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Risk sensitive asset allocation

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Abstract

This paper develops a continuous time modeling approach for making optimal asset allocation decisions. Macroeconomic and financial factors are explicitly modeled as Gaussian stochastic processes which directly affect the mean returns of the assets. We employ methods of risk sensitive control theory, thereby using an infinite horizon objective that is natural and features the long run expected growth rate and the asymptotic variance as two measures of performance, analogous to the mean return and variance, respectively, in the single period Markowitz model. The optimal strategy is a simple function of the factor levels, and, even with constraints on the portfolio proportions, it can be computed by solving a quadratic program. Explicit formulas can be obtained, as is illustrated by an example where the only factor is a Vasicek-type interest rate and where there are two assets: cash and a stock index. The methods are further illustrated by studies of two data sets: U.S. data with two assets and up to three factors, and Australian data with three assets and three factors. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

During recent years there has been a number of empirical studies providing evidence that macroeconomic and financial variables such as unemployment rates and market-to-book ratios can be useful for forecasting returns for asset categories. For example, Pesaran and Timmermann (1995) examined the robustness of the evidence on predictability of US stock returns with respect to seven factors: dividend yield, earnings-price ratio, 1 month T-bill rate, 12 month T-bill rate, inflation rate, change in industrial production, and monetary growth rate. Backtesting a simple switching strategy, they provided evidence that stock return predictability 'could have been exploited by investors in the volatile markets of the 1970s'. In a more recent paper, Pesaran and Timmermann (1998) extended and generalized their first paper's recursive modeling strategy. They focused their analysis on simulating 'investors' search in the 'real time' for a model that can forecast stock returns. Their new key idea was to divide the set of potential regressors into three categories expressing different degrees of forecasting importance. Once again their findings provided evidence of predictability of stock returns, this time in the UK's stock market, that can be exploited by investors. Patelis (1997) concluded US excess stock returns can be predicted by looking at five monetary policy factors along with dividend yield, an interest rate spread, and the one-month real interest rate. Ilmanen (1997) showed that the excess returns of long term T-bond are affected by term spread, real yield, inverse wealth, and momentum factors. Furthermore, the bibliographies in these papers cite numerous additional, similar studies.

Some of these studies proceeded from their statistical conclusions to investigate whether return predictability can be exploited with dynamic investment strategies in order to achieve significant profits. All such investigations involved backtests of relatively simple, *ad hoc* trading rules. For instance, Ilmanen (1997) showed two dynamic strategies would have provided excess returns well in excess of two benchmark static strategies. In one dynamic strategy the investor is long the T-bond if and only if its predicted excess return is positive. In the other, the position in the T-bond is proportional to the predicted excess return. This and other studies reinforce the view that dynamic asset allocation models which incorporate financial and economic factors can be useful for investors.

Meanwhile, there has been considerable research involving stochastic process models of assets combined with optimal consumption and/or investment decisions by economic agents. In some cases the models also include stochastic process models of factors. The famous study by Lucas (1978) has all the right elements: discrete time stochastic assets which are affected by factor levels, a Markovian factor process, and economic agents who choose portfolio positions and consumption levels so as to maximize the expected discounted utility of their consumption over an infinite planning horizon. Merton (1971), Karatzas (1996), and other researchers used stochastic control theory to develop continuous time portfolio management models where the assets are modeled as stochastic processes but financial and economic factors are ignored. Much more relevant to the present paper is the one by Merton (1973), because he provided a continuous time asset management model featuring stochastic factors. Using the necessary conditions emerging from the dynamic programming functional equation, he was able to establish some important financial economic principles. Merton's formulation was very general and abstract, and he did not provide any explicit calculations, concrete examples, or computational results.

Given Merton's groundwork and the empirical literature indicated above, one would logically expect there to be a large literature on continuous time, optimal portfolio management models which explicitly incorporate stochastic factors and the mechanisms by which they affect asset returns, but the opposite is the case. Apparently the theoretical and computational difficulties are too great. With the objective of either maximizing expected utility of terminal wealth or maximizing expected discounted utility of consumption over an infinite planning horizon, Merton's approach entails the solution of a partial differential equation. However, explicit solutions are known only for a few special cases, and the pde's can be solved numerically only for very small problems. Meanwhile, the implications of Merton's (1973) work for economic equilibrium have been investigated in a variety of papers, among which the studies by Breeden (1979) and Cox et al. (1985) are noteworthy.

Kim and Omberg (1996), Canestrelli (1998), and Canestrelli and Pontini (1998) studied some simple, special cases of Merton's (1973) model where the investor's objective is to maximize expected (HARA or power) utility of wealth at a fixed, finite date. They derived via a Riccati equation explicit solutions for the value function and optimal trading strategy. Brennan and Schwartz (1996) and Brennan et al. (1997) studied a similar model, but they used numerical methods to solve the dynamic programming functional equation (i.e., the Hamilton–Jacobi–Bellman partial differential equation) for the value function and optimal trading strategy. But with only three factors, the pde has three state variables and so they were already near the maximum that can be accommodated with a numerical approach.

Brandt (1998) and Campbell and Viceira (1999) worked directly with discrete time variations of Merton's (1973) model. Campbell and Viceira dealt with a model that is similar to Kim and Omberg's except that the investor's objective is to maximize expected utility of consumption over an infinite planning horizon. Brandt (1998), also in discrete time, worked with the objective of maximizing expected utility of consumption over a finite planning interval. The set of feasible portfolio and consumption decisions was allowed to vary according to an observable forecasting process which, unfortunately, was not explicitly specified. Brandt developed a computational procedure involving the conditional method of moments and the Euler equations that represent the first order optimality conditions. However, no proof was provided that this procedure is optimal or even approximately optimal.

In short, the literature demonstrates that for concrete applications of Merton's (1973) approach it is difficult to obtain explicit characterizations of the optimal strategies, even for simple models involving only a few factors. And these computational difficulties are not rescued by the modern approach which avoids dynamic programming by using risk neutral probability measures (see Karatzas (1996), Korn (1997), or Pliska (1986,1997)), because the inclusion of stochastic factors means the resulting securities market model is incomplete (in the sense of Magill and Quinzii (1996)). However, as will be argued in this paper, the computational difficulties can be ameliorated by adopting a new kind of portfolio optimization objective or criterion.

The mathematical theory in this paper was introduced in Bielecki and Pliska (1999). Our model resembles the one developed by Merton (1973) in that factors are modeled as stochastic processes which explicitly affect the asset returns. However, instead of maximizing the expected utility of terminal wealth or the expected utility of consumption, the objective now is to maximize the portfolio's long run growth rate adjusted by a measure of the portfolio's average volatility. This 'risk sensitive' criterion corresponds to an infinite horizon objective, and so the optimal strategies are simpler, depending on the factor levels but not on time. Moreover, the optimal strategies can be computed by solving simple quadratic programs, and so models with dozens or even hundreds of factors are tractable. An interesting feature of the theoretical results presented in Bielecki and Pliska (1999) is that investment strategies that are optimal for the infinite horizon objective are universally optimal, i.e., they are optimal within any finite planning horizon for the type of asset allocation problems considered here.

It should be mentioned that the optimal strategies which emerge from our risk sensitive dynamic asset management model resemble, at least for the case of a single risky asset, the proportional strategies studied by Ilmanen (1997) and others. Hence the ideas in this paper provide a sound, theoretical footing for dynamic investment strategies which previously have been selected only on an *ad hoc* basis.

This paper is not the first to apply a risk sensitive optimality criterion to a financial problem. Lefebvre and Montulet (1994) studied a model for a firm's optimal mix between liquid and non-liquid assets; the calculus of variations approach was used to derive an explicit expression for the optimal division. Fleming (1995) used risk sensitive methods to obtain two kinds of asymptotic results. In the first he considered a conventional, finite horizon portfolio model and studied certain limits as the coefficient of risk aversion tends to infinity. In the second he studied the long-term growth rate for conventional models with transaction costs and HARA utility functions. Finally, Carino (1987) used risk sensitive linear/quadratic/Gaussian control theory (see Whittle, 1990) to solve a particular discrete time, Lucas-type problem.

In summary, the purpose of this paper is to demonstrate that the risk sensitive dynamic asset management model of Bielecki and Pliska (1999) is a practical, tractable approach for making optimal asset allocation decisions. A precise formulation of this model as well as the main results will all be found in Section 3. First, however, Section 2 will discuss and explain the risk sensitive criterion that is a fundamental element of the model.

Section 4 is devoted to a special case of a simple asset allocation model featuring a Vasicek type short interest rate and a stock index that is affected by the level of interest rates. This example is completely solved; explicit formulas for the optimal trading strategies and the optimal objective value are obtained. In order to develop understanding and economic intuition, the effects of individual parameters in these mathematical expressions are studied. Not only does this example illustrate the main ideas of Sections 2 and 3, but it will also be of independent interest to financial economists because it is one of the few models in the literature to provide explicit results and formulas for a concrete version of Merton's (1973) model.

The Bielecki–Pliska methodology is further illustrated in the next two sections where it is applied to two sets of monthly economic/financial data. In Section 5 the model is applied to US data, the same data that Brennan et al. (1997) studied. Section 6 is devoted to Australian data. In both cases the statistical ability of our factors for forecasting asset returns is very limited, and yet the results are surprisingly good. This suggests that incorporation of better factors, such as those in the empirical studies cited above, would yield attractive strategies for dynamic asset allocation.

2. The risk sensitive criterion

In order to introduce and explain the risk sensitive criterion, let $V(t)$ denote the time-t value of a portfolio and consider the measure of performance

$$\liminf_{t\to\infty} (1/\gamma)t^{-1}\ln \mathrm{E}(V(t))^{\gamma}, \quad \gamma < 1,\ \gamma \neq 0.$$

This was used by Grossman and Zhou (1993) and Cvitanic and Karatzas (1994) to study a classical portfolio problem under a drawdown constraint. Note that letting $\gamma \to 0$ this becomes, in the limit, the same as the objective of maximizing the portfolio's long-run expected growth rate (the Kelly criterion), whereas for $\gamma > 1$ it is not clear how to meaningfully interpret this criterion, although it resembles expected utility with an isoelastic or power utility function.

However, suppose this expression is rewritten as

$$J_\theta := \liminf_{t\to\infty} (-2/\theta)t^{-1}\ln \mathrm{E}e^{-(\theta/2)\ln V(t)},$$

where $\theta > -2$, $\theta \neq 0$, and where ':=' means '*defined as*'. Substituting $C(t) = \ln V(t)$ enables one to establish a connection with the recently developed literature on *risk sensitive optimal control* (e.g., see Whittle (1990)), where $C(t)$ plays the role of a cumulative cost. This means that if we adopt, as we shall, the objective of maximizing J_θ, then many of the techniques that have recently been developed for risk sensitive control can potentially be applied to our portfolio management problem.

Moreover, as is well understood in the risk sensitive control literature, a power expansion [in powers of θ, for θ close to 0] of the quantity $-\frac{2}{\theta}\ln \mathrm{E}e^{-\frac{\theta}{2}\ln V(t)}$ yields

$$-\frac{2}{\theta}\ln \mathrm{E}e^{-\frac{\theta}{2}\ln V(t)} = \mathrm{E}\ln V(t) - \frac{\theta}{4}\mathrm{var}(\ln V(t)) + \mathrm{O}(\theta^2), \tag{2.1}$$

where $\mathrm{O}(\theta^2)$ will typically depend on t. Hence J_θ can be interpreted as the long-run expected growth rate minus a penalty term, with an error that is proportional to θ^2. Furthermore, the penalty term is proportional to the *asymptotic variance*, a quantity that was studied by Konno et al. (1993) in the case of a conventional, multivariate geometric Brownian motion model of securities. The penalty term is also proportional to θ, so θ should be interpreted as a *risk sensitivity parameter* or *risk aversion parameter*, with $\theta > 0$ and $\theta < 0$ corresponding to risk averse and risk seeking investors, respectively. Moreover, in the risk averse case maximizing J_θ protects an investor interested in maximizing the expected growth rate of the capital against large deviations of the realized rate from the expectations. The special case of $\theta = 0$ will be referred to as the *risk null case*[1], and note this is the same as the classical Kelly criterion, that is

$$J_0 = \liminf_{t\to\infty} t^{-1}\,\mathrm{E}\ln V(t).$$

Some insight into the risk sensitive criterion can be obtained by considering the case where the process $V(t)$ is a simple geometric Brownian motion with constant parameters μ and σ. A simple calculation gives

$$J_\theta = \mu - \frac{1}{2}\sigma^2 - \frac{\theta}{4}\sigma^2,$$

so the approximation mentioned above is, in this case, exact (which means that the term $\mathrm{O}(\theta^2)$ is in fact equal to 0), with the asymptotic variance being precisely the same as the square of the usual volatility.

[1] Whittle (1990) used the term *risk neutral* rather than *risk null* in this case. We chose the latter terminology in order to avoid a possible confusion with the term *risk neutral* used in the asset pricing context.

Additional insight about the risk sensitive criterion can be obtained by comparing it with the objective under the classical single period Markowitz model. Ignoring the higher order terms and interpreting θ as a Lagrange multiplier, it is apparent that the problem of choosing a trading strategy so as to maximize J_θ is the same as maximizing the growth rate subject to a constraint that the asymptotic volatility is equal to a fixed value. Hence maximizing J_θ for a range of θ will derive the 'risk sensitive frontier', exactly analogous to the efficient frontier in the Markowitz model. There are only two differences. First, the risk sensitive frontier lives in an asymptotic space corresponding to infinite horizon measures of mean return and variance rather than single period measures. Second, the asymptotic frontier that is computed may not be exactly equal to the true asymptotic frontier, due to the higher order terms in the Taylor series expansion of J_θ.

Naturally, if the investor has a very clear, specific planning horizon, then the expected utility of terminal wealth should probably be maximized (assuming that the results can be computed) and our risk adjusted performance measure should not be used. However, for many important problems, especially the management of mutual funds, our risk adjusted growth rate criterion is ideal, for it captures both the portfolio's growth rate and the portfolio's average volatility over an extended period of time.

3. Formulation of the model and the main results

We will develop a model consisting of $m \geq 2$ securities and $n \geq 1$ factors. Let $(\Omega, \{\mathscr{F}_t\}, \mathscr{F}, \mathrm{P})$ be the underlying probability space. Denoting by $S_i(t)$ the price of the ith security and by $X_j(t)$ the level of the jth factor at time t, we consider the following market model for the dynamics of the security prices and factors:

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = (a + AX(t))_i \,\mathrm{d}t + \sum_{k=1}^{m+n} \sigma_{ik} \,\mathrm{d}W_k(t), \quad S_i(0) = s_i > 0, \quad i = 1, 2, \ldots, m, \tag{3.1}$$

$$\mathrm{d}X(t) = (b + BX(t))\,\mathrm{d}t + \Lambda \,\mathrm{d}W(t), \quad X(0) = x, \tag{3.2}$$

where $W(t)$ is a R^{m+n} valued standard Brownian motion process with components $W_k(t)$, $X(t)$ is the R^n valued factor process with components $X_j(t)$, the market parameters a, A, $\Sigma := [\sigma_{ij}]$, b, B, $\Lambda := [\lambda_{ij}]$ are matrices of appropriate dimensions, and $(a + Ax)_i$ denotes the ith component of the vector $a + Ax$.

Let $h(t)$ denote an R^n valued investment process or strategy whose components are $h_i(t)$, $i = 1,2, \ldots, m$.

Definition 3.1. An investment process $h(t)$ is *admissible* if the following conditions are satisfied:
(i) $h(t)$ takes values in a given subset χ of R^m, and $\sum_{i=1}^m h_i(t) = 1$, and
(ii) $h(t)$ satisfies appropriate measurability and integrability conditions, as explained in Bielecki and Pliska (1999).

The class of admissible investment strategies will be denoted by $\mathscr{H}$.

Let now $h(t)$ be an admissible investment process and consider the solution $V(t)$ of the following stochastic differential equation:

$$\mathrm{d}V(t) = \sum_{i=1}^{m} h_i(t)V(t)\left[\mu_i(X(t))\,\mathrm{d}t + \sum_{k=1}^{m+n} \sigma_{ik}\,\mathrm{d}W_k(t)\right], \quad V(0) = v > 0, \tag{3.3}$$

where $\mu_i(x)$ is the ith coordinate of the vector $a + Ax$ for $x \in R^n$. The process $V(t)$ represents the investor's capital at time t, and $h_i(t)$ represents the proportion of capital that is invested in security i, so that $h_i(t)V(t)/S_i(t)$ represents the number of shares invested in security i, just as in, for example, Section 3 of Karatzas and Kou (1996).

In this paper we shall investigate the following family of risk sensitized optimal investment problems, labeled as (P_θ):

for $\theta \in (0, \infty)$, maximize the risk sensitized expected growth rate

$$J_\theta(v, x; h(\cdot)) := \liminf_{t \to \infty} (-2/\theta)t^{-1} \ln \mathrm{E}^{h(\cdot)}[\mathrm{e}^{-(\theta/2)\ln V(t)} | V(0) = v, X(0) = x] \tag{3.4}$$

over the class of all admissible investment processes $h(\cdot)$, subject to Eqs. (3.2) and (3.3),

where E is the expectation with respect to P. The notation $\mathrm{E}^{h(\cdot)}$ emphasizes that the expectation is evaluated for the process $V(t)$ generated by Eq. (3.3) under the investment strategy $h(t)$.

Before we can present the main results pertaining to these investment problems, we need to introduce the following notation, for $\theta \geq 0$ and $x \in R^n$:

$$K_\theta(x) := \inf_{h \in \chi,\ 1'h = 1} [(1/2)(\theta/2 + 1)h'\Sigma\Sigma' h - h'(a + Ax)]. \tag{3.5}$$

It is perhaps interesting to observe that Eq. (3.5) is a 'local' optimization problem which amounts to maximization of the instantaneous return (i.e., $h'(a + Ax)$) on the portfolio penalized by the portfolio's instantaneous volatility (i.e., the other term in Eq. (3.5)).

We also need to make the following assumptions:

Assumption (*A*1). *The investment constraint set* χ *satisfies one of the following two conditions*:
(a) $\chi = R^n$, *or*
(b) $\chi = \{h \in R^n\colon h_{1i} \le h_i \le h_{2i},\ i = 1,2,\dots,m\}$, *where* $h_{1i} < h_{2i}$ *are finite constants.*

Assumption (*A*2). *For* $\theta > 0$,

$$\lim_{||x|| \to \infty} K_\theta(x) = -\infty.$$

Assumption (*A*3). *The matrix* $\Lambda\Lambda'$ *is positive definite.*

Assumption (*A*4). *The matrix* $\Sigma\Lambda'$ *equals* 0.

Remark 3.1. (i) Note that if $\Sigma\Sigma'$ is positive definite and if $Ker(A) = 0$, then assumption (A2) is implied by assumption (A1)(a).

(ii) These assumptions are sufficient for the results below to be true, but, as will be seen for the example considered in the next section, Assumption (A2) is not necessary, in general.

(iii) Assumption (A4) says that the residuals of the factors are uncorrelated with the residuals of the assets. This assumption, which may be realistic for some applications, but not for others, is discussed further in the concluding section.

Theorems 3.1–3.4, which were proved in Bielecki and Pliska (1999), contain key results that will be used in this paper.

Theorem 3.1. Assume (*A*1)–(*A*4) *and fix* $\theta > 0$.

Let $H_\theta(x)$ *denote a minimizing selector in Eq.* (3.5), *which means that* $H_\theta(x)$ *satisfies the following equation*

$$K_\theta(x) = (1/2)(\theta/2 + 1)H_\theta(x)'\Sigma\Sigma' H_\theta(x) - H_\theta(x)'(a + Ax).$$

Then the investment process h_θ *is optimal for problem* (P_θ), *where for all* $t \ge 0$

$$h_\theta(t) := H_\theta(X(t)). \tag{3.6}$$

Theorem 3.2. Assume (*A*1)–(*A*4), *fix* $\theta > 0$, *and consider problem* P_θ. *Let* $h_\theta(t)$ *be as in Theorem* 3.1. *Then*

(a) *For all* $v > 0$ *and* $x \in R^n$ *we have*

$$J_\theta(v, x; h_\theta(\cdot)) = \lim_{t \to \infty} (-2/\theta)t^{-1} \ln \mathrm{E}^{h_\theta(\cdot)}[\mathrm{e}^{-(\theta/2)\ln V(t)} | V(0) = v,\ X(0) = x]$$
$$=: \rho(\theta).$$

(b) *The constant* $\rho(\theta)$ *in* (a) *is the unique non-negative constant which is a part of the solution* $(\rho(\theta), v(x;\theta))$ *to the following equation*:

$$\rho = (b + Bx)'\mathrm{grad}_x v(x)$$

$$- (\theta/4) \sum_{i,j=1}^{n} \frac{\partial v(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} + (1/2) \sum_{i,j=1}^{n} \frac{\partial^2 v(x)}{\partial x_i \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk}$$

$$- K_\theta(x),$$

$$v(x) \in C^2(R^n), \qquad \lim_{||x|| \to \infty} v(x) = \infty,$$

$$\rho = \text{const.}, \tag{3.7}$$

where $\mathrm{grad}_x v(x)$ *denotes the gradient of* $v(x)$.

The key point of the first equality in (a) is, of course, that the optimal objective value is given by an ordinary lim rather than the lim inf as in Eq. (3.4). The key point of the second equality in (a) is that the optimal objective value does not depend on either the initial amount of the investor's capital (v) or on the initial values of the underlying economic factors (x), although it depends, of course, on the investor's attitude towards risk (encoded in the value of θ). The key point of (b) is that the optimal objective value is characterized in terms of the Eq. (3.7). The example in the next section illustrates how to solve the system (3.7).

Notice that in the preceding two theorems we had $\theta > 0$. It remains to consider the case corresponding to $\theta = 0$. This is the classical problem of maximizing the portfolio's expected growth rate, that is, the growth rate under the log-utility function (see, e.g. Karatzas, 1996). This problem, which will be labeled P_0, is exactly the same as P_θ for $\theta > 0$ except that now[2]

$$J_0(v, x; h(\cdot)) := \liminf_{t \to \infty} t^{-1} \mathrm{E}^{h(\cdot)} [\ln V(t) | V(0) = v,\ X(0) = x].$$

It turns out that to solve P_0 it is necessary to make three additional assumptions:

Assumption (B1). For each $\theta \geq 0$ *the function* $K_\theta(x)$ (*see Eq.* (3.5)) *is quadratic and of the form*:

$$K_\theta(x) = (1/2)x'K_1(\theta)x + K_2(\theta)x + K_3(\theta),$$

[2] This criterion comes about by setting $\theta = 0$ in Eq. (2.1).

where $K_1(\theta)$, $K_2(\theta)$, and $K_3(\theta)$ are functions of appropriate dimension depending only on θ.

Assumption (B2). For each $\theta \geq 0$ the matrix $K_1(\theta)$ is symmetric and negative definite.

Assumption (B3). The matrix B in Eq. (3.2) *is stable.*

Remark 3.2. (a) Assumption (B1) is satisfied if, e.g. the matrix $\Sigma\Sigma'$ is non-singular and if $\chi = R^n$. As will be seen in the next section, non-singularity of $\Sigma\Sigma'$ is not a necessary condition for (B1) to hold.

(b) It follows from Section 5.5 in Bank et al. (1983) that $\lim_{\theta \downarrow 0} K_i(\theta) = K_i(0)$ for $i = 1, 2, 3$.

(c) Note the Assumption (B2) implies that Assumption (A2) is satisfied.

(d) Assumption (A1) was sufficient in the case $\theta > 0$ in order to provide for appropriate smoothness of the function $K_\theta(x)$. Here, we make a stronger Assumption (B1) and therefore the Assumption (A1) is no longer needed.

Theorem 3.3. Assume (A3), (A4), *and* (B1)–(B3). *Then the optimal strategy for* P_0 *is as in Theorem* 3.1 *with* $\theta = 0$ *and the optimal objective value* $\rho(0)$ *is as in Theorem* 3.2(b) *with* $\theta = 0$. *Moreover, the optimal objective values for problems* (P_θ), $\theta > 0$, *converge to the optimal objective value for the risk null problem* P_0 *when the risk-aversion parameter converges to zero.*

The next result characterizes the portfolio's expected growth rate under the optimal investment strategy for the risk aversion level $\theta > 0$; this will be used in the next section where the motivation behind establishing this result will become apparent. We denote this growth rate by ρ_θ, which is to be distinguished from the optimal objective value $\rho(\theta)$, as in Theorem 3.2.

Theorem 3.4. Assume (A3), (A4), *and* (B1)–(B3), *fix* $\theta > 0$, *let* $H_\theta(x)$ *be as in Theorem* 3.1, *and suppose that* $H_\theta(x)$ *is an affine function and that*

$$\lim_{||x|| \to \infty} [(1/2)H_\theta(x)'\Sigma\Sigma' H_\theta(x) - H_\theta(x)'(a + Ax)] = -\infty. \tag{3.8}$$

Consider the equation

$$\begin{aligned} \rho_\theta &= (b + Bx)' \operatorname{grad}_x v_{\theta,0}(x) + (1/2) \sum_{i,j=1}^{n} \frac{\partial^2 v_{\theta,0}(x)}{\partial x_i \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} \\ &\quad - [(1/2)H_\theta(x)'\Sigma\Sigma' H_\theta(x) - H_\theta(x)'(a + Ax)], \\ & v_{\theta,0}(x) \in C^2(R^n), \qquad \lim_{||x|| \to \infty} v_{\theta,0}(x) = \infty, \\ & \rho_\theta = \text{const.} \end{aligned} \tag{3.9}$$

Then there exists a solution $(\rho_\theta, v_{\theta,0})$ *to the preceding equation, the constant* ρ_θ *is unique, and we have*

$$J_0(v, x; h_\theta(\cdot)) = \rho_\theta \tag{3.10}$$

for all $(v, x) \in (0, \infty) \times R^n$, *where* $h_\theta(\cdot)$ *is defined as in Eq.* (3.6).

4. Example: Asset allocation with Vasicek interest rates

In this section we present a simple example which not only illustrates the ideas developed in the preceding sections, but also is of independent interest in its own right. We study a model of an economy where the mean returns of the stock market are affected by the level of interest rates. Consider a single risky asset, say a stock index, that is governed by the stochastic differential equation

$$\frac{\mathrm{d}S_1(t)}{S_1(t)} = (\mu_1 + \mu_2 r(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W_1(t), \quad S_1(0) = s > 0,$$

where the spot interest rate $r(\cdot)$ is governed by the classical 'Vasicek' process

$$\mathrm{d}r(t) = (b_1 + b_2 r(t))\,\mathrm{d}t + \lambda\,\mathrm{d}W_2(t), \quad r(0) = r > 0.$$

Here μ_1, μ_2, b_1, b_2, σ, and λ are fixed, scalar parameters, to be estimated, while W_1 and W_2 are two independent Brownian motions. We assume $b_1 > 0$ and $b_2 < 0$ in all that follows. We make no assumptions about the signs of μ_1 and μ_2; readers looking for the risk premium to be constant should expect $\mu_2 = 1$, whereas we obtained $\mu_2 < 0$ in our empirical studies reported below.

The investor can take a long or short position in the stock index as well as borrow or lend money, with continuous compounding, at the prevailing interest rate. It is therefore convenient to follow the common approach and introduce the 'bank account' process S_2, where

$$\frac{\mathrm{d}S_2(t)}{S_2(t)} = r(t)\,\mathrm{d}t.$$

Thus $S_2(t)$ represents the time-t value of a savings account when $S_2(0) = 1$ dollar is deposited at time-0. This enables us to formulate the investor's problem as in the preceding sections, for there are $m = 2$ securities S_1 and S_2, there is $n = 1$ factor $X = r$, and we can set $b = b_1$, $B = b_2$, $a = (\mu_1, 0)'$, $A = (\mu_2, 1)'$, $\Lambda = (0, 0, \lambda)'$, and

$$\Sigma = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With only two assets it is convenient to describe the investor's trading strategy in terms of the scalar valued function $H_\theta(r)$, which is interpreted as the proportion of capital invested in the stock index, leaving the proportion $1 - H_\theta(r)$ invested in the bank account. We suppose for simplicity that there are no special restrictions (e.g., short sales constraints, borrowing restrictions, etc.) on the investor's trading strategy, so the investment constraint set χ is taken to be the whole real line.

In view of Theorem 3.1 the optimal trading strategy is easy to work out. With (see Eq. (3.5))

$$K_\theta(r) = \inf_{h \in R} [(1/2)(\theta/2 + 1)(h, 1 - h)\Sigma\Sigma'(h, 1 - h)' - (h, 1 - h)(a + Ar)],$$

it follows that the optimal trading strategy is $h_\theta(t) = [\tilde{h}_\theta(t), 1 - \tilde{h}_\theta(t)]'$, where $\tilde{h}_\theta(t) = H_\theta(r(t))$ and

$$H_\theta(r) = \frac{\mu_1 + \mu_2 r - r}{(\theta/2 + 1)\sigma^2}, \tag{4.1}$$

in which case

$$K_\theta(r) = -r - \frac{(\mu_1 + \mu_2 r - r)^2}{(\theta + 2)\sigma^2}.$$

It is interesting to note the obvious similarity between this optimal strategy and the well known results (see Merton (1971) or Karatzas (1996)) for the case of conventional, complete models of securities markets and power utility functions. In particular, when $\mu_2 = 0$, so the mean returns of the stock market are independent of the interest rates, the expressions for the trading strategies are identical. Another special case of interest is when $\mu_2 = 1$, so that the 'market risk premium' $(\mu_1 + \mu_2 r - r)/\sigma$ is constant. Here the results are somewhat boring, in that H_θ is constant with respect to r and $K_\theta(r)$ is linear in r.

More interesting is the study of $\rho(\theta)$, our measure of performance under the optimal trading strategy (see Theorem 3.2). In view of Eq. (3.7) this is obtained as part of the solution (ρ, v) of the equation

$$\rho = \frac{1}{2}\lambda^2 v''(r) + (b_1 + b_2 r)v'(r) - (\theta/4)\lambda^2(v'(r))^2 - K_\theta(r), \tag{4.2}$$

where v is a unique (up to a constant) function satisfying $\lim_{|r| \to \infty} v(r) = \infty$. To solve this, we conjecture that a solution is obtained with v having the quadratic form

$$v(r) = \alpha r^2 + \beta r + \gamma$$

for suitable constants α, β, and γ. Substituting this and the expression for $K_\theta(r)$ into Eq. (4.2) and then collecting terms, we see that the quadratic terms cancel out if and only if

$$\lambda^2\theta\alpha^2 - 2b_2\alpha - \frac{(\mu_2-1)^2}{(\theta+2)\sigma^2} = 0.$$

This quadratic equation in α has two roots, one of which is positive, while the other is negative. However, the requirement that $\lim_{|r|\to\infty} v(r) = \infty$ is satisfied only for the positive root, so recalling our assumption that $b_2 < 0$ it follows that for the value of α we should take (for future purposes it is convenient to denote the dependence on θ and λ)

$$\alpha(\theta,\lambda) = \frac{b_2 + \sqrt{b_2^2 + \theta\lambda^2(\mu_2-1)^2/[(\theta+2)\sigma^2]}}{\lambda^2\theta}. \tag{4.3}$$

The linear terms on the right-hand side of Eq. (4.2) cancel if and only if the value of β is

$$\beta(\theta,\lambda) = \frac{1 + 2\mu_1(\mu_2-1)/[(\theta+2)\sigma^2] + 2b_1\alpha(\theta,\lambda)}{\sqrt{b_2^2 + \theta\lambda^2(\mu_2-1)^2/[(\theta+2)\sigma^2]}}. \tag{4.4}$$

Thus Eq. (4.2) does indeed have a solution with v as indicated; this solution is unique up to the constant γ, the value of which does not matter. The value of $\rho(\theta,\lambda)$ will then equal the remaining terms on the right-hand side of Eq. (4.2), i.e.

$$\rho(\theta,\lambda) = \lambda^2\alpha(\theta,\lambda) + b_1\beta(\theta,\lambda) - \frac{\lambda^2\theta[\beta(\theta,\lambda)]^2}{4} + \frac{\mu_1^2}{(\theta+2)\sigma^2}. \tag{4.5}$$

Remark 4.1. Note that the above results are valid also in the case when $\mu_2 = 1$. The assumption (A2) is not satisfied in this case since $\lim_{r\to-\infty} K_\theta(r) = \infty$.

It is interesting to consider the risk null case, because here $\rho(0)$ will turn out to be the long-run expected growth rate under the strategy that is optimal when $\theta = 0$. Using L'Hospital's rule we compute the limits

$$\begin{aligned} \alpha(0,\lambda) &= \lim_{\theta\downarrow 0}\alpha(\theta,\lambda) = -\frac{(\mu_2-1)^2}{4b_2\sigma^2},\\ \beta(0,\lambda) &= \lim_{\theta\downarrow 0}\beta(\theta,\lambda) = \frac{b_1(\mu_2-1)^2}{2b_2^2\sigma^2} - \frac{1}{b_2} - \frac{\mu_1(\mu_2-1)}{b_2\sigma^2},\\ \rho(0,\lambda) &= \lim_{\theta\downarrow 0}\rho(\theta,\lambda) = -\frac{b_1}{b_2} + \frac{[\mu_1 - (b_1/b_2)(\mu_2-1)]^2}{2\sigma^2} - \frac{\lambda^2(\mu_2-1)^2}{4b_2\sigma^2}. \end{aligned} \tag{4.6}$$

Note that each of the three terms is non-negative. The Vasicek interest rate has a limiting distribution with a mean equal to the so-called 'mean reversion' level $-b_1/b_2$, which is the first term. The second term equals the contribution to the long-run expected growth rate due to trading in the stock index, assuming the interest rate is the constant mean reversion level. The third term equals the contribution to the long-run expected growth rate due to the volatility of the interest rate.

Another quantity of interest is the long-run expected growth rate which results from using the strategy $h_\theta(t)$ that is optimal for a particular value of θ, a quantity that will be denoted by $\rho_\theta(\lambda)$. Of course, $\rho_0(\lambda) = \rho(0, \lambda)$, which is given by Eq. (4.6), whereas for $\theta > 0$ we use Theorem 3.4 and obtain the quantity $\rho_\theta(\lambda)$ by solving for the constant ρ and the function v such that $\lim_{|r| \to \infty} v(r) = \infty$ and

$$\rho = \frac{1}{2}\lambda^2 v''(r) + (b_1 + b_2 r)v'(r) - [(1/2)(H_\theta(r), 1 - H_\theta(r))\Sigma\Sigma(H_\theta(r),$$
$$1 - H_\theta(r))' - (H_\theta(r), 1 - H_\theta(r))(a + Ar)]. \tag{4.7}$$

We solve Eq. (4.7) in exactly the same way as Eq. (4.2), obtaining

$$\rho_\theta(\lambda) = -\frac{b_1}{b_2} + \frac{2(\theta+1)}{(\theta+2)^2\sigma^2}\left[[\mu_1 - (b_1/b_2)(\mu_2 - 1)]^2 - \frac{\lambda^2(\mu_2 - 1)^2}{2b_2}\right]. \tag{4.8}$$

Note that the second and third terms, respectively, of Eqs. (4.6) and (4.8) differ by the factor $4(\theta+1)/(\theta+2)^2$. This factor is strictly less than one for all $\theta > 0$, so $\rho(0, \lambda) > \rho_\theta(\lambda)$ for all $\theta > 0$. Thus the optimal expected growth rate when $\theta = 0$ is greater than when θ is positive, as anticipated.

Remark 4.2. At this point we want to emphasize one more time that the main advantage of the risk-sensitive approach to dynamic asset allocation over the classical log-utility approach is that the risk-sensitive approach provides an optimal compromise between maximization of the capital expected growth rate and controlling the investment risk, given the investor's attitude towards risk encoded in the value of θ. Even though the long-run expected growth rate of the capital under $h_0(\cdot)$ is greater than under $h_\theta(\cdot)$, if $\theta > 0$, the asymptotic risk of investment decreases with increasing values of θ (see the discussion below, as well as our numerical results that conclude this section).

Still another quantity of interest is $(4/\theta)[\rho_\theta - \rho(\theta)]$ which, by Eq. (2.1) can be interpreted as an estimate of the asymptotic variance of $\ln V(t)$ under the strategy that is optimal for the particular value of θ. In general, this is a messy formula when expressed in terms of the original data; no simplifications seem possible. However, there is interest in computing $\rho'(0, \lambda) := \frac{\partial^+ \rho(\theta, \lambda)}{\partial \theta}\big|_{\theta = 0}$, because when $\theta = 0$ the asymptotic variance under the optimal trading strategy will be $\lim_{\theta \downarrow 0} (4/\theta)[\rho_\theta - \rho(\theta)] = -4\rho'(0, \lambda)$. After lengthy, tedious calculations using

L'Hospital's rule and so forth, we obtained

$$\rho'(0,\lambda) = -\frac{[\mu_1-(b_1/b_2)(\mu_2-1)]^2}{4\sigma^2} + \frac{\lambda^2(\mu_2-1)^2}{32b_2^3\sigma^4}[\lambda^2(\mu_2-1)^2+4b_2^2\sigma^2] - \frac{\lambda^2}{4b_2^2\sigma^4}[\sigma^2+\mu_1(\mu_2-1)-(b_1/b_2)(\mu_2-1)^2]^2.$$

Note that each of the three terms is non-positive, as desired.

Our various calculations can be reconciled with classical continuous time optimal portfolio models (e.g., Merton, 1971; Karatzas, 1996) by considering various limits as the data parameter $\lambda \to 0$. This is because in the long-run when $\lambda = 0$ the interest rate is essentially equal to the constant mean reverting value $-b_1/b_2$, in which case the drift coefficient in the SDE for the stock index is the constant $\mu_1 - \mu_2 b_1/b_2$. Hence, for instance, we have

$$\lim_{\lambda\to 0}\alpha(\theta,\lambda) = -\frac{(\mu_2-1)^2}{2(\theta+2)b_2\sigma^2},$$

$$\lim_{\lambda\to 0}\beta(\theta,\lambda) = -\frac{1}{b_2} - \frac{2\mu_1(\mu_2-1)-(b_1/b_2)(\mu_2-1)^2}{b_2(\theta+2)\sigma^2},$$

$$\lim_{\lambda\to 0}\rho(\theta,\lambda) = -\frac{b_1}{b_2} + \frac{[\mu_1-(b_1/b_2)(\mu_2-1)]^2}{(\theta+2)\sigma^2},$$

$$\lim_{\lambda\to 0}\rho_\theta(\lambda) = -\frac{b_1}{b_2} + \frac{2(\theta+1)[\mu_1-(b_1/b_2)(\mu_2-1)]^2}{(\theta+2)^2\sigma^2}.$$

Note that when $\lambda = 0$ then our optimal strategy is the same as that which emerges from the classical models when the objective is to maximize expected utility of terminal wealth under a power utility function.

We now provide some numerical calculations that are intended to generate some economic intuition about our asset allocation problem. Throughout we envision time units in years and set $b_1 = 0.05$ and $b_2 = -1$, so the mean reverting interest rate is 5% per annum. We also set $\mu_1 = 0.1 + (b_1/b_2)\mu_2$ so that the stock index's drift coefficient is always 0.1 whenever the interest rate is at the mean reverting level. Finally, the volatility parameter for the stock index is always taken to be $\sigma = 0.2$. Thus if the interest rate is fixed at the mean reverting level, then the stock index evolves like ordinary geometric Brownian motion and has a long run expected growth rate equal of 8% per annum and an asymptotic variance equal to 0.04.

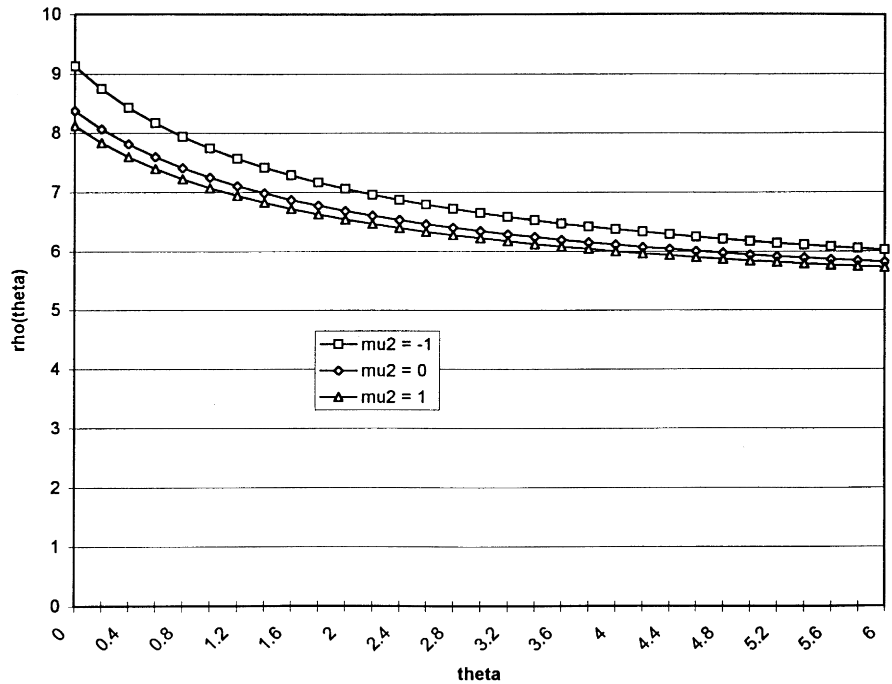

Fig. 1. Rho(theta) for selected mean return sensitivities.

This leaves two unspecified parameters: λ and μ_2. For Figs. 1–3 we fix $\lambda = 0.02$ and consider the effect of the interest rate sensitivity parameter μ_2.

Fig. 1 shows three graphs of the function $\rho(\theta)$ corresponding to, from top to bottom, respectively, $\mu_2 = -1$, $\mu_2 = 0$, and $\mu_2 = 1$. The numerical values are expressed as percentages; the value of θ varies between 0 and 6.0. Although the function $\rho(\theta)$ involves the factor $(\mu_2 - 1)$ raised to the first power, it turns out for our chosen parameters that the value of $\rho(\theta)$ when $\mu_2 = 1 + \delta$ is not much different when $\mu_2 = 1 - \delta$, for all $\delta > 0$ and $\theta > 0$. Hence, roughly speaking, the greater the sensitivity of the stock index risk premium $(\mu_1 + \mu_2 r - r)/\sigma$ to the interest rate, the greater the optimal objective value $\rho(\theta)$.

Fig. 2 shows three graphs of the function ρ_θ corresponding to, from top to bottom, respectively, $\mu_2 = -1$, $\mu_2 = 0$, and $\mu_2 = 1$. It is interesting to compare these values with 8%, the long run expected growth rate of the stock index itself when the interest rate is fixed at the mean reverting level. Note that μ_2 enters the equation for ρ_θ only as part of the factor $(\mu_2 - 1)^2$.

Fig. 3 shows three graphs of the estimated asymptotic variance corresponding to, from top to bottom, respectively, $\mu_2 = -1$, $\mu_2 = 0$, and $\mu_2 = 1$. Plotted is the quantity $(4/\theta)[\rho_\theta - \rho(\theta)]$, with ρ_θ and $\rho(\theta)]$ expressed as percentages. It is

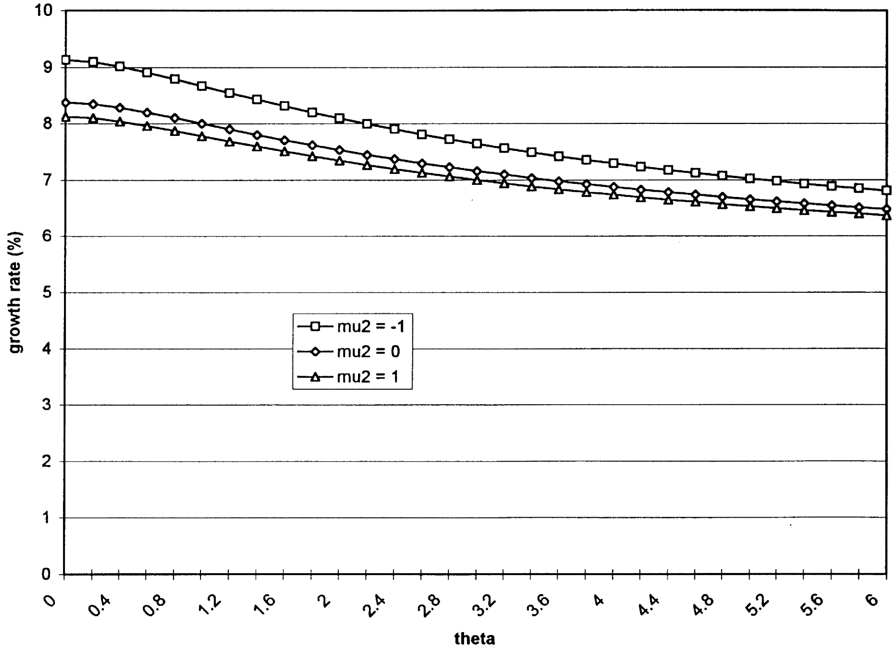

Fig. 2. Growth rate versus theta for selected mean return sensitivities.

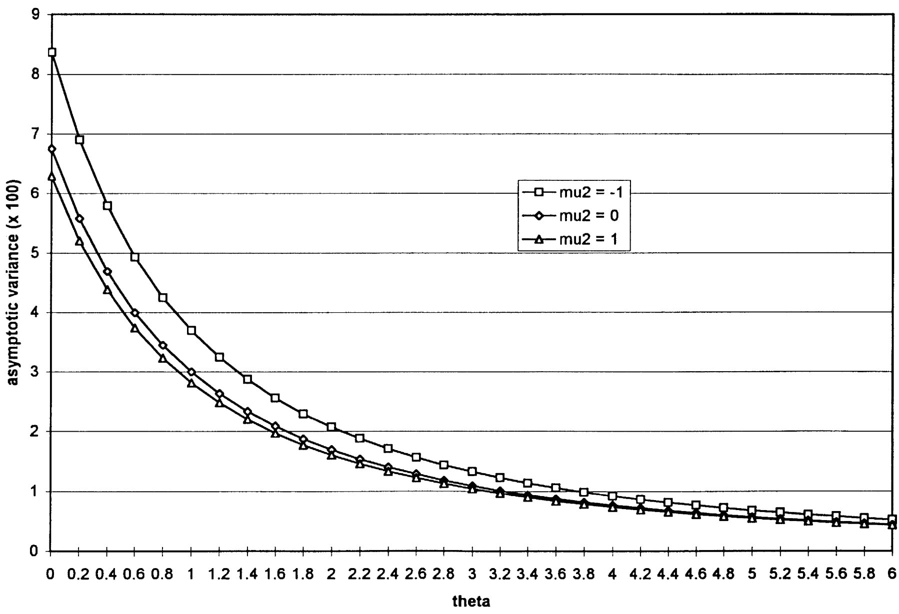

Fig. 3. Asymptotic variance versus theta for selected mean return sensitivities.

interesting to compare these values with 4.0, the asymptotic variance for the stock index itself when the interest rate is fixed at the mean reverting level. As with ρ_θ, the estimated asymptotic variances are more sensitive to the value of $|\mu_2 - 1|$ than to the value of $|\mu_2|$ itself.

We also fixed $\mu_2 = 0$ and studied the effect of the interest rate volatility parameter λ. We generated graphs of $\rho(\theta)$, ρ_θ, and the estimated asymptotic variances, respectively, for three different values of λ: 0, 0.02, and 0.04. The resulting three graphs turned out to be qualitatively almost identical to Figs. 1–3, respectively, except that the curves labeled $\mu_2 = -1,0$, and 1 should be relabeled $\lambda = 0.04$, 0.02, and 0, respectively. In other words, with all three figures, the bigger the value of λ, the bigger the value of the corresponding function. Hence it seems that the greater the volatility of the interest rate, the greater the investment opportunities, although these opportunities will be accompanied by greater volatilities.

5. Experiments with US data

In this section we backtest a two-asset model using data from the asset allocation study by Brennan et al. (1997). They had monthly data from January 1974 to December 1994 for the T-bill short rate, a long term T-bond rate, the monthly returns for an index of US equities, and the dividend yields for the same equity index. We augmented these data with similar numbers from January 1995 through November 1997. Our main objective for this and the following section is to illustrate how to implement our risk sensitive asset allocation model.

A secondary objective for our empirical work is to see if risk sensitive trading strategies do better than more conventional ones, in spite of the fact that our choice of three factors is very poor for the purpose of predicting stock returns. When we regressed next month's returns against the three factors, the R^2 and adjusted R^2 turned out to be only 0.038 and 0.026, respectively. Moreover, although the coefficients for the short rate and the dividend yield were statistically significant at the 95% level, the intercept and the coefficient for the long rate were not significant. On the other hand, Kandel and Stambaugh (1996) used a one-period optimization model for a Bayesian investor to conclude that weak regression results (such as ours) should nevertheless 'exert a substantial influence on the investor's portfolio decision'. If this is true in general, then we should expect our risk sensitive strategies to outperform conventional ones.

We compare four kinds of trading strategies, each of which starts with $1000 in January 1983. The data prior to 1983 were used for some of the strategies to make initial estimates of parameters. First are the constant proportion strategies, where each month the division of wealth between the stock index and cash

Table 1
Constant proportion strategies for US data

Stock proportion	Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
0.0	6.77	0.57	0.00	2,657	0.0
0.1	7.85	1.49	0.72	3,081	3.3
0.2	8.93	2.83	0.76	3,563	5.9
0.3	10.03	4.22	0.77	4,106	7.7
0.4	11.13	5.61	0.78	4,719	8.8
0.5	12.25	7.01	0.78	5,406	9.2
0.6	13.38	8.41	0.79	6,174	8.8
0.7	14.51	9.81	0.79	7,030	7.7
0.8	15.66	11.21	0.79	7,979	5.9
0.9	16.82	12.61	0.80	9,027	3.3
1.0	17.98	14.01	0.80	10,181	0.0
1.1	19.16	15.42	0.80	11,446	4.0
1.2	20.35	16.82	0.81	12,826	8.8
1.3	21.55	18.22	0.81	14,324	14.3
1.4	22.76	19.63	0.81	15,944	20.6
1.5	23.98	21.03	0.82	17,686	27.6
1.6	25.22	22.43	0.82	19,551	35.4
1.7	26.46	23.83	0.83	21,535	43.9
1.8	27.72	25.24	0.83	23,634	53.2
1.9	28.99	26.64	0.83	25,841	63.3
2.0	30.26	28.04	0.84	28,145	74.2

is rebalanced to a specified proportion. Different proportions were evaluated, ranging between 0 and 2.0. Table 1 shows for each proportion the corresponding portfolio's mean annual return, volatility, Sharpe ratio, final (November 1997) dollar value, and average annual turnover. The last measure is the percentage of the portfolio's value that is shifted between assets due to the rebalancing process; it is included in order to give some indication of the possible transaction costs. Fig. 4 shows a graph of the mean annual return versus the volatility for values of the stock proportion ranging from about 0.13 to about 1.50.

The second kind of trading strategies are 0-factor, risk sensitive strategies. These are the strategies resulting from our model when you take the matrix $A = 0$ in Eq. (3.1). In particular, for the purposes of this section, this is the same as taking $\mu_2 = 0$ in the Vasicek model of the preceding section. Consequently, the optimal proportion (4.1) reduces to the well known formula resulting from the portfolio management problem where the investor's objective is to maximize expected iso-elastic utility of wealth at a specified (finite) planning horizon (see Merton (1971), Karatzas (1996), and Eq. (41) of Kandel and Stambaugh (1996); the value of θ depends in a simple way on the risk aversion parameter in the

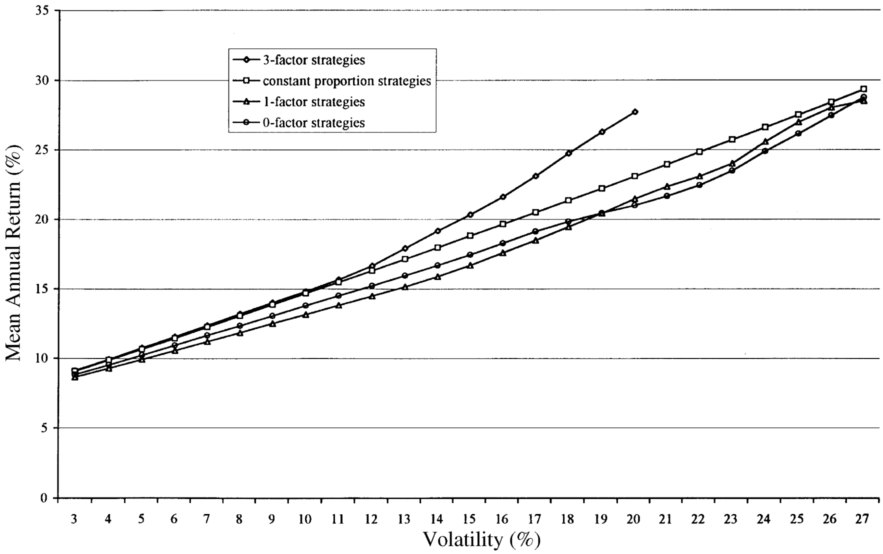

Fig. 4. Mean return versus volatility for US data.

utility function). For this reason, our 0-factor risk sensitive strategies can also be thought of as classical, Merton-type stochastic control strategies.

In our backtesting experiment we are interested in whether the Merton-type strategies fare better than the benchmark constant proportion strategies. If so, this would be evidence for the merits of the risk sensitive control approach (and stochastic control theory approaches, in general), irrespective of any opportunity to use factors for predicting asset returns. This is because the factor levels are ignored for the purpose of estimating the value of the parameters μ_1 and σ. In particular, only the stock index returns prior to 1983 were used for initial estimates of these two parameters. Moreover, each month on a rolling basis the most recently observed return was added to the data set for the purpose of updating the parameter estimates, and then the new portfolio proportions were computed from Eq. (4.1) and implemented. However, in order to facilitate a comparison with the benchmark constant proportion strategies described earlier, we imposed lower and upper bounds on the stock index proportion of 0 and 2.0, respectively. This means that if the proportion coming from expression (4.1) was lower (respectively, higher) than 0 (resp. 2.0), then the proportion actually implemented during the month was 0 (resp. 2.0).

We backtested the 0-factor risk sensitive strategies for different values of θ, as shown in Table 2. Fig. 4 shows a graph of the mean return versus the volatility as θ ranges from about 1 to 23.

Table 2
0-Factor risk sensitive strategies for US data

Theta	Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
0	29.47	27.62	0.82	26,071	89.3
1	28.53	26.81	0.81	24,162	122.5
2	26.45	25.25	0.78	20,082	138.7
3	24.76	23.90	0.75	17,217	138.3
4	22.89	22.57	0.71	14,365	139.3
6	20.29	18.71	0.72	12,027	119.3
8	18.23	15.96	0.72	10,050	104.2
10	16.19	13.33	0.71	8,216	84.6
12	14.80	11.42	0.70	7,113	69.0
15	13.34	9.40	0.70	6,068	54.1
20	11.82	7.26	0.70	5,091	39.6
25	10.87	5.91	0.69	4,541	31.2
30	10.22	4.99	0.69	4,191	25.8
40	9.39	3.81	0.69	3,773	19.1
50	8.88	3.08	0.69	3,533	15.1
75	8.19	2.11	0.68	3,226	10.0

The third kind of strategy is our risk sensitive control strategy where the short rate is the only factor, as in the preceding section. To backtest this kind of strategy we estimated σ as for the 0-factor strategies, but now instead of estimating μ_1 from just the sample of historical returns we estimated both μ_1 and μ_2 by regressing in a rolling, month-to-month fashion the stock index returns of prior months against the T-bill short rate at the beginning of the corresponding months. The parameters μ_1 and μ_2 were taken to be the regression intercept and slope, respectively. This risk sensitive control strategy was backtested for different values of θ, as shown in Table 3. Fig. 4 shows a graph of portfolio mean return versus volatility as θ ranges from about 2.5 to about 50.

Finally, the fourth kind of strategy we backtested is our risk sensitive control strategy where there are three factors: both interest rates and the stock index dividend yield. We included this kind of strategy in spite of the fact that our Assumption (A4) was violated in this case. In particular, although the residuals of the two interest rate factors are virtually uncorrelated with the residuals of the stock index, the residuals of the dividend yield factor are highly correlated with the stock index residuals, just as one would anticipate. Nevertheless, we thought there might be some interesting things to learn by proceeding with the backtest as if Assumption (A4) were realistic.

The backtest was conducted just like the backtest of the one factor, risk sensitive strategies, only now the estimates of μ_1 and μ_2 (the latter now being a 3-component vector) were updated each month by conducting a regression

Table 3
1-Factor risk sensitive strategies for US data

Theta	Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
0	28.86	27.43	0.81	24,492	123.1
1	28.30	26.76	0.80	23,563	151.3
2	28.02	25.97	0.82	23,528	177.0
3	27.14	25.12	0.81	21,912	196.9
4	25.85	24.15	0.79	19,469	199.9
6	23.13	22.05	0.74	15,215	205.3
8	20.49	19.05	0.72	12,232	197.2
10	17.80	16.25	0.68	9,432	175.9
12	15.99	14.16	0.65	7,855	154.2
15	14.27	11.70	0.64	6,600	123.6
20	12.53	9.04	0.64	5,470	92.2
25	11.44	7.37	0.63	4,832	73.5
30	10.70	6.22	0.63	4,426	61.2
40	9.75	4.74	0.63	3,941	45.7
50	9.17	3.84	0.63	3,663	36.5
75	8.39	2.61	0.62	3,309	24.4

Table 4
3-Factor risk sensitive strategies for US data

Theta	Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
0	28.60	20.60	1.06	31,550	484.7
1	27.18	19.64	1.04	27,461	499.7
2	25.55	18.48	1.02	23,375	506.1
3	23.86	17.45	0.98	19,585	490.4
4	22.01	16.34	0.93	16,059	473.2
6	19.55	14.30	0.89	12,403	429.0
8	16.81	12.16	0.83	9,125	379.7
10	15.03	10.28	0.80	7,472	332.5
12	13.88	8.84	0.80	6,564	291.1
15	12.59	7.28	0.80	5,645	240.3
20	11.25	5.63	0.79	4,790	186.2
25	10.41	4.60	0.79	4,311	152.0
30	9.83	3.88	0.79	4,005	128.4
40	9.10	2.97	0.78	3,640	98.0
50	8.64	2.42	0.77	3,430	79.2
75	8.03	1.68	0.75	3,160	53.6

with all three factors as independent variables. This kind of strategy was backtested for different values of θ, as shown in Table 4. Fig. 4 shows a graph of each portfolio's mean return versus its volatility as θ ranges from about 0 to about 50.

In conclusion, it should be clear that it is easy to implement risk sensitive strategies in a practical way. The continuous time theory is readily transformed to a discrete time context by proceeding on a rolling basis to use statistical methods for updating parameters, to use these updated parameters and new factor values to compute new asset proportions, and to rebalance accordingly. On the other hand, it remains unclear from these preliminary experiments whether risk sensitive strategies with underlying factors do better than more conventional strategies, especially when transaction costs are considered. As can be seen from Fig. 4, which ignores transaction costs, the 3-factor strategies did consistently better than the others, by margins which are noteworthy at the higher volatilities. On the other hand, the 0-factor and 1-factor risk sensitive strategies did worse than the benchmark constant proportion strategies at all levels of volatility. The performance of risk sensitive strategies will be discussed further in the following two sections.

6. Experiments with Australian data

In this section we back test some 3-asset, 3-factor risk sensitive strategies with monthly Australian data from January 1981 to July 1997. The three assets are the All Industrials index, the All Resources index (these are indexes of disjoint sets of stocks), and cash. The three factors are the 90 day Australian bank bill rate (which determines the short interest rate for the cash asset), the 10 year Australian Treasury bond rate, and the dividend yield for the All Ordinaries stock index.

We regressed each month's returns of the All Industrials against the month-beginning values of the three factors, giving $R^2 = 0.022$ and only the intercept significant at the 95% level. In a similar regression of the All Resources, $R^2 = 0.047$ and the intercept and two interest rate coefficients were significant at the 95% level. Thus our ability to forecast returns of Australian stocks is comparable to what it was for the US market.

We also conducted an (approximate) asymptotic test for the presence of sample correlations between the residuals of the asset returns and the factor changes. We thereby could not reject the null hypothesis of no correlation between the short interest rate changes and each of the two index residuals, so our Assumption (A4) is satisfied if the short rate is the only factor. However, the correlations between the dividend yield changes and each of the two index residuals are significantly negative, while the correlations between the 10 year bond yield changes and the index residuals are negative at a much lower level of significance. Hence our tests of strategies involving these last two factors must be viewed with caution.

In a fashion similar to what we did with our US data (see the preceding section), we compared three kinds of strategies by doing backtests assuming the

Table 5
Constant proportion efficient frontier for Australia (proportions ≥ 0)

Proportion (%)			Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
All industrials	All resources	Cash					
100	0	0	18.9	16.19	0.45	7,431	0.0
98.8	0	1.2	18.81	16	0.45	7,393	0.5
92.6	0	7.4	18.34	15	0.45	7,185	2.7
86.4	0	13.6	17.88	14	0.45	6,972	4.7
80.2	0	19.8	17.41	13	0.45	6,754	6.3
73.9	0	26.1	16.94	12	0.45	6,533	7.6
67.7	0	32.3	16.48	11	0.45	6,310	8.7
61.5	0	38.5	16.02	10	0.45	6,085	9.4
55.2	0	44.8	15.56	9	0.45	5,858	9.8
48.9	0	51.1	15.09	8	0.44	5,632	9.9
42.6	0	57.4	14.63	7	0.44	5,406	9.6
36.3	0	63.7	14.17	6	0.44	5,181	9.1
29.9	0	70.1	13.71	5	0.43	4,956	8.2
23.4	0	76.6	13.23	4	0.42	4,731	7.0
16.7	0	83.3	12.75	3	0.40	4,502	5.5
9.3	0	90.7	12.22	2	0.33	4,254	3.3
0	0	100	11.55	1.33	0.00	3,952	0.0

portfolios all start with $1000 in January 1985, using the data prior to 1985 for initial estimates of the parameters, as necessary. We also considered two sets of constraints on the portfolio proportions: (1) all three proportions are non-negative (i.e., no short sales or borrowing) and (2) all three proportions are greater than or equal to one (so, e.g. starting with a dollar you can borrow a dollar, go short one dollar in one index, and go long three dollars in the other).

For a benchmark we computed the efficient frontier with respect to all feasible constant proportion strategies with monthly rebalancing. Of course this is a very tough benchmark, because a priori the investor will not know what proportions will give rise to an outcome on the efficient frontier. Nevertheless, we computed for selected volatilities the feasible portfolio proportions that maximize the mean annual return. The results for proportions ≥ 0 are given in Tables 5–7 and for ≥ 1 in Tables 8–10. Tables 5 and 8 show the results for the two sets of constraints. Note that for the case of nonnegative proportions, the efficient frontier involves putting nothing in the All Resources, with all the funds divided between cash and the All Industrials.

The second kind of strategies are the 0-factor risk sensitive strategies. These correspond to setting the matrix $A = 0$ in Eq. (3.1), thereby giving rise to a geometric Brownian motion model for the pair of stock indexes. Just like

Table 6
0-Factor risk sensitive strategies for Australia (proportions ≥ 0)

Theta	Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
0	18.12	15.68	0.42	6,905	56.9
1	18.76	15.23	0.48	7,453	54.9
2	18.74	14.89	0.49	7,488	56.0
3	18.59	14.55	0.49	7,412	56.9
4	18.40	14.27	0.48	7,302	57.1
6	18.31	13.89	0.49	7,273	54.8
8	18.08	13.63	0.48	7,134	52.9
10	17.82	13.39	0.47	6,966	49.4
12	17.63	13.22	0.46	6,844	44.4
15	17.61	12.30	0.50	6,961	41.3
20	17.10	10.79	0.52	6,764	43.7
25	16.14	9.48	0.49	6,203	41.0
30	15.36	8.13	0.48	5,788	34.4
40	14.44	6.28	0.47	5,325	25.5
50	13.88	5.15	0.46	5,047	20.2
75	13.12	3.65	0.45	4,677	13.4

Table 7
3-Factor risk sensitive strategies for Australia (proportions ≥ 0)

Theta	Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
0	17.82	16.35	0.39	6,588	198.1
1	17.54	16.12	0.38	6,426	205.4
2	17.65	15.52	0.40	6,593	211.7
3	18.28	14.39	0.47	7,244	207.3
4	18.73	13.62	0.53	7,714	204.2
6	19.15	12.58	0.61	8,217	205.1
8	19.24	11.95	0.65	8,381	205.7
10	19.27	11.45	0.68	8,473	203.7
12	19.17	11.07	0.69	8,425	196.8
15	19.10	10.48	0.73	8,423	180.4
20	18.70	9.76	0.74	8,153	162.4
25	17.98	9.16	0.71	7,605	149.9
30	17.22	8.46	0.68	7,061	139.5
40	15.85	7.39	0.59	6,153	119.5
50	14.90	6.31	0.54	5,598	101.6
75	13.81	4.38	0.53	5,033	68.7

the 0-factor strategies that were described in the preceding section, the parameters Σ and a here are estimated from the monthly returns by computing simple sample means, variances, and covariances, and they are updated monthly on a rolling basis. Then each month the updated parameters are substituted into

Table 8
Constant proportion efficient frontier for Australia (proportions ≥ -1)

Proportion (%)			Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
All industrials	All resources	Cash					
300	− 100	− 100	30.1	36.76	0.50	10,813	182.6
289.2	− 89.2	− 100	29.73	36	0.51	10,742	168.2
269.8	− 78.4	− 91.4	28.66	34	0.50	10,702	141.0
254	− 73.4	− 80.6	27.59	32	0.50	10,652	119.2
238.1	− 69.2	− 68.9	26.53	30	0.50	10,513	99.4
222.2	− 64.6	− 57.6	25.47	28	0.50	10,292	81.7
206.4	− 60.1	− 46.3	24.43	26	0.50	10,001	66.0
190.4	− 55.4	− 35	23.39	24	0.49	9,649	52.0
174.6	− 50.9	− 23.7	22.36	22	0.49	9,247	40.0
158.7	− 46.2	− 12.5	21.33	20	0.49	8,805	30.0
142.8	− 41.6	− 1.2	20.32	18	0.49	8,331	22.3
126.8	− 37	10.2	19.31	16	0.49	7,836	20.6
110.9	− 32.4	21.5	18.31	14	0.48	7,327	19.8
95	− 27.8	32.8	17.31	12	0.48	6,812	19.2
79	− 23.2	44.2	16.32	10	0.48	6,297	18.4
63	− 18.5	55.5	15.34	8	0.47	5,787	16.9
46.8	− 13.8	67	14.35	6	0.47	5,287	14.4
30.3	− 9	78.7	13.35	4	0.45	4,794	10.6
12.3	− 3.8	91.5	12.27	2	0.36	4,279	4.9

Table 9
0-Factor risk sensitive strategies for Australia (proportions ≥ -1)

Theta	Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
0	27.18	33.78	0.46	9,229	312.0
1	26.87	32.38	0.47	9,469	291.1
2	26.39	31.61	0.47	9,294	273.3
3	25.54	30.92	0.45	8,747	270.5
4	25.53	30.20	0.46	8,918	248.0
6	24.11	29.16	0.43	7,954	212.4
8	23.62	26.53	0.46	8,852	182.9
10	23.09	24.62	0.47	9,153	166.5
12	22.53	22.65	0.49	9,256	149.2
15	21.12	19.53	0.49	8,761	118.8
20	18.95	15.46	0.48	7,648	88.6
25	17.56	12.70	0.48	6,932	70.3
30	16.60	10.76	0.47	6,433	57.1
40	15.38	8.27	0.47	5,803	41.9
50	14.64	6.75	0.47	5,424	33.3
75	13.63	4.70	0.46	4,922	22.1

Table 10
3-Factor risk sensitive strategies for Australia (proportions ≥ -1)

Theta	Mean return (%)	Volatility (%)	Sharpe ratio	Final wealth ($)	Annual turnover (%)
0	30.19	36.55	0.51	11,267	937.5
1	32.75	32.95	0.65	17,941	910.7
2	32.47	30.43	0.69	19,384	868.2
3	32.47	28.62	0.73	20,759	822.3
4	32.24	27.13	0.76	21,416	784.1
6	31.64	24.95	0.81	21,693	689.7
8	30.53	23.10	0.82	20,584	601.3
10	29.59	21.66	0.84	19,541	542.3
12	28.07	20.43	0.81	17,365	493.0
15	25.50	18.60	0.75	14,029	444.7
20	21.93	15.99	0.65	10,284	368.6
25	19.83	13.39	0.62	8,683	305.5
30	18.51	11.32	0.62	7,811	256.1
40	16.82	8.68	0.61	6,745	192.7
50	15.79	7.05	0.61	6,132	154.5
75	14.40	4.87	0.60	5,353	103.2

the right-hand side of Eq. (3.5) in order to compute the portfolio proportions for the coming month. The portfolio is rebalanced accordingly. These 0-factor risk sensitive strategies were backtested for selected values of θ and the two sets of constraints. The results are presented in Tables 6 and 9.

The third and final kind of strategies we backtested are 3-factor risk sensitive strategies. The variances and covariances were estimated in the same way as was done for the 0-factor strategies, but the a vector and the A matrix in Eq. (3.1) were updated monthly by doing regressions. Again, the 3-factor proportions were implemented monthly after computing the minimizing selector in Eq. (3.5). Thus our backtests of 3-factor strategies with Australian data were conducted in the same general way that was done in the experiments with US data, as described in the preceding section. The results for selected values of the risk aversion parameter θ and for the two sets of constraints are displayed in Tables 7 and 10.

In order to compare the three kinds of strategies, we constructed graphs showing, for each kind of strategy, how the portfolio's mean annual return varied with respect to its volatility. Thus Fig. 5 shows three graphs, all for the case where the portfolio proportions are required to be nonnegative. Fig. 6 shows a similar picture for the case where each proportion is only required to be greater than or equal to minus one (i.e., -100%). In both cases the 3-factor risk sensitive strategies generally perform better than the 0-factor

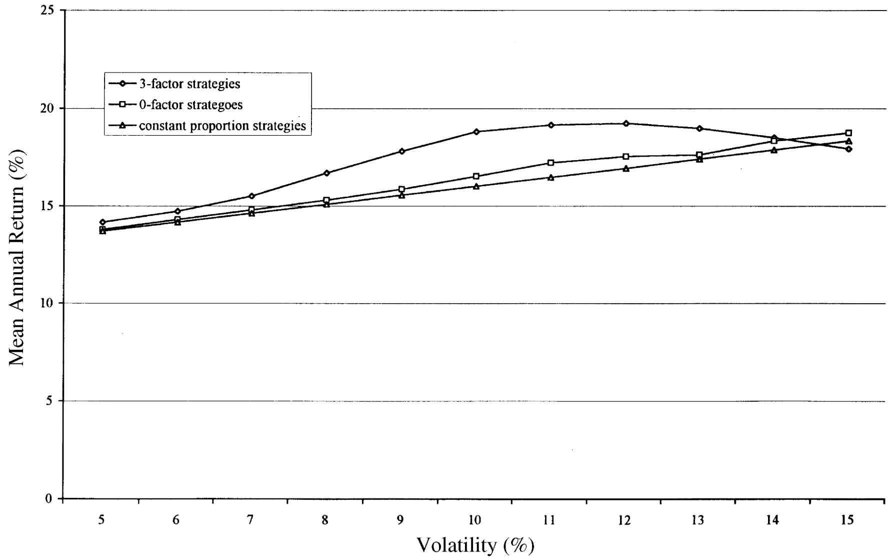

Fig. 5. Mean return versus volatility for Australian data (nonnegative proportions).

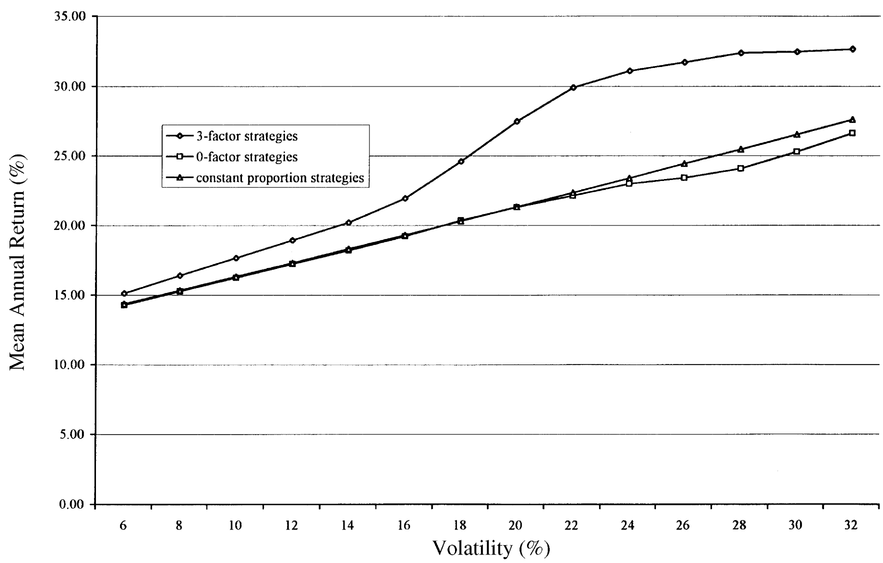

Fig. 6. Mean return versus volatility for Australian data (proportions ≥ -1).

strategies, and these, in turn, generally perform just as well as the benchmark strategies on the constant proportion efficient frontier. Of course, these comparisons ignore any transaction costs.

7. Discussion, final remarks, and future research

In this paper we have shown how a risk sensitive criteria can be applied to optimal asset allocation decision making. Expected returns are assumed to be driven by specific factors, in this case interest rates and dividend yields. The approach developed takes advantage of any predictability in expected returns arising from these factors. Expected returns on the asset classes are time varying. The optimal strategy is then determined using rolling regressions to estimate the dynamics for asset returns. The risk sensitive criterion is based on infinite horizon measures of mean return and variance. In contrast to approaches to asset allocation using factor models currently in the literature, the risk sensitive approach readily allows the use of any number of factors and asset classes in the asset allocation optimisation.

The results from applying the approach to both USA and Australian data in backtests are very encouraging. Using the mean-variance efficient frontier as a basis for comparison of various strategies shows that dynamic strategies outperform constant proportion strategies and using factor models can provide further enhancement in performance. Dynamic strategies lead to higher turnover and potentially higher transaction costs. This requires further investigation but use of futures rather than spot markets for physical assets, especially at the asset class level, will mean that transaction costs will be lower than otherwise.

Throughout this paper we have been maintaining the assumption that residuals of the price processes and the residuals of the factor processes are not correlated. In terms of our model of Section 3, this assumption amounts to requiring that $\Sigma\Lambda' = 0$. This assumption may or may not be reasonable, depending upon the circumstances. For instance, based on the data that we collected for our empirical studies, with equities for assets this assumption is very reasonable if the short interest rate is the only factor, it is marginally reasonable if a long-term interest rate is added as a factor, but it is clearly unreasonable if dividend yield is added as a factor.

However, it is important to keep in mind that our assumption is *not* the same as the assertion that changes in factor levels are uncorrelated with asset returns. This can be clearly seen by considering our Vasicek example. Suppose, for instance, that the short rate is below the mean reversion level, a situation that is bullish for the stock. Then over a coming time interval the stock's return is likely to be above average and the interest rate is likely to increase, and so our model will demonstrate a positive correlation between stock returns and interest rate movements. This positive correlation exists in spite of our $\Sigma\Lambda' = 0$ assumption being satisfied.

Nevertheless, it must be admitted it is very desirable to extend our present model to cases where our $\Sigma\Lambda' = 0$ assumption is not satisfied. Presently our trading strategies do not account for interactions (correlations) between the randomness underlying the price processes and the randomness underlying the factor process, such as interactions one would find between the short rate factor and bonds with medium or long maturities. In general, if such interactions are not negligible, then they will need to be accounted for in the optimal trading strategy via the so-called *hedging term*.

We shall briefly illustrate this point by slightly extending the simple asset allocation model of Section 4. Specifically, consider the following extension of this model:

$$\frac{dS_1(t)}{S_1(t)} = (\mu_1 + \mu_2 r(t))\,dt + \sigma_1\,dW_1(t) + \sigma_2\,dW_2(t), \quad S_1(0) = s > 0,$$

$$dr(t) = (b_1 + b_2 r(t))\,dt + \lambda_1\,dW_1(t) + \lambda_2\,dW_2(t), \quad r(0) = r > 0.$$

We conjecture (this will be studied in a future paper) that for this model the optimal trading strategy is $h_\theta(t) = [\tilde{h}_\theta(t), 1 - \tilde{h}_\theta(t)]'$, where $\tilde{h}_\theta(t) = H_\theta(r(t))$ and

$$H_\theta(r) = \frac{\mu_1 + \mu_2 r - r}{(\theta/2 + 1)\sigma^2} + (\theta/2)\frac{\phi'(r)(\lambda_1\sigma_1 + \lambda_2\sigma_2)}{(\theta/2 + 1)\sigma^2},$$

and where the function $\phi(r)$ is derived from an equation similar to (but more complicated than) Eq. (4.2). The second term in the above expression for $H_\theta(r)$ hedges against the interactions between randomness underlying the price formation block of the model, and the randomness underlying the factor dynamics. Note that in the model of Section 4 we postulated $\sigma_1 = \sigma$, $\sigma_2 = 0$, $\lambda_1 = 0$, and $\lambda_2 = \lambda$; this implies $\lambda_1\sigma_1 + \lambda_2\sigma_2 = 0$, thereby eliminating the hedging term from the formula for the optimal strategy. A challenging problem for future research is to develop a model along these lines where interest rates of different maturities are the factors and where there are fixed income asset categories corresponding to these same maturities.

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