c^- u$ on the right-hand side of (2.20). For any $t\in\mathbb{R}_+$, let $\overline{F}_t$ be the distribution function of $\overline{\varphi}_t(0,0)$ under $\mathbb{P}$. Using integration by parts, the assumption that $u\in C_c^1(\mathbb{R})$, and Fubini's theorem, we have

$$
\begin{aligned}
\big(\mathcal{E}_c^+ u\big)(a) &= c\,\mathbb{E}\bigg(\int_0^\infty u\big(\overline{\varphi}_t(0,a)\big)h(t)\,dt\bigg) = c\int_0^\infty\bigg(\int_0^\infty u(a+\ell)\,d\overline{F}_t(\ell)\bigg)h(t)\,dt\\
&= -c\int_0^\infty\bigg(\int_0^\infty u'(a+\ell)\mathbb{P}\big(\overline{\varphi}_t(0,0)\le\ell\big)\,d\ell\bigg)h(t)\,dt\\
&= -c\int_0^\infty u'(a+\ell)\bigg(\int_0^\infty \mathbb{P}\big(\tau_\ell^+(0)>t\big)h(t)\,dt\bigg)d\ell = -c\int_0^\infty u'(a+\ell)\mathbb{E}\bigg(\int_0^{\tau_\ell^+(0)} h(t)\,dt\bigg)d\ell.
\end{aligned}
$$

Recalling $h(t)=e^{-ct}$, it follows from integration by parts, the assumption that $u\in C_c^1(\mathbb{R})$, (4.35), and (4.36) that

$$\begin{aligned}\big(\mathcal{E}_c^+u\big)(a) &= \int_0^\infty u'(a+\ell)\Big(\mathbb{E}\big(e^{-c\tau_\ell^+(0)}\big)-1\Big)\,d\ell = \int_0^\infty u'(a+\ell)\big(\big(\mathcal{P}_\ell^+h\big)(0)-1\big)\,d\ell \\ &= -\big(\Gamma^+h\big)(0)\cdot\int_0^\infty u(a+\ell)\big(\mathcal{P}_\ell^+h\big)(0)\,d\ell = -\big(\Gamma^+h\big)(0)\cdot\int_0^\infty u(a+\ell)\,e^{\ell(\Gamma^+h)(0)}\,d\ell. \end{aligned} \tag{4.38}$$

A similar argument as above shows that

$$\big(\mathcal{E}_c^-u\big)(a) = -\big(\Gamma^-h\big)(0)\cdot\int_0^\infty u(a-\ell)\,e^{\ell(\Gamma^-h)(0)}\,d\ell. \tag{4.39}$$

Therefore, by combining (4.38) and (4.39), we deduce that

$$\big(\mathcal{E}_c^-\mathcal{E}_c^+u\big)(a) = \big(\Gamma^+h\big)(0)\big(\Gamma^-h\big)(0)\cdot\int_0^\infty\int_0^\infty u(a+x-y)\,e^{x(\Gamma^+h)(0)+y(\Gamma^-h)(0)}\,dx\,dy.$$

Finally, by solving $(\Gamma^\pm h)(0)$ from (3.13) (with $v(t)\equiv v$ and $\sigma(t)\equiv\sigma$), and noting that $(\Gamma^\pm h)(0)\le 0$ in light of (3.11), we obtain that

$$\big(\Gamma^\pm h\big)(0) = \pm\frac{v}{\sigma^2}-\sqrt{\frac{v^2}{\sigma^4}+\frac{2c}{\sigma^2}},$$

and thus

$$\big(\mathcal{E}_c^-\mathcal{E}_c^+u\big)(a) = 2c\sigma^{-2}\int_0^\infty\int_0^\infty u(a+x-y)\,e^{x(\Gamma^+h)(0)+y(\Gamma^-h)(0)}\,dx\,dy. \tag{4.40}$$

Combining (4.37) and (4.40) completes the proof of corollary. □

5 Example

In this section, we will present a nontrivial example of functions v and σ in (2.1), for which our Assumptions 2.2, 2.5, and 2.7 are all satisfied. More precisely, for some $v_0,v_1\in\mathbb{R}$ and $t_0,\sigma_0,\sigma_1\in(0,\infty)$, we consider

$$v(t)=\begin{cases} v_0, & t\in[0,t_0)\\ v_1, & t\in[t_0,\infty)\end{cases},\qquad \sigma(t)=\begin{cases} \sigma_0, & t\in[0,t_0)\\ \sigma_1, & t\in[t_0,\infty)\end{cases}, \tag{5.1}$$

which clearly satisfy Assumption 2.2. In what follows, we will verify that such v and σ also satisfy Assumptions 2.5 and 2.7. By induction, the result can then be extended to any càdlàg piecewise constant functions v and σ with finitely many jumps, with the proof omitted here to avoid extra technicalities. Note that by refining the partition we can always assume that v and σ share the same set of jump times (with possibly zero-size jumps).

Step 1. We first show that the functions v and σ, given as in (5.1), satisfy Assumption 2.5. We will only present the proof for the continuity of γ^+, since the continuity for γ^- can be verified in an analogous way.

To begin with, for any $\ell \in \mathbb{R}_+$, by (A.7) we first have

$$\mathbb{P}\big(\tau_\ell^+(s) > t\big) = \Phi\bigg(\frac{\ell - v_0(t-s)}{\sigma_0\sqrt{t-s}}\bigg) - e^{2v_0\ell/\sigma_0^2}\Phi\bigg(-\frac{\ell + v_0(t-s)}{\sigma_0\sqrt{t-s}}\bigg), \quad 0 \le s < t \le t_0,$$

and

$$\mathbb{P}\big(\tau_\ell^+(s) > t\big) = \Phi\bigg(\frac{\ell - v_1(t-s)}{\sigma_1\sqrt{t-s}}\bigg) - e^{2v_1\ell/\sigma_1^2}\Phi\bigg(-\frac{\ell + v_1(t-s)}{\sigma_1\sqrt{t-s}}\bigg), \quad 0 < t_0 \le s < t. \tag{5.2}$$

When $0 \le s < t_0 < t$, the Markov property of φ implies that

$$\begin{aligned}\mathbb{P}\big(\tau_\ell^+(s) > t\big) &= \mathbb{P}\big(\tau_\ell^+(s) > t,\, \tau_\ell^+(s) > t_0\big) = \mathbb{E}\Big(\mathbb{1}_{\{\tau_\ell^+(s)>t_0\}}\mathbb{E}\Big(\mathbb{1}_{\{\tau_\ell^+(s)>t\}}\,\Big|\,\mathscr{F}_{t_0}\Big)\Big)\\ &= \mathbb{E}\Big(\mathbb{1}_{\{\tau_\ell^+(s)>t_0\}}\mathbb{E}\Big(\mathbb{1}_{\{\tau_\ell^+(s)>t\}}\,\Big|\,\varphi_{t_0}(s)\Big)\Big) = \mathbb{E}\Big(\mathbb{1}_{\{\tau_\ell^+(s)>t_0\}}\,G\big(\varphi_{t_0}(s); t_0, t\big)\Big),\end{aligned}$$

where $G(y; t_0, t) := \mathbb{P}(\tau_{\ell-y}^+(t_0) > t)$, $y \in \mathbb{R}$. It follows from (A.10) and a standard simple function approximation procedure that, for any $0 \le s < t_0 < t$,

$$\mathbb{P}\big(\tau_\ell^+(s) > t\big) = \int_{-\infty}^{\ell} \frac{\mathbb{P}\big(\tau_{\ell-y}^+(t_0) > t\big)}{\sigma_0\sqrt{2\pi(t_0-s)}}\left(\exp\bigg(-\frac{\big(y - v_0(t_0-s)\big)^2}{2\sigma_0^2(t_0-s)}\bigg) - \exp\bigg(\frac{2v_0\ell}{\sigma_0^2} - \frac{\big(2\ell - y + v_0(t_0-s)\big)^2}{2\sigma_0^2(t_0-s)}\bigg)\right) dy.$$

Hence, we conclude that $\mathbb{P}(\tau_\ell^+(s) > t)$ is differentiable with respect to ℓ on $\mathbb{R}_+$ (with right-differentiability at $\ell = 0$). By Proposition 2.4 (ii), we deduce that

$$\gamma^+(s,t) = \begin{cases} \dfrac{e^{-v_0^2(t-s)/(2\sigma_0^2)}}{\sigma_0\sqrt{2\pi(t-s)}} - \dfrac{2v_0}{\sigma_0^2}\Phi\bigg(-\dfrac{v_0\sqrt{t-s}}{\sigma_0}\bigg), & 0 \le s < t \le t_0,\\[2ex] \displaystyle\int_{-\infty}^{0} \frac{-2y}{\sigma_0^3\sqrt{2\pi(t_0-s)^3}} \exp\bigg(-\frac{\big(y - v_0(t_0-s)\big)^2}{2\sigma_0^2(t_0-s)}\bigg) & \\ \quad\cdot\bigg(\Phi\bigg(\dfrac{-y - v_1(t-t_0)}{\sigma_1\sqrt{t-t_0}}\bigg) - e^{-2v_1y/\sigma_1^2}\,\Phi\bigg(-\dfrac{-y + v_1(t-t_0)}{\sigma_1\sqrt{t-t_0}}\bigg)\bigg)\,dy, & 0 \le s < t_0 < t,\\[2ex] \dfrac{e^{-v_1^2(t-s)/(2\sigma_1^2)}}{\sigma_1\sqrt{2\pi(t-s)}} - \dfrac{2v_1}{\sigma_1^2}\Phi\bigg(-\dfrac{v_1\sqrt{t-s}}{\sigma_1}\bigg), & t_0 \le s < t. \end{cases} \tag{5.3}$$

We now verify the continuity of $\gamma^+(\cdot, t)$ on $[0, t)$, for any $t \in (0, \infty)$. When $t \le t_0$, it is clear from the first equality in (5.3) that $\gamma^+(\cdot, t)$ is continuous on $[0, t)$. When $t > t_0$, the second and the third equalities in (5.3) ensure that $\gamma^+(\cdot, t)$ is continuous on $[0, t_0) \cup (t_0, t)$, and that $\gamma^+(\cdot, t)$ is right-continuous at t_0. It remains to show the left-continuity of $\gamma^+(\cdot, t)$ at t_0. To this end, in view of (5.2) and the second equality in (5.3), for any $0 \le s < t_0 < t$ we have

$$\begin{aligned}\gamma^+(s,t) &= \int_{-\infty}^{0} \frac{-2y}{\sigma_0^3\sqrt{2\pi(t_0-s)^3}} \exp\bigg(-\frac{\big(y - v_0(t_0-s)\big)^2}{2\sigma_0^2(t_0-s)}\bigg)\,\mathbb{P}\big(\tau_{-y}^+(t_0) > t\big)\,dy\\ &= \int_{-\infty}^{-v_0\sqrt{t_0-s}/\sigma_0} \frac{2\ell(s,z)}{\sigma_0^2\sqrt{2\pi}(t_0-s)}\mathbb{P}\big(\tau_{\ell(s,z)}^+(t_0) > t\big)e^{-z^2/2}\,dz,\end{aligned}$$

where $\ell(s,z) := -\sigma_0\sqrt{t_0-s}z - v_0(t_0-s)$. It follows from Proposition 2.4 (i) & (iii) and dominated convergence that

$$\lim_{s\to t_0-}\gamma^+(s,t) = \int_{-\infty}^0 \bigg(\lim_{s\to t_0-}\frac{2\ell^2(s,z)}{\sigma_0^2\sqrt{2\pi}(t_0-s)}\cdot\frac{1}{\ell(s,z)}\mathbb{P}\Big(\tau^+_{\ell(s,z)}(t_0)>t\Big)\bigg)e^{-z^2/2}\,dz$$
$$= \gamma^+(t_0,t)\int_{-\infty}^0 \frac{2z^2}{\sqrt{2\pi}}\,e^{-z^2/2}\,dz = \gamma^+(t_0,t).$$

Step 2. Next, we show that the functions v and σ in (5.1) satisfy Assumption 2.7 with $\lambda = 1$. In this case, using (5.3) and the analogous formula for γ^-, and recalling that $\gamma = \gamma^+ + \gamma^-$, we have

$$\gamma(s,t) = \begin{cases} \dfrac{2\sqrt{2}}{\sqrt{\pi\sigma_0^2(t-s)}}\exp\bigg(-\dfrac{v_0^2(t-s)}{2\sigma_0^2}\bigg), & 0\le s<t\le t_0, \\[2ex] \dfrac{2\sqrt{2}}{\sqrt{\pi\sigma_1^2(t-s)}}\exp\bigg(-\dfrac{v_1^2(t-s)}{2\sigma_1^2}\bigg), & t_0\le s<t. \end{cases} \tag{5.4}$$

The expression of $\gamma(s,t)$ when $0\le s<t_0<t$ is omitted here as it is not needed for the rest of the proof. Since $C_c^1(\mathbb{R}_+)$ is dense in $C_0(\mathbb{R}_+)$, in order to show that $\{(I-\Gamma)f : f\in C^{\rm ac}_{e,{\rm cdl}}(\mathbb{R}_+)\}$ is dense in $C_0(\mathbb{R}_+)$, it is sufficient to verify that this set is dense in $C_c^1(\mathbb{R}_+)$. In what follows, for any fixed $\varepsilon>0$ and $h\in C_c^1(\mathbb{R}_+)$, with ${\rm supp}(h)\subset[0,T)$ for some $T\in(0,\infty)$, we will construct $f^\varepsilon\in C^{\rm ac}_{e,{\rm cdl}}(\mathbb{R}_+)$ such that $\big\|h-(I-\Gamma)f^\varepsilon\big\|_\infty\le\varepsilon$.

To begin with, we first construct $f_1\in C^{\rm ac}_{e,{\rm cdl}}(\mathbb{R}_+)$ such that

$$\big((I-\Gamma)f_1\big)(s) = h(s), \quad \text{for any } s\in[t_0,\infty). \tag{5.5}$$

Note that if $T\in(0,t_0]$, (5.5) holds trivially with $f_1\equiv 0$. Hence, without loss of generality, hereafter we assume $T\in(t_0,\infty)$. To this end, for any $s\in\mathbb{R}_+$, we consider the process $\varphi(s,a)$ as in (2.1) with $a=0$ and $v(t)\equiv v_1$, $\sigma(t)\equiv\sigma_1$, for any $t\in\mathbb{R}_+$, which we now denote by $\varphi^1(s) = (\varphi^1_t(s))_{t\in[s,\infty)}$ in order to distinguish it from the process $\varphi(s)$ with coefficient functions (5.1). Note that those constant coefficients trivially satisfy Assumption 2.2. Accordingly, the passage times defined by (2.2), the functions defined by Proposition 2.4 (ii), and the operators defined by (2.5) with respect to $\varphi^1(s)$, are denoted respectively by $\tau^{1,\pm}_\ell(s)$, $\mathcal{P}^{1,\pm}_\ell$, and $\gamma^{1,\pm}$, for any $s\in\mathbb{R}_+$ and $\ell\in\mathbb{R}_+$. From the result of step 1, $\gamma^{1,\pm}(\cdot,t)$ is continuous on $[0,t)$ for any $t\in\mathbb{R}_+$ (i.e., Assumption 2.5 is satisfied), and by a version of Proposition 2.4 (ii) (with constant coefficients v_1 and σ_1) and (A.9), we have

$$\gamma^{1,\pm}(s,t) = \frac{\sqrt{2}}{\sqrt{\pi\sigma_1^2(t-s)}}\exp\bigg(-\frac{v_1^2(t-s)}{2\sigma_1^2}\bigg)\mp\frac{2v_1}{\sigma_1^2}\Phi\bigg(\mp\frac{v_1\sqrt{t-s}}{\sigma_1}\bigg). \tag{5.6}$$

Next, by a version of Proposition 3.4 (with constant coefficients v_1 and σ_1), $(\mathcal{P}^{1,\pm}_\ell)_{\ell\in\mathbb{R}_+}$ is a Feller semigroup, and in view of (A.8), for any $\ell\in\mathbb{R}_+$ and $f\in L^\infty(\mathbb{R}_+)$, we have

$$\big(\mathcal{P}^{1,\pm}_\ell f\big)(s) = \int_0^\infty \frac{\ell}{\sqrt{2\pi\sigma_1^2t^3}}\exp\bigg(-\frac{\big(\ell\mp v_1t\big)^2}{2\sigma_1^2t}\bigg)f(s+t)\,dt, \quad s\in\mathbb{R}_+.$$

Together with the commutativity between $(\mathcal{P}^{1,+}_\ell)_{\ell\in\mathbb{R}_+}$ and $(\mathcal{P}^{1,-}_\ell)_{\ell\in\mathbb{R}_+}$, given by a version of Lemma 4.2 (with constant coefficients v_1 and σ_1), we deduce that, for any $k,\ell\in\mathbb{R}_+$ and $f\in L^\infty(\mathbb{R}_+)$,

$$\big(\mathcal{P}^{1,+}_k\mathcal{P}^{1,-}_\ell f\big)(s) = \big(\mathcal{P}^{1,-}_\ell\mathcal{P}^{1,+}_k f\big)(s) = \int_0^\infty \rho^1_\ell(t)f(s+t)\,dt, \quad s\in\mathbb{R}_+, \tag{5.7}$$

where

$$\rho_\ell^1(t) := \frac{\ell^2 e^{-v_1^2 t/2}}{2\pi\sigma_1^2} \int_0^t \frac{1}{\sqrt{r^3(t-r)^3}} \exp\left(-\frac{2\ell^2 t}{2\sigma_1 r(t-r)}\right) dr, \quad t \in \mathbb{R}_+.$$

Moreover, by a version of Lemma 4.2 (with constant coefficients v_1 and σ_1) again, $(\mathcal{P}_\ell^{1,+}\mathcal{P}_\ell^{1,-})_{\ell\in\mathbb{R}_+}$ is a Feller semigroup which coincides with the Feller semigroup given by a version of Proposition 2.9 (i) (with constant coefficients v_1 and σ_1). In particular, the strong generator of $(\mathcal{P}_\ell^{1,+}\mathcal{P}_\ell^{1,-})_{\ell\in\mathbb{R}_+}$ is the closure of Γ^1 defined by a version of (2.7) (with constant coefficients v_1 and σ_1). Denoting the strong generator of $(\mathcal{P}_\ell^{1,\pm})_{\ell\in\mathbb{R}_+}$ by $\Gamma^{1,\pm}$, it follows from a version of Proposition 3.5 (with constant coefficients v_1 and σ_1) and (5.6) that, for any $f \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$,

$$\big(\Gamma^1 f\big)(s) = \big(\Gamma^{1,+} f + \Gamma^{1,-} f\big)(s) = \int_s^\infty \frac{2\sqrt{2}}{\sqrt{\pi\sigma_1^2(t-s)}} \exp\left(-\frac{v_1^2(t-s)}{2\sigma_1^2}\right) g_f(t)\, dt, \quad s \in \mathbb{R}_+. \tag{5.8}$$

This, together with (2.7) and (5.4), implies that, for any $f \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$,

$$\big(\Gamma f\big)(s) = \int_s^\infty \frac{2\sqrt{2}}{\sqrt{\pi\sigma_1^2(t-s)}} \exp\left(-\frac{v_1^2(t-s)}{2\sigma_1^2}\right) g_f(t)\, dt = \big(\Gamma^1 f\big)(s), \quad s \in [t_0,\infty). \tag{5.9}$$

We now define f_1 as the 1-resolvent operator associated with $(\mathcal{P}_\ell^{1,+}\mathcal{P}_\ell^{1,-})_{\ell\in\mathbb{R}_+}$ on h, namely

$$f_1(s) := \int_0^\infty e^{-\ell}\big(\mathcal{P}_\ell^{1,+}\mathcal{P}_\ell^{1,-} h\big)(s)\, d\ell = \int_0^\infty \left(\int_0^\infty e^{-\ell}\rho_\ell^1(t)\, d\ell\right) h(s+t)\, dt, \quad s \in \mathbb{R}_+,$$

where the second equality follows from (5.7). Since $h \in C_c^1(\mathbb{R}_+)$, we see that the last integral above is finite, and that $f_1 \in C_c^1(\mathbb{R}_+) \subset C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$, and thus $f_1 \in \mathscr{D}(\overline{\Gamma^1})$. Moreover, by [7, Lemma 1.27], we have $(I-\Gamma^1)f_1 = h$ on $\mathbb{R}_+$, which, together with (5.9), leads to (5.5).

Next, in view of (5.5), we have $h_0 := h - (I-\Gamma)f_1 \in C_c(\mathbb{R}_+)$ with $\mathrm{supp}(h_0) \subset [0,t_0)$. Hence, there exists $h_0^\varepsilon \in C_c^1(\mathbb{R}_+)$, with $\mathrm{supp}(h_0^\varepsilon) \subset [0,t_0)$, such that

$$\|h_0^\varepsilon - h_0\|_\infty = \|h_0^\varepsilon - h + (I-\Gamma)f_1\|_\infty \le \varepsilon. \tag{5.10}$$

We will construct $f_0^\varepsilon \in C_c^1(\mathbb{R}_+)$ such that

$$\big((I-\Gamma)f_0^\varepsilon\big)(s) = h_0^\varepsilon(s), \quad s \in \mathbb{R}_+. \tag{5.11}$$

The construction of f_0^ε is almost identical to that of f_1 above, with the constant coefficients v_1 and σ_1 replaced by v_0 and σ_0, respectively, and the function h replaced by h_0^ε. More precisely, let $(\mathcal{P}_\ell^{0,\pm})_{\ell\in\mathbb{R}_+}$ be the analogous operators of $(\mathcal{P}_\ell^{1,\pm})_{\ell\in\mathbb{R}_+}$ with (v_1,σ_1) replaced by (v_0,σ_0). With similar arguments leading to (5.7), we deduce that $(\mathcal{P}_\ell^{0,+}\mathcal{P}_\ell^{0,-})_{\ell\in\mathbb{R}_+}$ is a Feller semigroup, and that for any $k,\ell \in \mathbb{R}_+$ and $f \in L^\infty(\mathbb{R}_+)$,

$$\big(\mathcal{P}_k^{0,+}\mathcal{P}_\ell^{0,-} f\big)(s) = \big(\mathcal{P}_\ell^{0,-}\mathcal{P}_k^{0,+} f\big)(s) = \int_0^\infty \rho_\ell^0(t) f(s+t)\, dt, \quad s \in \mathbb{R}_+, \tag{5.12}$$

where

$$\rho_\ell^0(t) := \frac{\ell^2 e^{-v_0^2 t/2}}{2\pi\sigma_0^2} \int_0^t \frac{1}{\sqrt{r^3(t-r)^3}} \exp\left(-\frac{2\ell^2 t}{2\sigma_0 r(t-r)}\right) dr, \quad t \in \mathbb{R}_+.$$

Moreover, similar arguments leading to (5.8) imply that the strong generator of $(\mathcal{P}_\ell^{0,+}\mathcal{P}_\ell^{0,-})_{\ell\in\mathbb{R}_+}$ is the closure of Γ^0 defined by a version of (2.7) (with constant coefficients v_0 and σ_0), and that for any $f \in C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)$,

$$\big(\Gamma^0 f\big)(s) = \int_s^\infty \frac{2\sqrt{2}}{\sqrt{\pi\sigma_0^2(t-s)}} \exp\left(-\frac{v_0^2(t-s)}{2\sigma_0^2}\right) g_f(t)\,dt, \quad s\in\mathbb{R}_+. \tag{5.13}$$

We now define f_0^ε as 1-resolvent operator associated with $(\mathcal{P}_\ell^{0,+}\mathcal{P}_\ell^{0,-})_{\ell\in\mathbb{R}_+}$ on h_0^ε, namely,

$$f_0^\varepsilon(s) := \int_0^\infty e^{-\ell}\big(\mathcal{P}_\ell^{0,+}\mathcal{P}_\ell^{0,-}h_0^\varepsilon\big)(s)\,d\ell = \int_0^\infty\left(\int_0^\infty e^{-\ell}\rho_\ell^0(t)\,d\ell\right)h_0^\varepsilon(s+t)\,dt, \quad s\in\mathbb{R}_+,$$

where the second equality follow from (5.12). Since $h_0^\varepsilon \in C_c^1(\mathbb{R}_+)$ with $\operatorname{supp}(h_0^\varepsilon)\subset[0,t_0)$, we see that the last integral above is finite, and that $f_0^\varepsilon \in C_c^1(\mathbb{R}_+)\subset C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)$ with $\operatorname{supp}(f_0^\varepsilon)\subset[0,t_0)$, so that $f_0^\varepsilon\in\mathscr{D}(\overline{\Gamma^0})$ and $g_{f_0^\varepsilon} = (f_0^\varepsilon)'$. Together with (2.7), (5.4), and (5.13), we obtain that

$$\big(\Gamma f_0^\varepsilon\big)(s) = \big(\Gamma^0 f_0^\varepsilon\big)(s) = \begin{cases} \displaystyle\int_s^{t_0} \frac{2\sqrt{2}}{\sqrt{\pi\sigma_0^2(t-s)}}\exp\left(-\frac{v_0^2(t-s)}{2\sigma_0^2}\right)(f_0^\varepsilon)'(t)\,dt, & t\in[0,t_0),\\[2ex] 0, & t\in[t_0,\infty). \end{cases} \tag{5.14}$$

On the other hand, by [7, Lemma 1.27] again, we have $(I-\Gamma^0)f_0^\varepsilon = h_0^\varepsilon$ on $\mathbb{R}_+$, which, together with (5.14), leads to (5.11).

Finally, we define $f^\varepsilon := f_0^\varepsilon + f_1$. In view of (5.5) and since $\operatorname{supp}(f_0^\varepsilon)\subset[0,t_0)$, for any $s\in[t_0,\infty)$, we deduce that

$$\big|h(s) - \big((I-\Gamma)f^\varepsilon\big)(s)\big| = \big|h(s) - \big((I-\Gamma)f_1\big)(s)\big| = 0.$$

Moreover, for any $s\in[0,t_0)$, by (5.10) and (5.11) we have

$$\big|h(s) - \big((I-\Gamma)f^\varepsilon\big)(s)\big| = \big|h(s) - \big((I-\Gamma)f_1\big)(s) - h_0^\varepsilon(s)\big| \le \varepsilon.$$

which concludes the proof in step 2.

A Additional Lemmas and Proofs

This appendix includes additional proofs of technical results. For those results presented in both the "+" and the "−" cases, we will only provide the proof for the "+" scenario, as the "−" scenario can be proved in an analogous way.

A.1 Proof of Proposition 2.3

Without loss of generality, we will fix $s=0$, $a=0$, and any $\ell\in\mathbb{R}$ through out the proof, and will omit the variables $(0,0)$ in $\varphi(0,0)$, $\tau_\ell^+(0,0)$, and $\eta_\ell^+(0,0)$ for simplicity.

To start with note the following. By the fact that $(tW_{1/t})_{t>0}$ is also a standard Brownian motion with respect to $\mathbb{F}$ under $\mathbb{P}$ and the oscillation behavior of Brownian paths close to infinity (cf. [19, Chapter I, Lemma (3.6)]), we have

$$\mathbb{P}\Big(\forall\varepsilon\in(0,1),\ \exists\, t_1,t_2\in(0,\varepsilon),\ \text{s.t. } W_{t_1}<0<W_{t_2}\Big) = 1. \tag{A.1}$$

By Girsanov theorem, for any $\mu \in \mathbb{R}$, there exists a probability measure $\mathbb{Q}$ on $\mathscr{F}_1$, such that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ restricted on $\mathscr{F}_1$, and that $(\mu t + W_t)_{t\in[0,1]}$ is a standard Brownian motion under $\mathbb{Q}$. This, together with a version of (A.1), implies that

$$\mathbb{P}\Big(\forall \varepsilon \in (0,1),\, \exists\, t_1, t_2 \in (0,\varepsilon), \text{ s.t. } \mu t_1 + W_{t_1} < 0 < \mu t_2 + W_{t_2}\Big) = 1. \tag{A.2}$$

Next, we will show that $\mathbb{P}(\tau_\ell^+ = \eta_\ell^+) = 1$ when $\sigma \equiv 1$. Toward this end, note that

$$\begin{aligned}
\mathbb{1}_{\{\tau_\ell^+<\infty\}}\mathbb{P}\Big(\tau_\ell^+ = \eta_\ell^+ \,\Big|\, \mathscr{F}_{\tau_\ell^+}\Big) &= \mathbb{1}_{\{\tau_\ell^+<\infty\}}\mathbb{P}\Big(\forall \varepsilon \in (0,1),\, \exists t \in (0,\varepsilon), \text{ s.t. } \varphi_{\tau_\ell^+ + t} > \ell \,\Big|\, \mathscr{F}_{\tau_\ell^+}\Big)\\
&\geq \mathbb{1}_{\{\tau_\ell^+<\infty\}}\mathbb{P}\Big(\forall \varepsilon \in (0,1),\, \exists t \in (0,\varepsilon), \text{ s.t. } \ell - \|v\|_\infty t + W_{\tau_\ell^+ + t} - W_{\tau_\ell^+} > \ell \,\Big|\, \mathscr{F}_{\tau_\ell^+}\Big)\\
&= \mathbb{1}_{\{\tau_\ell^+<\infty\}}\mathbb{P}\Big(\forall \varepsilon \in (0,1),\, \exists t \in (0,\varepsilon), \text{ s.t. } - \|v\|_\infty t + W_t > 0\Big) = \mathbb{1}_{\{\tau_\ell^+<\infty\}},
\end{aligned}$$

where the last equity follows from (A.2). It follows immediately that

$$\mathbb{1}_{\{\tau_\ell^+<\infty\}}\mathbb{P}\Big(\tau_\ell^+ = \eta_\ell^+ \,\Big|\, \mathscr{F}_{\tau_\ell^+}\Big) = \mathbb{1}_{\{\tau_\ell^+<\infty\}},$$

and thus

$$\mathbb{P}(\tau_\ell^+ = \eta_\ell^+) = \mathbb{E}\Big(\mathbb{1}_{\{\tau_\ell^+<\infty\}}\mathbb{P}\Big(\tau_\ell^+ = \eta_\ell^+ \,\Big|\, \mathscr{F}_{\tau_\ell^+}\Big) + \mathbb{1}_{\{\tau_\ell^+=\infty\}}\Big) = \mathbb{P}(\tau_\ell^+ \leq \infty) = 1. \tag{A.3}$$

Finally, we use a time-change argument to complete the proof. Towards this end, let us consider the martingale $M := (M_t)_{t\in\mathbb{R}_+}$ defined as

$$M_t := \int_0^t \sigma(s)\, dW_s, \quad t \in \mathbb{R}_+.$$

The quadratic variation process of M is given as $\langle M\rangle_t = \int_0^t \sigma^2(r)dr$ for $t \in \mathbb{R}_+$. In light of the strict positivity of σ in Assumption 2.2 (ii), $\langle M\rangle$ is a strictly increasing (deterministic) bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$. Hence, the function β, given as

$$\beta(t) := \inf\big\{s \in \mathbb{R}_+ : \langle M\rangle_s > t\big\}, \quad t \in \mathbb{R}_+,$$

is its inverse, so that $\beta(\langle M\rangle_t) = \langle M\rangle_{\beta(t)} = t$. It follows from the Dambis-Dubins-Schwarz theorem (cf. [13, Chapter 3, Theorem 4.6]) that the process $B := (B_t)_{t\in\mathbb{R}_+}$ given via

$$B_t := M_{\beta(t)} = \int_0^{\beta(t)} \sigma(r)\, dW_r, \quad t \in \mathbb{R}_+. \tag{A.4}$$

is standard Brownian motion with respect to $(\mathscr{F}_{\beta(t)})_{t\in\mathbb{R}_+}$ under $\mathbb{P}$. Together with (2.1), (2.2), and (2.6), we obtain that

$$\begin{aligned}
\mathbb{P}(\tau_\ell^+ = \eta_\ell^+) &= \mathbb{P}\Big(\inf\big\{t \in \mathbb{R}_+ : \varphi_t \geq \ell\big\} = \inf\big\{t \in \mathbb{R}_+ : \varphi_t > \ell\big\}\Big)\\
&= \mathbb{P}\Big(\inf\big\{t \in \mathbb{R}_+ : \varphi_{\beta(t)} \geq \ell\big\} = \inf\big\{t \in \mathbb{R}_+ : \varphi_{\beta(t)} > \ell\big\}\Big)\\
&= \mathbb{P}\left(\inf\left\{t \in \mathbb{R}_+ : \int_0^{\beta(t)} v(r)\, dr + B_t \geq \ell\right\} = \inf\left\{t \in \mathbb{R}_+ : \int_0^{\beta(t)} v(r)\, dr + B_t > \ell\right\}\right)\\
&= \mathbb{P}\left(\inf\left\{t \in \mathbb{R}_+ : \int_0^t \frac{v(\beta(r))}{\sigma^2(\beta(r))}\, dr + B_t \geq \ell\right\} = \inf\left\{t \in \mathbb{R}_+ : \int_0^t \frac{v(\beta(r))}{\sigma^2(\beta(r))}\, dr + B_t > \ell\right\}\right) = 1,
\end{aligned}$$

where the last equality follows from a version of (A.3). The proof of Proposition 2.3 is now complete.

A.2 Proof of Proposition 2.4

The proof of Proposition 2.4 relies on the following two technical lemmas. The first lemma provides the estimates for the tail distribution of τ_ℓ^+ when both v and σ are functions of time. An analogous estimate for τ_ℓ^- can be obtained by replacing v with $-v$.

Lemma A.1. *For any $t > s \geq 0$ and $\ell \in \mathbb{R}_+$, we have*

$$\mathbb{P}\bigg(\sup_{r\in[0,\overline{\sigma}^2(t-s)]}\Big(\frac{\|v\|_\infty r}{\underline{\sigma}^2}+W_r\Big)<\ell\bigg)\leq\mathbb{P}\big(\tau_\ell^+(s)>t\big)\leq\mathbb{P}\bigg(\sup_{r\in[0,\underline{\sigma}^2(t-s)]}\Big(-\frac{\|v\|_\infty r}{\underline{\sigma}^2}+W_r\Big)<\ell\bigg).\tag{A.5}$$

Proof. Using the same time-change technique as in the proof of Proposition 2.3 (see Section A.1), for any $t > s \geq 0$ and $\ell \in \mathbb{R}_+$, we have

$$\begin{aligned}\mathbb{P}\big(\tau_\ell^+(s)>t\big)&=\mathbb{P}\big(\varphi_r(s)<\ell,\,\forall r\in[s,t]\big)=\mathbb{P}\Big(\varphi_{\beta(r)}(s)<\ell,\,\forall r\in\big[\langle M\rangle_s,\langle M\rangle_t\big]\Big)\\&=\mathbb{P}\bigg(\int_{\langle M\rangle_s}^r\frac{v\big(\beta(u)\big)}{\sigma^2\big(\beta(u)\big)}\,du+B_r-B_{\langle M\rangle_s}<\ell,\,\forall r\in\big[\langle M\rangle_s,\langle M\rangle_t\big]\bigg),\end{aligned}\tag{A.6}$$

where B is a standard Brownian motion under $\mathbb{P}$, given as in (A.4). It follows that, for any $t > s \geq 0$ and $\ell \in \mathbb{R}_+$,

$$\mathbb{P}\big(\tau_\ell^+(s)>t\big)\leq\mathbb{P}\left(\sup_{r\in[0,\langle M\rangle_t-\langle M\rangle_s]}\Big(-\frac{\|v\|_\infty r}{\underline{\sigma}^2}+B_r\Big)<\ell\right)\leq\mathbb{P}\left(\sup_{r\in[0,\underline{\sigma}^2(t-s)]}\Big(-\frac{\|v\|_\infty r}{\underline{\sigma}^2}r+B_r\Big)<\ell\right),$$

and that

$$\mathbb{P}\big(\tau_\ell^+(s)>t\big)\geq\mathbb{P}\left(\sup_{r\in[0,\langle M\rangle_t-\langle M\rangle_s]}\Big(\frac{\|v\|_\infty r}{\underline{\sigma}^2}+B_r\Big)<\ell\right)\geq\mathbb{P}\left(\sup_{r\in[0,\overline{\sigma}^2(t-s)]}\Big(\frac{\|v\|_\infty r}{\underline{\sigma}^2}+B_r\Big)<\ell\right),$$

which completes the proof of the lemma. □

The second lemma provides two formulas for the tail probability of $\tau_\ell^+(s)$ in the case when both v and σ are constant functions (i.e., when φ is a drifted Brownian motion). In particular, both the lower bound and the upper bound in (A.5) can be computed using those formulas.

Lemma A.2. *Suppose that $v(t)\equiv v\in\mathbb{R}$ and $\sigma(t)\equiv\sigma\in(0,\infty)$, for all $t\in\mathbb{R}_+$. Then, for any $t>s\geq 0$, $\ell\in\mathbb{R}_+$, we have*

$$\begin{aligned}\mathbb{P}\big(\tau_\ell^+(s)>t\big)&=\Phi\bigg(\frac{\ell-v(t-s)}{\sigma\sqrt{t-s}}\bigg)-e^{2v\ell/\sigma^2}\,\Phi\bigg(-\frac{\ell+v(t-s)}{\sigma\sqrt{t-s}}\bigg)&\text{(A.7)}\\&=\int_0^{t-s}\frac{\ell}{\sigma\sqrt{2\pi r^3}}\exp\bigg(-\frac{\big(\ell-vr\big)^2}{2\sigma^2 r}\bigg)dr,&\text{(A.8)}\end{aligned}$$

where Φ denotes the standard normal distribution function. Consequently, for any $t > s \geq 0$,

$$\lim_{\ell\to 0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t\big)=\frac{\sqrt{2}}{\sqrt{\pi\sigma^2(t-s)}}\exp\bigg(-\frac{v^2}{2\sigma^2}(t-s)\bigg)-\frac{2v}{\sigma^2}\,\Phi\Big(-\frac{v}{\sigma}\sqrt{t-s}\Big).\tag{A.9}$$

Proof. When both v and σ are constants, for any $t > s \geq 0$, $\ell \in \mathbb{R}_+$, and $A \in \mathcal{B}((-\infty, \ell))$,

$$\begin{aligned}
\mathbb{P}\big(\tau_\ell^+(s) > t,\, \varphi_t(s) \in A\big) &= \mathbb{P}\bigg(\sup_{r\in[s,t]} \Big(v(r-s) + \sigma\big(W_r - W_s\big)\Big) < \ell,\, v(t-s) + \sigma\big(W_t - W_s\big) \in A\bigg) \\
&= \mathbb{P}\bigg(\sup_{u\in[0,t-s]} \big(vu + \sigma W_u\big) < \ell,\, v(t-s) + \sigma W_{t-s} \in A\bigg) \\
&= \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \int_A \left(\exp\!\bigg(-\frac{\big(y - v(t-s)\big)^2}{2\sigma^2(t-s)}\bigg) - \exp\!\bigg(\frac{2v\ell}{\sigma^2} - \frac{\big(2\ell - y + v(t-s)\big)^2}{2\sigma^2(t-s)}\bigg)\right) dy, \quad \text{(A.10)}
\end{aligned}$$

where we have used (1.1.8) in [5, Part II, Section 2.1] for the last equality above. The identity (A.7) follows immediately by taking $A = (-\infty, \ell)$ in (A.10), and it leads directly to (A.9) by L'Hôpital's rule. Finally, the identity (A.8) is an immediate consequence of (2.0.2) in [5, Part II, Section 2.2]), which completes the proof of the lemma. □

Proof of Proposition 2.4. In view of Lemma A.1 and (A.9), for any $t > s \geq 0$, we have

$$\begin{aligned}
&\frac{\sqrt{2}}{\sqrt{\pi\overline{\sigma}^2(t-s)}} \exp\left(-\frac{\overline{\sigma}^2\|v\|_\infty^2}{2\underline{\sigma}^4}(t-s)\right) - \frac{2\|v\|_\infty}{\underline{\sigma}^2}\Phi\bigg(-\frac{\overline{\sigma}\|v\|_\infty}{\underline{\sigma}^2}\sqrt{t-s}\bigg) \\
&\quad\leq \liminf_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t\big) \leq \limsup_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t\big) \\
&\quad\leq \frac{\sqrt{2}}{\sqrt{\pi\underline{\sigma}^2(t-s)}} \exp\left(-\frac{\|v\|_\infty^2}{2\underline{\sigma}^2}(t-s)\right) + 2\frac{\|v\|_\infty}{\underline{\sigma}^2}\Phi\bigg(\frac{\|v\|_\infty}{\underline{\sigma}}\sqrt{t-s}\bigg). \qquad \text{(A.11)}
\end{aligned}$$

We first claim that, for any fixed $T > s \geq 0$, and for any $\varepsilon > 0$, there exist at most finitely many $t \in (s, T]$ such that

$$\limsup_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t\big) - \liminf_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t\big) > \varepsilon.$$

Otherwise, there exists $\varepsilon_0 > 0$ and an increasing sequence $(t_n)_{n\in\mathbb{N}} \subset (s, T]$ such that

$$\limsup_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t_n\big) - \liminf_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t_n\big) > \varepsilon_0, \quad \text{for any } n \in \mathbb{N}.$$

Since $\mathbb{P}(\tau_\ell^+(s) > t)$ is non-increasing in t, the above statement implies that

$$\begin{aligned}
&\limsup_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t_1\big) - \liminf_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > T\big) \\
&\quad > \varepsilon_0 + \liminf_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t_1\big) - \liminf_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > T\big) \\
&\quad \geq \varepsilon_0 + \limsup_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t_2\big) - \liminf_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > T\big) > \cdots \\
&\quad > n\varepsilon_0 + \limsup_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t_{n+1}\big) - \liminf_{\ell\to 0+} \frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > T\big) \to \infty, \quad \text{as } n \to \infty,
\end{aligned}$$

which is a clear contradiction to (A.11). For any $s \in \mathbb{R}_+$, let $\mathcal{T}_s$ be the collection of $t \in (s,\infty)$ such that $\lim_{\ell\to 0+} \ell^{-1}\mathbb{P}(\tau_\ell^+(s) > t)$ exists and is finite. Noting that

$$(s,\infty)\setminus\mathcal{T}_s = \Big\{t\in(s,\infty):\ \limsup_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t\big) - \liminf_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t\big) > 0\Big\}$$
$$= \bigcup_{T=\lfloor s\rfloor+1}^{\infty}\ \bigcup_{k=1}^{\infty}\Big\{t\in(s,T]:\ \limsup_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t\big) - \liminf_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t\big) > \frac{1}{k}\Big\},$$

we conclude that, for each $s\in\mathbb{R}_+$, $(s,\infty)\setminus\mathcal{T}_s$ is at most countable.

Next, we will show that, for any $s\in\mathbb{R}_+$, $\lim_{\ell\to0+}\ell^{-1}\mathbb{P}(\tau_\ell^+(s)>\cdot)$ is continuous on $\mathcal{T}_s$. In view of (A.6), for any $t\in(s,\infty)$,

$$\mathbb{P}\big(\tau_\ell^+(s)>t\big) = \mathbb{P}\bigg(\inf\Big\{r\in\big[\langle M\rangle_s,\infty):\int_{\langle M\rangle_s}^{r}\frac{v\big(\beta(u)\big)}{\sigma^2\big(\beta(u)\big)}\,du + B_r - B_{\langle M\rangle_s}\ge\ell\Big\} > \langle M\rangle_t\bigg).$$

Since $\langle M\rangle$ is continuous on $\mathbb{R}_+$, it is sufficient to prove the continuity of $\lim_{\ell\to0+}\ell^{-1}\mathbb{P}(\tau_\ell^+(s)>\cdot)$ on $\mathcal{T}_s$ when $\sigma\equiv1$. In what follows, we will fix $s\in\mathbb{R}_+$ and assume that $\sigma\equiv1$. For any $t,t'\in\mathcal{T}_s$ with $t'>t$, we first have

$$\mathbb{P}\big(\tau_\ell^+(s)>t\big)-\mathbb{P}\big(\tau_\ell^+(s)>t'\big) = \mathbb{P}\bigg(\inf\Big\{r\in[s,\infty):\int_s^r v(u)\,du + W_r - W_s\ge\ell\Big\}\in(t,t']\bigg)$$
$$= \mathbb{P}\bigg(\inf\Big\{r\in\mathbb{R}_+:\int_0^r v(s+u)du + W_r\ge\ell\Big\}\in(t-s,t'-s]\bigg). \tag{A.12}$$

For any fixed $T\in[t'-s,\infty)$, we define a probability measure $\overline{\mathbb{P}}_1$ on $(\Omega,\mathscr{F}_T)$ via

$$\frac{d\overline{\mathbb{P}}_1}{d\mathbb{P}|_{\mathcal{F}_T}} = \exp\bigg(-\int_0^T v(s+r)\,dW_r - \frac{1}{2}\int_0^T v^2(s+r)\,dr\bigg).$$

By Girsanov Theorem, the process $\overline{W} := (\overline{W}_t)_{t\in[0,T]}$ defined by

$$\overline{W}_t := \int_0^t v(s+r)\,dr + W_t,\quad t\in[0,T],$$

is a standard Brownian motion under $\overline{\mathbb{P}}_1$. With the help of Cauchy Schwarz inequality, we deduce from (A.12) that, for any $\ell\in(0,\infty)$,

$$\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t\big)-\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t'\big) = \frac{1}{\ell}\,\mathbb{P}\Big(\inf\Big\{r\in\mathbb{R}_+:\overline{W}_r\ge\ell\Big\}\in(t-s,t'-s]\Big)$$
$$= \frac{1}{\ell}\,\overline{\mathbb{E}}_1\bigg(\exp\bigg(\int_0^{t'-s} v(s+r)\,d\overline{W}_r - \frac{1}{2}\int_0^{t'-s} v^2(s+r)\,dr\bigg)\mathbb{1}_{\{\inf\{r\in\mathbb{R}_+:\overline{W}_r\ge\ell\}\in(t-s,t'-s]\}}\bigg)$$
$$\le\bigg(\frac{1}{\ell}\,\overline{\mathbb{E}}_1\bigg(\exp\bigg(2\int_0^{t'-s} v(s+r)\,d\overline{W}_r - \int_0^{t'-s} v^2(s+r)\,dr\bigg)\mathbb{1}_{\{\inf\{r\in\mathbb{R}_+:\overline{W}_r\ge\ell\}\in(t-s,t'-s]\}}\bigg)\bigg)^{1/2}$$
$$\cdot\bigg(\frac{1}{\ell}\,\overline{\mathbb{P}}_1\Big(\inf\{r\in\mathbb{R}_+:\overline{W}_r\ge\ell\}\in(t-s,t'-s]\Big)\bigg)^{1/2}. \tag{A.13}$$

Similarly, we define another probability measure $\overline{\mathbb{P}}_2$ on $(\Omega, \mathscr{F}_T)$ via

$$\frac{d\overline{\mathbb{P}}_2}{d\overline{\mathbb{P}}_1} = \exp\bigg(2\int_0^T v(s+r)\,d\overline{W}_r - 2\int_0^T v^2(s+r)\,dr\bigg).$$

The Girsanov Theorem implies that the process $\overline{B} := (\overline{B}_t)_{t\in[0,T]}$, where

$$\overline{B}_t := \overline{W}_t - 2\int_0^t v(s+r)\,dr, \quad t\in[0,T],$$

is a standard Brownian motion under $\overline{\mathbb{P}}_2$. Hence, the first factor in (A.13) can be estimated as

$$\begin{aligned}
&\frac{1}{\ell}\,\overline{\mathbb{E}}_1\bigg(\exp\bigg(2\int_0^{t'-s} v(s+r)\,d\overline{W}_r - \int_0^{t'-s} v^2(s+r)\,dr\bigg)\mathbb{1}_{\{\inf\{r\in\mathbb{R}_+:\overline{W}_r\geq\ell\}\in(t-s,t'-s]\}}\bigg)\\
&\quad = \exp\bigg(\int_0^{t'-s} v^2(s+r)\,dr\bigg)\cdot\frac{1}{\ell}\,\overline{\mathbb{P}}_2\bigg(\inf\bigg\{r\in\mathbb{R}_+ : \overline{B}_r + 2\int_0^r v(s+u)\,du \geq \ell\bigg\}\in(t-s,t'-s]\bigg)\\
&\quad \leq e^{\|v\|_\infty T}\cdot\frac{1}{\ell}\,\overline{\mathbb{P}}_2\Big(\inf\Big\{r\in\mathbb{R}_+ : \overline{B}_r - 2\|v\|_\infty r\geq\ell\Big\} > t-s\Big).
\end{aligned}\tag{A.14}$$

Combining (A.13) and (A.14), and using (A.8) and (A.9), for any $t, t' \in \mathcal{T}_s$ with $t' > t$, we have

$$\begin{aligned}
&\lim_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t\big) - \lim_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s) > t'\big)\\
&\quad\leq e^{\|v\|_\infty T/2}\bigg(\lim_{\ell\to0+}\frac{1}{\ell}\,\overline{\mathbb{P}}_2\Big(\inf\Big\{r\in\mathbb{R}_+ : \overline{B}_r - 2\|v\|_\infty r\geq\ell\Big\} > t-s\Big)\bigg)^{1/2}\\
&\qquad\cdot\bigg(\lim_{\ell\to0+}\frac{1}{\ell}\,\overline{\mathbb{P}}_1\Big(\inf\{r\in\mathbb{R}_+ : \overline{W}_r\geq\ell\}\in(t-s,t'-s]\Big)\bigg)^{1/2}\\
&\quad= e^{\|v\|_\infty T/2}\bigg(\frac{\sqrt{2}\,e^{-2\|v\|_\infty^2(t-s)}}{\sqrt{\pi(t-s)}} - 4\|v\|_\infty\Phi\Big(-2\|v\|_\infty\sqrt{t-s}\Big)\bigg)^{1/2}\bigg(\lim_{\ell\to0+}\int_{t-s}^{t'-s}\frac{e^{-\ell^2/(2r)}}{\sqrt{2\pi r^3}}\,dr\bigg)^{1/2}\\
&\quad\leq\frac{2^{1/4}e^{\|v\|_\infty T/2}}{\pi^{1/4}(t-s)^{1/4}}\bigg(\int_{t-s}^{t'-s}\frac{1}{\sqrt{2\pi r^3}}\,dr\bigg)^{1/2},
\end{aligned}$$

which completes the proof of the continuity of $\lim_{\ell\to0+}\ell^{-1}\mathbb{P}(\tau_\ell^+(s) > \cdot)$ on $\mathcal{T}_s$.

Finally, since $\mathcal{T}_s$ is dense in (s,∞), for any $t\in(s,\infty)$, there exist an increasing sequence $(\underline{t}_n)_{n\in\mathbb{N}}\subset\mathcal{T}_s$ and a decreasing sequence $(\overline{t}_n)_{n\in\mathbb{N}}\subset\mathcal{T}_s$ such that $\lim_{n\to\infty}\underline{t}_n = \lim_{n\to\infty}\overline{t}_n = t$, and so

$$\limsup_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t\big) - \liminf_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>t\big)\leq\lim_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>\underline{t}_n\big) - \lim_{\ell\to0+}\frac{1}{\ell}\,\mathbb{P}\big(\tau_\ell^+(s)>\overline{t}_n\big)\to0,$$

as $n\to\infty$, where the convergence follows from the continuity of $\lim_{\ell\to0+}\ell^{-1}\mathbb{P}(\tau_\ell^+(s) > \cdot)$ on $\mathcal{T}_s$. This completes the proof of part (i), which, together with (A.11), leads to the estimates for $\gamma^+(s,t)$ in part (iii). As for part (ii), since $\mathcal{T}_s = (s,\infty)$, we have shown that $\gamma^+(s,\cdot)$ is continuous on (s,∞) for every $s\in\mathbb{R}_+$. The non-increasing property of $\gamma^+(s,\cdot)$ on (s,∞) follows immediately from the same property for $\mathbb{P}(\tau_\ell^+(s) > \cdot)$. The proof of Proposition 2.4 is now complete. ∎

A.3 Proof of Lemma 2.6

We fix any $f \in C^{\mathrm{ac}}_{e,\mathrm{cdl}}(\mathbb{R}_+)$ throughout the proof. By Proposition 2.4 (iii), for any $s \in \mathbb{R}_+$, we have

$$\int_s^\infty |g_f(t)\gamma(s,t)|\,dt \le \widetilde{K}\int_s^\infty \Big(\frac{1}{\sqrt{t-s}}+1\Big)e^{-\kappa t}\,dt = \widetilde{K}e^{-\kappa s}\int_0^\infty \Big(\frac{1}{\sqrt{t}}+1\Big)e^{-\kappa t}\,dt,$$

where $\widetilde{K} := K(\sqrt{2}/\sqrt{\pi\underline{\sigma}^2}+2\|v\|_\infty/\underline{\sigma}^2)$. Since the last integral above is finite, we obtain that Γf is well-defined and vanishes at infinity with exponential rate.

It remains to verify the continuity of Γf on $\mathbb{R}_+$. For any $\varepsilon > 0$, we pick $\delta \in (0,1)$ so that $\int_0^\delta (t^{-1/2}+1)dt \le \varepsilon$. For any $s \in \mathbb{R}_+$ and $s' \in (s, s+\delta]$, by Proposition 2.4 (iii) and the boundedness of g_f, we deduce that

$$\begin{aligned}\Big|(\Gamma f)(s) - (\Gamma f)(s')\Big| &\le \int_s^{s+\delta} |g_f(t)|\gamma(s,t)dt + \int_{s'}^{s+\delta} |g_f(t)|\gamma(s',t)dt + \bigg|\int_{s+\delta}^\infty g_f(t)\big(\gamma(s,t)-\gamma(s',t)\big)dt\bigg| \\ &\le 2\widetilde{K}\int_s^{s+\delta}\Big(\frac{1}{\sqrt{t-s}}+1\Big)dt + \bigg|\int_{s+\delta}^\infty g_f(t)\big(\gamma(s,t)-\gamma(s',t)\big)dt\bigg| \\ &\le 2\widetilde{K}\varepsilon + \bigg|\int_0^\infty \mathbb{1}_{[s+\delta,\infty)}(t) g_f(t)\big(\gamma(s,t)-\gamma(s',t)\big)dt\bigg|.\end{aligned}$$

The second term above vanishes to zero, as $s' \to s+$, due to Assumption 2.5, the dominated convergence theorem, as well as the estimate

$$\mathbb{1}_{[s+\delta,\infty)}(t)\big|g_f(t)\big|\big|\gamma(s,t)-\gamma(s',t)\big| \le 2\widetilde{K}\delta^{-1/2}e^{-\kappa t},$$

which follows from Proposition 2.4 (iii) and the exponential decay of g_f. Thus Γf is right-continuous at any $s \in \mathbb{R}_+$. The left continuity of Γf on $\mathbb{R}_+$ can be shown using similar arguments. The proof of Lemma 2.6 is complete.

A.4 Proof of Lemma 3.3

We will only present the proof for the "plus" case, as the "minus" case can be verified in an analogous way. We will fix $f \in C_0(\mathbb{R}_+)$ throughout the proof, for which we 2 stipulate $f(\infty) = 0$.

(i) We fix any $\ell_1, \ell_2 \in \mathbb{R}_+$ with $\ell_1 < \ell_2$, and so $\widetilde{\tau}^+_{\ell_1} \le \widetilde{\tau}^+_{\ell_2}$. Hence, we have $f(Z^1_{\widetilde{\tau}^+_{\ell_2}}) = f(\infty) = 0$ on $\{\widetilde{\tau}^+_{\ell_1} = \infty\}$ and $Z^2_{\widetilde{\tau}^+_{\ell_1}} = \ell_1$ on $\{\widetilde{\tau}^+_{\ell_1} < \infty\}$. Together with Lemma 3.2 and (3.9), we deduce that

$$\begin{aligned}\widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell_2}}\Big)\Big) &= \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_{\ell_1}<\infty\}}\mathbb{1}_{\{\widetilde{\tau}^+_{\ell_1}\le\widetilde{\tau}^+_{\ell_2}\}}\widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell_2}}\Big)\Big|\widetilde{\mathscr{F}}_{\widetilde{\tau}^+_{\ell_1}}\Big)\Big) \\ &= \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_{\ell_1}<\infty\}}\mathbb{1}_{\{\widetilde{\tau}^+_{\ell_1}\le\widetilde{\tau}^+_{\ell_2}\}}\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\tau}^+_{\ell_1}},Z^2_{\widetilde{\tau}^+_{\ell_1}}}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell_2}}\Big)\Big)\Big) = \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_{\ell_1}<\infty\}}\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\tau}^+_{\ell_1}},\ell_1}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell_2}}\Big)\Big)\Big) \\ &= \widetilde{\mathbb{E}}_{s,0}\Big(\mathbb{1}_{\{\widetilde{\tau}^+_{\ell_1}<\infty\}}\,\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\tau}^+_{\ell_1}},0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\delta\ell}}\Big)\Big)\Big), \qquad \text{(A.15)}\end{aligned}$$

where $\delta\ell := \ell_2 - \ell_1$. Therefore, by (3.8), Lemma A.1, and (A.7), we obtain that

$$\begin{aligned}
&\Big|\widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell_2}}\Big)\Big) - \widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell_1}}\Big)\Big)\Big| \le \widetilde{\mathbb{E}}_{s,0}\left(\mathbb{1}_{\{\widetilde{\tau}^+_{\ell_1}<\infty\}}\Big|\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\tau}^+_{\ell_1}},0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\delta\ell}}\Big)\Big) - f\Big(Z^1_{\widetilde{\tau}^+_{\ell_1}}\Big)\Big|\right)\\
&\quad\le \sup_{t\in\mathbb{R}_+}\Big|\widetilde{\mathbb{E}}_{t,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\delta\ell}}\Big)\Big) - f(t)\Big| = \sup_{t\in\mathbb{R}_+}\Big|\mathbb{E}\Big(f\big(\tau^+_{\delta\ell}(t)\big)\Big) - f(t)\Big|\\
&\quad\le \sup_{s_1,s_2\in\mathbb{R}_+:\,|s_2-s_1|\le\delta\ell}\big|f(s_2)-f(s_1)\big| + 2\|f\|_\infty \sup_{t\in\mathbb{R}_+}\mathbb{P}\Big(\tau^+_{\delta\ell}(t) > \delta\ell + t\Big)\\
&\quad\le \sup_{\substack{s_1,s_2\in\mathbb{R}_+\\|s_2-s_1|\le\delta\ell}}\big|f(s_2)-f(s_1)\big| + 2\|f\|_\infty\left(\Phi\left(\frac{\sqrt{\delta\ell}}{\underline{\sigma}}(1-\|v\|_\infty)\right) - e^{-2\|v\|^2_\infty\delta\ell/\underline{\sigma}^2}\,\Phi\left(-\frac{\sqrt{\delta\ell}}{\underline{\sigma}}(1+\|v\|_\infty)\right)\right),
\end{aligned}$$

where the right-hand side tends to 0 as $\delta\ell = \ell_2 - \ell_1 \to 0+$, uniformly for all $s \in \mathbb{R}_+$. This finishes the proof of part (i).

(ii) Since $f \in C_0(\mathbb{R}_+)$, and $Z^1_{\widetilde{\tau}^+_\ell} \ge s$, $\widetilde{\mathbb{P}}_{s,0}$-a.s., we have

$$\Big|\widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}^\pm_\ell}\Big)\Big)\Big| \le \sup_{r\in[s,\infty)}\big|f(r)\big| \to 0, \quad \text{as } s\to\infty.$$

It remains to show that $s \mapsto \widetilde{\mathbb{E}}_{s,0}(f(Z^1_{\widetilde{\tau}^\pm_\ell}))$ is continuous on $\mathbb{R}_+$, for any $\ell \in \mathbb{R}_+$. Note that the continuity is trivial when $\ell = 0$ since $\widetilde{\mathbb{E}}_{s,0}(f(Z^1_{\widetilde{\tau}^\pm_0})) = f(s)$. We therefore fix any $\ell \in (0,\infty)$ for the rest of the proof. For any $s_1, s_2 \in \mathbb{R}_+$ with $s_1 < s_2$ and $\delta s := s_2 - s_1$, by Lemma 3.2 and the fact that $Z^1_{\delta s} = s_1 + \delta_s = s_2$, $\widetilde{\mathbb{P}}_{s_1,0}$-a.s., we first have

$$\begin{aligned}
\widetilde{\mathbb{E}}_{s_1,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) &= \widetilde{\mathbb{E}}_{s_1,0}\Big(\mathbb{1}_{\{\delta s\le\widetilde{\tau}^+_\ell\}} f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) + \widetilde{\mathbb{E}}_{s_1,0}\Big(\mathbb{1}_{\{\delta s>\widetilde{\tau}^+_\ell\}} f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\\
&= \widetilde{\mathbb{E}}_{s_1,0}\left(\mathbb{1}_{\{\delta s\le\widetilde{\tau}^+_\ell\}}\widetilde{\mathbb{E}}_{s_1,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big|\widetilde{\mathscr{F}}_{\delta s}\Big)\right) + \widetilde{\mathbb{E}}_{s_1,0}\Big(\mathbb{1}_{\{\delta s>\widetilde{\tau}^+_\ell\}} f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\\
&= \widetilde{\mathbb{E}}_{s_1,0}\left(\mathbb{1}_{\{\delta s\le\widetilde{\tau}^+_\ell\}}\widetilde{\mathbb{E}}_{s_2,Z^2_{\delta s}}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\right) + \widetilde{\mathbb{E}}_{s_1,0}\Big(\mathbb{1}_{\{\delta s>\widetilde{\tau}^+_\ell\}} f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)
\end{aligned}$$

It follows that

$$\begin{aligned}
\Big|\widetilde{\mathbb{E}}_{s_1,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - \widetilde{\mathbb{E}}_{s_2,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\Big| &\le \left|\widetilde{\mathbb{E}}_{s_1,0}\left(\mathbb{1}_{\{\delta s\le\widetilde{\tau}^+_\ell\}}\left(\widetilde{\mathbb{E}}_{s_2,Z^2_{\delta s}}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big) - \widetilde{\mathbb{E}}_{s_2,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\right)\right)\right|\\
&\quad + \Big|\widetilde{\mathbb{E}}_{s_1,0}\Big(\mathbb{1}_{\{\delta s>\widetilde{\tau}^+_\ell\}}\widetilde{\mathbb{E}}_{s_2,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\Big)\Big| + \Big|\widetilde{\mathbb{E}}_{s_1,0}\Big(\mathbb{1}_{\{\delta s>\widetilde{\tau}^+_\ell\}} f\Big(Z^1_{\widetilde{\tau}^+_\ell}\Big)\Big)\Big|\\
&\le \widetilde{\mathbb{E}}_{s_1,0}\Big(\overline{f}\Big(Z^2_{\delta s}\Big)\Big) + 2\|f\|_\infty\widetilde{\mathbb{P}}_{s,0}\big(\widetilde{\tau}^+_\ell < \delta s\big),
\end{aligned} \tag{A.16}$$

where we define

$$\overline{f}(r) := \sup_{\substack{s\in\mathbb{R}_+,\ell_1,\ell_2\in\mathbb{R}_+\\|\ell_1-\ell_2|<|r|}}\Big|\widetilde{\mathbb{E}}_{s,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell_1}}\Big)\Big) - \widetilde{\mathbb{E}}_{s_1,0}\Big(f\Big(Z^1_{\widetilde{\tau}^+_{\ell_2}}\Big)\Big)\Big|, \quad r\in\mathbb{R}.$$

In view of part (i), $\overline{f}$ is a bounded even function such that $\lim_{r\to0}\overline{f}(r) = 0$. Since $Z^2_{\delta s}$ admits a normal distribution under $\widetilde{\mathbb{P}}_{s_1,0}$ with mean $\int_{s_1}^{s_2} v(t)dt$ and variance $\int_{s_1}^{s_2}\sigma^2(t)dt$), which converges to 0 in distribution as $\delta s \to 0$, we obtain that

$$\lim_{\delta s\to0}\widetilde{\mathbb{E}}_{s_1,0}\Big(\overline{f}\Big(Z^2_{\delta s}\Big)\Big) = 0. \tag{A.17}$$

Moreover, by (3.8), Lemma A.1, and (A.7), we have

$$\widetilde{\mathbb{P}}_{s_1,0}\big(\widetilde{\tau}_\ell^+ < \delta s\big) = \mathbb{P}\big(\tau_\ell^+(s_1) < s_2\big) \leq \left(1 - \Phi\left(\frac{\ell - \|v\|_\infty \delta s}{\underline{\sigma}\sqrt{\delta s}}\right) + e^{2\|v\|_\infty \ell/\underline{\sigma}^2}\,\Phi\left(-\frac{\ell + \|v\|_\infty \delta s}{\underline{\sigma}\sqrt{\delta s}}\right)\right) \to 0, \tag{A.18}$$

as $\delta s \to 0$. Combining (A.16), (A.17), and (A.18) completes the proof of part (ii).

A.5 A Technical Lemma for the Proof of (3.15)

The identity (3.15) follows immediately from the following lemma.

Lemma A.3. *Let $f \in C(\mathbb{R}_+)$. Assume that f is right-differentiable on $\mathbb{R}_+$, and that its right-derivative, denoted by f'_+, is càdlàg and bounded on $\mathbb{R}_+$. Then, f is globally Lipschitz continuous, and we have*

$$f(t) - f(0) = \int_0^t f'_+(s)\,ds, \quad t \in \mathbb{R}_+.$$

Proof. We will fix any $t \in (0,\infty)$ for the rest of the proof. Since f'_+ is càdlàg, it has at most countably many jumps in $[0,t)$, denoted by $(r_n)_{n\in\mathbb{N}}$ (note that this sequence is not ordered in general). For each $n \in \mathbb{N}$, we define $s_n := \inf\{r > r_n : f'_+(r) \neq f'_+(r-)\} \wedge t$. The right-continuity of f'_+ implies that $I_n := [r_n, s_n)$ is nonempty and disjoint from each other, and that f'_+ is continuous on I_n (with right-continuity at r_n), $n \in \mathbb{N}$. Assume without loss of generality that $[0,t)\setminus(\cup_{n\in\mathbb{N}} I_n) \neq \emptyset$. From the construction of $(I_n)_{n\in\mathbb{N}}$, we see that $[0,t) \setminus (\cup_{n\in\mathbb{N}} I_n)$ is a countable union of intervals $J_m := [a_m, b_m)$, $m \in \mathbb{N}$, so that $[0,t) = (\cup_{n\in\mathbb{N}} I_n) \cup (\cup_{m\in\mathbb{N}} J_m)$. Moreover, f'_+ is continuous on each J_m (with right-continuity at a_m). By [17, Corollary 2.1.2], f is continuously differentiable on each I_n and J_m (with right-continuous differentiability at each left-end point). Hence, by the fundamental theorem of calculus and the continuity of f,

$$f(s_n) - f(r_n) = \int_{r_n}^{s_n} f'_+(r)\,dr, \quad f(b_m) - f(a_m) = \int_{a_m}^{b_m} f'_+(r)\,dr, \quad n, m \in \mathbb{N}. \tag{A.19}$$

In particular, f is Lipschitz continuous on $[0,t]$ with Lipschitz constant $\|f'_+\|_\infty < \infty$. Hence, for any $N \in \mathbb{N}$, by rearranging the end points of $(I_n)_{n=1}^N$ and $(J_m)_{m=1}^N$ into

$$0 =: t_0 \leq t_1 < t_2 \leq t_3 < t_4 \leq \cdots \leq t_{4N-1} < t_{4N} \leq t_{4N+1} := t,$$

and denoting by $A_N := (\cup_{n=1}^N I_n) \cup (\cup_{m=1}^N J_m) = \cup_{k=1}^{2N} [t_{2k-1}, t_{2k})$, we deduce from (A.19) that

$$\left| f(t) - f(0) - \int_{A_N} f'_+(r)\,dr \right| = \left| \sum_{k=0}^{2N} \big(f(t_{2k+1}) - f(t_{2k})\big) \right| \leq \|f'_+\|_\infty \cdot \mathrm{Leb}\big([0,t) \setminus A_N\big) \to 0, \quad N \to \infty.$$

Finally, we obtain from the dominate convergence that

$$f(t) - f(0) = \lim_{N\to\infty} \int_{A_N} f'_+(r)\,dr = \int_0^t f'_+(r)\,dr,$$

which completes the proof of the lemma. □

A.6 Proof of Lemma 4.1

In view of Proposition 2.4 (iii), for any $\varepsilon > 0$, there exists $c \in (0, \infty)$, depending only on $\|v\|_\infty$, $\overline{\sigma}$, and $\underline{\sigma}$, such that for any $s \in \mathbb{R}_+$ and $r \in [0, c]$, $\gamma(s, s+r) \geq \varepsilon$. Assuming first $T \geq c/2$, we define

$$f_{2T}(t) := (2T - t)\mathbb{1}_{[0,2T]}(t) = -\int_t^\infty \big(-\mathbb{1}_{[0,2T)}(r)\big)\, dr. \quad t \in \mathbb{R}_+. \tag{A.20}$$

Clearly, $f_{2T} \in C^{\text{ac}}_{e,\text{cdl}}(\mathbb{R}_+)$. By (2.7) and (A.20) we have, for any $t \in \mathbb{R}_+$,

$$\begin{aligned}
-\big(\Gamma f_{2T}\big)(t) &= \int_t^\infty \mathbb{1}_{[0,2T)}(r)\gamma(t, r)\, dr = \mathbb{1}_{[0,2T)}(t) \int_t^{2T} \gamma(t, r)\, dr \\
&\geq \mathbb{1}_{[0,2T-c)}(t) \int_t^{t+c} \gamma(t, r)\, dr + \mathbb{1}_{[2T-c,2T)}(t) \int_t^{2T} \gamma(t, r)\, dr \\
&\geq c\varepsilon \mathbb{1}_{[0,2T-c)}(t) + (2T - t)\varepsilon \mathbb{1}_{[2T-c,2T)}(t) \geq \frac{2T - t}{2T}\, c\varepsilon \mathbb{1}_{[0,2T)}(t) = \frac{c\varepsilon}{2T} f_{2T}(t).
\end{aligned}$$

Together with [9, Chapter 1, Proposition 1.5 (b)] and the positivity of $(\mathcal{P}_\ell)_{\ell \in \mathbb{R}_+}$ (recalling from Proposition 2.9 (i) that $(\mathcal{P}_\ell)_{\ell \in \mathbb{R}_+}$ is a Feller semigroup), we obtain that, for any $\ell \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$,

$$\frac{\partial}{\partial \ell}\big(\mathcal{P}_\ell f_{2T}\big)(t) = \big(\mathcal{P}_\ell \Gamma f_{2T}\big)(t) \leq -\frac{c\varepsilon}{2T}\big(\mathcal{P}_\ell f_{2T}\big)(t).$$

Since $\mathbb{1}_{[0,T]} \leq f_{2T}/T$, the positivity of $(\mathcal{P}_\ell)_{\ell \in \mathbb{R}_+}$ with Grönwall's inequality implies that

$$\big\|\mathcal{P}_\ell \mathbb{1}_{[0,T]}\big\|_\infty \leq \frac{1}{T}\big\|\mathcal{P}_\ell f_{2T}\big\|_\infty \leq \frac{e^{-c\varepsilon\ell/(2T)}}{T}\big\|f_T\big\|_\infty = 2e^{-c\varepsilon\ell/(2T)}, \quad T \geq \frac{c}{2}. \tag{A.21}$$

Finally, when $T \in (0, c/2)$, it follows from the positivity property of $(\mathcal{P}_\ell)_{\ell \in \mathbb{R}_+}$ and (A.21) that

$$\big\|\mathcal{P}_\ell \mathbb{1}_{[0,T]}\big\|_\infty \leq \big\|\mathcal{P}_\ell \mathbb{1}_{[0,c/2]}\big\|_\infty \leq 2e^{-\varepsilon\ell},$$

which completes the proof of the lemma.

A.7 Proof of Lemma 4.2

When both v and σ are constants, the time-homogeneity of φ implies that, for any $s \in \mathbb{R}_+$, $\tau^\pm(s) - s$ has the identical law as $\tau^\pm(0)$. It follows from (2.5) that, for any $\ell \in \mathbb{R}_+$ and $f \in L^\infty(\mathbb{R}_+)$,

$$\big(\mathcal{P}^\pm_\ell f\big)(s) = \mathbb{E}\big(f\big((s + \tau^\pm_\ell(0)\big)\big), \quad s \in \mathbb{R}_+. \tag{A.22}$$

Hence by Fubini's theorem, for any $k, \ell \in \mathbb{R}_+$ and $f \in L^\infty(\mathbb{R}_+)$, we have

$$\begin{aligned}
\big(\mathcal{P}^+_k \mathcal{P}^-_\ell f\big)(s) &= \Big(\mathcal{P}^+_k\, \mathbb{E}\big(f\big(\,\cdot + \tau^-_\ell(0)\big)\big)\Big)(s) = \mathbb{E}\bigg(\mathbb{E}\Big(f\big(r + \tau^-_\ell(0)\big)\Big)\Big|_{r = s + \tau^+_k(0)}\bigg) \\
&= \mathbb{E}\bigg(\mathbb{E}\Big(f\big(s + r + \tau^-_\ell(0)\big)\Big)\Big|_{r = \tau^+_k(0)}\bigg) = \mathbb{E}\bigg(\mathbb{E}\Big(f\big(s + r + \tau^+_k(0)\big)\Big)\Big|_{r = \tau^-_\ell(0)}\bigg) \\
&= \mathbb{E}\bigg(\mathbb{E}\Big(f\big(r + \tau^+_k(0)\big)\Big)\Big|_{r = s + \tau^-_\ell(0)}\bigg) = \Big(\mathcal{P}^-_\ell\, \mathbb{E}\big(f\big(\,\cdot + \tau^+_k(0)\big)\big)\Big)(s) = \big(\mathcal{P}^-_\ell \mathcal{P}^+_k f\big)(s), \quad s \in \mathbb{R}_+,
\end{aligned}$$

that is, $\mathcal{P}_k^+$ and $\mathcal{P}_\ell^-$ are commutative. This, together with Proposition 3.4, implies that $(\mathcal{P}_\ell^+\mathcal{P}_\ell^-)_{\ell\in\mathbb{R}_+}$ is a Feller semigroup. Moreover, by (2.7) and Proposition 3.5, for any $f\in C_{e,\mathrm{cdl}}^{\mathrm{ac}}(\mathbb{R}_+)$, we have

$$\lim_{\ell\to 0+}\frac{1}{\ell}\big\|\mathcal{P}_\ell^+\mathcal{P}_\ell^- f-f\big\|_\infty=\lim_{\ell\to 0+}\frac{1}{\ell}\big\|\mathcal{P}_\ell^+\big(\mathcal{P}_\ell^- f-f\big)\big\|_\infty+\lim_{\ell\to 0+}\frac{1}{\ell}\big\|\mathcal{P}_\ell^+ f-f\big\|_\infty=\big(\Gamma^+ +\Gamma^-\big)f=\Gamma f,$$

namely, the strong generator of $(\mathcal{P}_\ell^+\mathcal{P}_\ell^-)_{\ell\in\mathbb{R}_+}$ coincides with Γ on $C_{e,\mathrm{cdl}}^{\mathrm{ac}}(\mathbb{R}_+)$. In view of Proposition 2.9 (i), we deduce that

$$\mathcal{P}_\ell f=\mathcal{P}_\ell^+\mathcal{P}_\ell^- f,\quad \text{for any } f\in L^\infty(\mathbb{R}_+),\quad \ell\in\mathbb{R}_+.$$

Next, for any $\ell\in\mathbb{R}_+$, $f\in C_{e,\mathrm{cdl}}^{\mathrm{ac}}(\mathbb{R}_+)$, and $s\in\mathbb{R}_+$, we deduce from (2.4), (A.22), and Fubini's theorem that

$$\big(\mathcal{P}_\ell^\pm f\big)(s)=\mathbb{E}\bigg(-\int_{s+\tau_\ell^\pm(0)}^\infty g_f(r)dr\bigg)=-\int_0^\infty\bigg(\int_s^\infty g_f(t+r)dt\bigg)dF_\ell^\pm(r)=-\int_s^\infty\bigg(\int_0^\infty g_f(t+r)dF_\ell^\pm(r)\bigg)dt,$$

where $F_\ell^\pm$ denotes the distribution function of $\tau_\ell^\pm$ under $\mathbb{P}$. Since g_f is càdlàg and vanishes at infinity with exponential rate, so is $h(t):=\int_0^\infty g_f(t+r)dF_\ell^\pm(r)$, $t\in\mathbb{R}_+$. Hence, we have $\mathcal{P}_\ell^\pm f\in C_{e,\mathrm{cdl}}^{\mathrm{ac}}(\mathbb{R}_+)$.

Finally, by the fact that $C_{e,\mathrm{cdl}}^{\mathrm{ac}}(\mathbb{R}_+)\subset\mathscr{D}(\Gamma^\pm)$ (given as in Proposition 3.5), the commutativity of $\mathcal{P}_k^+$ and $\mathcal{P}_\ell^-$, for any $k,\ell\in\mathbb{R}_+$, as well as the strong continuity of $(\mathcal{P}_\ell^\pm)_{\ell\in\mathbb{R}_+}$ (given as in Proposition 3.4), we obtain that, for any $f\in C_{e,\mathrm{cdl}}^{\mathrm{ac}}(\mathbb{R}_+)$ and $\ell\in\mathbb{R}_+$,

$$\Gamma^+\mathcal{P}_\ell^- f=\lim_{\delta\ell\to 0+}\frac{1}{\ell}\big(\mathcal{P}_{\delta\ell}^+-I\big)\mathcal{P}_\ell^- f=\lim_{\delta\ell\to 0+}\frac{1}{\ell}\mathcal{P}_\ell^-\big(\mathcal{P}_{\delta\ell}^+-I\big)f=\mathcal{P}_\ell^-\Gamma^+ f.$$

The other identity can be shown using similar arguments. The proof of the lemma is now complete.

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