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Risk-Sensitive Dynamic Asset Management

T. R. Bielecki[1] and S. R. Pliska[2]

[1]Department of Mathematics, The Northeastern Illinois University,
5500 North St. Louis Avenue, Chicago, IL 60625-4699, USA
t-bielecki@neiu.edu

[2]Department of Finance, University of Illinois at Chicago,
601 S. Morgan St., Chicago, IL 60607-7124, USA
srpliska@uic.edu

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Abstract. This paper develops a continuous time portfolio optimization model where the mean returns of individual securities or asset categories are explicitly affected by underlying economic factors such as dividend yields, a firm's return on equity, interest rates, and unemployment rates. In particular, the factors are Gaussian processes, and the drift coefficients for the securities are affine functions of these factors. We employ methods of risk-sensitive control theory, thereby using an infinite horizon objective that is natural and features the long run expected growth rate, the asymptotic variance, and a single risk-aversion parameter. Even with constraints on the admissible trading strategies, it is shown that the optimal trading strategy has a simple characterization in terms of the factor levels. For particular factor levels, the optimal trading positions can be obtained as the solution of a quadratic program. The optimal objective value, as a function of the risk-aversion parameter, is shown to be the solution of a partial differential equation. A simple asset allocation example, featuring a Vasicek-type interest rate which affects a stock index and also serves as a second investment opportunity, provides some additional insight about the risk-sensitive criterion in the context of dynamic asset management.

Key Words. Risk-sensitive stochastic control, Optimal portfolio selection, Incomplete markets, Large deviations.

AMS Classification. Primary 90A09, Secondary 60H30, 60G35, 93E20.

1. Introduction

Beginning with the fundamental work by Merton [23], a number of very sophisticated stochastic control models have been proposed for making optimal investment decisions. A typical approach takes diffusion process models of securities and looks for the trading strategy which maximizes the expected utility of consumption and/or terminal wealth over a finite planning horizon. The optimal strategy is obtained by solving the dynamic programming equation, which is a PDE that must be solved numerically except for a few simple cases.

The computational difficulties associated with PDEs have been circumvented somewhat by taking a more modern approach (see [16], [26], and [27]) using convex optimization theory and risk-neutral probability measures. The idea is to decompose the portfolio problem into two parts: first an optimization problem for the random variable representing the optimal terminal wealth (or its optimal consumption analog) is solved, and then a martingale representation problem for the optimal trading strategy is solved. Explicit results can be obtained for the first step in a wide variety of circumstances, provided the model is complete, which means, roughly speaking, that the number of underlying Brownian motions is equal to the number of securities (see [13]). However, the second step remains difficult in many cases. Moreover, it is often more realistic to consider incomplete models, but here the results are extremely limited (see [16] for the most recent results).

So while portfolio management seems like a natural application of stochastic control theory, the fact is that this technology is rarely used in practice. While there are several possible reasons for this lack of use, two primary explanations seem to be computational tractability and the statistical difficulties associated with the estimation of model parameters. The aim of this paper is to introduce a new stochastic control approach that is intended to address these two problems and thereby reduce the gap between theory and practice. This approach is distinguished by two key features: underlying economic factors such as accounting ratios, dividend yields, and macroeconomic measures are explicitly incorporated in the model, and an infinite horizon, risk-sensitive criterion is used for the investor's objective.

Practitioners commonly use economic factors to forecast the returns of securities, so in this paper we explicitly model such factors as well as the dependence of security prices on such factors. While our approach does not circumvent the statistical difficulties of making good estimates, it sheds better light on this problem, because the variables which are used to forecast returns now reside within the model. Both the estimation and optimization parts of the portfolio management process can be analyzed in an integrated fashion.

Finite horizon optimization criteria often lead to time-dependent optimal strategies, in which case the computational difficulties may be great. The alternative, namely, the adoption of an infinite horizon optimization objective, offers the possibility of stationary policies being optimal and thus of less severe computational difficulties. In addition, an infinite horizon objective frequently is very appropriate for practical investment problems such as the problem of managing a mutual fund. However, the choices of infinite horizon criteria are quite limited. About the only such criterion that has been studied in the portfolio management context is that of maximizing the portfolio's long-run growth rate (i.e., the Kelly criterion, associated with expected log utility). This criterion is not conservative enough for most investors, and investors vary widely in regard to their attitudes about risk. Meanwhile, the two classical infinite horizon criteria in the control theory literature, namely, average cost/reward and discounted cost/reward, are poorly suited for portfolio management applications.

However, consider the criterion (here $V(t)$ is time-t portfolio value)

$$J_\theta := \liminf_{t\to\infty} \left(\frac{-2}{\theta}\right) t^{-1} \ln \mathbf{E} e^{-(\theta/2)\ln V(t)},$$

where $\theta > -2$, $\theta \neq 0$. Substituting $R(t) = \ln V(t)$ enables one to establish a connection with the recently developed literature on *risk-sensitive optimal control* (e.g., see [28]), where $R(t)$ plays the role of a cumulative reward. This means that if we adopt, as we shall, the objective of maximizing J_θ, then many of the techniques that have recently been developed for risk-sensitive control can be potentially applied to our portfolio management problem.

Moreover, as is well understood in the risk-sensitive control literature, a Taylor series expansion about $\theta = 0$ yields

$$-\frac{2}{\theta} \ln \mathbf{E} e^{-\theta/2 \ln V(t)} = \mathbf{E} \ln V(t) - \frac{\theta}{4} \mathbf{var}(\ln V(t)) + O(\theta^2). \tag{1.1}$$

Hence J_θ can be interpreted as the long-run expected growth rate minus a penalty term, with an error that is proportional to θ^2. Furthermore, the penalty term is proportional to the *asymptotic variance*, a quantity that was studied in [19] in the case of a conventional, multivariate geometric Brownian motion model of securities. The penalty term is also proportional to θ, so θ should be interpreted as a *risk-sensitivity parameter* or *risk-aversion parameter*. The special case of $\theta = 0$ is referred to as the *risk-null case*; this is the classical Kelly criterion, that is, $J_0 = \liminf_{t\to\infty} t^{-1}\mathbf{E} \ln V(t)$.

Note that J_θ has the form of the large-deviations-type functional for the capital process $V(t)$. Consequently, maximizing J_θ for $\theta > 0$ protects an investor interested in maximizing the expected growth rate of the capital against large deviations of the actually realized rate from the expectations.

Some insight into our risk-sensitive criterion can be obtained by considering the case where the process $V(t)$ is a simple geometric Brownian motion with parameters μ and σ. A simple calculation gives $J_\theta = \mu - \frac{1}{2}\sigma^2 - (\theta/4)\sigma^2$, so the approximation mentioned above is, in this case, exact.

This paper is not the first to apply a risk-sensitive optimality criterion to a financial problem. Lefebvre and Montulet [22] used the calculus of variations approach to study a firm's optimal mix between liquid and nonliquid assets. Fleming [9] used risk-sensitive methods to obtain asymptotic results for two kinds of portfolio management problems.

In summary, in this paper we develop a portfolio optimization model where securities explicitly depend on underlying economic factors and where the objective is to maximize the risk-sensitive criterion J_θ that was introduced above. A precise formulation of our model as well as the main results are all found in Section 2. Various preliminary and auxiliary results are located in Section 3, while Section 4 has the principal arguments and proofs of our main results.

Sections 2–4 are all for the case where the risk-aversion parameter $\theta > 0$. The risk-null case $\theta = 0$ is the subject of Section 5. It should be pointed out here, however, that when our model has no factors and $\theta = 0$ it collapses to a well-studied, complete model, but when factors are included our model is incomplete and our expected growth rate maximization results are new. Indeed, most results in this paper, both when $\theta = 0$ and $\theta > 0$, are significant contributions to the financial economic theory of incomplete models of security markets.

Section 6 is devoted to providing some additional, interesting insight into the nature of risk-sensitive optimality in the context of dynamic asset management. In Section 7 we provide some thoughts about our future research. The paper concludes with an Appendix in which we prove Lemmas 5.1 and 5.2.

2. Formulation of the Problem and the Main Results

We consider a market consisting of $m \geq 2$ securities and $n \geq 1$ factors. The set of securities may include stocks, bonds, cash, and derivative securities, as in [7], for example. The set of factors may include dividend yields, price-earning ratios, short-term interest rates, the rate of inflation, etc., as in [25], for example.

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbf{P})$ be the underlying probability space. Denoting by $S_i(t)$ the price of the ith security and by $X_j(t)$ the level of the jth factor at time t, we consider the following market model for the dynamics of the security prices and factors:

$$\frac{dS_i(t)}{S_i(t)} = (a + AX(t))_i\, dt + \sum_{k=1}^{m+n} \sigma_{ik}\, dW_k(t), \qquad S_i(0) = s_i, \quad i = 1, 2, \ldots, m, \tag{2.1}$$

$$dX(t) = (b + BX(t))\, dt + \Lambda\, dW(t), \qquad X(0) = x, \tag{2.2}$$

where $W(t)$ is an R^{m+n}-valued standard Brownian motion process with components $W_k(t)$, $X(t)$ is the R^n-valued factor process with components $X_j(t)$, the market parameters $a,\ A,\ \Sigma := [\sigma_{ij}],\ b,\ B,\ \Lambda := [\lambda_{ij}]$ are matrices of appropriate dimensions, and $(a + Ax)_i$ denotes the ith component of the vector $a + Ax$. It is well known that a unique, strong solution exists for (2.1), (2.2), and that the processes $S_i(t)$ are positive with probability 1 (see, e.g., Chapter 5 of [18]).

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$, where $S(t) = (S_1(t), S_2(t), \ldots, S_m(t))$ is the security price process. Let $h(t)$ denote an R^n-valued investment process or strategy whose components are $h_i(t),\ i = 1, 2, \ldots, m$.

Definition 2.1. An investment process $h(t)$ is *admissible* if the following conditions are satisfied:

(i) $h(t)$ takes values in a given measurable subset χ of R^m, and $\sum_{i=1}^m h_i(t) = 1$,
(ii) $h(t)$ is measurable, $\mathcal{G}_t$-adapted,
(iii) $\mathbf{P}[\int_0^t h'(s)h(s)\, ds < \infty] = 1$, for all finite $t \geq 0$.

The class of admissible investment strategies will be denoted by $\mathcal{H}$.

Let now $h(t)$ be an admissible investment process. Then there exists a unique, strong, and almost surely positive solution $V(t)$ to the following equation:

$$dV(t) = \sum_{i=1}^{m} h_i(t)V(t)\left[\mu_i(X(t))\,dt + \sum_{k=1}^{m+n}\sigma_{ik}\,dW_k(t)\right], \qquad V(0) = v > 0, \quad (2.3)$$

where $\mu_i(x)$ is the ith coordinate of the vector $a + Ax$ for $x \in R^n$. The process $V(t)$ represents the investor's capital at time t, and $h_i(t)$ represents the proportion of capital that is invested in security i, so that $h_i(t)V(t)/S_i(t)$ represents the number of shares invested in security i, just as in, for example, Section 3 of [17].

In this paper we investigate the following family of risk-sensitized optimal investment problems, labeled as $(\mathbf{P}_\theta)$:

for $\theta \in (0, \infty)$, *maximize the risk-sensitized expected growth rate*

$$J_\theta(v, x; h(\cdot)) := \liminf_{t\to\infty}\left(\frac{-2}{\theta}\right)t^{-1}\ln \mathbf{E}^{h(\cdot)}\left[e^{-(\theta/2)\ln V(t)}|V(0) = v, X(0) = x\right] \quad (2.4)$$

over the class of all admissible investment processes $h(\cdot)$, *subject to* (2.2) *and* (2.3),

where $\mathbf{E}$ is the expectation with respect to $\mathbf{P}$. The notation $\mathbf{E}^{h(\cdot)}$ emphasizes that the expectation is evaluated for process $V(t)$ generated by (2.3) under the investment strategy $h(t)$.

Remark 2.1. As mentioned in the Introduction, the positive value of the risk-sensitivity parameter θ corresponds to a risk-averse investor. The techniques used in this paper can also be used to study problems $(\mathbf{P}_\theta)$ for negative values of θ, corresponding to risk-seeking investors. The risk-null case, for $\theta = 0$, is studied in Section 5 as the limit of the risk-averse situation when the risk-sensitivity parameter θ goes to zero.

Before we can present the main results contributed by this paper, we need to introduce the following notation, for $\theta \geq 0$ and $x \in R^n$:

$$K_\theta(x) := \inf_{h\in\chi,\ \mathbf{1}'h=1}\left[\tfrac{1}{2}\left(\frac{\theta}{2}+1\right)h'\Sigma\Sigma'h \;-\; h'(a + Ax)\right]. \quad (2.5)$$

We also need to introduce the following assumptions:

Assumption (A1). *The investment constraint set* χ *satisfies one of the following two conditions*:

(a) $\chi = R^n$, *or*
(b) $\chi = \{h \in R^n : h_{1i} \leq h_i \leq h_{2i},\ i = 1, 2, \ldots, m\}$, *where* $h_{1i} < h_{2i}$ *are finite constants*.

Assumption (A2). *For* $\theta > 0$,

$$\lim_{\|x\|\to\infty} K_\theta(x) = -\infty.$$

Assumption (A3). *The matrix* $\Lambda\Lambda'$ *is positive definite, and* $\Sigma\Lambda' = 0$.

Remark 2.2. (i) Note that if $\Sigma\Sigma'$ is positive definite and $\ker(A) = 0$, then assumption (A2) is implied by assumption (A1)(a).

(ii) These assumptions are sufficient for the results below to be true, but, as will be seen for the example considered in Section 6, Assumption (A2) is not necessary, in general.

Theorems 2.1 and 2.2 below contain the main results of this paper.

Theorem 2.1. *Assume* (A1)–(A3). *Fix* $\theta > 0$. *Let* $H_\theta(x)$ *denote a minimizing selector in* (2.5), *that is,*

$$K_\theta(x) = \tfrac{1}{2}\left(\frac{\theta}{2} + 1\right) H_\theta(x)'\Sigma\Sigma' H_\theta(x) \; - \; H_\theta(x)'(a + Ax).$$

Then the investment process

$$h_\theta(t) := H_\theta(X(t)) \tag{2.6}$$

is optimal sure in the sense of Foldes (*see* [11]). *That is, letting, for each* $\tau \geq 0$,

$$J_\theta^\tau(v, x; h(\cdot)) := \frac{-2}{\theta} \ln \mathbf{E}^{h(\cdot)}[V^{-\theta/2}(\tau) | V(0) = v, X(0) = x],$$

we have

$$J_\theta^\tau(v, x; h(\cdot)) \leq J_\theta^\tau(v, x; h_\theta(\cdot)) \tag{2.7}$$

for all admissible strategies $h(\cdot)$, $v > 0$, $x \in R^n$, *and all* $\tau \geq 0$.

Corollary 2.1. *The investment process* $h_\theta(t)$ *is optimal for problem* $(\mathbf{P}_\theta)$, *that is,*

$$J_\theta(v, x; h(\cdot)) \leq J_\theta(v, x; h_\theta(\cdot))$$

holds for all $h(\cdot) \in \mathcal{H}$, $v > 0$, $x \in R^n$.

Theorem 2.2. *Assume* (A1)–(A3), *fix* $\theta > 0$, *and consider problem* $(\mathbf{P}_\theta)$. *Let* $h_\theta(t)$ *be as in Theorem* 2.1. *Then*:

(a) *For all* $v > 0$ *and* $x \in R^n$ *we have*

$$\begin{aligned} J_\theta(v, x; h_\theta(\cdot)) &= \lim_{t\to\infty} \left(\frac{-2}{\theta}\right) t^{-1} \ln \mathbf{E}^{h_\theta(\cdot)}[e^{-(\theta/2)\ln V(t)} | V(0) = v, X(0) = x] \\ &=: \rho(\theta). \end{aligned}$$

(b) *The constant* $\rho(\theta)$ *in* (a) *is the unique nonnegative constant which is a part of* *the solution* $(\rho(\theta), v(x;\theta))$ *to the following equation*:

$$\rho = (b+Bx)' \operatorname{grad}_x v(x) - \frac{\theta}{4}\sum_{i,j=1}^{n} \frac{\partial v(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}$$
$$+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 v(x)}{\partial x_i\,\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk} - K_\theta(x),$$
$$v(x)\in C^2(R^n), \qquad \lim_{\|x\|\to\infty} v(x)=\infty, \qquad \rho = const. \tag{2.8}$$

The key point of the first equality in (a) is, of course, that the optimal objective value is given by an ordinary lim rather than the lim inf as in (2.4). The key point of the second equality in (a) is that the optimal objective value does not depend on either the initial amount of the investor's capital (v) or on the initial values of the underlying economic factors (x), although it depends, of course, on the investor's attitude toward risk (encoded in the value of θ). The key point of (b) is that the optimal objective value is characterized in terms of (2.8). It is important to observe that for the problem studied in this paper the part $v(x;\theta)$ of the solution to (2.8) is a classical, smooth solution of a PDE. This is mainly because the diffusion term in the factor equation (2.2) is nondegenerate. For more general formulations, viscosity solutions may have to be considered.

3. Auxiliary Results

In this section we formulate several technical results that will be needed later. Let $K(x)$ be a real-valued function on R^n. Throughout this section it is assumed that $K(x)$ has the following properties:

Assumption (B1).

(a) $K(x) \le 0$.
(b) $\lim_{\|x\|\to\infty} K(x) = -\infty$.
(c) $|K(x)| \le c(1+\|x\|^2)$, *where c is a positive constant.*
(d) *Let* $K_i \subset R^n$, $i = 1, 2, \ldots, I$, I *finite, be disjoint, open sets, such that* $\bigcup_{i=1}^{I} \bar{K}_i = R^n$, *where* $\bar{K}_i$ *is the closure of* K_i. *Then* $K(x)$ *is smooth on each of the* K_i*'s.*
(e) $K(x)$ *is locally Lipschitz on* R^n.

We begin by considering the following Cauchy problem:

$$\frac{\partial f(t,x)}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f(t,x)}{\partial x_i\,\partial x_j}\sum_{k=1}^{m+n}\lambda_{ik}\lambda_{jk}$$
$$+ (b+Bx)' \operatorname{grad}_r f(t,r) + K(x) f(x,t), \tag{3.1}$$
$$f(0,x) = 1,$$

for $t \in (0,T]$, $T < \infty$, and $x \in R^n$. The above problem has been extensively studied in the literature, and its properties are well known (see [12] or [21] for classical expositions).

The classical solution to (3.1) can, for example, be constructed as follows:

Let $Z(x, \xi, t, \tau)$ be given as in (11.13) of [21] for $x, \xi \in R^n$ and $0 \leq \tau < t \leq T$, and define

$$g(t, x) := \int_{R^n} Z(x, \xi, t, 0)\, d\xi, \qquad (t, x) \in [0, T] \times R^n. \tag{3.2}$$

Using the above estimates and formulas (11.8) and (11.9) on p. 358 of [21], it follows that

$$g \in C^{1,2}((0, T), R^n) \cap C([0, T], R^n) \tag{3.3}$$

and

$$g \in C^{1+\alpha/2, 2+\alpha}((0, T), B_r) \cap C([0, T], R^n), \tag{3.4}$$

where $B_r = \{x \in R^n : \|x\| < r\}$, and $r > 0$ is arbitrary. In addition, we also have (see the discussion in Section IV.14 of [21]) that g is a solution to the Cauchy problem (3.1). Finally, using the Feynmann-Kac formula (see, e.g., [18]) we obtain the following stochastic representation for g:

$$g(t, x) = \mathbf{E}[e^{\int_0^t K(X(s))ds} | X(0) = x], \tag{3.5}$$

where $X(t)$ is our factor process. Since every smooth solution for the Cauchy problem (3.1) has the above representation, then g is the unique solution to the problem (3.1). In view of conditions (B1)(a),(c), representation (3.5) implies the following estimates for g:

$$0 < g(t, x) \leq 1, \qquad (t, x) \in [0, T] \times R^n, \tag{3.6}$$

$$\frac{\partial g(t, x)}{\partial t} \leq 0, \qquad (t, x) \in (0, T) \times R^n. \tag{3.7}$$

We now fix $\theta > 0$ and define

$$u_{\theta,T}(t, x) := -\frac{2}{\theta} \ln g(t, x), \qquad (t, x) \in [0, T] \times R^n. \tag{3.8}$$

The following lemma summarizes properties of $u_{\theta,T}$ that we need.

Lemma 3.1. *Assume* (A2) *and* (B1). *Then the function* $u_{\theta,T}$ *defined by* (3.8) *enjoys the following properties*:

(a) $u_{\theta,T} \geq 0$,
(b) $\partial u_{\theta,T}/\partial t \geq 0$,
(c) $u_{\theta,T}$ *is the only, nonnegative, classical solution of*

$$\frac{\partial u(t, x)}{\partial t} = (b + Bx)' \operatorname{grad}_x u(t, x)$$
$$+ \tfrac{1}{2} \left[\frac{-\theta}{2} \sum_{i,j=1}^{n} \frac{\partial u(t, x)}{\partial x_i} \frac{\partial u(t, x)}{\partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} \right.$$

$$+\sum_{i,j=1}^{n}\frac{\partial^2 u(t,x)}{\partial x_i\,\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}\Bigg]-\frac{2}{\theta}K(x), \tag{3.9}$$

$u(0,x)=0,$

for $(t,x)\in(0,T]\times R^n$,

(d) $u_{\theta,T}(t,x)$ *is three times continuously differentiable in* x *on* $\bigcup_{i=1}^{I}K_i$.

Proof. Properties (a), (b), and (c) are obvious consequences of (3.12)–(3.14) and the fact that $g(t,x)$ is the unique solution to (3.1). Property (d) follows since $K(x)$ is smooth on $\bigcup_{i=1}^{I}K_i$. □

In view of the above lemma and the fact that $\operatorname{grad}_x u_{\theta,T}$ and $\partial u_{\theta,T}/\partial t$ are continuous, we can now apply the argument used in the proof of Lemma 1.5 of [24] in order to obtain the following important estimate (compare (1.28) in [24]):

$$t\left(\|\operatorname{grad}_x u_{\theta,T}\|^2+\gamma\frac{\partial u_{\theta,T}}{\partial t}\right)\le tK_{r,\gamma}+L_{r,\gamma},\qquad \text{on}\quad (0,T]\times B_r, \tag{3.10}$$

where γ, $K_{r,\gamma}$, and $L_{r,\gamma}$ are some positive constants that are independent of t and T.

We want to extend the solution $u_{\theta,T}(t,x)$ from $[0,T]\times R^n$ to $[0,\infty)\times R^n$. Toward this end, following the argument of Section 1.4 in [24] with $u_R(t,x)=u_1(t,x)=u_2(t,x)=u_{\theta,T}(t,x)$ on $[0,T]\times\bar{B}_R$ (using the notation of [24]), we arrive at the following result (compare Theorem 1.1 in [24]):

Lemma 3.2. *Assume* (A2) *and* (B1). *Then*:

(a) *The equation*

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t}&=(b+Bx)'\operatorname{grad}_x u(t,x)\\&\quad+\tfrac{1}{2}\Bigg[\frac{-\theta}{2}\sum_{i,j=1}^{n}\frac{\partial u(t,x)}{\partial x_i}\frac{\partial u(t,x)}{\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}\\&\qquad+\sum_{i,j=1}^{n}\frac{\partial^2 u(t,x)}{\partial x_i\,\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}\Bigg]\\&\quad-\frac{2}{\theta}K(x),\qquad (t,x)\in(0,\infty)\times R^n,\\u(0,x)&=0,\qquad x\in R^n,\end{aligned}\tag{3.11}$$

has a nonnegative solution $u_\theta\in C^{1,2}((0,\infty),R^n)\cap C([0,\infty),R^n)$. *This* u_θ *is an extension of* $u_{\theta,T}$ *to* $[0,\infty)\times R^n$.

(b) $\partial u_\theta/\partial t\ge 0$.

(c) $$t\left(\|\operatorname{grad}_x u_\theta\|^2+\gamma\frac{\partial u_\theta}{\partial t}\right)\le tK_{r,\gamma}+L_{r,\gamma},\qquad \textit{on}\quad (0,\infty)\times B_r, \tag{3.12}$$

for some positive constants γ, $K_{r,\gamma}$ *and* $L_{r,\gamma}$ *that are independent of* t.

We see now that Theorem 3.4 of [24] applies in our context. We state the version of this theorem, appropriate for the situation considered here, as

Lemma 3.3. *Assume* (A2) *and* (B1). *Then*:

(a) *As* $t \to \infty$ *then the function* $u_\theta(t, x) - u_\theta(t, 0)$ *converges to a function* $\nu_\theta(x)$ *in* $W^1_{2,\mathrm{loc}}$ *uniformly on each compact subset of* R^n, *and the function* $\partial u_\theta(t, x)/\partial t$ *converges to a constant* ρ_θ.

(b) *The pair* $(\nu_\theta, \rho_\theta)$ *is the unique solution to equation* (ν_θ *is unique up to an additive constant*)

$$\rho = (b + Bx)' \operatorname{grad}_x \nu(x) - \frac{\theta}{4} \sum_{i,j=1}^{n} \frac{\partial \nu(x)}{\partial x_i} \frac{\partial \nu(x)}{\partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk}$$
$$+ \tfrac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \nu(x)}{\partial x_i\, \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} - \frac{2}{\theta} K(x), \tag{3.13}$$
$$\nu(x) \in C^2(R^n), \qquad \lim_{\|x\| \to \infty} \nu(x) = \infty, \qquad \rho = \mathit{const}.$$

Since $\partial u_\theta(t, x)/\partial t$ converges to a constant ρ_θ, we thus have an obvious

Corollary 3.1.

$$\lim_{t \to \infty} \frac{u_\theta(t, x)}{t} \equiv \rho_\theta. \tag{3.14}$$

4. Proofs of the Main Results

In this section we verify validity of the results stated in Section 2. Assumptions (A1)–(A3) are supposed to hold throughout the section.

Fix $\theta > 0$ and consider the following Bellman–Hamilton–Jacobi equation:

$$\begin{aligned} &0 = \inf_{h \in \chi} [L^h \varphi(t, x, v)], \\ &\varphi(0, x, v) = v^{-(\theta/2)}, \end{aligned} \tag{4.1}$$

for $t > 0,\ x \in R^n,\ v > 0$, where

$$L^h \varphi(t, x, v) := -\frac{\partial \varphi(t, x, v)}{\partial t} + \frac{\partial \varphi(t, x, v)}{\partial v} h'(a + Ax)v$$
$$+ (b + Bx)' \operatorname{grad}_x \varphi(t, x, v)$$
$$+ \tfrac{1}{2} \frac{\partial^2 \varphi(t, x, v)}{\partial v^2} h' \Sigma \Sigma' h v^2 + \tfrac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \varphi(t, x, v)}{\partial x_i\, \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk}.$$

We seek a solution to (4.1) in the form

$$\Phi(t, x, v; \theta) = v^{-(\theta/2)} e^{-(\theta/2) U(t, x; \theta)}, \tag{4.2}$$

for some suitable function $U(t, x; \theta)$. To this end, we consider first the following two equations:

$$\begin{aligned}\frac{\partial U(t,x)}{\partial t} &= (b+Bx)'\operatorname{grad}_x U(t,x)\\ &\quad + \frac{1}{2}\left[\frac{-\theta}{2}\sum_{i,j=1}^{n}\frac{\partial U(t,x)}{\partial x_i}\frac{\partial U(t,x)}{\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}\right.\\ &\quad\left.+\sum_{i,j=1}^{n}\frac{\partial^2 U(t,x)}{\partial x_i\,\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}\right]\\ &\quad - K_\theta(x), \qquad (t,x)\in(0,\infty)\times R^n,\\ U(0,x) &= 0, \qquad x\in R^n,\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\frac{\partial \bar{U}(t,x)}{\partial t} &= (b+Bx)'\operatorname{grad}_x \bar{U}(t,x)\\ &\quad + \frac{1}{2}\left[\frac{-\theta}{2}\sum_{i,j=1}^{n}\frac{\partial \bar{U}(t,x)}{\partial x_i}\frac{\partial \bar{U}(t,x)}{\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}\right.\\ &\quad\left.+\sum_{i,j=1}^{n}\frac{\partial^2 \bar{U}(t,x)}{\partial x_i\,\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}\right]\\ &\quad - \bar{K}_\theta(x), \qquad (t,x)\in(0,\infty)\times R^n,\\ \bar{U}(0,x) &= 0, \qquad x\in R^n,\end{aligned} \tag{4.4}$$

where

$$\bar{K}_\theta(x) = K_\theta(x) - K_\theta$$

and $K_\theta(x)$ and K_θ are defined in (2.5) and (2.9), respectively. We now have the following:

Proposition 4.1. *The constant K_θ is finite, and the function $\bar{K}_\theta(x)$ satisfies Assumption* (B1).

Proof. Invoking the results from Section 5.5 of [1] we conclude that $K_\theta(x)$ satisfies conditions (c) and (d) of Assumption (B1). In view of this and Assumption (A2), we see that the constant K_θ is finite, and that $\bar{K}_\theta(x)$ satisfies conditions (a)–(d) of Assumption (B1). Condition (e) is satisfied in view of problem (11)(a) in [10]. □

From the above proposition and from Lemma 3.2 it follows that there exists a nonnegative classical solution $\bar{U}(t, x; \theta)$ to (4.4). Now, by letting

$$U(t,x;\theta) = \bar{U}(t,x;\theta) - tK_\theta, \qquad (t,x)\in[0,\infty)\times R^n, \tag{4.5}$$

we see that $U(t, x; \theta)$ is a classical solution to (4.3). We thus have the following:

Proposition 4.2. *Let $\Phi(t, x, v; \theta)$ be as in* (4.2)*, with $U(t, x; \theta)$ as in* (4.5)*. Then $\Phi(t, x, v; \theta)$ is a classical solution to the Bellman–Hamilton–Jacobi equation* (4.1).

Proof. The result follows by direct inspection. □

We proceed with the statement and proof of a verification result, from which Theorem 2.1 and Corollary 2.1 follow immediately:

Proposition 4.3. *Let $\Phi_\theta(t, x, v)$ denote any classical solution to* (4.1)*. For each $h(\cdot) \in \mathcal{H}$ we have*

$$\begin{aligned}&\Phi_\theta(t, x, v) \le \mathbf{E}^{h(\cdot)}[V(t)^{-(\theta/2)} | V(0) = v, X(0) = x],\\ &t \ge 0, \quad (x, v) \in R^n \times (0, \infty),\end{aligned} \tag{4.6}$$

For $h_\theta(\cdot)$ defined in (2.6) *we have*

$$\begin{aligned}&\Phi_\theta(t, x, v) = \mathbf{E}^{h_\theta(\cdot)}[V(t)^{-(\theta/2)} | V(0) = v, X(0) = x],\\ &t \ge 0, \quad (x, v) \in R^n \times (0, \infty).\end{aligned} \tag{4.7}$$

Proof. The results are clearly true for $t = 0$.

Fix $t > 0$ and a strategy $h(\cdot) \in \mathcal{H}$. Applying Ito's formula to $\Psi_\theta(s, x, v) = \Phi_\theta(t - s, x, v)$ for $0 \le s \le t$, we get for each sufficiently small $\varepsilon > 0$ the following equality:

$$\begin{aligned}&\mathbf{E}^{h(\cdot)}[\Phi_\theta(\varepsilon, X(t-\varepsilon), V(t-\varepsilon)) | V(0) = v, X(0) = x] - \Phi_\theta(t, x, v)\\ &\quad = \mathbf{E}^{h(\cdot)}\left[\int_\varepsilon^t L^{h(\cdot)}\Phi_\theta(r, X(t-r), V(t-r))\,dr \,|\, V(0) = v, X(0) = x\right]\end{aligned} \tag{4.8}$$

for all $x \in R^n$ and $v > 0$, where $L^{h(\cdot)}\Phi_\theta(s, X(s), V(s))$ is defined similarly as $L^h\Phi_\theta(s, X(s), V(s))$ with $h(s)$ substituting for h.

It follows from (4.1) that the expression on the right-hand side of (4.8) is nonnegative. Thus, letting ε go to zero, we obtain (4.6).

It follows from the results of Section 5.5 in [1] that $H_\theta(x)$ (defined in Theorem 2.1) is a piecewise affine function on R^n. Thus $h_\theta(\cdot)$ is an admissible strategy, and the conclusion (4.7) follows since the right-hand side of (4.8) is equal to zero for $h(\cdot) \equiv h_\theta(\cdot)$. □

We are ready now to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\Phi(t, x, v; \theta)$ be as in (4.2), with $U(t, x; \theta)$ as in (4.5). Then it follows from Propositions 4.2 and 4.3 that $\Phi(t, x, v; \theta)$ is the unique solution to the Bellman–Hamilton–Jacobi equation (4.1), and that it satisfies (4.6) and (4.7). This implies (2.7). □

It remains to demonstrate Theorem 2.2.

Proof of Theorem 2.2. As in the proof of Theorem 2.1 we first observe that $\Phi(t, x, v; \theta)$ is the unique solution to (4.1). Thus $\bar{U}(t, x, v; \theta)$ is the unique nonnegative solution to (4.4), and we have

$$-\frac{2}{\theta} \ln \Phi(t, x, v; \theta) = \ln v + \bar{U}(t, x; \theta) - tK_\theta, \tag{4.9}$$

for $(t, x, v) \in [0, \infty) \times R^n \times (0, \infty)$. Applying Lemma 3.3 and Corollary 3.1 to (2.8) and (4.4) we conclude that

$$\lim_{t\to\infty} \frac{\bar{U}(t, x; \theta)}{t} = \rho(\theta) + K_\theta. \tag{4.10}$$

The conclusions of Theorem 2.2 follow now from (4.9) and (4.10) since $\Phi(t, x, v; \theta)$ satisfies (4.6) and (4.7). □

5. Risk-Null Problem ($\theta = 0$)

In this section we study the limit when $\theta \downarrow 0$ of problems $(\mathbf{P}_\theta)$. This leads to consideration of the classical problem of maximizing the portfolio's expected growth rate, that is, the growth rate under the log-utility function (see, e.g., [15] and [16]). We label this problem as $(\mathbf{P}_0)$ and formulate it as follows:

maximize the expected growth rate

$$J_0(v, x; h(\cdot)) := \liminf_{t\to\infty} t^{-1}\, \mathbf{E}^{h(\cdot)}[\ln V(t) | V(0) = v, X(0) = x] \tag{5.1}$$

over the class of all admissible investment processes $h(\cdot)$, *subject to* (2.2) *and* (2.3).

Throughout the section we impose the following three assumptions:

Assumption (C1). *For each* $\theta \ge 0$ *the function* $K_\theta(x)$ (*see* (2.5)) *is quadratic and of the form*

$$K_\theta(x) = \tfrac{1}{2} x' K_1(\theta) x + K_2(\theta) x + K_3(\theta).$$

Assumption (C2). *For each* $\theta \ge 0$ *the matrix* $K_1(\theta)$ *is symmetric and negative definite.*

Assumption (C3). *The matrix* B *in* (2.2) *is stable.*

Remark 5.1. (a) Assumption (C1) will be relaxed in a future paper. We briefly discuss at the end of this section how our results can be generalized to the case of general functions $K_\theta(x)$, as defined in (2.5).

(b) Assumption (C1) is satisfied if, for example, the matrix $\Sigma\Sigma'$ is nonsingular and if $\chi = R^n$. As will be seen in Section 6, nonsingularity of $\Sigma\Sigma'$ is not a necessary condition for (C1) to hold.

(c) It follows from Section 5.5 in [1] that $\lim_{\theta\downarrow 0} K_i(\theta) = K_i(0)$ for $i = 1, 2, 3$.

(d) Note that we did not assume stability of B in order to prove Corollary 2.1 and Theorem 2.2. This is because positivity of the risk-sensitivity parameter θ (which was assumed there) was enough to enforce "good" behavior of our objective functionals for large t.

(e) Note that Assumption (A2) is satisfied. Assumption (A1) is no longer needed.

We intend to establish a relationship between the *risk-null problem* $(\mathbf{P}_0)$ and the *risk-sensitive problems* $(\mathbf{P}_\theta)$, $\theta > 0$. Toward this end we consider the following equation:

$$\begin{aligned} &\rho(0) = (b + Bx)' \operatorname{grad}_x v_0(x) + \tfrac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 v_0(x)}{\partial x_i \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} - K_0(x), \\ &v_0(x) \in C^2(R^n), \qquad \lim_{\|x\|\to\infty} v_0(x) = \infty, \qquad \rho(0) = \text{const.} \end{aligned} \tag{5.2}$$

We now have the following two results:

Lemma 5.1. *Assume* (A3) *and* (C1)–(C3). *Then there exists a solution pair* $(\rho(0), v_0)$ *to* (5.2).

Proof. See the Appendix. □

Proposition 5.1. *Let $H_0(x)$ be a minimizing selector on the right-hand side of* (2.5). *Define a strategy $h_0(\cdot)$ as in* (2.6) *with* 0 *replacing θ. If $(\rho(0), v_0)$ is a solution to* (5.2), *then we have*:

(a) *The strategy $h_0(\cdot)$ is optimal for $(\mathbf{P}_0)$, and*

$$J_0(v, x; h_0(\cdot)) = \lim_{t\to\infty} t^{-1}\, \mathbf{E}^{h_0(\cdot)}[\ln V(t) | V(0) = v, X(0) = x] = \rho(0). \tag{5.3}$$

(b) *The constant $\rho(0)$ is unique.*

Proof. Fix an arbitrary admissible strategy $h(\cdot)$ and $(v, x) \in (0, \infty) \times R^n$. Applying Ito's formula to v_0 and using (2.5) and (5.2) we get

$$\begin{aligned} &v_0(X(t)) - v_0(v) \\ &\quad = \int_0^t \left[(b + Bx)' \operatorname{grad}_x v_0(X(s)) + \tfrac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 v_0(X(s))}{\partial x_i\, \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} \right] ds \\ &\qquad - \int_0^t \operatorname{grad}'_x v_0(X(s)) \Lambda\, dW(s) \\ &\quad \le t\rho(0) + \int_0^t [\tfrac{1}{2} h'(X(s)) \Sigma \Sigma' h(X(s)) - h'(X(s))(a + AX(s))]\, ds \\ &\qquad - \int_0^t \operatorname{grad}'_x v_0(X(s)) \Lambda\, dW(s), \qquad t \ge 0. \end{aligned} \tag{5.4}$$

Under the strategy $h(\cdot)$ the capital process $V(t)$ is the geometric Brownian motion (see, e.g., p. 361 of [18])

$$V(t) = v\cdot \exp\left\{-\int_0^t [\tfrac{1}{2}h'(X(s))\Sigma\Sigma' h(X(s)) - h'(X(s))(a + AX(s))]\,ds \right.$$
$$\left. + \int_0^t h'(X(s))\Sigma\, dW(s)\right\}, \qquad t \geq 0.$$

Therefore, we obtain from (5.4)

$$\begin{aligned} &\mathbf{E}^{h(\cdot)}[v_0(X(t))|V(0) = v, X(0) = x] - v_0(v) \\ &\quad \leq t\rho(0) + \ln v - \mathbf{E}^{h(\cdot)}[\ln V(t)|V(0) = v, X(0) = x], \qquad t \geq 0. \end{aligned} \tag{5.5}$$

Now, it follows from the proof of Lemma 5.1 that $v_0(x)$ is a quadratic function. Since the factor process $X(t)$ is ergodic (because B is stable) then it can be shown (see, e.g., [4]) that for all $(v, x) \in (0, \infty) \times R^n$ and for all $h(\cdot) \in \mathcal{H}$ we have that

$$\lim_{t\to\infty} \left(\frac{1}{t}\right) \mathbf{E}^{h(\cdot)}[v_0(X(t))|V(0) = v, X(0) = x] = 0. \tag{5.6}$$

It follows from the results of Section 5.5 in [1] that $h_0(\cdot)$ is an admissible strategy. Thus, applying Ito's formula to $v_0(x)$ we obtain from (5.2)

$$\begin{aligned} &v_0(X(t)) - v_0(v) \\ &\quad = \int_0^t \left[(b + Bx)' \operatorname{grad}_x v_0(X(s)) + \tfrac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 v_0(X(s))}{\partial x_i\, \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk}\right] ds \\ &\qquad - \int_0^t \operatorname{grad}_x' v_0(X(s))\Lambda\, dW(s) \\ &\quad = t\rho(0) + \int_0^t [\tfrac{1}{2}h_0'(X(s))\Sigma\Sigma' h_0(X(s)) - h_0'(X(s))(a + AX(s))]\,ds \\ &\qquad - \int_0^t \operatorname{grad}_x' v_0(X(s))\Lambda\, dW(s), \qquad t \geq 0. \end{aligned} \tag{5.7}$$

Thus

$$\begin{aligned} &\mathbf{E}^{h_0(\cdot)}[v_0(X(t))|V(0) = v, X(0) = x] - v_0(v) \\ &\quad = t\rho(0) + \ln v - \mathbf{E}^{h_0(\cdot)}[\ln V(t)|V(0) = v, X(0) = x], \qquad t \geq 0. \end{aligned} \tag{5.8}$$

The result in (a) follows now from (5.5), (5.6), and (5.8).

Uniqueness of $\rho(0)$ follows from its stochastic representation in (5.8). □

Remark 5.2. When $\Sigma\Sigma'$ is positive definite, the optimal strategy $h_0(\cdot)$ agrees with the one derived by Karatzas (see (9.19) in [15]).

In order to relate the risk-sensitive problems $(\mathbf{P}_\theta)$, $\theta > 0$, and the risk-null problem $(\mathbf{P}_0)$ we need the following result:

Lemma 5.2. *The constants* $\rho(\theta)$, $\theta > 0$, *in the solutions to* (2.8) *converge to the constant* $\rho(0)$ *in the solution to* (5.2) *when* θ *converges to zero.*

Proof. See the Appendix. □

We now have the following proposition, which says that the optimal objective values for problems $(\mathbf{P}_\theta)$ converge to the optimal objective value for the risk-null problem (P_0) when the risk-aversion parameter decays to zero:

Proposition 5.2. *Assume* (A3) *and* (C1)–(C3). *Then*

$$\lim_{\theta\downarrow 0}\max_{h(\cdot)\in\mathcal{H}}\left[\lim_{t\to\infty}\left(\frac{-2}{\theta}\right)t^{-1}\ln\mathbf{E}^{h(\cdot)}[e^{-(\theta/2)\ln V(t)}|V(0)=v,\,X(0)=x]\right]$$
$$=\max_{h(\cdot)\in\mathcal{H}}\left[\lim_{t\to\infty}t^{-1}\,\mathbf{E}^{h(\cdot)}[\ln V(t)|V(0)=v,\,X(0)=x]\right]. \tag{5.9}$$

Proof. It follows from Corollary 2.1 and Theorem 2.1 that the left-hand side of (5.9) is equal to $\lim_{\theta\downarrow 0}\rho(\theta)$. Proposition 5.1 implies that the right-hand side of (5.9) is equal to $\rho(0)$. This proves the result in view of the Lemma 5.2. □

The following result characterizes the portfolio expected growth rate corresponding to the optimal investment strategy for the risk-aversion level $\theta > 0$.

Lemma 5.3. *Assume* (*A*3) *and* (*C*3). *Fix* $\theta > 0$. *Let* $H_\theta(x)$ *be as in Theorem* 2.1 *and assume that* $H_\theta(x)$ *is a linear function and that*

$$\lim_{\|x\|\to\infty}[\tfrac{1}{2}H_\theta(x)'\Sigma\Sigma' H_\theta(x) - H_\theta(x)'(a+Ax)] = -\infty. \tag{5.10}$$

Consider the equation

$$\rho_\theta = (b+Bx)'\,\mathrm{grad}_x\, v_{\theta,0}(x) + \tfrac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 v_{\theta,0}(x)}{\partial x_i\,\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}$$
$$-\ [\tfrac{1}{2}H_\theta(x)'\Sigma\Sigma' H_\theta(x) - H_\theta(x)'(a+Ax)], \tag{5.11}$$
$$v_{\theta,0}(x)\in C^2(R^n),\qquad \lim_{\|x\|\to\infty} v_{\theta,0}(x)=\infty,\qquad \rho_\theta = const.$$

Then there exists a solution $(\rho_\theta, v_{\theta,0})$ *to the above equation, the constant* ρ_θ *is unique, and we have*

$$J_0(v,x;h_\theta(\cdot)) = \rho_\theta \tag{5.12}$$

for all $(v,x)\in(0,\infty)\times R^n$, *where* $h_\theta(\cdot)$ *is defined as in* (2.6).

Proof. The proof is analogous to the proofs of Lemma 5.1 and Proposition 5.2 and therefore is omitted. □

It can be demonstrated that condition (5.10) is sufficient but not necessary.

Remark 5.3. The results of Bensoussan and Frehse [2] can be used in order to prove that (5.2) admits a solution for general $K_0(x)$, as defined in (2.5), if one relaxes smoothness requirements for $v_0(x)$ (e.g., $v_0(x)$ is only assumed to belong to an appropriate Sobolev class). Then the generalized Ito lemma [20] can be used to obtain a result analogous to Proposition 5.1. In a future paper we shall endeavor to extend Lemma 5.1 and the other results of this section to the case where $K_\theta(x)$ is not necessarily a quadratic function.

6. Up-Side Chance and Down-Side Risk

In this section we present a simple example which provides yet another justification of the superiority of the risk-sensitive criterion over the (classical) log-utility one. Specifically, for the example considered here we demonstrate that the asymptotic ratio of "up-side chance" to "down-side risk" is maximal for some θ positive.

We consider a model of an economy where the mean returns of the stock market are affected by the level of interest rates. Consider a single risky asset, say a stock index, that is governed by the SDE

$$\frac{dS_1(t)}{S_1(t)} = (\mu_1 + \mu_2 r(t))\,dt + \sigma\,dW_1(t), \qquad S_1(0) = s,$$

where the spot interest rate $r(\cdot)$ is governed by the classical "Vasicek" process

$$dr(t) = (b_1 + b_2 r(t))\,dt + \lambda\,dW_2(t), \qquad r(0) = r > 0.$$

Here $\mu_1,\ \mu_2,\ b_1,\ b_2,\ \sigma$, and λ are fixed, scalar parameters, to be estimated, while W_1 and W_2 are two independent Brownian motions. We assume $b_1 > 0$ and $b_2 < 0$ in all that follows.

The investor can take a long or short position in the stock index as well as borrow or lend money, with continuous compounding, at the prevailing interest rate. It is therefore convenient to follow the common approach and introduce the "bank account" process S_2, where

$$\frac{dS_2(t)}{S_2(t)} = r(t)\,dt.$$

Thus $S_2(t)$ represents the time-t value of a savings account when $S_2(0) = 1$ dollar is deposited at time 0. This enables us to formulate the investor's problem as in the preceding sections, for there are $m = 2$ securities S_1 and S_2, there is $n = 1$ factor $X = r$, and we can set $b = b_1,\ B = b_2,\ a = (\mu_1, 0)',\ A = (\mu_2, 1)'$, $\Lambda = (0, 0, \lambda)$, and

$$\Sigma = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With only two assets it is convenient to describe the investor's trading strategy in terms of the scalar-valued function $\tilde{H}_\theta(r)$, which is interpreted as the proportion of capital invested in the stock index, leaving the proportion $1 - \tilde{H}_\theta(r)$ invested in the bank account. Thus here we have $H_\theta(r) = [\tilde{H}_\theta(r), 1 - \tilde{H}_\theta(r)]$. We suppose for simplicity that there are no special restrictions (e.g., short sales constraints, borrowing restrictions, etc.) on the investor's trading strategy, so the investment constraint set χ is taken to be the whole real line.

In order to simplify the calculations that follow we reduce our Vasicek model to the case that corresponds to the classical Merton [23] model of 1971 by assuming that

$$\mu_2 = 0, \qquad b_1 = b_2 = \lambda = 0, \qquad \mu_1 > r.$$

Observe that we have assumed a constant spot rate (that is, $r(t) = r$ for all t) and a constant rate of return (μ_1) on the stock index. We thus have a very conventional, two-asset model. Our assumptions are imposed in order to simplify the following development of a new interpretation of the risk-sensitive optimality criterion. In particular, we analyze dependence on $\theta \geq 0$ of the following quantity:

$$R(\theta) := \lim_{t\to\infty} \left(\frac{1}{t}\right) \ln \frac{\mathbf{P}^{h_\theta(\cdot)}((1/t)\ln V(t) > \rho(0))}{\mathbf{P}^{h_\theta(\cdot)}((1/t)\ln V(t) < r)}.$$

To interpret this quantity, using the strong law of large numbers for the Brownian motion process it can easily be shown that, under the parametrization considered here,

$$\rho_\theta := \lim_{t\to\infty} \left(\frac{1}{t}\right) \mathbf{E}^{h_\theta(\cdot)}[\ln V(t) | V(0) = v] = \lim_{t\to\infty} \left(\frac{1}{t}\right) \ln V(t), \qquad \mathbf{P}^{h_\theta(\cdot)} \;\; a.s.,$$

for all $v \in (0, \infty)$ and $\theta \geq 0$. In particular, $\rho_0 = \rho(0) = \frac{1}{2}((\mu_1 - r)^2/\sigma^2) + r$ is the maximal (expected) growth rate of the investor's portfolio. Thus the quantity $R(\theta)$ above can be interpreted as the asymptotic, logarithmic ratio of the chance that the actual growth rate of the investor's portfolio under the strategy $h_\theta(\cdot) = [\tilde{h}_\theta(\cdot), 1 - \tilde{h}_\theta(\cdot)]'$ will exceed the maximal limit, to the risk that the growth rate will fall below the spot rate.

It is interesting to see which value of $\theta \geq 0$ maximizes $R(\theta)$. Toward this end we first note that in the current situation we have

$$\tilde{h}(t) = H_\theta(r) := \frac{\mu_1 - r}{\sigma^2}$$

for all $t \geq 0$. Thus, under $h_\theta(\cdot)$, we have

$$\left(\frac{1}{t}\right) \ln V(t) = \mu_1 H_\theta(r) + (1 - H_\theta(r))r - \tfrac{1}{2} H_\theta^2(r)\sigma^2 + H_\theta(r)\frac{W_1(t)}{t}.$$

Consequently (using (1.1.4) of [8]) we obtain

$$R(\theta) = \inf\left\{\frac{x^2}{2} : x \in \Lambda(\theta)\right\} - \inf\left\{\frac{x^2}{2} : x \in \Gamma(\theta)\right\},$$

where

$$\Lambda(\theta) := \left(\infty, -\tfrac{1}{2}(\mu_1 - r)\frac{\theta+1}{\theta/2+1}\right) \quad \text{and} \quad \Gamma(\theta) := \left(\frac{1}{8}\frac{\theta^2(\mu_1 - r)}{\theta/2+1}, \infty\right).$$

Thus

$$R(\theta) = \frac{1}{4}\frac{(\mu_1 - r)^2}{(\theta/2+1)^2}\left[(\theta+1)^2 - \frac{\theta^4}{16}\right].$$

Finally we see that $R(\theta)$ is maximized over $[0, \infty)$ by $\theta^* = 2$. This means that the ratio of *up-side chance* to *down-side risk* is maximized if the investor maximizes the growth of $-(1/t)\ln \mathbf{E}(1/V(t))$!

7. Future Research

It is important to study risk-sensitive investment problems with partial information. Typically, the values of the market parameters a, A, Σ, b, B, and Λ are not known to an investor. So how should our model be implemented? One possibility is that the investor obtains initial estimates of the market parameters based on historical time series, and then holds onto these estimates throughout the entire future investment horizon. A potentially better approach for the investor would be to select her or his investment decisions *adaptively* based on currently available market information and the optimal decision strategies (perhaps the ones developed in this paper). This means that the estimates of market parameters are updated as time goes by and new market information is acquired, and subsequently the updated estimates are used instead of the "true" values of those market parameters in the formulas for optimal risk-sensitive investment rules. Adaptive, risk-sensitive investment rules should be investigated in the future, based on the results presented in this paper as well as some of the ideas developed in [3], [5], and [31]. The guiding rule should be to develop a simple estimation scheme for the model parameters which, when combined with (2.5), should lead to the development of practical, feasible algorithms for real-time dynamic asset management. The incorporation of partial state observation (imprecise measurements of the security prices and/or of the factor levels) is another desirable research objective.

Further, it is important to study the risk-sensitive optimal portfolio selection problem for a generalization of the basic market model that we introduced in Section 2. Specifically, one should relax the linearity assumption on the drift coefficients and also allow for dependence on factors of the diffusion coefficients in the market SDEs. Thus these equations would take the following form:

$$\begin{aligned} \frac{dS_i(t)}{S_i(t)} &= \mu_i(X(t))\,dt + \sum_{k=1}^{m+n}\sigma_{ik}(X(t))\,dW_k(t), \\ S_i(0) &= s_i, \quad i = 1, 2, \ldots, m \end{aligned} \tag{7.1}$$

$$dX(t) = B(X(t))\,dt + \Lambda(X(t))\,dW(t), \qquad X(0) = x, \tag{7.2}$$

where, as before, $W(t)$ is an R^{m+n}-valued standard Brownian motion process with components $W_k(t)$, and the market functions $\mu(x) := [\mu_i(x)]_{1\times m}$, $\Sigma(x) := [\sigma_{ik}(x)]_{m\times(m+n)}$, $B(x)$, and $\Lambda(x)$ are such that a unique strong solution exists to the above equations. Finally, it would be desirable to relax the assumption about the lack of correlation between residuals in the security prices and factor equations, that is, to drop the requirement that $\Sigma(x)\Lambda'(x)$ is the zero matrix.

Appendix. Proof of Lemmas 5.1 and 5.2

Proof of Lemma 5.1. Consider the equation

$$\frac{\partial \mathcal{V}(t,x)}{\partial t} = (b+Bx)' \operatorname{grad}_x \mathcal{V}(t,x) + \tfrac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{V}(t,x)}{\partial x_i\,\partial x_j} \sum_{k}^{m+n} \lambda_{ik}\lambda_{jk} - K_0(x), \qquad \mathcal{V}(0,x) = 0, \tag{8.1}$$

for $x \in R^n$ and $t > 0$. It can be easily verified that a solution $\mathcal{V}_0(t,x)$ to (8.1) exists and is given by

$$\mathcal{V}_0(t,x) = \tfrac{1}{2} x'\mathcal{P}_0(t)x + \mathcal{Q}_0(t)x + \mathcal{G}_0(t), \tag{8.2}$$

provided the functions $\mathcal{P}_0(t)$, $\mathcal{Q}_0(t)$ and $\mathcal{G}_0(t)$ satisfy the ordinary differential equations:

$$\dot{\mathcal{P}}_0(t) = B'\mathcal{P}_0(t) + \mathcal{P}_0(t)B - K_1(0), \qquad \mathcal{P}_0(0) = 0, \tag{8.3}$$

$$\dot{\mathcal{Q}}_0(t) = B'\mathcal{Q}_0(t) + b'\mathcal{P}_0(t) - K_2(0), \qquad \mathcal{Q}_0(0) = 0, \tag{8.4}$$

$$\dot{\mathcal{G}}_0(t) = b'\mathcal{Q}_0(t) + \tfrac{1}{2}\operatorname{tr}(\mathcal{P}_0(t)\Lambda\Lambda') - K_3(0), \qquad \mathcal{G}_0(0) = 0, \tag{8.5}$$

for $t > 0$. Now, it is well known (see, e.g., [30] and [29]) that there exists a unique, nonnegative definite solution to the Lyapunov equation (8.3), given by

$$\mathcal{P}_0(t) = -\int_0^t S'(t-s)K_1(0)S(t-s)\,ds, \qquad t \geq 0, \tag{8.6}$$

where

$$S(t) := \exp\{Bt\}, \qquad t \geq 0. \tag{8.7}$$

The other two differential equations are standard, so, indeed, there exists a unique solution to (8.3)–(8.5). Moreover, since B is stable there exist limits

$$\mathcal{P}_0 := \lim_{t\to\infty} \mathcal{P}_0(t), \qquad \mathcal{Q}_0 := \lim_{t\to\infty} \mathcal{Q}_0(t), \tag{8.8}$$

and $\mathcal{P}_0$ is nonnegative definite. Letting $t \to \infty$ in (8.3)–(8.5) we thus have (also compare Section 12.4 of [29])

$$0 = B'\mathcal{P}_0 + \mathcal{P}_0 B - K_1(0), \tag{8.9}$$

$$0 = B'\mathcal{Q}_0 + b'\mathcal{P}_0 - K_2(0), \tag{8.10}$$

$$\rho(0) := \lim_{t\to\infty} \dot{\mathcal{G}}_0(t) = b'\mathcal{Q}_0 + \tfrac{1}{2}\operatorname{tr}(\mathcal{P}_0\Lambda\Lambda') - K_3(0). \tag{8.11}$$

It can be easily verified now that the constant $\rho(0)$ defined here and the function $v_0(x) := \frac{1}{2}x'\mathcal{P}_0 x + \mathcal{Q}_0 x$ satisfy (5.2). This completes the proof of Lemma 5.1. □

Proof of Lemma 5.2. Fix $\theta > 0$ and consider the following equation (which is the same as (4.3)):

$$\begin{aligned}\frac{\partial\mathcal{V}(t,x;\theta)}{\partial t} &= (b+Bx)'\operatorname{grad}_x \mathcal{V}(t,x;\theta)\\ &\quad - \frac{\theta}{4}\sum_{i,j=1}^{n}\frac{\partial\mathcal{V}(t,x;\theta)}{\partial x_i}\frac{\partial\mathcal{V}(t,x;\theta)}{\partial x_j}\sum_{k}^{m+n}\lambda_{ik}\lambda_{jk}\\ &\quad + \tfrac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\mathcal{V}(t,x;\theta)}{\partial x_i\,\partial x_j}\sum_{k}^{m+n}\lambda_{ik}\lambda_{jk} - K_\theta(x),\\ \mathcal{V}(0,x;\theta) &= 0,\end{aligned} \tag{8.12}$$

for $x \in R^n$ and $t > 0$. Because of Assumption (C1), it can be easily verified that a solution $\mathcal{V}_0(t,x;\theta)$ to (8.12) exists and is given by

$$\mathcal{V}_0(t,x;\theta) = \tfrac{1}{2}x'\mathcal{P}_\theta(t)x + \mathcal{Q}_\theta(t)x + \mathcal{G}_\theta(t), \tag{8.13}$$

provided the functions $\mathcal{P}_\theta(t)$, $\mathcal{Q}_\theta(t)$, and $\mathcal{G}_\theta(t)$ satisfy the ordinary differential equations:

$$\dot{\mathcal{P}}_\theta(t) = B'\mathcal{P}_\theta(t) + \mathcal{P}_\theta(t)B - \frac{\theta}{2}\mathcal{P}_\theta(t)\Lambda\Lambda'\mathcal{P}_\theta(t) - K_1(\theta), \qquad \mathcal{P}_\theta(0) = 0, \tag{8.14}$$

$$\dot{\mathcal{Q}}_\theta(t) = \left(B - \frac{\theta}{2}\mathcal{P}_\theta(t)\Lambda\Lambda'\right)'\mathcal{Q}_\theta(t) + b'\mathcal{P}_\theta(t) - K_2(\theta), \qquad \mathcal{Q}_\theta(0) = 0, \tag{8.15}$$

$$\begin{aligned}\dot{\mathcal{G}}_\theta(t) &= b'\mathcal{Q}_\theta(t) + \tfrac{1}{2}\operatorname{tr}(\mathcal{P}_\theta(t)\Lambda\Lambda') - \frac{\theta}{4}\mathcal{Q}_\theta(t)\Lambda\Lambda'\mathcal{Q}_\theta(t)\\ &\quad - K_3(\theta), \qquad \mathcal{G}_\theta(0) = 0,\end{aligned} \tag{8.16}$$

for $t > 0$. Now, it is well known (see, e.g., [30] and [29]) that there exists a unique, nonnegative definite solution to the Riccati equation (8.14), given by

$$\mathcal{P}_\theta(t) = -\int_0^t S'(t-s)\left[K_1(\theta) + \frac{\theta}{2}\mathcal{P}_\theta(s)\Lambda\Lambda'\mathcal{P}_\theta(s)\right]S(t-s)ds, \qquad t \geq 0, \tag{8.17}$$

where $S(t)$ is as in (8.7). Thus there exists a unique solution to (8.14)–(8.16). Moreover, since B is stable there exists a nonnegative definite limit

$$\mathcal{P}_\theta := \lim_{t\to\infty}\mathcal{P}_\theta(t). \tag{8.18}$$

We now consider still another differential equation:

$$\dot{\bar{\mathcal{Q}}}_\theta(t) = \left(B - \frac{\theta}{2}\mathcal{P}_\theta\Lambda\Lambda'\right)'\bar{\mathcal{Q}}_\theta(t) + b'\mathcal{P}_\theta(t) - K_2(\theta), \qquad \bar{\mathcal{Q}}_\theta(0) = 0. \tag{8.19}$$

It is well known (see, e.g., Lemma 10 of [6]) that the convergence in (8.18) is exponentially fast, that is, there exist positive constants δ_1 and δ_2 such that

$$\|\mathcal{P}_\theta - \mathcal{P}_\theta(t)\| \leq \delta_1 e^{-\delta_2 t}, \qquad t \geq 0. \tag{8.20}$$

Lemma 4.2 in [14] and (8.15) imply that

$$\sup_{t\geq 0} \|\mathcal{Q}_\theta(t)\| < \infty. \tag{8.21}$$

Now note that the difference $\tilde{\mathcal{Q}}_\theta(t) := \mathcal{Q}_\theta(t) - \bar{\mathcal{Q}}_\theta(t)$ satisfies the equation

$$\dot{\tilde{\mathcal{Q}}}_\theta(t) = \left(B - \frac{\theta}{2}\mathcal{P}_\theta\Lambda\Lambda'\right)'\tilde{\mathcal{Q}}_\theta(t) - \frac{\theta}{2}((\mathcal{P}_\theta(t) - \mathcal{P}_\theta)\Lambda\Lambda')'\mathcal{Q}_\theta(t), \qquad \tilde{\mathcal{Q}}_\theta(0) = 0. \tag{8.22}$$

Since B is stable, then $B - (\theta/2)\mathcal{P}_\theta\Lambda\Lambda'$ is also stable (the easy proof is omitted). Thus, it follows from (8.20)–(8.22) and Lemma 4.2 in [14] that

$$\lim_{t\to\infty} \|\tilde{\mathcal{Q}}_\theta(t)\| = 0. \tag{8.23}$$

In view of the above, we have

$$\lim_{t\to\infty} \mathcal{Q}_\theta(t) = \mathcal{Q}_\theta, \tag{8.24}$$

where $\mathcal{Q}_\theta := \lim_{t\to\infty} \bar{\mathcal{Q}}_\theta(t)$, which exists by virtue of (8.19) and the fact that $B - (\theta/2)\mathcal{P}_\theta\Lambda\Lambda'$ is stable. Letting $t \to \infty$ in (8.15) we thus have

$$0 = \left(B - \frac{\theta}{2}\mathcal{P}_\theta\Lambda\Lambda'\right)'\mathcal{Q}_\theta + b'\mathcal{P}_\theta - K_2(\theta), \tag{8.25}$$

in which case

$$\rho(\theta) := \lim_{t\to\infty} \dot{\mathcal{G}}_\theta(t) = b'\mathcal{Q}_\theta + \tfrac{1}{2}\operatorname{tr}(\mathcal{P}_\theta\Lambda\Lambda') - \frac{\theta}{4}\mathcal{Q}_\theta\Lambda\Lambda'\mathcal{Q}_\theta - K_3(\theta). \tag{8.26}$$

It is left to the reader to verify that this constant $\rho(\theta)$ and the function

$$v(x;\theta) := \tfrac{1}{2}x'\mathcal{P}_\theta x + \mathcal{Q}_\theta x$$

satisfy (2.8).

To complete the proof it suffices to show that the following convergences hold uniformly with respect to $t \geq 0$:

$$\lim_{\theta\downarrow 0} \mathcal{P}_\theta(t) = \mathcal{P}_0(t) \tag{8.27}$$

and

$$\lim_{\theta\downarrow 0} \mathcal{Q}_\theta(t) = \mathcal{Q}_0(t). \tag{8.28}$$

This is because these uniform limits imply

$$\lim_{\theta\downarrow 0} \mathcal{P}_\theta = \mathcal{P}_0, \qquad \lim_{\theta\downarrow 0} \mathcal{Q}_\theta = \mathcal{Q}_0 \tag{8.29}$$

which, together with (8.11) and (8.26), give the desired result.

To demonstrate that the limits in (8.27) and (8.28) exist uniformly in $t \geq 0$, we first observe that from (8.6) and (8.17) we obtain

$$\begin{aligned}\mathcal{P}_\theta(t) - \mathcal{P}_0(t) = &\int_0^t S'(t-s)(K_1(0) - K_1(\theta))S(t-s)\,ds \\ &- \frac{\theta}{2}\int_0^t S'(t-s)\mathcal{P}_\theta(t)\Lambda\Lambda'\mathcal{P}_\theta(t)S(t-s)\,ds,\end{aligned} \tag{8.30}$$

for all $t \geq 0$. Thus

$$\|\mathcal{P}_\theta(t) - \mathcal{P}_0(t)\| \leq c_1\|K_1(0) - K_1(\theta)\| + c_2\theta, \qquad t \geq 0, \tag{8.31}$$

for some positive constants c_1 and c_2 independent of t. Next, observe from (8.4) and (8.15) that the function $\hat{Q}_\theta(t) := Q_\theta(t) - Q_0(t)$ satisfies the equation

$$\begin{aligned}\dot{\hat{Q}}_\theta(t) = &B'\hat{Q}_\theta(t) - \frac{\theta}{2}\mathcal{P}_\theta(t)\Lambda\Lambda' Q_\theta(t) - (K_2(\theta) - K_2(0)) \\ &+ b'(\mathcal{P}_\theta(t) - \mathcal{P}_0(t)), \qquad \widehat{Q}_\theta(0) = 0,\end{aligned} \tag{8.32}$$

for $t > 0$. Thus,

$$\|\hat{Q}_\theta(t)\| \leq c_3\theta + c_4\|K_2(\theta) - K_2(0)\| + c_5\|K_1(\theta) - K_1(0)\|, \qquad t \geq 0, \tag{8.33}$$

for some positive constants c_3–c_5 independent of t.

The proof of Lemma 5.2 is now complete. □

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