Finance Stochast. 4, 1–33 (2000)

Finance and
Stochastics

Risk sensitive asset management with transaction costs

Tomasz R. Bielecki[1], **Stanley R. Pliska**[2]

[1] Department of Mathematics, The Northeastern Illinois University, 5500 North St. Louis Avenue, Chicago, IL 60625-4699, USA (e-mail: t-bielecki@neiu.edu)
[2] Department of Finance, University of Illinois at Chicago, 601 S. Morgan St., Chicago, IL 60607-7124, USA (e-mail: srpliska@uic.edu)

Abstract. This paper develops a continuous time risk-sensitive portfolio optimization model with a general transaction cost structure and where the individual securities or asset categories are explicitly affected by underlying economic factors. The security prices and factors follow diffusion processes with the drift and diffusion coefficients for the securities being functions of the factor levels. We develop methods of risk sensitive impulsive control theory in order to maximize an infinite horizon objective that is natural and features the long run expected growth rate, the asymptotic variance, and a single risk aversion parameter. The optimal trading strategy has a simple characterization in terms of the security prices and the factor levels. Moreover, it can be computed by solving a *risk sensitive quasi-variational inequality*. The Kelly criterion case is also studied, and the various results are related to the recent work by Morton and Pliska.

Key words: Risk-sensitive impulsive stochastic control, quasi-variational inequalities, optimal portfolio selection, incomplete markets, transaction costs

JEL Classification: G11

Mathematics Subject Classification (1991): 90A09, 93E20, 49N25

1 Introduction

The mathematical problem of optimally managing a portfolio of securities when there are transaction costs has received considerable research attention in recent years. For the classical problem where the objective is to maximize expected utility of terminal wealth, most of the attention has been devoted to the case of

Mansucript received: July 1998; final version received: January 1999

proportional transaction costs, that is, to the case where the cost associated with a transaction is proportional to the amount of money that is shifted between the securities. Representative of work in this category are the papers by Cvitanic and Karatzas [10], Davis and Norman [11], Taksar et al. [33], Fleming et al. [15], and Shreve and Soner [31]. Typically, the optimal strategy is characterized by a no-trade region, with trading that is essentially continuous on its boundary used to keep a certain process contained in the region.

Other authors assumed the transaction cost has a fixed component, thereby precluding the optimality of continuous trading. For example, Morton and Pliska [26] (see also Pliska and Selby [28]) assumed the transaction cost is proportional to the current value of the portfolio, and they showed the optimal strategy is characterized by a fixed vector of portfolio proportions together with an optimal stopping rule, a pair which can be computed by solving a free boundary problem. Eastham and Hastings [12] and Hastings [17] considered a more general set-up for transaction costs having a fixed component, although they obtained less specific results. Instead of using optimal stopping theory, they characterized the optimal strategies in terms of quasi-variational inequalities. Indeed, the approach that will be presented in this paper was largely inspired by the Eastham and Hastings work which, in turn, seems to have been inspired by Bensoussan and Lions [3].

In very recent work Korn [20] applied the theories of optimal stopping and quasi-variational inequalities to two portfolio management problems, both involving two assets (a bank account with interest rate zero and a geometric Brownian motion stock) and transaction costs of the form

$$K + k|\Delta S|,$$

where ΔS is the amount of funds added to the stock position when a transaction occurs. One problem is to maximize expected utility of wealth at a fixed, finite time horizon; the other is to maximize expected discounted utility of consumption over an infinite planning horizon, where consumption occurs in discrete lumps at discrete intervention times. Korn characterized the optimal solutions and showed how they may be computed with iterative schemes. His asset and transaction cost models are special cases of ours, but his two portfolio management problems do not overlap with our infinite horizon risk sensitive objective.

In all of this literature the assets are modeled with conventional stochastic differential equations as in Merton [24], for example. In other words, the only physical processes being modeled are the assets themselves along with the bank account and its short term interest rate. Moreover, there is no explicit dependence of the asset processes on this interest rate. Meanwhile, Bielecki and Pliska [5], Bielecki et al. [6], Brennan et al. [9], Brennan and Schwartz [8], and Merton [25] have developed transaction-free optimal portfolio models where the asset processes explicitly depend on underlying economic factors which are also explicitly modeled with stochastic differential equations. This allows the return processes for the assets to be affected by economic factors such as interest rates, unemployment rates, and dividend yields. The optimal strategies therefore depend on the levels of these factors.

As explained more fully in Bielecki et al. [6], there are at least two reasons why it is desirable to explicitly include factor processes in the optimization model. First, factors are often used to make forecasts of asset returns, so their inclusion facilitates understanding of the statistical issues and estimation difficulties. Second, the optimal strategies that are obtained when the factor processes are included are often different from, and thus superior to, those obtained with the certainty equivalence approach. In other words, the naive approach of first computing statistical estimates of asset drift and diffusion coefficients by conducting, say, linear regressions of returns against factor levels and then substituting these statistical estimates in formulas that emerge from conventional optimization models will lead to strategies that are not optimal. This difference is sometimes called the "hedging effect" in the financial economics literature.

The aim of this paper is to combine these two streams of research and develop a portfolio optimization model that features transaction costs as well as economic factors which affect the assets. The transaction costs will have a fixed component, so we will follow the impulse control approach taken by Eastham and Hastings [12] and Hastings [17]. We will also take the risk sensitive control theory approach developed by Bielecki and Pliska [5] in order to solve models which have factor processes included. As explained in Bielecki et al. [6], the risk sensitive objective is to maximize the risk adjusted exponential growth rate (i.e., the volatility adjusted geometric mean return), and so the resulting set-up is analogous to the Markowitz single period model, except the risk and mean return measures are with respect to an infinite planning horizon, and points on the efficient frontier are computed only approximately.

The main result in this paper is our characterization of optimal trading strategies in terms of what we call *risk sensitive quasi-variational inequalities* (RS-QVI). This is presented in Sect. 4 along with two corollaries which give sufficient conditions for ruling out optimal strategies where one should rebalance more than once at the same point in time. The problem formulation and some preliminary results are presented in Sects. 2 and 3, respectively.

The results in Sect. 4 are for cases where the risk adjustment to the exponential growth rate is nontrivial. With no adjustment one has the classical objective of maximizing the exponential growth rate, that is, maximizing expected log utility. Also called the Kelly criterion, this is what we call the risk null criterion and is the subject of Sect. 5. We characterize an optimal solution corresponding to this criterion in terms of what we call *risk null quasi-variational inequalities* (RN-QVI).

Finally, in Sect. 6 we revisit the Morton and Pliska [26] problem, showing explicitly how it is solved via risk sensitive quasi-variational inequalities. Various proofs are relegated to an appendix.

It is important to stress that in this paper we do not study the existence and uniqueness of solutions of our quasi-variational inequalities. This will be an objective for a future publication. However, the examples discussed in Sect. 6 do lead to explicit solutions for both the RS-QVI and the RN-QVI.

2 Formulation of the problem

We shall consider a market consisting of $m \geq 1$ risky securities, one risk-free security (a bank account) and $n \geq 1$ factors. The set of risky securities may include stocks, bonds and derivative securities, as in [8] for example. The set of factors may include dividend yields, price-earning ratios, short-term interest rates, the rate of inflation, etc., as in Pesaran and Timmermann [27] for example.

Let $(\Omega, \{\mathscr{F}_t\}, \mathscr{F}, \mathbf{P})$ be the underlying probability space. Denoting by $S_i(t)$ the price of the i-th security and by $X_j(t)$ the level of the j-th factor at time t, we consider the following market model for the dynamics of the security prices and factors:

$$\begin{aligned} \frac{dS_0(t)}{S_0(t)} &= A_0(X(t))dt \quad (risk-free\ security) \\ \frac{dS_i(t)}{S_i(t)} &= A_i(X(t))dt + \sum_{k=1}^{m+n} \sigma_{ik}(X(t))dW_k(t), \quad i = 1, 2, \dots, m \\ S_i(0) &= s_i > 0, \quad i = 0, 1, 2, \cdots, m, \end{aligned} \tag{2.1}$$

$$dX(t) = B(X(t))dt + \Lambda(X(t))dW(t), \quad X(0) = x, \tag{2.2}$$

where $W(t)$ is a R^{m+n} valued standard Brownian motion process with components $W_k(t)$, $X(t)$ is the R^n valued factor process with components $X_j(t)$, and the market parameter functions $A(x) := [A_0(x)\ A_1(x) \dots A_m(x)]^T$, $\tilde{\Sigma}(x) := [\sigma_{ik}(x)]_{i=1,2,\dots,m}^{k=1,2,\dots,m+n}$, $\Lambda(x) := [\lambda_{jk}(x)]_{j=1,2,\dots,n}^{k=1,2,\dots,m+n}$, and $B(x)$ are matrix-valued functions of appropriate dimensions, which satisfy standard conditions as in Borodin and Salminen [7], III.4.17, page 45. It is well known that under such conditions a unique, non-explosive, strong solution exists for (2.1), (2.2). Moreover, the processes $S_i(t)$ are positive with probability 1 (see e. g. [19], chapter 5). This is an extension of the model that was studied by Bielecki and Pliska [5]. For the purpose of this paper, however, we shall need to assume a stronger condition about $\tilde{\Sigma}(x)$, namely, that it is a bounded function.

Let $\mathscr{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$, where $S(t) = (S_1(t), S_2(t), \dots, S_m(t))$ is the security price process. As usual, all filtrations considered here are assumed to be completed.

Due to the nature of transaction costs that are going to be considered, it will be appropriate to study impulsive investment strategies, rather than singular control strategies (see e.g. Fleming and Soner [16], Karatzas [18] and references therein for descriptions of the latter). An impulsive investment strategy $u = ((\tau_k, N_k),\ k = 0, 1, 2, \dots)$ is defined as follows:

(a) $\tau_0 \equiv 0 \leq \tau_1 \leq \dots \leq \tau_k \leq \tau_{k+1} \leq \dots$ are $(\mathscr{G}_t)$-stopping times (these are portfolio rebalancing times),

(b) $\tau_k \to \infty$ almost surely as $k \to \infty$,

(c) $N_k := [N_{k,0}\ N_{k,1} \dots N_{k,m}]^T$ is $\mathscr{G}_{\tau_k}$ measurable, where $N_{k,i}$ is the number of shares of security i to which the investor rebalances her/his portfolio at the transaction time τ_k, and

(d) $N_{k,i} \geq 0,\ i = 0, 1, 2, \dots$ for all $k \geq 0$.

Similarly as in Eastham and Hastings [12] we define for each impulsive investment strategy u a random sequence:

$$m_k^u := \begin{cases} \inf\{l \geq 1 : \ \tau_k < \tau_{k+l}\}, & \text{if } \tau_k < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.1 Note that in view of the condition (b) above we have that $m_k^u < \infty$ almost surely for each $k \geq 0$. Although some transaction cost structures will allow the investor to rebalance multiple times at the same point in time, sooner or later she/he must let the clock run. Thus for each ω such that $\tau_k(\omega) < \infty$ we shall have $\tau_k(\omega) = \tau_{k+1}(\omega) = \ldots = \tau_{k+m_k(\omega)-1}(\omega) < \tau_{k+m_k(\omega)}(\omega)$.

We shall need to consider in what follows the share holding process $N^u(t)$ defined for each impulsive investment strategy u as

$$N^u(t) = N_{k+m_k^u-1}, \quad t \in [\tau_{k+m_k^u-1}, \tau_{k+m_k^u}[, \quad k = 0, 1, 2, \ldots, m.$$

Let us also denote by $H(s, N)$ a vector valued function of $(s, N) \in (0, \infty)^{m+1} \times [0, \infty)^{m+1}$ with components

$$H_i(s, N) = \frac{s_i N_i}{s^T N}, \quad i = 0, 1, 2, \ldots, m \ , \ \textit{if} \ s^T N > 0,$$

otherwise

$$H_0(s, N) = 1, \quad H_i(s, N) = 0, \quad i = 1, 2, \ldots, m.$$

In order to simplify the exposition we shall write $h^u(t)$ instead of $H(S(t), N^u(t))$. Thus $h_i^u(t)$ represents the fraction of the investor's time-t wealth (generated by the impulsive investment strategy u) that is held in security i. (Compare also Morton and Pliska [26], Bielecki and Pliska [5].)

Remark 2.2 Observe that for any impulsive investment strategy, condition (d) implies that short selling of the securities is prohibited, and consequently it holds that

$$0 \leq \sup_{i=0,1,2,\ldots,m; \ t \geq 0} h_i^u(t) \leq 1.$$

Remark 2.3 Note that by Remark 2.2 and our assumption that $\tilde{\Sigma}(x)$ is bounded, for each impulsive investment strategy u and for each $t \geq 0$ we have

$$\mathbf{E} e^{(1/2) \int_0^t \|h^u(r))^T \Sigma(X(r))\|^2 dr} < \infty$$

where

$$\Sigma(x) = \begin{pmatrix} 0 & 0^T \\ 0 & \tilde{\Sigma}(x) \end{pmatrix}.$$

and where $\mathbf{E}$ is the expectation with respect to $\mathbf{P}$.

The transaction costs will be modeled by means of a function $C(s, N, N')$ representing the cost of the transaction when the security prices are $s \in (0, \infty)^{(m+1)}$ and the portfolio changes from $N \in [0, \infty)^{m+1}$ to the rebalanced portfolio $N' \in [0, \infty)^{m+1}$. We assume the following about C:
(c1) C is lower-semi-continuous (see Bertsekas and Shreve [4], Definition 7.13),
(c2) $C(s, N, N') \geq 0$ for all $s \in (0, \infty)^{m+1}$ and $N, N' \in [0, \infty)^{m+1}$,
(c3) For each $0 < \kappa < \infty$ there exists some number $\delta > 0$ such that $\inf_{\{s,N,N':s^T N \leq \kappa\}} \left[\frac{C(s,N,N')}{s^T N}\right] \geq \delta$.

Remark 2.4 A popular example of a cost function of the above type would be the *"proportional to the transaction volume"* cost function: $C(s, N, N') := c + c_1|s^T(N - N')|, \quad c > 0, \quad c_1 \geq 0$, for example. (Frequently it will also hold that $C(s, N, N) = c$, as in this example.) Another important example is the *"proportional to the investor's wealth level"* cost function considered in Morton and Pliska [26]: $C(s, N, N') := \alpha s^T N$, for some constant $\alpha \in (0, 1)$.

To the transaction cost function C there is associated a multifunction $\mathscr{A}$ representing sets of admissible transactions:

$$\begin{aligned} \mathscr{A}(s, N) &= \{N' \in [0, \infty)^{m+1} \ : \ s^T N - C(s, N, N') \geq s^T N'\}, \\ &\text{for} \quad (s, N) \in (0, \infty)^{(m+1)} \times [0, \infty)^{m+1}. \end{aligned}$$

Each set $\mathscr{A}(s, N)$ is compact. Boundedness is obvious. In order to see closedness, let N'_k be a convergent sequence so that $N'_k \in \mathscr{A}(s, N)$ for each k, and $\lim N'_k = N'$. Since $s^T(N - N'_k) \geq C(s, N, N'_k)$ for each k, then taking lim inf on both sides and using $(c1)$ we obtain that $s^T(N - N') \geq C(s, N, N')$, and thus $N' \in \mathscr{A}(s, N)$.

The following natural condition is analogous to (1.4) in [12]:

$$\mathscr{A}(s, N) \neq \emptyset \iff [0\ 0\ 0 \ldots 0]^T \in \mathscr{A}(s, N). \tag{2.3}$$

We shall be assuming (2.3) in what follows. In particular this condition is satisfied for the examples of the two cost functions given in Remark 2.4.

Remark 2.5 Conditions $(c1)$ and (2.3) imply that the set $\mathbf{A} = \{(s, N) : \mathscr{A}(s, N) \neq \emptyset\}$ is closed. Similarly as in Lemma 2.7 of [12] it can be easily demonstrated that the multifunction $\mathscr{A}$ is upper-semi-continuous (u.s.c.) in the sense of Kuratowski (see [23]) on the set $\mathbf{A}$.

We can now define an admissible impulsive investment strategy:

Definition 2.1 *An impulsive strategy* $u = ((\tau_k, N_k),\ k = 0, 1, 2, \ldots)$ *is* **admissible** *if and only if*

$$N_k \in \mathscr{A}(S(\tau_k), N_{k-1}), \quad S^T(\tau_k)N_k > 0, \quad k = 0, 1, 2, \ldots,$$

where N_{-1} *denotes the portfolio held prior to the first rebalancing time* $\tau_0 = 0$*, about which we assume that* $S^T(0)N_{-1} > 0$*. We let* $\mathscr{U}$ *denote the set of admissible impulsive investment strategies.*

In this paper we shall investigate the following family of risk sensitized optimal investment problems, labeled as $(\mathbf{P}_\theta)$:

for $\theta \in (0, \infty)$, maximize the risk sensitized expected exponential growth rate of the investor's portfolio, namely

$$J_\theta(s, x; u) := \liminf_{t\to\infty}(-2/\theta)t^{-1} ln\ \mathbf{E}\ [(S^T(t)N^u(t))^{-\frac{\theta}{2}}|S(0) = s, X(0) = x], \quad (2.4)$$

over the class of all $u \in \mathcal{U}$, subject to (2.1) and (2.2).

Remark 2.6 We are using the terminology *expected exponential growth rate of the investor's portfolio* due to the fact that for each t the expected value of the investor portfolio is of the order of magnitude equal to $e^{(expected\ \ exponential\ \ growth\ \ rate)t}$. For the interpretation and the discussion of the functional (2.4) please see Bielecki and Pliska [5] for example.

Remark 2.7 Let us denote by $V^u(t)$ the investor's wealth at time t corresponding to the impulsive investment strategy u, that is, $V^u(t) = S^T(t)N^u(t)$. The time-t cumulative return on the investor's portfolio, adjusted for the portfolio's volatility, is given by

$$AR^u(t) := \int_0^t \frac{dV^u(r)}{V^u(r-)} - (\frac{1}{2})\int_0^t h^u(r)^T \Sigma(X(r))\Sigma^T(X(r))h^u(r)dr.$$

According to $It\hat{o}'s$ formula we have, for $t \geq 0$,

$$\begin{aligned} lnV^u(t) - lnV^u(0) &= AR^u(t) + \sum_{0<r\leq t}\Big[ln\big(V^u(r)/V^u(r-)\big) \\ &\quad -\Big(\big(V^u(r) - V^u(r-)\big)/V^u(r-)\Big)\Big]. \end{aligned}$$

Because $(V^u(t))^{-(\frac{\theta}{2})} = e^{\left\{-(\frac{\theta}{2})lnV^u(t)\right\}}$, then maximizing the risk sensitized expected exponential growth rate of the investor's portfolio, as in (2.4), is related to maximizing the risk sensitized expected growth rate of the investor's portfolio cumulative return adjusted for the portfolio's volatility.

It is perhaps interesting to observe that maximizing the risk sensitized expected exponential growth rate of the investor's portfolio, under the restriction that the resulting wealth processes are continuous, is **equivalent** to maximizing the risk sensitized expected growth rate of the investor's portfolio cumulative return adjusted for the portfolio's volatility, because then we have

$$lnV^u(t) - lnV^u(0) = AR^u(t).$$

This is not the case here, however, since implementation of non-trivial admissible impulsive strategies prohibits the resulting wealth processes from being continuous.

Remark 2.8 The positive value of the risk sensitivity parameter θ corresponds to a risk averse investor (see Bielecki and Pliska [5] for example). The risk null case, where $\theta = 0$, will be studied in Sect. 5. This case can be thought of as the limit of the risk averse situations as the risk sensitivity parameter θ goes to zero (compare discussion at the end of Sect. 4).

The main result of this paper, that will be presented in Theorem 4.1, is that the solution to the above family of problems is characterized in terms of the risk sensitive quasi-variational inequality (RS-QVI) (4.1). Before that, however, we need some preliminary results.

3 Auxiliary results

In this section we shall obtain a convenient transformation of the functional $J_\theta(s, x; u)$.

Towards this end we first observe that:

$$\begin{aligned} ln(S^T(t)N^u(t)) &= \sum_{k=0}^{\infty} \chi_{\{\tau_k \le t\}} \Bigg[\int_{\tau_k}^{(\tau_{k+1}\wedge t)-} d(lnS^T(r)N_k) \\ &\quad + \ln(S^T(\tau_k)N_k) - \ln(S^T(\tau_k)N_{k-1}) \Bigg] , \end{aligned} \tag{3.1}$$

where $\chi_{\{\tau \le t\}} = 1$ if $\tau \le t$ and $= 0$ otherwise.

Let us now define $g_1(s, N) := ln(s^T N)$, $g_2(s, N) := -ln(s^T N)$, the scalar valued function $f(x, h) := A^T(x)h$, and the $1 \times (m + n)$ vector valued function $\gamma(x, h) = \tilde{h}^T \tilde{\Sigma}(x)$, where $h = [h_0\ h_1\ \dots\ h_m] \in R^{m+1}$ and $\tilde{h} := [h_1\ h_2\ \dots\ h_m]$. Using $It\hat{o}'s$ formula we thus obtain:

$$\begin{aligned} ln(S^T(t)N^u(t)) &= \int_0^t f(X(r), h^u(r))dr \\ &\quad + \sum_{k=0}^{\infty} \chi_{\{\tau_k \le t\}} \Big[g_1(S^T(\tau_k), N_k) + g_2(S^T(\tau_k), N_{k-1}) \Big] \\ &\quad - (1/2) \int_0^t \|\gamma(X(r), h^u(r))\|^2 dr \\ &\quad + \int_0^t \gamma(X(r), h^u(r)) dW(r). \end{aligned} \tag{3.2}$$

Consequently, we obtain

$$\begin{aligned} (S^T(t)N^u(t))^{-\frac{\theta}{2}} &= \exp\Bigg\{ -\frac{\theta}{2} \Bigg(\int_0^t f_\theta(X(r), h^u(r))dr \\ &\quad + \sum_{k=0}^{\infty} \chi_{\{\tau_k \le t\}} \Big[g_1(S^T(\tau_k), N_k) + g_2(S^T(\tau_k), N_{k-1}) \Big] \Bigg) \end{aligned}$$

$$-(1/2)\int_0^t \|\gamma_\theta(X(r), h^u(r))\|^2 dr$$
$$+\int_0^t \gamma_\theta(X(r), h^u(r))dW(r)\Bigg\}, \qquad (3.3)$$

where

$$f_\theta(x,h) := f(x,h) - (1/2)(\frac{\theta}{2}+1)\|\gamma(x,h)\|^2$$

and

$$\gamma_\theta(x,h) := -(\frac{\theta}{2})\gamma(x,h).$$

Remark 3.1 Observe that the function $f_\theta(x,h)$ is equal to the negative of the argument minimized in $K_\theta(x)$ of Bielecki and Pliska [5].

Owing to Remark 2.3, for each $u \in \mathscr{U}$ and $\theta > 0$ we can define an equivalent measure $\mathbf{P}^{u,\theta}$ by:

$$\frac{d\mathbf{P}^{u,\theta}}{d\mathbf{P}}\bigg|_{\mathscr{F}_t} = \exp\Bigg\{-(1/2)\int_0^t \|\gamma_\theta(X(r), h^u(r))\|^2 dr$$
$$+\int_0^t \gamma_\theta(X(r), h^u(r))dW(r)\Bigg\}. \qquad (3.4)$$

(Note that we do not claim the uniqueness of the measure $\mathbf{P}^{u,\theta}$.) Thus we can finally reformulate the functional $J_\theta(s,x;u)$ in terms of $\mathbf{P}^{u,\theta}$ as follows:

$$J_\theta(s,x;u) = \liminf_{t\to\infty}(-2/\theta)t^{-1} ln\ \mathbf{E}^{u,\theta}$$
$$\times\Bigg[\exp\Bigg\{-\frac{\theta}{2}\Bigg[\int_0^t f_\theta(X(r), h^u(r))dr$$
$$+\sum_{k=0}^{\infty}\chi_{\{\tau_k\le t\}}\Big[g_1(S^T(\tau_k), N_k)$$
$$+g_2(S^T(\tau_k), N_{k-1})\Big]\Bigg]\Bigg\}\Bigg|S(0)=s, X(0)=x\Bigg], \qquad (3.5)$$

where $\mathbf{E}^{u,\theta}$ denotes expectation under $\mathbf{P}^{u,\theta}$. Observe that the term under the exponent is written in the form typical for the theory of impulsive stochastic control (see e.g. Bensoussan and Lions [3], Bensoussan [2], Robin [29], Stettner [32] and references therein).

4 The characterization theorem

In this section we shall characterize optimal, risk-sensitive, impulsive investment strategies for the family of optimization problems P_θ introduced in Sect. 2.

For each $u \in \mathscr{U}$ it will be convenient to introduce a piece-wise *Itô* process: $Y^u(t) := (S(t), X(t),$

$N^u(t))^T$. We shall be denoting the state of the process $Y^u(t)$ by $y = (s, x, N)^T \in \mathscr{O} := (0, \infty)^{m+1} \times R^n \times [0, \infty)^{m+1}$. For future use we shall also denote $\mathscr{O}' := (0, \infty)^{m+1} \times R^n$.

Between two consecutive but separated transaction times τ_{k+m_k-1} and τ_{k+m_k} the process $Y^u(t)$ is an Itô process with a differential operator, denoted by L^θ, under the measure $\mathbf{P}^{u,\theta}$. Before we give the expression for the operator, we need to introduce the notation:

$$\mathbf{A}(s,x) := Diag[A_i(x), i = 0, 1, 2, \ldots, m]s,$$

$$\mathbf{S}(s,x) := \begin{pmatrix} 0\ 0 \quad \ldots \quad 0\ 0 \\ s_1\tilde{\Sigma}_1(x) \\ s_2\tilde{\Sigma}_2(x) \\ . \\ . \\ . \\ s_m\tilde{\Sigma}_m(x) \end{pmatrix}, \quad \tilde{\Sigma}_i(x) := \text{the i-th row of } \tilde{\Sigma}(x),$$

$$\alpha(y) := \begin{pmatrix} \mathbf{A}(s,x) \\ B(x) \\ 0 \end{pmatrix},$$

$$\beta(y) := \begin{pmatrix} \mathbf{S}(s,x) \\ \Lambda(x) \\ 0 \end{pmatrix},$$

$$\alpha_\theta(y) := \alpha(y) + \beta(y)\gamma_\theta^T(s, H(x,N)).$$

Observe that the dimensions of the above matrices are:

$$\mathbf{A}(s,x) \quad - \quad ((m+1) \times 1),$$

$$\mathbf{S}(s,x) \quad - \quad ((m+1) \times (m+n)),$$

$$\alpha(y) \quad - \quad ((2m+2+n) \times 1),$$

$$\beta(y) \quad - \quad ((2m+2+n) \times (m+n)).$$

Remark 4.1 Let $W^{u,\theta}(t)$ be a Brownian motion under $\mathbf{P}^{u,\theta}$. If $\tau_k < \tau_{k+1}$ then the process $Y^u(t)$ satisfies the following SDE for $t \in [\tau_k, \tau_{k+1}[$:

$$dY^u(t) = \alpha_\theta(Y^u(t))dt + \beta(Y^u(t))dW^{u,\theta}(t),$$

with the initial condition

$$Y^u(\tau_k) = (S(\tau_k), X(\tau_k), N_k).$$

The operator L^θ can now be defined as

$$L^\theta\phi(y) := \alpha_\theta^T(y)\phi_y(y) + (1/2)tr\left(\beta(y)\beta^T(y)\phi_{yy}(y)\right).$$

Remark 4.2 Note that L^θ does not really act on the component N of y, that is, $L^\theta\phi(y)$ does not depend on N. This is because the process $N^u(t)$ is fixed between two consecutive impulse times.

Let us also define another operator:

$$M^\theta\phi(y) := \sup_{N'\in\mathscr{A}(s,N)}\left[\left(\frac{\theta}{2}\right)(g_1(s,N') + g_2(s,N)) + \phi(s,x,N')\right],$$

where, as usual, the *supremum* taken over the *empty set* equals $-\infty$. It is perhaps interesting to observe that the operator M^θ is a *"local optimization"* operator acting on a *"bias function"* ϕ. The terminology "bias function" is borrowed here from the theory of stochastic control with long-run-average optimization criteria.

We shall be considering the following risk-sensitive quasi-variational inequality (RS-QVI), which is to be solved for a constant λ and a function ϕ on $\mathscr{O}$:

$$\begin{aligned}
L^\theta\phi(y) - (1/2)\|\phi_y^T(y)\beta(y)\|^2 - \left(\frac{\theta}{2}\right)\left(\lambda - \overline{f}_\theta(y)\right) &\leq 0, \quad y\in int\mathscr{O},\\
-M^\theta\phi(y) + \phi(y) &\geq 0, \quad y\in\mathscr{O},\\
\left[L^\theta\phi(y) - (1/2)\|\phi_y^T(y)\beta(y)\|^2 - \left(\frac{\theta}{2}\right)\left(\lambda - \overline{f}_\theta(y)\right)\right]& \\
\left[-M^\theta\phi(y) + \phi(y)\right] &= 0, \quad y\in int\mathscr{O},
\end{aligned}\tag{4.1}$$

where $\overline{f}_\theta(y) := f_\theta(x, H(s,N))$, and where $int\mathscr{O}$ is the interior of the domain $\mathscr{O}$. Related to the RS-QVI (4.1) is the *continuation set*:

$$\mathscr{K}_{\mathscr{R}\mathscr{S}} := \{y\in\mathscr{O} : -M^\theta\phi(y) + \phi(y) > 0\}.$$

We now suppose $(\lambda^\theta, \phi^\theta)$ is a solution pair for (4.1) with ϕ^θ being sufficiently regular. We denote the continuation set corresponding to this pair by $\mathscr{K}^\theta$.

Remark 4.3 The function ϕ^θ does not have to be a classical (i.e. smooth) solution. What we require is that ϕ^θ is upper-semi-continuous on $\mathscr{O}$ and that we can apply a (generalized) Itô formula to it in the proofs below. This will be satisfied if, for example, $\phi^\theta(\cdot,\cdot,N)$ is in the Sobolev space $W^{2,p}(\mathscr{O}')$ for a sufficiently large p, for every N. In addition we require that ϕ_y^θ can be used to construct the measure $\mathscr{P}^{u,\theta}$ for the purpose of stating Theorem 4.1 below.

The study of existence of a pair $(\lambda^\theta, \phi^\theta)$ satisfying the above properties will be the subject of future work. Note that once existence of such a pair is demonstrated, then it will follow from the verification Theorem 4.1 below that λ^θ (but not necessarily ϕ^θ) is unique.

Whenever the generalized Itô formula is to be used in the proofs below, then it will be assumed that the non-degeneracy condition in Krylov [21], Sect. 2.10, is satisfied for the function $\tilde{\beta}(y)$, which is obtained from $\beta(y)$ by deleting the first row and the last $m+1$ rows.

Remark 4.4 The function $M^\theta\phi^\theta(y)$ is an upper-semi-continuous function of $y = (s, x, N)$ with $(s, N) \in \mathbf{A}$ and $x \in R^n$ (compare Lemma 2.8 in [12], or Proposition 7.32 in [4]).

We shall now construct what will turn out to be an optimal sequence $u^\theta = ((\tau_k^\theta, N_k^\theta),\ k = 0, 1, 2, \ldots)$ and a corresponding process $Y^{u^\theta}(t)$, which we shall denote for simplicity as $Y^\theta(t)$. To this end it will be convenient to introduce a measurable selector, which exists due to Remark 2.5 and Remark 4.4 (see e.g [12] and [30] for details):

$$\mathscr{N}^\theta(s, x, N) := \mathrm{argmax}_{N' \in \mathscr{A}(s,N)} \left[\left(\frac{\theta}{2}\right) (g_1(s, N') + g_2(s, N)) + \phi^\theta(s, x, N') \right].$$

Next we set $\tau_0^\theta = 0$. In addition if $(S(0), X(0), H(S(0), N_{-1}), N_{-1}) \in \mathscr{K}^\theta$ then we set $N_0^\theta = N_{-1}$; otherwise we set

$$N_0^\theta = \mathscr{N}^\theta(S(0), X(0), N_{-1}).$$

In general, for $k \geq 1$ we define (as usual, the inf taken over an empty set is $+\infty$):

$$\tau_k^\theta = \mathit{inf}\{t \geq \tau_{k-1}^\theta :\ (S(t), X(t), N_{k-1}^\theta) \notin \mathscr{K}^\theta\},$$

and

$$N_k^\theta = \mathscr{N}^\theta(S(\tau_k^\theta), X(\tau_k^\theta), N_{k-1}^\theta).$$

For ease of exposition we also define

$$m_k^\theta := m_k^{u^\theta}, \quad k = 0, 1, 2, \ldots$$

It will follow from Lemma 4.1 below that $m_k^\theta < \infty$ for each k. Finally we define

$$N^\theta(t) := N_{k+m^\theta-1}, \quad \text{for} \quad t \in [\tau^\theta_{k+m_k^\theta-1},\ \tau^\theta_{k+m_k^\theta}), \quad k = 0, 1, 2, \ldots\ ,$$

and

$$Y^\theta(t) := (S(t), X(t), N^\theta(t)), \quad t \geq 0.$$

Remark 4.5 Observe that the following important equality follows from the definition of the operator M^θ and from the above construction:

$$\phi^\theta(S(\tau_k^\theta), X(\tau_k^\theta), N_{k-1}^\theta) = \left(\frac{\theta}{2}\right) \Bigg(g_1(S(\tau_k^\theta), N_k^\theta) +$$

$$+ g_2(S(\tau_k^\theta), N_{k-1}^\theta) \Bigg) + \phi^\theta((S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta).$$

Moreover, if $\tau_{k-1}^\theta < \tau_k^\theta$, then

$$\phi^\theta(Y^\theta(\tau_k^\theta -)) = \left(\frac{\theta}{2}\right) \left(g_1(S(\tau_k^\theta), N_k^\theta) + g_2(S(\tau_k^\theta), N_{k-1}^\theta) \right) + \phi^\theta(Y^\theta(\tau_k^\theta)).$$

Remark 4.6 Observe that almost surely for each $k = 0, 1, 2, \dots$ we have that

$$S(\tau_k^\theta)^T N_k^\theta > 0.$$

In fact, let us fix $k \geq 0$ and assume that $S(\tau_k^\theta)^T N_{k-1}^\theta > 0$. Next we observe that $S(\tau_k^\theta)^T N_k^\theta = 0$ if and only if $g_1(S(\tau_k^\theta)^T N_k^\theta) = -\infty$. But then of course the equalities in Remark 4.5 are violated, a contradiction. Since we initially assumed in definition 2.1 that $S(0)^T N_{-1} > 0$ we thus conclude that $S(\tau_k^\theta)^T N_k^\theta > 0$ holds almost surely for each $k = 0, 1, 2, \dots,$.

We shall demonstrate in Lemma 4.1 that the sequence u^θ constructed as above is an admissible impulsive investment strategy. Then in Theorem 4.1 we shall demonstrate that this is in fact an optimal impulsive investment strategy for P^θ and that λ^θ is the optimal value of our risk-sensitive objective criterion.

Lemma 4.1 *Let us assume all the conditions of Sect. 2. Then the sequence u^θ constructed above constitutes an admissible impulsive investment strategy for P^θ.*

Proof. See the Appendix. □

For each $u \in \mathscr{U}$ we define a new measure $\mathscr{P}^{u,\theta}$ by

$$\left.\frac{d\mathscr{P}^{u,\theta}}{d\mathbf{P}^{u,\theta}}\right|_{\mathscr{F}_t} = \exp\bigg\{ -(1/2)\int_0^t \|\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))\|^2 dr - \int_0^t \phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r)\bigg\},$$

where $W^{u,\theta}(t)$ is a Brownian motion under $\mathbf{P}^{u,\theta}$.

Remark 4.7 In Remark 4.3 we postulated that the function $\phi^\theta(y)$ is such that $\mathscr{P}^{u,\theta}$ is well defined as a probability measure. This will be the case if, for example, the folowing condition is satisfied:

$$\mathbf{E}^{u,\theta} e^{(1/2)\int_0^t \|\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))\|^2 d(r)} < \infty, \quad \textit{for all} \quad t \geq 0.$$

As stated in Remark 4.3, existence of such a $\phi^\theta(y)$ will be the subject of future study. Section 6 below provides a non-trivial, interesting example of a problem for which the corresponding function $\phi^\theta(y)$ satifies the required property.

The following characterization theorem is the main result of this paper.

Theorem 4.1 *Let us assume all the conditions of Sect. 2. Consider the pair $(\lambda^\theta, \phi^\theta)$ and assume additionally that the function ϕ^θ satisfies the following conditions for each $u \in \mathscr{U}$, s and x:*

$$\liminf_{t\to\infty}(-2/\theta)t^{-1} ln\ \mathscr{E}^{u,\theta}\Big[\exp\Big(\phi^\theta(Y^u(t))\Big)\Big| S(0) = s,\ X(0) = x\Big] \leq 0,$$

and

$$\liminf_{t\to\infty}(-2/\theta)t^{-1}ln\ \mathscr{E}^{u^{\theta},\theta}\Big[\exp\Big(\phi^{\theta}(Y^{\theta}(t))\Big)\Big|S(0)=s,\ X(0)=x\Big]=0,$$

where $\mathscr{E}^{u,\theta}$ denotes the expectation with respect to $\mathscr{P}^{u,\theta}$. Then the sequence u^{θ} constructed above is an optimal impulsive investment strategy for $\mathbf{P}_{\theta}$. Moreover, the constant λ^{θ} is the optimal value of the risk-sensitive objective criterion (2.3).

Proof. We have already demonstrated in Lemma 4.1 that u^{θ} is an admissible impulsive investment strategy.
To show that u^{θ} is optimal, fix an arbitrary admissible impulsive investment strategy $u=((\tau_k,N_k),\ k=0,1,2,\ldots)$, and let $Y^u(t)$ correspond to this u.
Fix $k\geq 1$ so that $\tau_{k-1}<\tau_k<\infty$, and take $t\in[\tau_{k+m_k^u-1},\tau_{k+m_k^u}[$. From $It\hat{o}'s$ formula and from our RS-QVI we deduce that

$$\exp\left\{\left(-\frac{\theta}{2}\right)\int_{\tau_{k+m_k^u-1}}^{t}(\lambda^{\theta}-\overline{f}_{\theta}(Y^u(r)))dr\right\}$$

$$=\exp\left\{(1/2)\int_{\tau_{k+m_k^u-1}}^{t}\|\phi_y^{\theta^T}(Y^u(r))\beta(Y^u(r))\|^2dr-\int_{\tau_{k+m_k^u-1}}^{t}L^{\theta}\phi^{\theta}(Y^u(r))dr\right\}$$

$$=\exp\left\{\phi^{\theta}(Y^u(\tau_{k+m_k^u-1}))-\phi^{\theta}(Y^u(t))+(1/2)\int_{\tau_{k+m_k^u-1}}^{t}\|\phi_y^{\theta^T}(Y^u(r))\beta(Y^u(r))\|^2dr\right.$$

$$\left.+\int_{\tau_{k+m_k^u-1}}^{t}\phi_y^{\theta^T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r)\right\}.$$

This implies that, using the definition of the operator M^{θ} and the fact that $\tau_k=\tau_{k+m_k^u-1}$,

$$\exp\left\{\left(-\frac{\theta}{2}\right)\int_{\tau_k}^{t}(\lambda^{\theta}-\overline{f}_{\theta}(Y^u(r)))dr\right\}$$

$$\leq\exp\left\{\phi(Y^u(\tau_k-))+\sum_{l=k}^{k+m_k^u-1}\left[\left(-\frac{\theta}{2}\right)(g_1(S(\tau_k),N_l)+g_2(S(\tau_k),N_{l-1}))\right]\right.$$

$$-\phi(Y^u(t))+(1/2)\int_{\tau_k}^{t}\|\phi_y^{\theta^T}(Y^u(r))\beta(Y^u(r))\|^2dr$$

$$\left.+\int_{\tau_k}^{t}\phi_y^{\theta^T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r)\right\}.$$

Next, using $It\hat{o}$ again to substitute for $\phi^{\theta}(Y^u(\tau_k-))$, and then combining integrals, we get

$$\exp\left\{\left(-\frac{\theta}{2}\right)\int_{\tau_k}^{t}(\lambda^{\theta}-\overline{f}_{\theta}(Y^u(r)))dr\right\}$$

$$\leq\exp\left\{\phi^{\theta}(Y^u(\tau_{k-1}))+\int_{\tau_{k-1}}^{\tau_k}L^{\theta}\phi^{\theta}(Y^u(r))dr\right.$$

$$+\sum_{l=k}^{k+m_k^u-1}\left[\left(-\frac{\theta}{2}\right)(g_1(S(\tau_k),N_k)+g_2(S(\tau_k),N_{k-1}))\right]$$

$$-\phi(Y^u(t))+(1/2)\int_{\tau_k}^{t}\|\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))\|^2dr$$

$$+\int_{\tau_{k-1}}^{t}\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r)\bigg\}$$

$$=\exp\bigg\{\phi(Y^u(\tau_{k-1}))+(\frac{\theta}{2})\int_{\tau_{k-1}}^{\tau_k}(\lambda^\theta-\overline{f}_\theta(Y^u(r)))dr$$

$$+\sum_{l=k}^{k+m_k^u-1}\left[\left(-\frac{\theta}{2}\right)(g_1(S(\tau_k),N_k)+g_2(S(\tau_k),N_{k-1}))\right]$$

$$-\phi(Y^u(t))+(1/2)\int_{\tau_{k-1}}^{t}\|\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))\|^2dr$$

$$+\int_{\tau_{k-1}}^{t}\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r)\bigg\},$$

where the equality follows from the RS-QVI and a further combining of integrals. Continuing until $k=1$ we eventually obtain

$$\exp\left\{\left(-\frac{\theta}{2}\right)\int_{\tau_k}^{t}(\lambda^\theta-\overline{f}_\theta(Y^u(r)))dr\right\}$$

$$\leq\exp\bigg\{\phi^\theta(Y^\theta(0))+(\frac{\theta}{2})\int_0^{\tau_k}(\lambda^\theta-\overline{f}_\theta(Y^u(r)))dr$$

$$-\left(\frac{\theta}{2}\right)\sum_{l=0}^{k+m_k^u-1}\left[g_1(S(\tau_l),N_l)+g_2(S(\tau_l),N_{l-1})\right]$$

$$-\phi(Y^u(t))+(1/2)\int_0^{t}\|\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))\|^2dr$$

$$+\int_0^{t}\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r)\bigg\}.$$

In other words for arbitrary $t\geq 0$ we get

$$\exp\left\{-\left(\frac{\theta}{2}\right)\int_0^t\overline{f}_\theta(Y^u(r))dr-\left(\frac{\theta}{2}\right)\sum_{l=0}^{\infty}\chi_{\tau_l\leq t}\left[g_1(S(\tau_l),N_l)+g_2(S(\tau_l),N_{l-1})\right]\right\}$$

$$\geq\exp\bigg\{-\phi(Y^\theta(0))-\left(\frac{\theta}{2}\right)t\lambda^\theta+\phi(Y^u(t))$$

$$-(1/2)\int_0^t\|\phi^{\theta\,T}(Y^u(r))\beta(Y^u(r))\|^2dr-\int_0^t\phi_y^{\theta\,T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r)\bigg\}.$$

This means by (3.6) that

$$J_\theta(s,x;u) \quad \leq \quad \lambda^\theta + \liminf_{t\to\infty}(-2/\theta)t^{-1} ln \ \mathscr{E}^{u,\theta}$$
$$\left[\exp\left(\phi^\theta(Y^u(t))\right)\middle| S(0)=s,\ X(0)=x\right],$$

which implies by our hypothesis that

$$J_\theta(s,x;u) \leq \lambda^\theta.$$

The above inequality becomes an equality for $u = u^\theta$. This completes the proof of the theorem. □

Note that under our assumptions the risk-sensitive optimal investment problem that we consider here admits an optimal solution, with a finite value of the objective criterion, for each value of the risk-sensitivity parameter $\theta > 0$. As we have already indicated above, we are not dealing with questions of existence/uniqueness/finiteness of solutions to our RS-QVI's in this paper. This is left for a future publication which, in particular, will study dependence of the RS-QVI on θ. In the example that we present in Sect. 6, the corresponding RS-QVI admits a solution satisfying all our requirements.

In general the sequence of optimal transaction times $\{\tau_k^\theta\}$ does not have to be strictly increasing, that is, with $\tau_k^\theta < \tau_{k+1}^\theta$ almost surely for every k such that $\tau_k^\theta < \infty$. However, with a suitable additional condition imposed on the cost function C, then it is true that an optimal impulsive investment strategy exists for which the sequence of optimal transaction times is strictly increasing. Before we formulate a proposition to this effect let us first introduce the following definition:

Definition 4.1 *A strategy* $u = \{\tau_k, N_k\} \in \mathscr{U}$ *is called* **separated** *if* $\tau_k^\theta < \tau_{k+1}^\theta$ *almost surely for every* k *such that* $\tau_k^\theta < \infty$.

Proposition 4.1 *In addition to all the conditions of Lemma 4.1, assume*

(c4) $C(s,N,N') - C(s,N,N'') + C(s,N',N'') \geq 0$ *for all* $(s,N,N') \in (0,\infty)^{m+1} \times [0,\infty)^{2m+2}$.

Then a separated admissible impulsive investment strategy u^θ *can be constructed in the way described prior to Remark 4.5.*

Proof. See the Appendix. □

Observe that the proportional to the volume transaction cost function (see Remark 2.4) satisfies condition *(C4).*

In case of a more stringent assumption on the cost structure and admissible transaction sets, like the one considered by Morton and Pliska [26], we have even a stronger result.

Proposition 4.2 *Assume (C1)-(C4). In addition assume that the cost function satisfies:*

$(C5)$ *there exists a constant* $\alpha \in (0,1)$ *such that* $C(s,N,N') = \alpha s^T N$,

for all $(s,N,N') \in (0,\infty)^{m+1} \times (0,\infty)^{m+1} \times [0,\infty)^{m+1}$.

Moreover, suppose the sets of admissible transactions are defined as

$$\begin{aligned} \mathscr{A}(s,N) &= \{N' \in [0,\infty)^{m+1} \ : \ s^T N - C(s,N,N') = s^T N'\}, \\ &\textit{for} \quad (s,N) \in (0,\infty)^{(m+1)} \times [0,\infty)^{m+1}, \end{aligned}$$

and assume all other conditions imposed on the system (2.1)-(2.3). Then every impulsive investment strategy u^θ *constructed in the way described prior to Remark 4.5 is separated.*

Proof. See the Appendix. □

Before we conclude this section let us briefly discuss a formal limit of the RS-QVI (4.1) when the risk-sensitivity parameter θ decreases to 0 (the *risk-null limit*). Towards this end let us first observe that (4.1) can be rewritten as

$$\begin{aligned} L^\theta \phi_\theta(y) - (\theta/4)\|(\phi_\theta)_y^T(y)\beta(y)\|^2 - \left(\lambda - \overline{f}_\theta(y)\right) &\leq 0, \quad y \in int\mathscr{O}, \\ -M\phi_\theta(y) + \phi_\theta(y) &\geq 0, \quad y \in \mathscr{O}, \\ \left[L^\theta \phi_\theta(y) - (\theta/4)\|(\phi_\theta)_y^T(y)\beta(y)\|^2 - \left(\lambda - \overline{f}_\theta(y)\right)\right] & \\ \left[-M\phi_\theta(y) + \phi_\theta(y)\right] &= 0, \quad y \in int\mathscr{O}, \end{aligned} \tag{4.2}$$

where

$$M\phi_\theta(y) := \sup_{N' \in \mathscr{A}(s,N)} \left[g_1(s,N') + g_2(s,N) + \phi_\theta(s,x,N')\right],$$

and

$$\phi_\theta \equiv (2/\theta)\phi.$$

Now, consider a quasi-variational inequality which is like (4.2) above, except that ϕ_θ is replaced with a generic function Φ:

$$\begin{aligned} L^\theta \Phi(y) - (\theta/4)\|\Phi_y^T(y)\beta(y)\|^2 - \left(\lambda - \overline{f}_\theta(y)\right) &\leq 0, \quad y \in int\mathscr{O}, \\ -M\Phi(y) + \Phi(y) &\geq 0, \quad y \in \mathscr{O}, \\ \left[L^\theta \Phi(y) - (\theta/4)\|\Phi_y^T(y)\beta(y)\|^2 - \left(\lambda - \overline{f}_\theta(y)\right)\right] & \\ \left[-M\Phi(y) + \Phi(y)\right] &= 0, \quad y \in int\mathscr{O}. \end{aligned} \tag{4.3}$$

Letting $\theta = 0$ in (4.3) produces the risk-null quasi-variational inequality (5.3) studied in the next section.

5 Risk-null criterion ($\theta = 0$)

In this section we consider the risk-null investment problem in the presence of transaction costs. In the framework of Sect. 2 the objective functional to be maximized now is the classical long-run log-utility (Kelly) criterion:

$$J_0(s,x;u) := \liminf_{t\to\infty} t^{-1}\mathbf{E}[ln(S(t)^T N^u(t))|S(0)=s, X(0)=x]. \tag{5.1}$$

Similarly as in Sect. 3 one easily demonstrates (compare with (3.6)) that for every $u \in \mathscr{U}$ the above functional can be represented as

$$\begin{aligned} J_0(s,x;u) \quad := \quad & \liminf_{t\to\infty} t^{-1}\mathbf{E}\Bigg[\int_0^t F(X(r),h^u(r))dr + \\ & + \sum_{k=0}^{\infty} \chi_{\{\tau_k \le t\}}\Bigg[g_1(S^T(\tau_k),N_k) \\ & + g_2(S^T(\tau_k),N_{k-1})\Bigg]\Bigg| S(0)=s, X(0)=x\Bigg], \end{aligned} \tag{5.2}$$

where

$$F(x,h) = f(x,h) - (1/2)\|\gamma(x,h)\|^2.$$

Formally, $J_0(s,x;u)$ is the limit as $\theta \downarrow 0$ of $J_\theta(s,x;u)$. This can be made precise, but we shall not do so in this paper. In addition, it is interesting to observe that no change of measure is needed in order to obtain (5.2), which is the counter-part of (3.5) for the risk-null case.

Let us now define the operator:

$$L\phi(y) := \alpha^T(y)\phi_y(y) + (1/2)tr(\beta(y)\beta^T(y)\phi_{yy}(y)).$$

For the present case we shall need to consider a risk-null quasi-variational inequality (RN-QVI), which is to be solved for constant λ and function ϕ:

$$\begin{aligned} L\phi(y) - (\lambda - \bar{F}(y)) \quad &\le \quad 0, \quad y \in int\mathscr{O}, \\ -M\phi(y) + \phi(y) \quad &\ge \quad 0, \quad y \in \mathscr{O}, \\ \Big[L\phi - (\lambda - \bar{F}(y)\Big]\Big[-M\phi(y) + \phi(y)\Big] \quad &= \quad 0, \quad y \in int\mathscr{O}, \end{aligned} \tag{5.3}$$

where $\bar{F}(y) := F(x, H(s,N))$ and the operator M was defined at the end of the preceding section. Related to the RN-QVI (5.3) is the continuation set:

$$\mathscr{K}_{\mathscr{R}\mathscr{N}} := \{y \in \mathscr{O} : -M\phi(y) + \phi(y) > 0\}.$$

We now suppose (λ^0, ϕ^0) is a solution pair for (5.3) with ϕ^0 being sufficiently regular. We denote the continuation set corresponding to this pair by $\mathscr{K}^0$.

Remark 5.1 Similarly as in Sect. 4, the function ϕ^0 does not have to be a classical solution. What we require is that ϕ^0 is upper-semi-continuous on $\mathscr{O}$ and that we can apply a (generalized) Itô formula to it. Other conditions imposed on ϕ^0 will be stated in Theorem 5.1 below.

A remark analogous to Remark 4.4 applies to the function $M\phi^0(y)$. Thus, similarly as in Sect. 4, a measurable selector $\mathscr{N}^0$, an impulsive strategy $u^0 = \{(\tau_k^0, N_k^0),\ k = 0, 1, 2, \ldots\}$, and the corresponding process $Y^{u^0}(t)$ (which we shall be denoting as $Y^0(t)$) can be constructed based on the solution pair (λ^0, ϕ^0). To this end it will be convenient to introduce a measurable selector

$$\mathscr{N}^0(s, x, N) := \text{argmax}_{N' \in \mathscr{A}(s,N)} \left[g_1(s, N') + g_2(s, N) + \phi^0(s, x, N') \right].$$

Next we set $\tau_0^0 = 0$. In addition, if $(S(0), X(0), N_{-1}) \in \mathscr{K}^0$ then we set $N_0^0 = N_{-1}$, otherwise we set

$$N_0^0 = \mathscr{N}^0(S(0), X(0), N_{-1}).$$

In general, for $k \geq 1$ we define :

$$\tau_k^0 = inf\{t \geq \tau_{k-1}^0 :\ (S(t), X(t), N_{k-1}^0) \notin \mathscr{K}^0\},$$

and

$$N_k^0 = \mathscr{N}^0(S(\tau_k^0), X(\tau_k^0), N_{k-1}^0).$$

For ease of exposition we also define

$$m_k^0 := m_k^{u^0}, \quad k = 0, 1, 2, \ldots$$

It will follow from Lemma 5.1 below that $m_k^0 < \infty$ for each k. Finally we define

$$N^0(t) := N_{k+m^0-1}, \quad \text{for} \quad t \in [\tau^0_{k+m_k^0-1},\ \tau^0_{k+m_k^0}), \quad k = 0, 1, 2, \ldots\ ,$$

and

$$Y^0(t) := (S(t), X(t), N^0(t)), \quad t \geq 0.$$

Remark 5.2 Observe that the following important equality follows from the definition of the operator M and the above construction:

$$\begin{aligned}\phi^0(S(\tau_k^0), X(\tau_k^0), N_{k-1}^0) &= g_1(S(\tau_k^0), N_k^0) + g_2(S(\tau_k^0), N_{k-1}^0) \\ &\quad + \phi^0(S(\tau_k^0), X(\tau_k^0), N_k^0).\end{aligned}$$

Moreover, if $\tau_{k-1}^0 < \tau_k^0$, then

$$\phi^0(Y^0(\tau_k^0-)) = g_1(S(\tau_k^0), N_k^0) + g_2(S(\tau_k^0), N_{k-1}^0) + \phi^0(Y^0(\tau_k^0)).$$

The proof of the following lemma is analogous to the proof of Lemma 4.1 and therefore will be omitted.

Lemma 5.1 *Let us assume all the conditions of Sect. 2. Then the sequence u^0 constructed above constitutes an admissible impulsive investment strategy.*

In the following theorem we shall demonstrate that under some additional (mild) conditions on ϕ^0 the impulsive investment strategy u^0 is optimal for the criterion (5.1).

Theorem 5.1 *Let us assume all the conditions of Sect. 2. Consider the pair* (λ^0, ϕ^0) *and assume that the function* ϕ^0 *satisfies two additional conditions:*

- *for each* $u \in \mathscr{U}$, s *and* x *we have*

$$\liminf_{t\to\infty} t^{-1}\mathbf{E}\big[\phi^0(Y^u(t))\Big| S(0) = s,\ X(0) = x\big] \geq 0,$$

and

$$\liminf_{t\to\infty} t^{-1}\mathbf{E}\big[\phi^0(Y^0(t))\Big| S(0) = s,\ X(0) = x\big] = 0,$$

- *for each* $u \in \mathscr{U}$, $t \geq 0$, s *and* x *we have*

$$\mathbf{E}\Big[\int_0^t \|\phi_y^{0\,T}(Y^u(r))\beta(Y^u(r))\|^2 dr\Big| S(0) = s,\ X(0) = x\Big] < \infty.$$

Then the sequence u^0 *constructed above is an optimal impulsive investment strategy for the Kelly criterion (5.1). Moreover, the constant* λ^0 *is the optimal value of the objective criterion (5.1).*

Proof. It follows from Lemma 5.1 that u^0 is an admissible impulsive investment strategy.

To show that u^0 is optimal, fix an arbitrary admissible impulsive investment strategy $u = ((\tau_k, N_k),\ k = 0, 1, 2, \ldots)$, and let $Y^u(t)$ correspond to this u.

Fix $k \geq 1$ so that $\tau_{k-1} < \tau_k < \infty$, and take $t \in [\tau_{k+m_k^u-1}, \tau_{k+m_k^u}[$. From *Itô's* formula and from our RN-QVI we deduce that

$$\begin{aligned}
&\int_{\tau_{k+m_k^u-1}}^t (\lambda^0 - \overline{F}(Y^u(r)))dr \\
&= \int_{\tau_{k+m_k^u-1}}^t L\phi^0(Y^u(r))dr \\
&= -\phi^0(Y^u(\tau_{k+m_k^u-1})) + \phi^0(Y^u(t)) - \int_{\tau_{k+m_k^u-1}}^t \phi_y^{0\,T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r).
\end{aligned}$$

This implies that, using the definition of the operator M and the fact that $\tau_k = \tau_{k+m_k^u-1}$,

$$\begin{aligned}
&\int_{\tau_k}^t (\lambda^0 - \overline{F}(Y^u(r)))dr \\
&\geq -\phi^0(Y^u(\tau_k-)) + \sum_{l=k}^{k+m_k^u-1}\Big[g_1(S(\tau_k), N_l) + g_2(S(\tau_k), N_{l-1})\Big] \\
&\quad +\phi(Y^u(t)) - \int_{\tau_k}^t \phi_y^{0\,T}(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r).
\end{aligned}$$

Next, using *Itô* again to substitute for $\phi^0(Y^u(\tau_k-))$, and then combining integrals, we get

$$\int_{\tau_k}^{t}(\lambda^0 - \bar{F}(Y^u(r)))dr$$

$$\geq -\phi^0(Y^u(\tau_{k-1})) - \int_{\tau_{k-1}}^{\tau_k} L\phi^0(Y^u(r))dr + \sum_{l=k}^{k+m_k^u-1}\Big[g_1(S(\tau_k),N_k)+g_2(S(\tau_k),N_{k-1})\Big]$$

$$+\phi^0(Y^u(t)) - \int_{\tau_{k-1}}^{t} {\phi_y^0}^T(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r)$$

$$= -\phi^0(Y^u(\tau_{k-1})) - \int_{\tau_{k-1}}^{\tau_k}(\lambda^0 - \bar{F}(Y^u(r)))dr + \sum_{l=k}^{k+m_k^u-1}\Big[g_1(S(\tau_k),N_k)+g_2(S(\tau_k),N_{k-1})\Big]$$

$$+\phi^0(Y^u(t)) - \int_{\tau_{k-1}}^{t} {\phi_y^0}^T(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r),$$

where the equality follows from (I) in the RN-QVI and a further combining of integrals. Continuing until $k = 1$ we eventually obtain

$$\int_{\tau_k}^{t}(\lambda^0 - \bar{F}(Y^u(r)))dr$$

$$\geq -\phi(Y^u(0)) - \int_{0}^{\tau_k}(\lambda^0 - \bar{F}(Y^u(r)))dr + \sum_{l=0}^{k+m_k^u-1}\Big[g_1(S(\tau_l),N_l) + g_2(S(\tau_l),N_{l-1})\Big]$$

$$+\phi(Y^u(t)) - \int_{0}^{t} {\phi_y^0}^T(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r).$$

In other words for arbitrary $t \geq 0$ we get

$$\int_0^t \bar{F}(Y^u(r))dr + \sum_{l=0}^{\infty}\chi_{\tau_l\leq t}\Big[g_1(S(\tau_l),N_l) + g_2(S(\tau_l),N_{l-1})\Big]$$

$$\leq \phi(Y^u(0)) + t\lambda^0 - \phi(Y^u(t)) + \int_0^t {\phi_y^0}^T(Y^u(r))\beta(Y^u(r))dW^{u,\theta}(r).$$

This means by (5.2) and by our hypotheses that

$$J_0(s,x;u) \leq \lambda^0 - \liminf_{t\to\infty} t^{-1}\mathbf{E}\big[\phi^0(Y^u(t))\big|S(0)=s,\ X(0)=x\big] \leq \lambda^0.$$

The last inequalities become equalities for $u = u^0$, so this completes the proof of the theorem. □

6 Relation with results of Morton and Pliska

In their paper Morton and Pliska [26] considered a risk-null investment problem under proportional (fixed) transaction costs. In our framework the transaction cost structure considered by these authors can be described by taking the function C in the form (see also Remark 2.4):

$$C(s, N, N') := \alpha s^T N,$$

where $\alpha \in (0, 1)$ represents the fraction paid to the broker (e.g., $\alpha = 0.01$). As was already observed at the end of Sect. 4, this cost function satisfies conditions $(c1) - (c5)$. The price formation model assumed in [26] corresponds to our equations (2.1) with all coefficients constant (so no dependence on x) and with no factors (so equation (2.2) should be omitted). Additionally it is assumed in [26] that $A_0(x) \equiv r > 0$, where by r these authors denoted the short term interest rate. Finally, Morton and Pliska assumed that $N_0(t) > 0$ for all $t \geq 0$, but we do not need to do so here. The proper state space for this setting is therefore (compare with the beginning of Sect. 4) $\mathscr{O} := (0, \infty)^{m+1} \times [0, \infty)^{m+1}$. This is because the state of the process $Y^u(t)$ is now $y = (s, N) \in (0, \infty)^{m+1} \times [0, \infty)^{m+1}$. Theorem 5.1 is still applicable in this setting.

In this section we shall first apply the results of Sect. 5 in order to solve the optimal investment example stated in Sect. 4 of Morton and Pliska [26]; later we'll study a risk sensitive version (where $\theta > 0$) of this problem. In this example only two securities are considered: a risk-free security and a risky security (with an immediate mean return rate $\mu > r$ and volatility $\sigma > 0$). Specifically, our equations (2.1) take the following form for this example:

$$\begin{aligned} \frac{dS_0(t)}{S_0(t)} &= rdt \quad (risk - free\ security) \\ \frac{dS_1(t)}{S_1(t)} &= \mu dt + \sigma dW(t) \quad (risky\ security) \\ S_i(0) &= s_i > 0, \quad i = 0, 1. \end{aligned} \tag{6.1}$$

Thus, to summarize, the data for this example are:

- $m = 1,\ n = 0$
- $A(x) \equiv [r\ \ \mu]^T,\ \tilde{\Sigma}(x) \equiv \sigma$
- $B(x) \equiv 0,\ \Lambda(x) \equiv 0.$

The solution approach provided in Sect. 4 of [26] does not apply when $\frac{\mu - r}{\sigma^2} = 1/2$, because then the presented solution of a differential equation is degenerate. However, we can solve the Morton-Pliska example in this case using our quasi-variational inequality approach, as will now be demonstrated. And it will be seen that our solution is consistent with the results of Morton and Pliska derived for cases where $\frac{\mu - r}{\sigma^2} \neq 1/2$.

In the present example the state of the process $Y^u(t)$ is (with a slight change of notation) $y = (s_0, s_1, n_0, n_1) \in \mathscr{O} := (0, \infty)^2 \times [0, \infty)^2$, where n_i denotes the number of shares of security i. The RN-QVI (5.3) takes the following form:

$$(I) \quad s_0 r \phi_{s_0}(y) + s_1 \mu \phi_{s_1}(y) + \frac{1}{2} s_1^2 \sigma^2 \phi_{s_1 s_1} - \lambda$$

$$+ r \frac{s_0 n_0}{s_0 n_0 + s_1 n_1} + \mu \frac{s_1 n_1}{s_0 n_0 + s_1 n_1} - \frac{1}{2} \sigma^2 \left(\frac{s_1 n_1}{s_0 n_0 + s_1 n_1} \right)^2 \leq 0, \quad (s_0, s_1, n_0, n_1) \in int\mathscr{O}$$

$$(II) \quad \phi(y) \geq ln(1-\alpha) + \sup_{\{n_0', n_1' \geq 0 | s_0 n_0' + s_1 n_1' = (1-\alpha)(s_0 n_0 + s_1 n_1)\}} \phi(s_0, s_1, n_0', n_1'),$$

$$(s_0, s_1, n_0, n_1) \in \mathscr{O}$$

$$(I) \times (II) = 0, \quad (s_0, s_1, n_0, n_1) \in int\mathscr{O}. \tag{6.2}$$

We shall not analyze this RN-QVI directly. Instead, we shall consider an equivalent quasi-variational-inequality (which provides a natural counterpart to the approach taken by Morton and Pliska in [26]). The following QVI is to be solved for a function $\psi(b),\ b \in [0,1]$ and a constant λ:

$$(I) \quad \frac{\sigma^2}{2} b^2 (1-b)^2 \psi_{bb} + \sigma^2 \left(\frac{\mu - r}{\sigma^2} - b \right) b(1-b)\psi_b - \lambda + \mu b$$

$$+ r(1-b) - \frac{\sigma^2}{2} b^2 \leq 0, \quad b \in (0,1),$$

$$(II) \quad \psi(b) \geq ln(1-\alpha) + \sup_{0 \leq b' \leq 1} \psi(b'), \quad b \in [0,1],$$

$$(I) \times (II) = 0, \quad b \in (0,1), \tag{6.3}$$

which (using our assumption that $\frac{\mu - r}{\sigma^2} = 1/2$) can be restated as

$$(I) \quad \frac{\sigma^2}{2} b^2 (1-b)^2 \psi_{bb} + \sigma^2 \left(\frac{1}{2} - b \right) b(1-b)\psi_b - \lambda + r + \frac{\sigma^2}{2} b(1-b) \leq 0, \quad b \in (0,1),$$

$$(II) \quad \psi(b) \geq ln(1-\alpha) + \sup_{0 \leq b' \leq 1} \psi(b'), \quad b \in [0,1],$$

$$(I) \times (II) = 0, \quad b \in (0,1). \tag{6.4}$$

Note that the first partial differential inequality in (6.3) expresses the dynamics of what Morton and Pliska called the *risky fraction process*, which is the fraction of wealth in the risky asset when the share holdings are fixed.

The following lemma relates problems (6.2) and (6.3). In its formulation, $H^2(0,1)$ and $H^2(int\mathscr{O})$ denote the second Sobolev spaces on $(0,1)$ and $int\mathscr{O}$, respectively (see e.g. Bensoussan [2], chapter II, or Kufner et al. [22]).

Lemma 6.1 *Suppose a constant λ^0 and a function $\psi^0 \in H^2(0,1)$ satisfy (6.3). Also suppose that for some $b^* \in [0,1]$ we have*

$$\sup_{0\le b'\le 1} \psi^0(b') = \psi^0(b^*).$$

Define a function $\mathbf{b}(s_0,s_1,n_0,n_1)$ on $\mathscr{O}$ by

$$\mathbf{b}(s_0,s_1,n_0,n_1) = \begin{cases} \frac{s_1n_1}{s_0n_0+s_1n_1}, & \text{if } s_0n_0+s_1n_1 > 0 \\ 0, & \text{if } s_0n_0+s_1n_1 = 0. \end{cases}$$

Then the function $\phi^0(s_0,s_1,n_0,n_1)$ defined by

$$\phi^0(s_0,s_1,n_0,n_1) = \psi^0(\mathbf{b}(s_0,s_1,n_0,n_1)), \quad (s_0,s_1,n_0,n_1) \in \mathscr{O}$$

belongs in $H^2(int\mathscr{O})$ and, together with the constant λ^0, satisfies the QVI (6.2).

Proof. Since $\mathbf{b}(s_0,s_1,n_0,n_1)$ is smooth for $n_0,n_1 \neq 0$, then $\phi^0(s_0,s_1,n_0,n_1) \in H^2(int\mathscr{O})$. Also, it is easily seen by direct inspection that the function $\phi^0(s_0,s_1,n_0,n_1)$ and the constant λ^0 satisfy (I) in (6.2).

Now, from (II) in (6.3) we see that

$$\psi(\mathbf{b}(s_0,s_1,n_0,n_1)) \ge ln(1-\alpha) + \sup_{0\le b'\le 1} \psi(b')$$

$$\ge ln(1-\alpha) + \sup_{\{n_0',n_1'\ge 0 | s_0n_0'+s_1n_1' = (1-\alpha)(s_0n_0+s_1n_1)\}} \psi(\mathbf{b}(s_0,s_1,n_0',n_1')),\ (s_0,s_1,n_0,n_1) \in \mathscr{O},$$

which implies that (II) of (6.2) is satisfied by $\phi^0(s_0,s_1,n_0,n_1)$.

Next, let

$$n_1^*(s_0,s_1,n_0,n_1) := \frac{b^*(1-\alpha)(s_0n_0+s_1n_1)}{s_1},$$

and

$$n_0^*(s_0,s_1,n_0,n_1) := \frac{(1-b^*)(1-\alpha)(s_0n_0+s_1n_1)}{s_0}.$$

Fix arbitrary $(s_0,s_1,n_0,n_1) \in \mathscr{O}$. If equality holds in (II) of (6.3) for $b = \mathbf{b}(s_0,s_1,n_0,n_1)$, then equality holds in (II) of (6.2) for (s_0,s_1,n_0,n_1), since the *supremum* on the right hand side of (II) in (6.2) is realized by (n_0^*,n_1^*). On the other hand, if equality does not hold in (II) of (6.3), then it must hold in (I) of (6.3), in which case it also holds in (I) of (6.2). We thus see that (III) of (6.2) is satisfied by the function $\phi^0(s_0,s_1,n_0,n_1)$ and the constant λ^0. The proof of the lemma is complete. □

Thus it suffices to determine a solution pair (λ^0,ψ^0) for (6.4). Towards this end please consider a function $\overline{\psi}(b)$ defined as

$$\overline{\psi}(b) = \begin{cases} ln\left(\frac{2}{(1-\alpha)}\right) + \frac{\overline{\lambda}-r}{\sigma^2}\left(ln\left(\frac{b}{1-b}\right)\right)^2 \\ \qquad\qquad + \frac{1}{2}ln\left(b(1-b)\right), & \text{if } \overline{b} < b < 1-\overline{b} \\ 0, & \text{if } 0 \le b \le \overline{b} \text{ or } 1-\overline{b} \le b \le 1, \end{cases}$$

where $\bar{\lambda}$ and $\bar{b} \in (0, 1/2)$ are constants (depending on α) yet to be determined. Note that $\bar{\psi}$ is symmetric about $b = 1/2$.

We require that $\bar{\psi} \in H^2(0,1)$. Due to the Sobolev imbedding theorem (see e.g. Kufner et al. [22]) this implies that $\bar{\psi} \in C^1(0,1)$ (a class of functions on $(0,1)$ once differentiable in the classical sense, and with continuous first derivatives). In particular, in order for $\bar{\psi}$ and its derivative to be continuous at $\bar{b}$, the following conditions must be satisfied by $\bar{\lambda}$ and $\bar{b}$:

$$ln\left(\frac{2}{(1-\alpha)}\right) + \frac{\bar{\lambda}-r}{\sigma^2}\left(ln\left(\frac{\bar{b}}{1-\bar{b}}\right)\right)^2 + \frac{1}{2}ln\left(\bar{b}(1-\bar{b})\right) = 0, \qquad (6.5)$$

and

$$4\frac{\bar{\lambda}-r}{\sigma^2}ln\left(\frac{\bar{b}}{1-\bar{b}}\right) - 2\bar{b} + 1 = 0. \qquad (6.6)$$

Moreover, in order to ensure that (I) of (6.4) is satisfied by all $b < \bar{b}$ or $b > 1-\bar{b}$, we require that $\bar{b}$ satisfies the inequality

$$0 \leq \bar{b} \leq \frac{1-\sqrt{1-8\frac{\bar{\lambda}-r}{\sigma^2}}}{2}, \qquad (6.7)$$

which necessitates that

$$r \leq \bar{\lambda} \leq r + \frac{\sigma^2}{8}. \qquad (6.8)$$

It can be shown that for any $\bar{\lambda}$ satisfying (6.8), if there exists a corresponding solution $\bar{b}$ to (6.6), then $\bar{b}$ satisfies (6.7). This leaves us with the problem of solving (6.5) and (6.6) so that (6.8) is satisfied. Equations (6.5) and (6.6) can easily be solved numerically for specified values of $\alpha \in (0,1)$, $r \geq 0$, and $\sigma > 0$. For example, if $\alpha = 0.001$, $r = 0.07$, and $\sigma = 0.4$ (so that $\mu = 0.15$), then we obtain:

$$\bar{\lambda} = 0.0893 \quad and \quad \bar{b} = 0.3381.$$

It is interesting to note, by the way, that these numbers are very close to the computations one obtains, namely, $\bar{\lambda} = 0.0893$ and $\bar{b} = 0.3345$, with the asymptotic approach developed by Atkinson and Wilmott [1].

In summary, suppose that constants $\bar{\lambda}$ and $\bar{b}$ satisfying (6.5)-(6.8) have been found. Then it is easy to verify that the resulting pair $(\bar{\lambda}, \bar{\psi})$ satisfies the QVI (6.4). We can thus set

$$\lambda^0 = \bar{\lambda}, \quad \text{and} \quad \psi^0 \equiv \bar{\psi}. \qquad (6.9)$$

In addition we see that $b^* = 0.5$ realizes the *supremum* on the right-hand side of (II) in (6.4) for the above choice of (λ^0, ψ^0).

In order to conclude the discussion of this risk-null example, let us now observe that in view of Lemma 6.1 the pair λ^0 and $\phi^0 := \psi^0 \circ \mathbf{b}$, where λ^0 and ψ^0 are chosen according to (6.9), satisfies the QVI (6.2). Since ϕ^0 is bounded we can apply the generalized $It\hat{o}'s$ lemma to it (see Krylov [21]), although some minor modifications to the argument used in the proof of Theorem 5.1 are needed here since in our example the price process $S_0(t)$ is a degenerate diffusion process. Finally, again using boundedness of ϕ^0 we see that both *liminf* conditions of Theorem 5.1 are satisfied for ϕ^0. Consequently we may apply Theorem 5.1 to conclude the following:

The optimal impulsive investment strategy u^0 is determined for the considered example by

- the optimal continuation region

$$\mathscr{K}^0 = \left\{(s_0, s_1, n_0, n_1) \in \mathscr{O} | \bar{b} < \frac{s_1 n_1}{s_0 n_0 + s_1 n_1} < 1 - \bar{b}\right\},$$

- the optimal portfolio selector

$$\mathscr{N}^0(s_0, s_1, n_0, n_1) = \left(\frac{0.5(1-\alpha)(s_0 n_0 + s_1 n_1)}{s_0}, \frac{0.5(1-\alpha)(s_0 n_0 + s_1 n_1)}{s_1}\right).$$

This means that it is optimal for the investor to rebalance if the price/shareholdings process $Y^0(t)$ is outside of the region $\mathscr{K}^0$, and the rebalancing should be made to the share-holding levels determined by $\mathscr{N}^0(Y^0(t))$, resulting in a position with exactly half of the wealth in each asset. Note that this result is consistent with results obtained by Morton and Pliska in [26] via a combination of renewal theory and optimal-stopping theory for the case $\frac{\mu - r}{\sigma^2} \neq 0.5$. In particular, the optimal rebalancing times are separated (as they should be according to our Proposition 4.2).

We now conclude this section by considering a risk-sensitive version of the Morton/Pliska problem. That is, the objective criterion to be maximized now is (2.4), with all the other relevant data as in the example considered previously in this section. According to the remarks made at the end of Sect. 4, and using a result analogous to the one stated in Lemma 6.1, it can be easily shown that the quasi-variational inequalities one should consider in order to characterize an optimal solution to this risk-sensitive version of the Morton/Pliska problem can be written as:

$$(I) \quad \frac{\sigma^2}{2} b^2(1-b)^2(\psi_{bb} - (\theta/2)\psi_b^2) + \sigma^2\left(\frac{1}{2} - b(1+\theta/2)\right) b(1-b)\psi_b - \lambda + r$$
$$+ \frac{\sigma^2}{2} b(1 - b(1+\theta/2)) \leq 0, \quad b \in (0,1),$$

$$(II) \quad \psi(b) \geq ln(1-\alpha) + \sup_{0 \leq b' \leq 1} \psi(b'), \quad b \in [0,1],$$

$$(I) \times (II) = 0, \quad b \in (0,1). \tag{6.10}$$

For the numerical values of all the parameters as above, that is, if $\alpha = 0.001$, $r = 0.07$, and $\sigma = 0.4$, and for the value of the risk-sensitivity parameter $\theta = 0.1$, we obtained the following solution pair to the RS-QVI (6.10):

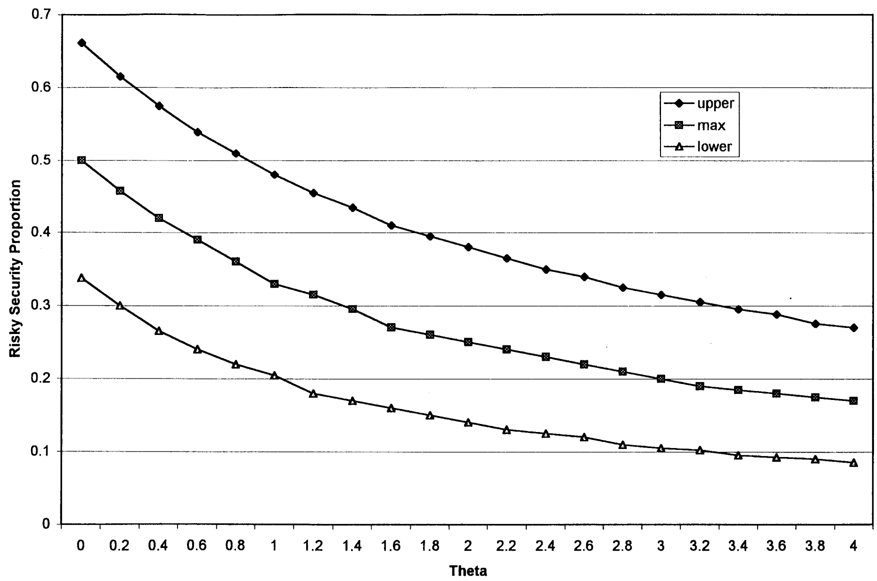

Fig. 1. Continuation region versus risk sensitivity parameter

$$\bar{\psi}_\theta(b) = \begin{cases} (\frac{1}{\theta})\left(\theta ln(1-b) + ln\left(1 + tan^2\left(\sqrt{\frac{\theta}{2}}\sqrt{c_1}[ln\left(\frac{b}{(1-b)}\right) + c_2^2]\right)\right)\right) + c_3, \\ \qquad \text{if } \bar{b}_{1,\theta} < b \leq \bar{b}_{2,\theta} \\ 0, \qquad \text{if } 0 \leq b \leq \bar{b}_{1,\theta} \text{ or } \bar{b}_{2,\theta} \leq b \leq 1, \end{cases}$$

and

$$\lambda_\theta = 0.0883,$$

where $c_1 = 0.2289$, $c_2 = 2.14225$, $c_3 = 0.1642$, $\bar{b}_{1,\theta} = 0.32$, and $\bar{b}_{2,\theta} = 0.64$. The maximum value of the function $\bar{\psi}_\theta(b)$ on the interval $[0, 1]$ is attained at $b_\theta^* = 0.485$.

The optimal impulsive investment strategy u^θ is determined for the considered example by

– the optimal continuation region

$$\mathscr{K}^\theta = \left\{(s_0, s_1, n_0, n_1) \in \mathscr{O} | \bar{b}_{1,\theta} < \frac{s_1 n_1}{s_0 n_0 + s_1 n_1} < \bar{b}_{2,\theta}\right\},$$

– the optimal portfolio selector

$$\mathscr{N}^\theta(s_0, s_1, n_0, n_1) = \left(\frac{0.515(1-\alpha)(s_0 n_0 + s_1 n_1)}{s_0}, \frac{0.485(1-\alpha)(s_0 n_0 + s_1 n_1)}{s_1}\right).$$

This means that it is optimal for the investor to rebalance if the price/share-holdings process $Y^\theta(t)$ is outside of the region $\mathscr{K}^\theta$, and the rebalancing should be made to the share-holding levels determined by $\mathscr{N}^\theta(Y^\theta(t))$, resulting in a position where 51.5% of the current wealth is allocated in the bond, and the remaining 48.5% of the current wealth is allocated in the risky asset. This result is consistent with economic intuition: since the investor is more risk averse that with $\theta = 0$, she/he is expected to allocate a larger proportion of the current wealth in the risk free asset at the time of rebalancing. Note that even such a small value of the risk-sensitivity parameter as $\theta = 0.1$ causes a significant change in the optimal rebalancing proportions from 50%-50% to 51.5%-48.5%.

We also solved the problem for various larger values of θ. Figure 1 illustrates the dependence on $\theta \in [0, 4]$ of the optimal continuation region $(\bar{b}_{1,\theta}, \bar{b}_{2,\theta})$, and of the optimal rebalancing point b^*_θ. The values of $\bar{b}_{1,\theta}$, $\bar{b}_{2,\theta}$, and b^*_θ are denoted by *lower*, *upper* and *max*, respectively, on the graph.

Acknowledgement. We are grateful to an anonymous referee for a careful reading and very useful comments and suggestions.

Appendix

Proof of Lemma 4.1 Since $\mathscr{N}^\theta$ is a measurable selector then N^θ_k is $\mathscr{G}_{\tau_k}$ measurable for $k = 0, 1, 2, \ldots$. It is straightforward to verify that τ^θ_k is a $\mathscr{G}_t$ stopping time using a *debut* theorem (see e.g. [13], chapter 6) and the fact that the continuation set $\mathscr{K}_{RS}$ is measurable. The condition contained in Definition 2.1 follows directly from the the construction of u^θ. We have also observed in Remark 4.6 that $S(\tau^\theta_k)^T N^\theta_k > 0$ for each $k = 0, 1, 2, \ldots$ almost surely. Therefore, it suffices to demonstrate that almost surely we have $\tau_k \to \infty$ as $k \to \infty$.

To prove this claim let $V(t)$ denote the optimal (maximal) wealth at time t of a risk-sensitive investor allowed to use $\mathscr{G}_t$-adapted trading strategies $h(t)$ for which the wealth equation analogous to equation (2.4) in Bielecki and Pliska [5] is not explosive. Note that in view of the Remark 2.2 the fraction process $h^\theta(t)$ corresponding to the impulsive strategy u^θ is admissible in the sense of Bielecki and Pliska [5]. Clearly, then, we have $S^T(t)N^\theta(t) \le V(t) < \infty$ almost surely for each finite t, and thus $\zeta(t) := \sup_{0 \le r \le t}[S^T(r)N^\theta(r)] < \infty$ almost surely for each finite t.

Let, for a finite integer $\mathscr{T}$,

$$\Omega^{\theta,\mathscr{T}} := \{\omega | \lim_{k\to\infty} \tau^\theta_k(\omega) \le \mathscr{T}\}.$$

Take $\omega \in \Omega^{\theta,\mathscr{T}}$ and denote

$$\mathscr{T}(\omega) := \lim_{k\to\infty} \tau^\theta_k(\omega).$$

Next, since

$$S^T(\tau^\theta_k)N^\theta_{k-1} - S^T(\tau^\theta_k)N^\theta_k \ge C(S^T(\tau^\theta_k), N^\theta_{k-1}, N^\theta_k),$$

we also have for all k (we suppress dependence on ω to simplify notation)

$$1 - \frac{C(S^T(\tau^\theta_k), N^\theta_{k-1}, N^\theta_k)}{S^T(\tau^\theta_k)N^\theta_{k-1}} \ge \frac{S^T(\tau^\theta_k)N^\theta_k}{S^T(\tau^\theta_k)N^\theta_{k-1}}.$$

Since $\zeta(\mathscr{T}(\omega)) < \infty$, we obtain from assumption (c3) that there exists a number $0 < c'(\omega) < 1$ such that, for all $k \geq 0$,

$$\begin{aligned} \frac{S^T(\tau_k^\theta)N_k^\theta}{S^T(\tau_k^\theta)N_{k-1}^\theta} &\leq 1 - \inf_{\{s,N,N':s^TN\leq\zeta(\mathscr{T}(\omega))\}}\left[\frac{C(s,N,N')}{s^TN}\right] \\ &\leq 1 - \delta(\zeta(\mathscr{T}(\omega))) \leq c'(w). \end{aligned} \tag{7.1}$$

From (3.1) we obtain

$$\begin{aligned} ln(S^T(\mathscr{T}(\omega))N^\theta(\mathscr{T}(\omega))) - \int_0^{\mathscr{T}(\omega)} f(X(r), H(S(r), N^\theta(r)))dr \\ + (1/2)\int_0^{\mathscr{T}(\omega)} \|\gamma(X(r), H(S(r), N^\theta(r)))\|^2 dr \\ - \int_0^{\mathscr{T}(\omega)} \gamma(X(r), H(S(r), N^\theta(r)))dW(r) \\ = \sum_{k=0}^{\infty}\left[g_1(S^T(\tau_k^\theta), N_k^\theta) + g_2(S^T(\tau_k^\theta), N_{k-1}^\theta)\right]. \end{aligned}$$

The left hand side of the above equality is finite almost surely. In view of (7.1) we have for our selected ω and for all $k \geq 0$

$$\begin{aligned} g_1(S^T(\tau_k^\theta), N_k^\theta)(\omega) + g_2(S^T(\tau_k^\theta), N_{k-1}^\theta)(\omega) &= ln\left(\frac{S^T(\tau_k^\theta)N_k^\theta}{S^T(\tau_k^\theta)N_{k-1}^\theta}(\omega)\right) \\ &\leq ln\left(c'(\omega)\right) < 0. \end{aligned}$$

Thus the right hand side of the above equality is equal to negative infinity, a contradiction. Thus the set $\Omega^{\theta,\mathscr{T}}$ has probability zero. This concludes the proof of the lemma.

Proof of Proposition 4.1 Fix $\omega \in \Omega$. In what follows we shall suppress dependence on ω for the ease of exposition. Fix k so that $\tau_{k-1}^\theta < \tau_k^\theta < \infty$. (If $k = 0$ then we set $\tau_{k-1}^\theta = 0$.)

We begin by observing that $\mathscr{A}(S(\tau_k^\theta), N_k^\theta) \subset \mathscr{A}(S(\tau_k^\theta), N_{k-1}^\theta)$. To see this, consider arbitrary $N' \in \mathscr{A}(S(\tau_k^\theta), N_k^\theta)$, that is, N' satisfying

$$S(\tau_k^\theta)^T N_k^\theta - C(S(\tau_k^\theta), N_k^\theta, N') \geq S(\tau_k^\theta)^T N'.$$

In view of the condition $(c4)$ we thus have

$$\begin{aligned} C(S(\tau_k^\theta), N_{k-1}^\theta, N_k^\theta) - C(S(\tau_k^\theta), N_{k-1}^\theta, N') + S(\tau_k^\theta)^T N_k^\theta \\ = C(S(\tau_k^\theta), N_{k-1}^\theta, N_k^\theta) - C(S(\tau_k^\theta), N_{k-1}^\theta, N') \\ + C(S(\tau_k^\theta), N_k^\theta, N') + S(\tau_k^\theta)^T N_k^\theta - C(S(\tau_k^\theta), N_k^\theta, N') \\ \geq S(\tau_k^\theta)^T N'. \end{aligned}$$

Meanwhile, since $N_k^\theta \in \mathscr{A}(S(\tau_k^\theta), N_{k-1}^\theta)$, we have

$$S(\tau_k^\theta)^T \left(N_{k-1}^\theta - N_k^\theta\right) \geq C(S(\tau_k^\theta), N_{k-1}^\theta, N_k^\theta).$$

Combining the two above inequalities yields

$$S(\tau_k^\theta)^T N_{k-1}^\theta - C(S(\tau_k^\theta), N_{k-1}^\theta, N') \geq S(\tau_k^\theta)^T N',$$

that is, $N' \in \mathscr{A}(S(\tau_k^\theta), N_{k-1}^\theta)$.

Next, since $c > 0$ we see that $N_k^\theta \notin \mathscr{A}(S(\tau_k^\theta), N_k^\theta)$. From the definition of the impulsive strategy u^θ it follows that (compare Remark 4.5)

$$\begin{aligned}
\phi^\theta((S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta)) &= \left(-\frac{\theta}{2}\right)\Bigg(g_1(S(\tau_k^\theta), N_k^\theta) \\
&\quad + g_2(S(\tau_k^\theta), N_{k-1}^\theta)\Bigg) + \phi^\theta((S(\tau_k^\theta), X(\tau_k^\theta), N_{k-1}^\theta)) \\
&= \left(-\frac{\theta}{2}\right)\left(g_1(S(\tau_k^\theta), N_k^\theta) + g_2(S(\tau_k^\theta), N_{k-1}^\theta)\right) \\
+ \sup_{N' \in \mathscr{A}(S(\tau_k^\theta), N_{k-1}^\theta)} &\left[\left(\frac{\theta}{2}\right)(g_1(S(\tau_k^\theta), N') + g_2(S(\tau_k^\theta), N_{k-1}^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N')\right].
\end{aligned}$$

We now consider two cases. Suppose that there is no $N' \in \mathscr{A}(S(\tau_k^\theta), N_k^\theta)$ for which the supremum is achieved on the right hand side of the last equality. Then we obviously have

$$\begin{aligned}
&\phi^\theta((S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta)) \\
&= \left(-\frac{\theta}{2}\right)\left(g_1(S(\tau_k^\theta), N_k^\theta) + g_2(S(\tau_k^\theta), N_{k-1}^\theta)\right) \\
&+ \sup_{N' \in \mathscr{A}(S(\tau_k^\theta), N_{k-1}^\theta)} \left[\left(\frac{\theta}{2}\right)(g_1(S(\tau_k^\theta), N') + g_2(S(\tau_k^\theta), N_{k-1}^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N')\right] \\
&> \left(-\frac{\theta}{2}\right)\left(g_1(S(\tau_k^\theta), N_k^\theta) + g_2(S(\tau_k^\theta), N_{k-1}^\theta)\right) \\
&+ \sup_{N' \in \mathscr{A}(S(\tau_k^\theta), N_k^\theta)} \left[\left(\frac{\theta}{2}\right)(g_1(S(\tau_k^\theta), N') + g_2(S(\tau_k^\theta), N_{k-1}^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N')\right] \\
&= M^\theta \phi^\theta((S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta)).
\end{aligned}$$

Thus in this case $m_k^\theta = 1$, so that $\tau_k^\theta < \tau_{k+1}^\theta$.

For the other case, suppose there exists $\tilde{N}_k^\theta \in \mathscr{A}(S(\tau_k^\theta), N_k^\theta)$ realizing the maximum in

$$\sup_{N' \in \mathscr{A}(S(\tau_k^\theta), N_{k-1}^\theta)} \left[\left(\frac{\theta}{2}\right)(g_1(S(\tau_k^\theta), N') + g_2(S(\tau_k^\theta), N_{k-1}^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N')\right].$$

Observe that the same $\tilde{N}_k^\theta$ realizes the maximum in

$$\sup_{N' \in \mathscr{A}(S(\tau_k^\theta), N_{k-1}^\theta)} \left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), N') + g_2(S(\tau_k^\theta), N_k^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N') \right],$$

and thus it also realizes the maximum in

$$\sup_{N' \in \mathscr{A}(S(\tau_k^\theta), N_k^\theta)} \left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), N') + g_2(S(\tau_k^\theta), N_k^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N') \right]$$

as well. It follows that

$$\begin{aligned} &\phi^\theta((S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta)) \\ &= \sup_{N' \in \mathscr{A}(S(\tau_k^\theta), N_k^\theta)} \left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), N') + g_2(S(\tau_k^\theta), N_k^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N') \right] \\ &= \left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), \tilde{N}_k^\theta)) + g_2(S(\tau_k^\theta), N_k^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), \tilde{N}_k^\theta)) \right], \end{aligned}$$

and so we may take $N_{k+1}^\theta := \tilde{N}_k^\theta$. Thus the portfolio jumps from N_k^θ to $N_{k+1}^\theta = \tilde{N}_k^\theta$ at the transaction time τ_{k+1}^θ.

Now if $m_k^\theta = 1$, then again we conclude that $\tau_k^\theta < \tau_{k+1}^\theta$. Alternatively, suppose that $m_k^\theta > 1$. We already know from Lemma 4.1 that $m_k^\theta < \infty$ (except possibly when ω is in a set of measure zero). It is thus enough to consider $m_k^\theta = 2$. According to the above considerations we see that rather than going from N_{k-1}^θ to $\tilde{N}_k^\theta$ in two consecutive transactions with one immediately following the other (recall that here $\tau_k^\theta = \tau_{k+1}^\theta$), we may as well go from N_{k-1}^θ to $\tilde{N}_k^\theta$ in just one jump. Since $m_k^\theta = 2$ we know that $\tau_{k+1}^\theta < \tau_{k+2}^\theta$. It is now clear that we can use the construction of Sect. 4 in order to construct an (optimal) impulsive investment strategy for which the transaction times form a strictly increasing sequence. Since ω was arbitrary (outside of a zero measure set), this strategy is separated and so the proof of the proposition is complete.

Proof of Proposition 4.2 With the notation used in the preceding proof, suppose N_k^θ and $\tilde{N}_k^\theta$ both realize the maximum in

$$\sup_{N' \in \mathscr{A}(S(\tau_k^\theta), N_{k-1}^\theta)} \left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), N') + g_2(S(\tau_k^\theta), N_{k-1}^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N') \right],$$

and $\tilde{N}_k^\theta \in \mathscr{A}(S(\tau_k^\theta), N_k^\theta)$. We shall show that given our present hypotheses the last inclusion cannot happen. In fact, since both N_k^θ and $\tilde{N}_k^\theta$ realize the maximum, then we have by $(C5)$

$$\begin{aligned} &\left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), N_k^\theta) + g_2(S(\tau_k^\theta), N_{k-1}^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta) \right] \\ &= \left(\frac{\theta}{2}\right) ln(1-\alpha) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta) \end{aligned}$$

$$= \left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), \tilde{N}_k^\theta) + g_2(S(\tau_k^\theta), N_{k-1}^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), \tilde{N}_k^\theta) \right]$$

$$= \left(\frac{\theta}{2}\right) ln(1-\alpha) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), \tilde{N}_k^\theta),$$

and thus

$$\phi(S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta) = \phi(S(\tau_k^\theta), X(\tau_k^\theta), \tilde{N}_k^\theta).$$

But it is also true that

$$\left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), N_k^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), N_k^\theta) \right]$$

$$= \left[\left(\frac{\theta}{2}\right) (g_1(S(\tau_k^\theta), \tilde{N}_k^\theta)) + \phi(S(\tau_k^\theta), X(\tau_k^\theta), \tilde{N}_k^\theta) \right]$$

and thus

$$g_1(S(\tau_k^\theta), N_k^\theta) = g_1(S(\tau_k^\theta), \tilde{N}_k^\theta).$$

However, this last equality cannot be satisfied since we assumed that $\tilde{N}_k^\theta \in \mathscr{A}(S(\tau_k^\theta), N_k^\theta)$. This is a contradiction, so the proof of the proposition is complete.

References

1. Atkinson, C., Wilmott, P.: Portfolio management with transaction costs: an asymptotic analysis of the Morton and Pliska model. Math. Finance **5**, 357–367 (1995)
2. Bensoussan, A.: Stochastic Control by Functional Analysis Methods. Amsterdam: North-Holland 1982
3. Bensoussan, A., Lions, J. L.: Impulse Control and Quasi-Variational Inequalities. Paris: Bordas 1984
4. Bertsekas, D. P., Shreve, S. E.: Stochastic Optimal Control: The Discrete Time Case. New York: Academic Press 1978
5. Bielecki, T. R., Pliska, S. R.: Risk sensitive dynamic asset management. Applied Math. Optimization (forthcoming, 1999)
6. Bielecki, T. R., Pliska, S. R., Sherris, M.: Risk sensitive asset allocation. J. Econ. Dyn. Control (forthcoming, 1999)
7. Borodin, A. N., Salminen, P.: Handbook of Brownian Motion – Facts and Formulae. Basel: Birkhäuser 1996
8. Brennan, M. J., Schwartz, E. S.: The use of treasury bill futures in strategic asset allocation programs. IFA working paper 226–1996, London Business School (1996)
9. Brennan, M. J., Schwartz, E. S., Lagnado, R.: Strategic asset allocation. J. Econ. Dyn. Control **21**, 1377–1403 (1997)
10. Cvitanic, J., Karatzas, I.: Hedging and portfolio optimization under transaction costs: a martingale approach. Math. Finance **6**, 133–165 (1996)
11. Davis, M. H. A., Norman, A. R.: Portfolio selection with transaction costs. Math. Oper. Res. **15**, 676–713 (1990)
12. Eastham, J., Hastings, K.: Optimal impulse control of portfolios. Math. Oper. Res. **13**, 588–605 (1988)
13. Elliott, R. J.: Stochastic Calculus and Applications. Berlin Heidelberg New York: Springer 1982
14. Fleming W. H.: Optimal investment models and risk sensitive stochastic control. In: Davis, M. H. A. et al. (eds.) Math. Finance. Berlin Heidelberg Berlin Heidelberg New York: Springer 1995, pp 75–88

15. Fleming, W. H., Grossman, S. G., Vila, J. L., Zariphopolou, T.: Optimal portfolio rebalancing with transaction costs. Preprint 1990
16. Fleming, W. H., Soner, H. M.: Controlled Markov Processes and Viscosity Solutions. Berlin Heidelberg New York: Springer 1993
17. Hastings, K. J.: Impulse control of portfolios with jumps and transaction costs. Commun. Stat. – Stochastic Models **8**, 59–72 (1992)
18. Karatzas I.: Lectures on the Mathematics of Finance. CRM Monograph Series **8**. Providence, RI: American Mathematical Society 1996
19. Karatzas I., Shreve, S. E.: Brownian Motion and Stochastic Calculus. Berlin Heidelberg New York: Springer 1988
20. Korn, R.: Portfolio optimisation with strictly positive transaction costs and impulse control. Finance Stochast. **2**, 85–114 (1998)
21. Krylov, N.V.: Controlled Diffusion Processes. Berlin Heidelberg New York: Springer 1980
22. Kufner, A., John, O., Fucik, S.: Function Spaces. Leyden: Noordhoff 1977
23. Kuratowski, K.: Topology I. New York: Academic Press 1966
24. Merton, R. C.: Optimum consumption and portfolio rules in a continuous time model. J. Econ. Theory **3**, 373–413 (1971)
25. Merton, R. C.: An intertemporal capital asset pricing model. Econometrica **41**, 866–887 (1973)
26. Morton, A. J., Pliska, R. S.: Optimal portfolio management with transaction costs. Math. Finance **5**, 337–356 (1995)
27. Pesaran M. H., Timmermann, A.: Predictability of stock returns: robustness and economic significance. J. Finance **50**, 1201–1228 (1995)
28. Pliska, S. R., Selby, M. J. P.: On a free boundary problem that arises in portfolio management. Philos. Trans. R. Soc. Lond. A **347**, 555–561 (1994)
29. Robin, M.: On some impulse control problems with long run average cost. SIAM J. Control Optimization **19**, 333–358 (1981)
30. Schäl, M.: A selection theorem for optimization problems. Arch. Math. **25**, 219–224 (1974)
31. Shreve, S. E., Soner, H. M.: Optimal investment and consumption with transaction costs. Ann. Appl. Probab. **4**, 609–692 (1994)
32. Stettner, L.: On ergodic stopping and impulsive control problems for nonuniformly ergodic Markov processes. Appl. Math. Optimization **19**, 75–95 (1989)
33. Taksar, M., Klass M., Assaf, D.: A diffusion model for optimal portfolio selection in the presence of brokerage fees. Math. Oper. Res. **13**, 277–294 (1988)
34. Whittle P.: Risk Sensitive Optimal Control. New York: John Wiley 1990